Effects of aberration on paraxial wave beams: beam tracing versus quasi-optical solutions

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Abstract. This paper aims to clarify the role of aberration effects on the propagation and absorption of wave beams in inhomogeneous dispersive and dissipative media. We consider models in which aberration effects can be caused by the presence of either caustics or spatially dispersive absorption, with reference to the propagation near a cut-off or to the electron cyclotron (EC) resonance, respectively. For such models, the standard beam tracing description of paraxial wave beams and the recently proposed quasi-optical method, which accounts for aberration, are compared and verified on the basis of the analytical exact solutions. We find that the presence of a cut-off implies no significant aberration of the beam, while significant aberration is found when dispersive absorption is so strong that different wavenumbers in the beam spectrum are damped at different locations. This phenomenon is well described by the quasi-optical method. At last, an extrapolation of this simple two-dimensional model to the case of the ITER upper EC port is addressed with the result that the broadening of the power deposition profiles never exceeds 10%.

1. Introduction

An accurate description of propagation and absorption of wave beams in fusion plasmas is a difficult multi-scales problem which requires significant computational resources; therefore, when fast or real-time calculations are needed, one relies on asymptotic solutions of the (integro-differential) equation that describes the considered wave. More specifically, for electromagnetic wave beams of fixed frequency ω that propagate in an inhomogeneous (stationary) medium, the dimensionless parameter $\kappa = \omega L/c = k_0 L$ is large, $\kappa \gg 1$, with L being the scale length of the medium spatial variations and c the speed of light in free space. Hence, one can construct asymptotic solutions of the relevant equation in the high-frequency limit $\kappa \rightarrow +\infty$; when κ is finite but large such asymptotic solutions give good approximations of the exact wave field.

For narrow and/or focussed wave beams, however, in addition to the wave length ω/c and the inhomogeneity scale L , the beam width W should be considered: When $W \ll L$ diffraction effects set in and must be dealt with in the asymptotics.

One of the most convenient and powerful methods for the description of diffracting wave beams in fusion plasmas is the beam tracing (or paraxial WKB) method developed by Pereverzev [1, 2] and successively implemented for both electron cyclotron waves [3] and lower-hybrid waves [4]. The main idea at the basis of the beam tracing method comes from the case of a focused beam [5]: the new scale W is assumed to satisfy the ordering $W^2 \sim L\lambda$ which means $W/L \sim 1/\sqrt{\kappa}$, thus, the introduction of the novel scale length corresponds to half-integer powers of the parameter κ ; this is essentially the paraxial approximation.

In its generality, the beam tracing method [1, 2] allows us to approximate the relevant equation for the wave field by an eigenvalue problem for a simpler second-order partial differential operator, the eigenfunctions of which are just the Hermite-Gaussian modes. This procedure has been developed for a class of scalar partial differential equations [1], as well as for electromagnetic waves in spatially non-dispersive media (cold plasmas) [2]; more recently, however, it has been shown that spatial dispersion can be accounted for as well [6]. On the other hand, up to now, the construction of the beam tracing solution can be rigorously justified only under the assumption of weakly non-Hermitian media, i.e., the anti-Hermitian part of the dielectric tensor, which is responsible for wave energy dissipation, should be small enough, precisely, $O(\kappa^{-1})$. This condition can be violated in fusion plasmas, e.g., near the electron-cyclotron resonance layer.

Another powerful description of diffractive wave beams can be achieved by means of the complex geometrical optics method [7]. The eikonal based form of complex geometrical optics, in particular, can be treated in terms of extended rays [8, 9] and such a technique has been successfully implemented for electron cyclotron waves [10, 11]. In the considered regime, however, beam tracing and extended rays yield the same asymptotic solutions and only the former is addressed, here.

Recently, Balakin *et al* [12, 13] have advanced arguments against the validity of the beam tracing solution, claiming that *aberration effects* can significantly alter both the propagation and the absorption of wave beams. Here, the term “aberration” refers to any deviation of the beam from the paraxial approximation upon which the beam tracing method relies; it is worth noting that such a definition does not include astigmatism, which is considered an aberration in optics. In principle, aberration of the beam can be caused, e.g., by strong inhomogeneities, by the formation of caustics, by spatial dispersion or by strong absorption as well as by the concurrence of such effects. Furthermore, Balakin *et al* proposed an alternative quasi-optical method for the description of wave beams under such critical conditions. The quasi-optical method generalizes the ideas of the “parabolic wave equation” method [14, 15, 16] by accounting for terms in the wave equation that are usually neglected in the paraxial approximation; spatial dispersion and strong absorption are also retained. Such a procedure shows significant aberration effects in the resonant absorption of EC wave beams in hot plasmas: In this case aberration is caused by the large anti-Hermitian part of the dielectric tensor which depends on the wave vector (dispersive absorption) through the

Doppler shift.

In the aberration-free limit and in absence of absorption, one can prove that the beam tracing method and the quasi-optical method should give the same result [12, Section 6], within the accuracy of the solution. For sake of completeness, we shall distinguish two versions of the quasi-optical method: the aberration-free (AF) quasi-optics [12] and the full quasi-optics (QO) [13]. The application of such new methods to electron-cyclotron wave beams in fusion plasmas yields a significant broadening of the power deposition profiles with possible consequences on the stabilization efficiency of MHD modes [13]. These effects have not been detected by any of the other main codes available for the description of electron cyclotron waves [17], and not even by the recent method of virtual beams [18] which can account for aberration effects, at least to some extent (near the resonance layer, formal applicability conditions of geometrical optics, which is used to trace each virtual beam, break down). Indeed, the results of Ref. [18] are in good agreement with those of Ref. [17], which suggests negligible aberration effects; nonetheless, the deposition profiles obtained by the virtual beams method exhibit in some cases non-Gaussian tails, but significantly smaller than in quasi-optical power deposition profiles.

The discussion on the relevance of aberration effects triggered a recent analysis of the beam tracing solutions which has been compared to the corresponding exact solutions for two simplified models of plasmas [19]. The first model is the classical linear layer problem [20], which is the paradigm for the reflection of a beam at a cut-off as relevant, e.g., to reflectometry. The second case is the absorbing half-plane model, which is a simplified model for the electron-cyclotron resonance layer in which the absorption is strong and spatially inhomogeneous so that one side of a Gaussian beam cross-section is damped before the main part of the beam; no spatial dispersion was considered. The results of such a verification do not support the idea of significant aberration effects, showing, in particular, that the beam tracing calculations give accurate predictions of physically relevant quantities, i.e., position and width of the caustic for the linear layer model, or power deposition profiles for the absorbing half-plane model; sources of major errors are encountered for very critical beams only, mainly because of high curvature of the beam trajectory in the case of the linear layer, or because of power losses due to reflection at the vacuum-medium interface in the case of the absorbing half-plane model; such effects, however, have nothing to do with aberration. For the specific case of the absorbing half-plane model, it was found that the spatially asymmetric damping of one tail of the beam corresponds to deformations of the Gaussian shape below 1% of the maximum amplitude [19, Figure 7b].

The analysis of Ref.[19], however, does not address the issue of which physical effect can cause a significant aberration in the beam: the only conclusion of that work in this respect was that strong spatial inhomogeneity does not imply aberration. The important case of spectral inhomogeneity (spatial dispersion) was not addressed. Moreover, in the previous work [19] only the beam tracing solution has been addressed, but neither a direct analysis of aberration effects nor a comparison between the beam tracing and

quasi-optical solution is attempted.

In this paper, such a comparative analysis is carried out for models in which aberration effects are expected to play an important role. Furthermore, both the beam tracing and the quasi-optical solutions are checked on the basis of exact solutions. We consider the same linear layer problem addressed in Ref.[19], but with different launching conditions chosen in order to emphasize the effects of the caustic near the cut-off. Spatial dispersion is introduced in the absorbing half-plane model, in such a way that each wavenumber composing the spectrum of the beam “feels” a different position of the absorbing layer; in addition, for each wavenumber the profile of the absorption coefficient is assumed to grow linearly with the distance from the interface [21]. This model of dispersive absorption includes the effects of both spatial and spectral inhomogeneity of absorption [22]. Models are discussed in section 2 whilst the corresponding solutions are addressed in section 3; the reader who is not interested in the details of the derivation of analytical solutions can skip to section 4 in which the results of our analysis are presented. Specifically, we show that aberration effects due to caustics in the linear layer model are negligible, whereas, for the absorption problem, spatial dispersion combined with strong absorption can produce significant aberration effects. In the latter case we find a broadening of the quasi-optical power deposition profiles projected in the direction normal to the resonance layer, and such a broadening is in agreement with the exact solution. Nonetheless, the parameters of the model are tuned in order to emphasize these effects. In section 5 a more realistic model of absorption coefficient, inferred from the case of injection from the ITER upper EC port [23, 24], is adopted in order to estimate quantitatively such effects. It is found that the broadening of the power deposition profiles never exceeds 10% for the considered cases.

2. Models

In (dimensionless) Cartesian coordinates (x, y, z) , normalized to the inhomogeneity scale L , and for a beam of fixed frequency ω in an isotropic stationary medium homogeneous in (y, z) , we assume that the wave electric field $\mathbf{E}(\mathbf{r}, \omega)$ is constant in z and polarized along the the homogeneity direction $\mathbf{e}_z$, namely, $\mathbf{E}(\mathbf{r}, \omega) = u(x, y)\mathbf{e}_z$; then, the wave equation takes the scalar form,

$$\Delta u(x, y) + \kappa^2 \mathcal{E}u(x, y) = 0, \quad (1)$$

where $\kappa = \omega L/c$, $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator in two-dimensions, and $\mathcal{E}$ is an integral operator which, in general, accounts for the non-local response of the medium, i.e., spatial dispersion.

We assume that the medium is dispersive in the y direction only so that

$$\mathcal{E}u(x, y) = \frac{\kappa}{2\pi} \int_{-\infty}^{+\infty} e^{i\kappa y p_y} \varepsilon(x, p_y) \hat{u}(x, p_y) dp_y, \quad (2)$$

where $\varepsilon(x, p_y)$ is the dielectric function of the medium and $\hat{u}$ denotes the Fourier transform,

$$\hat{u}(x, p_y) = \int_{-\infty}^{+\infty} u(x, y) e^{-i\kappa y p_y} dy, \quad (3)$$

properly normalized so that

$$u(x, y) = \frac{\kappa}{2\pi} \int_{-\infty}^{+\infty} e^{i\kappa y p_y} \hat{u}(x, p_y) dp_y. \quad (4)$$

Here, $p_y = \frac{\omega}{c} k_y = N_y$ is the momentum conjugate to the coordinate y , and it amounts to the y -component of the refractive index vector $\mathbf{N} = \frac{\omega}{c} \mathbf{k}$; therefore, the dielectric function $\varepsilon(x, p_y)$ fully describes the properties of the medium: inhomogeneity (x -dependence) and dispersion (p_y -dependence).

The two models we shall consider correspond to the following choices of the dielectric function.

(a) The *linear layer model*,

$$\varepsilon(x, p_y) = n^2(x) = 1 - x, \quad (5a)$$

describes a lossless non-dispersive medium (ε is real-valued and does not depend on p_y). This is the classical model of propagation near a cut-off which has been extensively studied [19, 20].

(b) The *dispersive absorbing half-plane*,

$$\varepsilon(x, p_y) = 1 + i\gamma(x, p_y), \quad \gamma(x, p_y) = \gamma_1 \begin{cases} 0, & \text{for } x \leq a(p_y), \\ x - a(p_y), & \text{for } x > a(p_y), \end{cases} \quad (5b)$$

where $a(p_y) = -p_y^2$. The real part of the dielectric function (5b) corresponds to free-space propagation, whereas the imaginary part, that accounts for dissipation, is expressed as a piecewise-defined function of x with a p_y -dependent boundary: each wavenumber in (4) starts to be absorbed at the position $x = a(p_y)$, wherefrom $\text{Im}\varepsilon(x, p_y) \geq 0$. This models the EC resonance layer in hot plasmas, with the y direction identified with the magnetic field direction so that $p_y \equiv N_{\parallel}$ is the parallel refractive index which enters the expression of the EC absorption coefficient through the Doppler shift.

On one hand, the linear layer model (a) allows us to address the possible aberration of the beam due to the formation of a fold caustic where the beam is reflected from the cut-off. On the other hand, the dispersive absorption model (b) allows us to study the combined effects of spatial and spectral inhomogeneities as discussed in the introduction. It is worth noting that both cases should be regarded as very simple models tailored to the specific effect we are addressing, e.g., Kramers-Kronig relations are not satisfied.

3. Exact and approximate solutions

In this section, the exact solution for the linear layer (section 3.1) and the dispersive absorbing half-plane (section 3.2) are discussed along with the corresponding beam tracing and quasi-optical approximations.

As a premise, we note that, for both models of section 2, the local dispersion function is given by

$$H(x, p_x, p_y) = p_x^2 + p_y^2 - \text{Re}[\varepsilon(x, p_y)], \quad (6)$$

and it plays a crucial role in the general semiclassical analysis of equation (1) and, more specifically, in the construction of the beam tracing solution.

3.1. Solution of the linear layer model and its quasi-optical approximations

The linear profile of the dielectric function (5a) models the propagation of a beam near the cut-off located at $x = 1$. This is relevant, for instance, to the description of reflectometry diagnostic in fusion plasmas [19].

Exact solution. The exact solution of the linear layer problem has been extensively studied in the classical book by Ginzburg [20] and in the recent paper [19]. On making use of the Fourier representation (3) the exact solution amounts to [19]

$$u^{Ex}(x, y) = \frac{\kappa}{2\pi} \int_{-\infty}^{+\infty} e^{i\kappa y p_y} \text{Ai}(-\kappa^{2/3}(1 - x - p_y^2)) f(p_y) dp_y, \quad (7)$$

where $\text{Ai}(\zeta)$ denotes the Airy function [25] and

$$f(p_y) = \begin{cases} 2\sqrt{\pi}\kappa^{1/6}(1 - p_y^2)^{1/4} e^{i\frac{2}{3}(1 - p_y^2)^{3/2} - i\pi/4} U(p_y), & \text{for } p_y^2 \leq 1, \\ 0, & \text{for } p_y^2 > 1, \end{cases}$$

$U(p_y)$ being the Fourier transform of the launched field which depends on the physical launching conditions for the beam; the appropriate form of the launched spectrum $U(p_y)$ will be specified in section 4.

Beam tracing. The corresponding beam tracing solution for a Gaussian beam (fundamental mode) has been fully worked out in Ref. [19]. The only novel issue here concerns the launching conditions that are such that the beam travels a longer distance before it finally undergoes reflection from the cut-off. The trajectory of the beam, called *reference ray*, is the solution to the Hamilton's equations with Hamiltonian (6) which for the case under consideration reads $H(x, p_x, p_y) = p_x^2 + p_y^2 + x - 1$; explicitly, that is

$$\begin{aligned} x(\tau) &= x_0 + 2p_{x0}\tau - \tau^2, & p_x(\tau) &= p_{x0} - \tau, \\ y(\tau) &= y_0 + 2p_{y0}\tau, & p_y(\tau) &= p_{y0}, \end{aligned} \quad (8)$$

where (x_0, y_0) and (p_{x0}, p_{y0}) are the initial position and momentum of the reference ray. The curve $(x(\tau), y(\tau))$ amounts to a parabola with the symmetry axis parallel to the x axis. We assume that the beam is launched before the cut-off, i.e., $x_0 < 1$, and we set $y_0 = -2p_{x0}p_{y0}$ so that the parabola of the reference ray is symmetric with respect to the x axis. Furthermore, let us set $p_{x0} = \sqrt{1 - x_0} \sin \theta$, $p_{y0} = -\sqrt{1 - x_0} \cos \theta$ so that

the local dispersion relation $H = 0$ is satisfied for $\tau = 0$ and, thus, for $\tau \geq 0$. The parameter θ is the injection angle of the beam onto the cut-off line.

Quasi-optics. In contrast to the beam tracing solution the quasi-optical description of the beam is obtained by solving numerically an evolution equation derived from equations (1)-(5a).

One considers a system of curvilinear coordinates (τ, ξ) around the reference ray (8) defined by $x(\tau, \xi) = x(\tau) + g_x(\tau)\xi$, $y(\tau, \xi) = y(\tau) + g_y(\tau)\xi$, here, the vector $(g_x(\tau), g_y(\tau))$ is uniquely determined as the unit vector normal to the reference ray, *viz.*,

$$(g_x(\tau), g_y(\tau)) = (p_y(\tau), -p_x(\tau))/\chi(\tau), \quad \chi(\tau) = \sqrt{p_x(\tau)^2 + p_y(\tau)^2}. \quad (9)$$

Such coordinates are flat in the transverse direction ξ , hence, the only non-trivial metric coefficient is

$$h(\tau, \xi) = 2\chi - \Theta(\tau)\xi, \quad \Theta = -p_y/\chi^2. \quad (10)$$

The Helmholtz equation (1)-(5a) takes the form [16]

$$\frac{1}{h} \frac{\partial}{\partial \tau} \left[\frac{1}{h} \frac{\partial u(\tau, \xi)}{\partial \tau} \right] + \frac{1}{h} \frac{\partial}{\partial \xi} \left[h \frac{\partial u(\tau, \xi)}{\partial \xi} \right] + \kappa^2(1 - x(\tau, \xi))u(\tau, \xi) = 0. \quad (11)$$

Equations of the form (11) have already been treated in the framework of the parabolic wave equation method by Permitin and Smirnov [16], but here a slightly different approach is considered. Let us start from the slowly varying envelope ansatz, which is the basis of the parabolic wave equation method, namely,

$$u(\tau, \xi) = a(\tau, \xi)e^{i\kappa\phi(\tau, \xi)}, \quad \phi(\tau, \xi) = \int^\tau \chi(\tau')h(\tau', \xi)d\tau', \quad (12)$$

where a is the slowly varying envelope and the phase ϕ has been fixed *a priori*; it is worth noting that ϕ depends on both τ and ξ . On substituting (12) into (11) one finds

$$\begin{aligned} i\kappa^{-1} \frac{\chi}{h} \frac{\partial a}{\partial \tau} + \frac{\kappa^{-1}}{2h} \frac{\partial}{\partial \xi} \left(h \frac{\partial a}{\partial \xi} \right) - (\chi^2 + x(t, \xi) - 1)a \\ + \frac{1}{\kappa^2} \left\{ \frac{1}{h^2} \frac{\partial^2 a}{\partial \tau^2} - \frac{1}{h^2} \frac{\partial h}{\partial \tau} \left[\frac{\partial a}{\partial \tau} + i\kappa \frac{\partial \phi}{\partial \tau} a \right] + i\kappa \left[\left(\frac{1}{h} \frac{\partial h}{\partial \xi} \frac{\partial \phi}{\partial \xi} + \frac{1}{h^2} \frac{\partial^2 \phi}{\partial \tau^2} \right) a + 2 \frac{\partial \phi}{\partial \xi} \frac{\partial a}{\partial \xi} \right] \right\} = 0. \end{aligned}$$

The parabolic wave equation relevant to the full quasi-optics method is then obtained by neglecting the term in brace brackets, and, on making use of a rescaled time-like coordinate $t = \int^\tau d\tau'/\chi(\tau')$ with $\chi^2 = n^2(x(\tau)) = 1 - x(\tau)$, that yields

$$i\kappa^{-1} \frac{\partial a}{\partial t} = -\frac{\kappa^{-2}}{2} \frac{\partial}{\partial \xi} \left(h \frac{\partial a}{\partial \xi} \right) + h(t, \xi)g_x(t)\xi a = 0. \quad (13)$$

This is a Schödinger-type partial differential equation, with κ^{-1} playing the role of the Planck constant, and it is solved numerically by a spectral method.

The aberration-free approximation of the full quasi-optics equation (13) is obtained by replacing the potential-like term $h(t, \xi)g_x(t)\xi$ by $\frac{1}{2}\partial_\xi^2(hg_x\xi)\xi^2$. As a result one gets,

$$i\kappa^{-1} \frac{\partial a}{\partial t} = -\frac{\kappa^{-2}}{2} \frac{\partial}{\partial \xi} \left(h \frac{\partial a}{\partial \xi} \right) - g_x(t)\Theta(t)\xi^2 a = 0, \quad (14)$$

thus, with respect to (13), the term $g_x(t)\xi a$ has been neglected.

3.2. Solution of the dispersive absorbing half-plane model and its quasi-optical approximations

Let us now address the model (1)-(5b) for the inhomogeneous dispersive absorbing layer.

Exact solution. In terms of the Fourier transformed field (3) the relevant equation reads

$$\partial_x^2 \hat{u}(x, p_y) + \kappa^2(1 - p_y^2 + i\gamma(x, p_y))\hat{u}(x, p_y) = 0, \quad (15)$$

where $\gamma(x, p_y)$ is the absorption coefficient defined in equation (5b). This amounts to an ordinary differential equation in x with a piecewise-defined coefficient depending parametrically on p_y . For $x < a(p_y) = -p_y^2$, the general solution is a superposition of plane waves $e^{\pm i\kappa x \sqrt{1-p_y^2}}$ that become evanescent for $p_y^2 > 1$; we shall always assume that the spectrum does not include such evanescent wavenumbers. For $x \geq a(p_y) = -p_y^2$ absorption sets in. From the physical point of view, it is worth noting that the position $x = a(p_y)$ at which the p_y -wavenumber starts to be absorbed depends on p_y . Therefore, the carrier wavenumber of a beam, say with $p_y = p_{y0}$, feels the presence of an absorbing half-plane at $x = a(p_{y0})$, but a significant part of the energy is lost before that position and, after, the energy dissipation is quicker than expected for the carrier, both effects being consequences of different absorption of the tails of the spectrum with $p_y^2 > p_{y0}^2$. Such a spectrally asymmetric absorption should not be confused with the usual idea of spatially asymmetric absorption addressed in Ref.[19]: the latter is a purely geometric effect due to oblique launching of the beam, while the position of the absorbing half-plane is the same for all the wavenumbers.

In the absorbing region $x \geq a(p_y)$, the solution of (15) is proportional to the Airy function $\text{Ai}(\zeta)$ of complex argument $\zeta = (\kappa/\gamma_1)^{2/3} e^{4\pi i/3} (1 - p_y^2 + i\gamma_1(x - a(p_y)))$. Continuity of the field and its derivative at the boundary allows us to find the unknown coefficients, C_t and C_r , in the exact solution

$$\hat{u}(x, p_y) = U(p_y) e^{-i\kappa x_0 p_x} \begin{cases} e^{i\kappa x p_x} + C_r e^{-i\kappa x p_x}, & x < a(p_y), \\ C_t \text{Ai}((\kappa/\gamma_1)^{2/3} e^{4\pi i/3} (p_x^2 + i\gamma_1(x - a(p_y)))), & x \geq a(p_y), \end{cases} \quad (16)$$

where $p_x = \sqrt{1 - p_y^2}$ and $U(p_y)$ is the spectrum of the launched beam which will be specified in section 4. Explicitly, one gets

$$C_t = C_t(p_y) = \frac{2p_x e^{i\kappa a p_x}}{p_x \text{Ai}(\zeta_*) + (\gamma_1/\kappa)^{1/3} e^{4\pi i/3} \text{Ai}'(\zeta_*)}, \quad (17)$$

and

$$C_r = C_r(p_y) = e^{2i\kappa a p_x} \frac{p_x \text{Ai}(\zeta_*) - (\gamma_1/\kappa)^{1/3} e^{4\pi i/3} \text{Ai}'(\zeta_*)}{p_x \text{Ai}(\zeta_*) + (\gamma_1/\kappa)^{1/3} e^{4\pi i/3} \text{Ai}'(\zeta_*)}, \quad (18)$$

where $\zeta_* = (\kappa/\gamma_1)^{2/3} e^{4\pi i/3} p_x^2$. By inspection of the exact solution (16) we see that $|C_r(p_y)|^2$ gives the ratio of the wave energy densities in the (x, p_y) -space carried by the reflected beam and the injected beam, that is, the reflection coefficient for the p_y -wavenumber.

Figure 1. The reflection coefficient $|C_r(p_y)|^2$ according to equation (18). The only relevant parameters are $\kappa = 50$ and γ_1 which is varied from $\gamma_1 = 1$ (continuous line), up to $\gamma_1 = 5$ (fine-dashed line) with unit increments.

Figure 1 shows that the reflection coefficient is symmetric with respect to $p_y = 0$ (it depends on p_y^2 only) and it grows from $\approx 10^{-4}$ at $p_y^2 = 0$ up to 1 at $p_y^2 = 1$, the growth being substantially localized at $p_y^2 \approx 1$; therefore, as expected, the reflection is stronger for wavenumbers corresponding to high incidence angle, i.e., with the refractive index vector (p_x, p_y) being almost parallel to the absorbing half-plane.

Beam tracing. For the dispersive absorbing model the dispersion function (6) amounts to $H(x, p_x, p_y) = p_x^2 + p_y^2 - 1$ which is just the vacuum dispersion function, therefore, the only non-trivial physics is related to dissipation, the reference ray, the phase and the width of the beam tracing solution being the same as in free space; hence, one can write

$$u^{BT}(x, y) = A(\tau(x))u_{\text{free}}(x, y), \quad (19)$$

where $A(\tau)$ is the amplitude, the function $\tau(x)$ is a parameter along the reference ray and $u_{\text{free}}(x, y)$ is the beam tracing solution for free space which has been extensively studied [1, 2, 19]. The reference ray is a straight line that can be conveniently represented in the form

$$y(x) = \tan \theta (x - a(p_{y0})), \quad (20)$$

where θ is the angle of incidence of the beam on the absorbing layer, and $\tau(x) = \frac{x-x_0}{2 \cos \theta}$, x_0 being the launching position of the beam. At last, the amplitude $A(\tau)$ along the reference ray is determined by the classical transport equation

$$\frac{dA(\tau)}{d\tau} = -\kappa\gamma(\tau)A(\tau), \quad (21)$$

where the absorption coefficient γ is evaluated on the reference ray and multiplied by κ in order to account for strong dissipation. The solution is

$$A(\tau) = A(0) \begin{cases} 1, & x_0 \leq x < a(p_{y0}), \\ \exp \left\{ -\frac{\kappa\gamma_1(x - a(p_{y0}))^2}{4 \cos \theta} \right\}, & x \geq a(p_{y0}), \end{cases} \quad (22)$$

Nevertheless the application of such a beam tracing construction to the case under consideration requires some more comments, as the model (1)-(5b) violates two crucial applicability conditions: the model is (i) spatially dispersive and (ii) strongly non-Hermitian. As discussed in the introduction it has been proven recently that spatial dispersion does not invalidate the standard beam tracing construction [6]; such a proof requires just minor changes in the derivation of the beam tracing equations. On the other hand, strongly non-Hermitian media, i.e., strong dissipation, can be a serious concern. Specifically, according to the amplitude transport equation (21), the amplitude varies on the scale of order $1/\kappa$ which is the scale of the wave length; although this construction is expected to yield the correct asymptotic of the wave field in the limit $\kappa \rightarrow +\infty$, for finite values of κ the field envelope thus obtained can be affected by a significant error. As discussed in Ref.[19], the proper description of absorption should be supported by additional physical informations, e.g., the energy continuity equation.

Quasi-optics. In the quasi-optical method, one considers orthogonal coordinates (t, ξ) constructed around the reference ray (20); these are just rotated with respect to Cartesian coordinates, namely,

$$x - x_0 = t \cos \theta - \xi \sin \theta, \quad (23a)$$

$$y - y_0 = t \sin \theta + \xi \cos \theta, \quad (23b)$$

thus, in particular, $\partial\tau/\partial y = \sin \theta = p_{y0}$ and $\partial\xi/\partial y = \cos \theta = p_{x0}$. In the new coordinates equation (1)-(5b) reads

$$\frac{1}{\kappa^2}(\partial_t^2 + \partial_\xi^2 + 1)u(t, \xi) + i\gamma\left(x, \frac{p_{y0}}{i\kappa}\partial_t + \frac{p_{x0}}{i\kappa}\partial_\xi\right)u(t, \xi) = 0, \quad (24)$$

where the integral operator due to the imaginary part in (5b) has been written in the form $\gamma(x, \frac{1}{i\kappa}\frac{\partial}{\partial y})$.

On following the usual ideas of the parabolic wave equation method, one writes

$$u(t, \xi) = a(t, \xi)e^{i\kappa t}, \quad (25)$$

where $a(t, \xi)$ is the slowly varying envelope. On substituting (25) into (24) the Laplace operator is approximated according to

$$\frac{e^{-i\kappa t}}{\kappa^2}(\partial_t^2 + \partial_\xi^2)u(t, \xi) \approx -a(t, \xi) + \frac{2i}{\kappa}\partial_t a(t, \xi) + \frac{1}{\kappa^2}\partial_\xi^2 a(t, \xi), \quad (26)$$

thus neglecting the second-order derivative $\frac{1}{\kappa^2}\partial_t^2 a(t, \xi)$ as appropriate to the slowly varying envelope approximation on which the parabolic wave equation method relies [15]. As for the operator $\gamma(x, \frac{1}{i\kappa}\frac{\partial}{\partial y})$, one notes that

$$\frac{1}{i\kappa}\frac{\partial}{\partial y}u(t, \xi) \approx e^{i\kappa t}\left(p_{y0} + \frac{p_{x0}}{i\kappa}\frac{\partial}{\partial \xi}\right)a(t, \xi),$$

where (25) has been accounted for and the term proportional to $\partial_t a$ has been neglected; then, one makes use of the approximation

$$e^{-i\kappa t}\gamma\left(x, \frac{1}{i\kappa}\frac{\partial}{\partial y}\right)u(t, \xi) \approx \gamma\left(x(t, \xi), p_{y0} + \frac{p_{x0}}{i\kappa}\frac{\partial}{\partial \xi}\right)a(t, \xi). \quad (27)$$

Roughly speaking in approximation (27), the envelope $a(t, \xi)$ is regarded as constant in t , while in the approximation of the Laplacian (26), the first-order derivative $\partial_t a(t, \xi)$ has been retained. On making use of approximations (26) and (27), equation (24) reduces to the quasi-optical evolution equation

$$i\kappa^{-1} \frac{\partial}{\partial t} a(t, \xi) = -\frac{\kappa^{-2}}{2} \frac{\partial^2}{\partial \xi^2} a(t, \xi) + i\gamma\left(x(t, \xi), p_{y0} + \frac{p_{x0}}{i\kappa} \frac{\partial}{\partial \xi}\right) a(t, \xi), \quad (28)$$

which is a Schrödinger-type equation, with κ^{-1} playing the role of the Planck constant. In the aberration-free approximation, the operator $\gamma(x, \frac{1}{i\kappa} \frac{\partial}{\partial y})$ is simply replaced by the multiplier $\gamma(t) = \gamma(x(t), p_{y0})$ which is the absorption coefficient evaluated on the reference ray (20); this yields the evolution equation

$$i\kappa^{-1} \frac{\partial}{\partial t} a(t, \xi) = -\frac{\kappa^{-2}}{2} \frac{\partial^2}{\partial \xi^2} a(t, \xi) + i\gamma(t) a(t, \xi). \quad (29)$$

The spectral method is used for the numerical integration of both quasi-optics evolution equations; for equation (28), however, the replacement

$$\gamma\left(x(t, \xi), p_{y0} + \frac{p_{x0}}{i\kappa} \frac{\partial}{\partial \xi}\right) \equiv \gamma(t, \xi, \partial_\xi) \longrightarrow \gamma(t, \xi, 0) + \gamma(t, 0, \partial_\xi) - \gamma(t, 0, 0), \quad (30)$$

has been used in the numerical implementation, in order to overcome some limitations of the spectral method; one should note that the left- and right-hand sides of equation (30) agree on the reference ray.

4. Results and discussion

In this section the beam tracing, quasi-optical, and quasi-optical aberration-free description of a wave beam are compared and benchmarked against the corresponding exact solution. The main issues under investigation are the effects of aberration due to either the presence of a caustic (linear layer model) or the spectrally inhomogeneous dissipation of the wave energy.

First, we have to specify the form of the launched spectrum for the exact solution of both the linear layer problem, equation (7), and the dispersive absorption model, equation (16), as well as the corresponding initial conditions for the beam tracing and quasi-optical solutions. As a launching condition the beam cross-section on the line $x = x_0$ is prescribed in the form

$$u|_{x=x_0}(y) = u_0 e^{-\kappa \frac{(y-y_0)^2}{\alpha^2}} e^{i\kappa[(y-y_0)p_{y0} + \frac{1}{2}\beta(y-y_0)^2]}, \quad (31)$$

which corresponds to a Gaussian beam; here, the parameters α, β correspond to the initial beam width and phase-front curvature in a non-trivial way since the considered cross-section is not normal to the reference ray. The corresponding spectrum is obtained by means of the Fourier transform (3), namely,

$$U(p_y) \equiv \hat{u}(x_0, p_y) = u_0 \sqrt{\frac{2\pi}{\kappa\mu}} e^{-\frac{\kappa}{2\mu}(p_y - p_{y0})^2} e^{-i\kappa p_y y_0}, \quad (32)$$

where $\mu = 2/\alpha^2 - i\beta$. Equation (7) together with (32) gives the exact solution of the linear layer problem, whereas (16) together with (32) gives the exact solution of the dispersive absorption problem. It is worth noting that, in both models, the initial value p_{y0} is related to the injection angle θ , cf., section 3.

As for the beam tracing solution, the parameters α and β together with the injection angle θ , are enough to define uniquely the initial conditions for the beam tracing equations as addressed in Ref. [19].

As for the solution of the evolutionary equations for the slowly varying envelope of the quasi-optical method, the relevant initial conditions should be given on the normal (to the reference ray) straight line passing through $\tau = 0$; in order to find the initial condition corresponding to (31) we have used the exact solution written in (τ, ξ) coordinates and restricted to the normal line in $\tau = 0$; such a procedure has been carried out numerically.

4.1. Linear layer

Let us first discuss the results for the linear layer model. We are particularly interested in the comparison between the beam tracing solution and the full quasi-optical solution. As for the aberration-free quasi-optical solution, it has been proved that it should agree with the beam tracing solution for lossless media [12] and such an agreement is confirmed by the numerical results within the accuracy of the solutions, cf., figure 2. The beam is launched from $x = -4$ and $y = 3$, and propagates toward the cut-off located at $x = 1$, cf., equation (5a); initially, diffraction effects broaden the beam which is then focused near the fold caustic where the beam is reflected back and further broadened by diffraction. The bottleneck shape of amplitude contours can be understood on noting that the caustic's width in geometric optics would be zero: both the beam tracing and the aberration-free solution account for diffraction effects that keep finite, but still small, the width near the reflection point.

Figure (3) shows the amplitude contours for both the beam tracing and quasi-optical solutions superposed to the exact solution, together with their errors. One can see that the exact field distribution is broader, effectively filling up the bottleneck shape; let us recall that the applicability of beam tracing and quasi-optical methods near caustics of fold type is marginal for the considered injection angle. The bottleneck-type profile of the amplitude indicates that in all approximate solutions the interference of the incident and reflected branches of the solution is not taken into account (for more comments on such interference we refer to the paper [19]); this also implies that, in all three cases, the error attains its maximum near the turning point where the two branches of the solution merge. The beam tracing solution, however, achieves a better agreement, while the two quasi-optical solutions exhibit a shift of the inner amplitude contours.

More quantitative informations come from the profiles of the (normalized) line

Figure 2. Contour plot (left) of the amplitude $|u(x,y)|/u_0$ according to both the beam tracing solution and the aberration-free quasi-optical solution together with the density plot (right) of their difference. Parameters are $\kappa = 50$, $\theta = 71.56^\circ$ (the angle for which the reference ray crosses the line $x = 0$ with an angle of 45°), $\alpha = 0.3\sqrt{\kappa}$, and $\beta = 0$, while the launching position is $x_0 = -4$. The outermost contour corresponds to 0.1 of the maximum amplitude.

Figure 3. Amplitude contours (upper line) and the density plots of the corresponding errors (lower line) of the beam tracing solution (left), the aberration-free quasi-optical solution (center), and the full quasi-optical solution (right). Parameters are as in figure 2. The outermost contour corresponds to 0.1 of the maximum amplitude.

Figure 4. Linear energy density computed from the exact, beam tracing, aberration-free quasi-optical, and full quasi-optical solutions. Parameters are $\kappa = 50$, $\theta = 71.56^\circ$, $\alpha = 0.3\sqrt{\kappa}$, and $\beta = 0$ (left) or $\beta = -0.4$ (right). The two values of the parameter β correspond to an almost flat phase front and to a focussed phase front, respectively. The beam tracing and the aberration-free quasi-optical profiles are superposed within the accuracy of the plot.

energy density, [19],

$$W(x) = \frac{1}{|u_0|^2} \int_{-\infty}^{+\infty} |u(x, y)|^2 dy, \quad (33)$$

which allows us to identify clearly the position and width of the caustic region. Figure 4, shows the line energy density profiles for two cases that differ for the value of the phase front curvature. The line energy density $W(x)$ has a peak that corresponds to the caustic region: for reflectometry applications one is particularly interested in the evaluation of the position and width of the peak, rather than on its height. In both cases the beam tracing and aberration-free solutions are superposed within the accuracy of the plot as expected and their maximum (the position of the caustic) corresponds to the classical turning point, i.e., the point where the parabola of the reference ray (8) has its vertex. The classical turning points, however, are closer to the cut-off line $x = 1$ than the caustic location of the exact solution, this shift being widely discussed in Ref.[19]. The full quasi-optical solution, on the other hand, overcompensates such a shift with the result that the quasi-optical peak location is too much on the left of the exact peak. We note that such effects are much more noticeable in the focussed case (right plot).

On recalling that the position of the cut-off is $x_{\text{cut-off}} = 1$, it appears convenient to normalize the error in the position of the caustic to the distance $1 - x_{\text{exact}}$ of the exact caustic location x_{exact} from the cut-off, that is, the relative error of the peak position x_σ is estimated by

$$\Delta x_\sigma = \frac{|x_\sigma - x_{\text{exact}}|}{1 - x_{\text{exact}}}, \quad (34)$$

where σ =beam tracing, or σ =full quasi-optics labels the two relevant descriptions.

Case	$\Delta x_{\text{QO}}(\%)$	$\Delta x_{\text{BT}}(\%)$	$\Delta w_{\text{QO}}(\%)$	$\Delta w_{\text{BT}}(\%)$
$\beta = 0$	5	5	24	26
$\beta = -0.4$	7	2.5	37	30

Table 1. Relative errors for the position and width of the caustic region according to definitions (34) and (36), for the two cases of figure 4.

Analogously, one can define the widths of the caustic region by

$$w_\sigma = \frac{\int_0^{+\infty} (x - x_\sigma)^2 W(x) dx}{\int_0^{+\infty} W(x) dx}, \quad (35)$$

with σ = beam tracing, full quasi-optics and exact; then, the relative error

$$\Delta w_\sigma = \frac{|w_\sigma - w_{\text{exact}}|}{w_{\text{exact}}}. \quad (36)$$

Table 1 gives the values of Δx_σ and Δw_σ for the two cases of figure 4. One can see that the errors for the beam tracing and quasi-optical solution are fairly close one to the other for the case of the (almost) flat beam ($\beta = 0$), while the beam tracing yields a better accuracy for the more critical case of a focussed beam ($\beta = -0.4$).

The foregoing analysis supports the idea that aberration effects due to the formation of a caustic are negligible.

4.2. Dispersive absorption

In this section we consider the aberration effects due to the presence of a strong dispersive anti-Hermitian part of the dielectric operator, i.e., dispersive absorption. In the presence of absorption, the beam tracing and the aberration-free quasi-optical solutions need no longer to agree; figure 5 shows the differences in the amplitude contours for the beam tracing and aberration-free solutions.

A first insight into the physical content of each approximate solutions is obtained on looking at the amplitude contours displayed in figure 6. One can note that the exact solution is characterized by a broadening of the amplitude contours with respect to the corresponding beam tracing solution; such a broadening is the main source of error for the beam tracing solution and it can be attributed to the effects of dispersion in the absorption coefficient. The corresponding aberration free quasi-optical solution fails to match the amplitude profile in the absorbing layer due to both the effect of dispersion (not accounted for in the aberration-free limit) and the specific coordinates used in the derivation of the quasi-optical equation: the transversal coordinates ξ must be orthogonal to the reference ray, thus, it cannot be aligned to the direction of homogeneity of the medium; on the other hand the beam tracing solution has a better accuracy due to the flexibility in the choice of local coordinates (not addressed here). The full quasi-optical solution, on the other hand, matches well the exact amplitude profile, showing, in particular, that it can account for the dispersive broadening of the beam. In all the

Figure 5. Contour plot of the amplitude $|u(x, y)|/u_0$ according to both the beam tracing solution (19) and the aberration-free quasi-optical solution (29). Parameters are $\kappa = 50$, $\theta = 30^\circ$, $\alpha = 0.5\sqrt{\kappa}$, and $\beta = -0.25$, while the launching position is $x_0 = -2$. The slope of the absorption coefficient is set to $\gamma_1 = 1$. The outermost contour corresponds to 0.1 of the maximum amplitude.

Figure 6. Amplitude contours (upper line) and the density plots of the corresponding errors versus the exact solution (lower line) of the beam tracing solution (left), the aberration-free quasi-optical solution (center), and the full quasi-optical solution (right). Parameters are as in figure 5. The largest errors are found on the outermost contour, which corresponds to 0.1 of the maximum amplitude.

Figure 7. Power deposition profiles in the direction perpendicular (left) and parallel (right) to the absorbing layer, computed from equation (37) and normalized to their maximum; the values of parameters are as in figure 5.

	$\Delta x_{\text{QO}}(\%)$	$\Delta x_{\text{BT}}(\%)$	$\Delta x_{\text{aBT}}(\%)$	$\Delta w_{\text{QO}}(\%)$	$\Delta w_{\text{BT}}(\%)$	$\Delta w_{\text{aBT}}(\%)$
perp.	0	2.5	1	3.5	25	9.2
paral.	2	3	2	4	9	6.9

Table 2. Relative errors for the position of the maximum x_σ and width w_σ of the deposition profiles displayed in figure 7, for the full-quasi-optical (QO), standard beam tracing (BT), and adapted beam tracing (aBT) solutions. The relative error in position is defined as $(x_\sigma - x_{\text{exact}})/w_{\text{exact}}$, while the relative error on the width is defined in the standard way $(w_\sigma - w_{\text{exact}})/w_{\text{exact}}$.

plots of figure 6 the reflected wave is beyond the resolution being of the order 10^{-3} , as it follows from the analysis of the reflection coefficient, cf., figure 1.

In order to be more quantitative, let us consider the power deposition profiles, with respect to both the directions perpendicular and parallel to the boundary of the layer. More specifically, we define the power deposition profiles as the derivative of energy fluxes through the lines $x = \text{constant}$ and $y = \text{constant}$, for the perpendicular and parallel deposition profiles, respectively. Explicitly, we write

$$P_x(x) = \frac{d}{dx} \int_{-\infty}^{+\infty} S_x(x, y) dy, \quad P_y(y) = \frac{d}{dy} \int_{-\infty}^{+\infty} S_y(x, y) dx, \quad (37)$$

where (S_x, S_y) is the Poynting flux. Figure 7 shows the power deposition profiles normalized to their own maximum, for the case considered in figures 5 and 6.

The normalization of the deposition profiles to their maximum is chosen for convenience of representation, as we are mainly interested in the width of the profiles. One can see that the beam tracing power deposition profile in the perpendicular direction is narrower than the exact deposition profile. On the other hand, the quasi-optical solution appears to yield a better description. The reason for the lack of accuracy of the beam tracing description can be understood on looking at equation (21): in the beam tracing formulation the wave energy density starts to be absorbed at the same point for all the wavenumbers comprising the beam, that is, where the reference ray

crosses the boundary of the absorption layer defined by the carrier momentum p_{y0} ; in the exact solution, however, wave energy starts to be dissipated a bit before, where the wavenumber with $p_y^2 \geq p_{y0}^2$ are absorbed. This explains the left tail ($x < 0$) of the perpendicular deposition profile. Analogously, the wavenumber with $p_y^2 < p_{y0}^2$ are absorbed a bit later with respect to the carrier, and this explains the right tail ($x > 0$) of the deposition profile. On the other hand, the parallel deposition profile, which gives information on the region of the layer that is illuminated by the beam, is well described by both the quasi-optical and beam tracing description. More quantitatively, the errors on both the position of the maximum and the width of the power deposition profiles are reported in table 2.

The observed differences between the exact and the standard beam tracing power deposition profiles can be attributed entirely to the dispersive effect. This is clearly proven by switching off dispersion in the model, that is, by setting $a(p_y) = 0$. Then, despite the presence of a large absorption coefficient which formally violates the applicability conditions, the standard beam tracing description matches very well both the amplitude contours and the power deposition profiles.

Nonetheless, for the specific model under consideration, it is possible to modify the beam tracing solution in such a way that, to some extent, the absorption of the tails of the spectrum can be accounted for. The main idea at the basis of such an *adapted* beam tracing solution is the replacement of the amplitude transport equation (21) with a more accurate transport equation which can be derived from the energy conservation. Indeed, the Helmholtz equation (1) implies the energy flux balance,

$$\frac{d\Phi_x(x)}{dx} = -\frac{\kappa^2}{2\pi} \int_{-\infty}^{+\infty} \gamma(x, p_y) |\hat{u}(x, p_y)|^2 dp_y, \quad (38)$$

where

$$\Phi_x(x) = \int_{-\infty}^{+\infty} S_x(x, y) dy, \quad (39)$$

is the wave energy flux across $x=\text{constant}$ lines. On one hand, equation (38) is an exact equation which gives a direct integral expression for the transversal power deposition profile $P_x(x) = d\Phi_x(x)/dx$ in terms of the absorption coefficient $\gamma(x, p_y)$ and of the spectrum of the wave field $\hat{u}(x, p_y)$. On the other hand, when (19) is substituted into (38) and the flux (39) is computed to the leading order in κ , one finds

$$\frac{d\Phi_{x,\text{BT}}(x)}{dx} = -\frac{\kappa}{p_x} \gamma_{\text{BT}}(x) \Phi_{x,\text{BT}}(x), \quad (40)$$

where,

$$\gamma_{\text{BT}}(x) = \sqrt{\frac{\kappa}{\pi \Delta p_y(x)^2}} \int_{-\infty}^{+\infty} \gamma(x, p_y) e^{-\kappa \frac{(p_y - p_{y0})^2}{\Delta p_y(x)^2}} dp_y, \quad (41)$$

is the average of the absorption coefficient $\gamma(x, p_y)$ with respect to the spectral distribution of the beam tracing solution; here, $\Delta p_y(x)$ is the spectral width of the beam tracing solution in the y -direction evaluated at the position x . Equation (40) is

then readily solved for the beam tracing flux $\Phi_{x,\text{BT}}$ from which one gets the amplitude $A(x)$. The corresponding power deposition profiles are also given in figure 7.

The crucial point is that the absorption coefficient (41) describes the wave energy dissipation as the mean result of different absorption rates for the different wavenumbers. As a consequence, the adapted beam tracing solution describes pretty well the initial phase of the power deposition, i.e., the left tail at $x < 0$, which is the region where dispersion plays a major role. The final phase of the power deposition, i.e., the right tail at $x > 0$, is still affected by the problem of a slightly overestimated dissipation. This problem is common to both the standard and adapted beam tracing solutions: indeed, numerically, one finds $\gamma_{\text{BT}}(x) \sim \gamma(x, p_{y0})$ for $x > -0.1$ with the parameters of figure 7. This also explains the agreement on the right-tails of the power deposition profiles obtained out of the adapted and the standard beam tracing solutions. Quantitatively, the errors in position and width of the adapted beam tracing profiles are given in table 2.

5. Application to the ITER upper-port launcher

Although the model described above is too simple (two dimensions only, no refraction, and simplified absorption) to reproduce the same conditions as those foreseen for the ITER upper-port launcher, some indications on the relevance of aberration effects (dispersive absorption) to the wave beams envisaged for ITER can be obtained by appropriately choosing the physical parameters to be considered.

The first issue is, of course, the geometry: with two dimensions available we can only give a very rough description. The effect we are looking at is related to the dependence of the absorption coefficient on the refractive index; for fusion plasmas that is essentially due to the Doppler-shift term in the electron cyclotron resonance which introduces a dependence on the parallel component, $N_{\parallel}$, of the refractive index. This implies that the conjugate momentum p_y should be identified with $N_{\parallel}$, and, thus, upon neglecting the poloidal component of the magnetic field, the y -axis should be roughly identified with the toroidal direction of the tokamak. Then, the x -axis is identified with the projection of the line of sight of the steering mirror into the poloidal section. Therefore, the launching angle θ amounts to the toroidal steering angle, while the poloidal steering angle gives the tilt of the plane of the model. The geometry is qualitatively sketched in figure 8.

Such a two-dimensional geometry implies that we neglect the effects of both the finite poloidal width of the beam and the poloidal steering angle (i.e., that the beam impinges obliquely on the resonance layer); however, these two effects have already been studied separately in Ref.[19] with the result that no relevant aberration effects should come from that. Of course, this does not exclude the possibility of a synergy between high poloidal steering angles and dispersive absorption; a precise analysis of this possibility, however, requires fully three-dimensional models.

Even in this adapted geometry, the absorption coefficient $\gamma(x, p_y)$ must be modified in order to fit, even approximately, the ITER profiles. Specifically, we replace the

Figure 8. Application of the two-dimensional model to the ITER upper EC port. The poloidal steering angle α corresponds to the orientation of the x - y plane, while the toroidal steering angle corresponds to the beam injection angle θ .

constant γ_1 in equation (5b) by an envelope function depending only on p_y , namely,

$$\gamma_1 \mapsto \gamma_1 g(p_y), \quad g(p_y) = \frac{(1 - |p_y|^2)^2}{1 + |p_y/p_{y\text{ref}}|^3}. \quad (42)$$

The analytical exact solution for this slightly different absorption coefficient follows from the analytical solution (16) again via the formal substitution (42). It is worth noting that in this modified form of the model, the linear growth of the absorption coefficient γ_1 depends on the parallel refractive index $N_{\parallel} = p_y$ in such a way that absorption is suppressed for high values of $N_{\parallel}$. Both the functional form of the envelope function $g(p_y)$ and the values of parameters γ_1 , and $p_{y\text{ref}}$ have been roughly determined by fitting the linear growth of the numerically computed weakly relativistic absorption coefficient for EC waves in ITER; the latter has been computed by means of the routine **DAMPBQ** [26] for the ordinary mode. The values of temperature and density in the absorption region are $T_{e0} = 7\text{keV}$ and $n_{e0} = 10^{14}\text{cm}^{-3}$, respectively. The tokamak minor radius coordinate r is related to the model coordinate x by

$$x = (r_0 - r)/L, \quad (43)$$

where r_0 is the position of the resonance layer for $N_{\parallel} = 0$, and L is the normalization scale length. The fit of the numerical absorption coefficient thus obtained yields

$$\gamma_1 = 0.017, \quad a_1 = 55, \quad p_{y\text{ref}} = 0.22. \quad (44)$$

Let us note that the curvature a_1 of the resonance layer in the x - p_y space is rather high as compared to the values used in the foregoing analysis; its effect is, however, balanced by the envelope function $g(p_y)$.

In order to avoid extremely fine mesh, the value of the dimensionless parameter $\kappa = \omega L/c$ cannot be too large. Upon setting $\kappa = 200$, and recalling that for the ITER

Figure 9. Power deposition profiles normalized to their maximum value as a function of normalized position x for $w_f = 3\text{cm}$ (left-hand plot) and $w_f = 2\text{cm}$ (central plot) both with $\theta = 22^\circ$. The right-hand side plot shows the power deposition profiles for $w_f = 2\text{cm}$ and $\theta = 28.04^\circ$ which corresponds to $N_{\parallel} = 0.47$. The beam tracing calculation (magenta) is very similar to the exact calculation (black) except in a small interval at the boundary of the resonance layer.

upper-port launcher $\omega = 1.06 \times 10^{12}\text{rad/sec}$, or, equivalently, $k_0 = \omega/c \approx 35\text{cm}^{-1}$, one finds $L = 5.71\text{cm}$ which is rather small as compared to the plasma size.

Due to the large value of the curvature a_1 , we need to enlarge the computational domain. The beam propagates for a distance $L_p \approx 10L = 57.1\text{cm}$ before being absorbed at the EC resonance. This is a rather short distance as compared to the whole path of the beam which amounts to 213cm . This means that our model can just describe the final part of the propagation near the waist of the beam and, therefore, the launching conditions of the model cannot be given by the parameters of the beam at the launching mirror. In order to work out the correct parameters, let us compute the Rayleigh distance, $L_R = \frac{1}{2}k_0w_f^2$ where w_f is the typical beam width at the waist in ITER. We have considered two values $w_f = 3\text{cm}$ and $w_f = 2\text{cm}$, which are typical of the upper steering mirror and of the lower steering mirror, respectively, and which correspond to $L_R = 157.5\text{cm}$ and $L_R = 70\text{cm}$, respectively. In both cases, this is much larger than the propagation length $L_p \ll L_R$, hence, we can assume that the computational domain of the model stays within the near field region of the beam waist, and this, to some extent, justify the approximation of neglecting the effect of refraction that should be relevant on a longer scale length. On making use of free-space formulas [2, 7, 8, 16] for the the beam width and the phase-front curvature we have $w = w_f\sqrt{1 + (L_p/L_R)^2}$ and $R = (L_R^2 + L_p^2)/L_p$.

According to the geometry of the model, the toroidal steering angle is identified with the launching angle, hence, $\theta = 22^\circ$, [17], which corresponds to $N_{\parallel} = p_{0y} = 0.37$. For the case of a 2cm beam we have also considered the value $\theta = 28.04^\circ$ which gives the larger value $N_{\parallel} = 0.47$. Then, the identities, [19],

$$\alpha = \sqrt{\kappa} \frac{w/L}{\cos \theta}, \quad \beta = p_{x0}^2 \frac{L}{R}, \quad (45)$$

provide the relevant launching parameters in (31).

The results for the power deposition profiles are reported in figure 9. One can see that in all the considered cases the quasi-optical solution is essentially superposed to the exact power deposition profile. On the other hand, the beam tracing calculation differs

	x_{dep}	w_{dep}	x_{dep}	w_{dep}	x_{dep}	w_{dep}
BT	-5.872	0.9559	-5.871	0.9612	-9.47185	1.39944
Ex	-5.876	1.0002	-5.881	1.0598	-9.4787	1.50558
error	0.004	-0.0443	0.01	-0.093	0.007	-0.07

Table 3. Average position x_{dep} and width w_{dep} of the deposition profiles for the three cases reported in figure 9, respectively. The error in the position is defined as the difference of the beam tracing and the exact values (which are already normalized to L) while the error of the width is relative to the exact value.

from the exact solutions, the difference being, however, limited to the very left tail of the deposition profile where the beam starts to be absorbed. The sharp increase of the beam tracing profile is a consequence of the piecewise-defined absorption coefficient which reflects itself into the amplitude (22). Quantitatively, the power deposition position and width of the beam tracing solution are summarized in table 3. In all the considered cases the beam tracing solution gives acceptable errors which suggest that aberration effects do not play a crucial role, at least within the many limitations of this two-dimensional model. With respect to the model studied in section 4.2, the effects of dispersive absorption is drastically reduced for the ITER-like case as a consequence of the envelope function in equation (42) introduced here in order to fit the numerically computed absorption coefficient; such an envelope suppresses the absorption of the high- $N_{\parallel}$ tail of the beam spectrum, thus, reducing the spectral asymmetry of the beam.

6. Conclusions

Recently, the question has been raised on whether the, nowadays standard, beam tracing description of high-frequency wave beams in fusion plasmas can be reliable in the presence of aberration effects. Balakin and co-workers [13] have recently developed a novel code which describes the propagation and diffraction of high-frequency wave beams in fusion plasmas taking into account aberration effects and their results show a significant broadening of power deposition profiles which has been attributed to aberration effects.

On the other hand, the detailed physical mechanism that yields such a broadening of power deposition profiles needs to be clarified, and, with this aim, one studies simplified models for which analytical solutions are available. In this respect a recent work [19] has demonstrated that spatial inhomogeneity alone (at least for typical media parameters) is insufficient for aberration effects to change significantly the propagation.

In this paper, we have considered aberration effects due to spatial inhomogeneity (linear layer model for propagation near cut-offs) as well as spatial inhomogeneity combined with both spectral inhomogeneity (spatial dispersion) and strong absorption. As our main result, we show that the presence of absorption and spatial dispersion, called dispersive absorption for short, is the crucial point for the generation of aberration effects. Analytical solutions for such models have been obtained and, in addition, a

detailed comparative analysis between the beam tracing solution and Balakin's quasi-optical code has been carried out.

The analysis of the linear layer problem has confirmed the results of Ref.[19] even under more critical conditions, showing that aberration effects associated to spatial inhomogeneity only are negligible; in addition, one can see that the quasi-optical code slightly overestimates such effects, e.g., in the shift of the caustic position.

On the other hand, dispersive absorption is found to be the source of significant aberration effects and consequent broadening of power deposition profiles in the normal direction to the resonance layer. The quasi-optical approach in this case gives an accurate description of the beam, while the standard beam tracing method cannot account for such a broadening. When dispersion is switched off, accuracy of the standard beam tracing solution is recovered, thus, proving the crucial role of spectral inhomogeneity.

For the specific model considered here, however, a way to adapt the beam tracing solution to the case of dispersive absorption has been found, upon exploiting the wave energy conservation in order to obtain a more accurate description of the wave amplitude. The results are quite encouraging as the errors in the beam tracing power deposition profiles thus obtained are significantly reduced, cf. table 2, and that opens the way to a further improvement of beam tracing codes (TORBEAM, [3]). In order to clarify the importance of aberration effects under realistic conditions, we finally adapted the dispersive absorption model to the case of the electron cyclotron upper-port launcher in ITER with the result that the errors in the standard beam tracing solution are acceptable, thus, suggesting a limited role of aberration for the considered regimes. On the other hand the quasi-optical solution yields a very precise description of the power deposition profiles including the tails where aberration can have an visible effect.

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References

- [1] G. V. Pereverzev, Reviews of Plasma Physics **19**, B. B. Kadomtsev ed., Consultants Bureau, New York, 1996, p.1.
- [2] G. V. Pereverzev, Phys. Plasmas **5**, 3529 (1998).
- [3] E. Poli, A. G. Peeters, G. V. Pereverzev, Comp. Phys. Comm. **136**, 90 (2001).

- [4] N. Bertelli, *A beam tracing code for the description of LH wave propagation in a tokamak plasma*, Ph.D thesis, University of Pavia (2007). ISBN 978-88-95767-04-8; N. Bertelli, G. V. Pereverzev and E. Poli, 34th EPS Conf. on Plasma Phys. (Warsaw) Vol **31F** (ECA) P5.051 (2007).
- [5] E. Poli, A. G. Peeters, G. V. Pereverzev, M. Bornatici, Fusion Engineering and Design **53**, 9 (2001).
- [6] G. V. Pereverzev, 34th EPS Conf. on Plasma Phys. (Warsaw) Vol **31F** (ECA) P5.061 (2007).
- [7] Yu. A. Kravtsov, *Geometrical Optics in Engineering Physics*, Alpha Science, 2005.
- [8] E. Mazzucato, Phys. Fluids B **1**, 1855 (1989).
- [9] A. Peeters, Phys. Plasmas **3**, 4386 (1996).
- [10] S. Nowak, and A. Orefice, Phys. fluids B **5**, 1945 (1993).
- [11] D. Farina, Fusion Sci. Technol. **52**, 154 (2007).
- [12] A. A. Balakin, M. A. Balakina, G. V. Permitin, A. I. Smirnov, J. Phys. D: Appl. Phys. **40**, 4285 (2007).
- [13] A. A. Balakin, M. A. Balakina, E. Westerhof, Nucl. Fusion **48**, 065003 (2008).
- [14] V. A. Fock, *Electromagnetic Diffraction and Propagation Problems*, Pergamon Press, New York, 1965.
- [15] V. M. Babič and V. S. Buldyrev, *Short-Wavelength Diffraction Theory* (Springer-Verlag, Berlin 1991).
- [16] G. V. Permitin and A. I. Smirnov, JETP **83**, 395 (1996).
- [17] R. Prater *et al* , Nucl. Fusion **48**, 035006 (2008).
- [18] A. N. Saveliev, Plasma Phys. Control. Fusion **51**, 075004 (2009).
- [19] O. Maj, G. V. Pereverzev, E. Poli, Phys. Plasmas **16**, 062105 (2009).
- [20] V. L. Ginzburg, *Propagation of electromagnetic waves in plasma*, Translated from the Russian by Royer and Roger; edited by Walter L. Sadowski, Gordon and Breach, Inc., New York, 1961.
- [21] A. A. Balakin, M. A. Balakina, and A. G. Shalashov, Plasma Physics Reports **33**, 659 (2007).
- [22] A. A. Balakin, M. A. Balakina, G. V. Permitin, and A. I. Smirnov, Plasma Physics Reports **34**, 486 (2008).
- [23] G. Ramponi *et al* , Nucl. Fusion **48**, 054012 (2008).
- [24] M. A. Henderson *et al* , Nucl. Fusion **48**, 054013 (2008).
- [25] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55.
- [26] E. Westerhof, Implementation of Toray at JET (1989), Rijnhuizen Report RR-89-183.