Magnetic surface quality in non-axisymmetric plasma equilibria

Carolin Nührenberg*
Max-Planck-Institut für Plamaphysik, IPP-Euratom Association, Teilinstitut Greifswald, 17491 Greifswald, Germany

Allen H. Boozer
Columbia University, New York, New York 10027, USA

Stuart R. Hudson
Princeton Plasma Physics Laboratory, P.O. Box 451, Princeton, New Jersey 08543, USA
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The confinement of plasmas by magnetic fields with non-axisymmetric shaping can be degraded or destroyed by the breakup of the magnetic surfaces through effects that are intrinsic to the shaping. An efficient perturbation method of determining this drive for islands was developed and applied to stellarator equilibria.

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A central requirement in magnetic confinement fusion is to balance the pressure force with the Lorentz force, as described by the ideal magnetohydrodynamic (MHD) equilibrium condition $\nabla p = \mathbf{j} \times \mathbf{B}$. When the pressure gradient is non-zero, both the magnetic field $\mathbf{B}$ and the current density $\mathbf{j}$ must lie on the constant pressure surfaces, since $\mathbf{B} \cdot \nabla p = 0$ and $\mathbf{j} \cdot \nabla p = 0$. The existence of spatially bounded magnetic surfaces, which means a function $p(\mathbf{r})$ exists such that $\mathbf{B} \cdot \nabla p = 0$, is only topologically possible when these surfaces are toroidal. In an axisymmetric plasma equilibrium, as in an ideal tokamak, the existence of magnetic surfaces is assured. However, this is not the case if the torus is asymmetric. Nevertheless, non-axisymmetric plasma shaping has benefits. For example, experiments using plasma equilibria with strong non-axisymmetric shaping (the stellarator concept) have shown immunity to the catastrophic loss of plasma equilibrium, called disruptions, and can sustain magnetic surfaces without a net plasma current. The benefits of non-axisymmetric shaping are of increasing importance as fusion energy research moves from confinement scaling studies to the broader issues required for a successful demonstration of fusion power [1].

A constraint and major challenge on non-axisymmetric shaping is that magnetic surfaces be maintained, which is a longstanding question in stellarator design [2]. The helical path followed by the magnetic field lines as they encircle the torus can resonate with the non-axisymmetric shaping to split the magnetic surfaces into islands and stochastic regions. Resonances are defined by the rotational transform ι of a magnetic field line, which is the average number of poloidal (short way) transits of the torus a field line makes per toroidal transit. If N is the number of toroidal periods of the non-axisymmetric device, then natural resonances of the system occur on magnetic surfaces on which $\iota = n/m$ with m an integer and n an integer multiple of N.

The breakup of magnetic surfaces in a given equilibrium can be studied using codes such as PIES [3] and HINT [4]. Approximate non-axisymmetric equilibria can be calculated with far less computational effort using the VMEC code [5], which exploits the assumption of nested magnetic surfaces. VMEC extremizes the plasma energy $W = \int (B^2/2\mu_0 - p) d^3r$ by varying the shape of the magnetic surfaces while holding the rotational transform $\iota(s)$ and the pressure $p(s)$ fixed. In this code the normalized toroidal flux is used as the surface label, $0 \leq s \leq 1$, so the toroidal flux enclosed by a magnetic surface is $sF_T(1)$ with $F_T(1)$ the toroidal magnetic flux enclosed by the outermost magnetic surface. The VMEC equilibrium code has become the worldwide standard for calculating non-axisymmetric equilibria but does not address the critical issue of the quality of the magnetic surfaces, which can be characterized by the fraction of toroidal flux not occupied by island chains and stochastic regions.

In this Letter, we show that any set of magnetic surfaces that define an *approximate* equilibrium, such as the surfaces found by VMEC, can be used to construct a more *exact* equilibrium with the plasma boundary unchanged. To shield out the island producing error fields, ideal MHD allows for localized surface currents at the rational surfaces, and these surface currents determine the size of the magnetic islands that would arise if the flux surface topology were allowed to break.

The magnetic surfaces given by VMEC do not exactly solve force balance, but the deviation can be found using quantities that are obtained from VMEC. In an equilibrium plasma with magnetic surfaces, $\mathbf{B} \cdot \nabla s = 0$ and $\mathbf{j} \cdot \nabla s = 0$. Magnetic angles can be chosen so the magnetic field has the representations [5–9]

$$\mathbf{B} = -\frac{F_T'(s)}{\sqrt{g}} \left(\frac{\partial \mathbf{r}}{\partial \phi} + \iota(s) \frac{\partial \mathbf{r}}{\partial \theta} \right) \quad ; \qquad (1)$$

$$\mathbf{B} \quad = I(s)\boldsymbol{\nabla}\phi + J(s)\boldsymbol{\nabla}\theta + \beta_s\boldsymbol{\nabla} s \quad , \tag{2}$$

where where $0 \leq \theta \leq 1$ is a poloidal and $0 \leq \phi \leq 1$ is a toroidal angle, $\iota(s)$ is the rotational transform, $\sqrt{g(s,\theta,\phi)} < 0$ is the Jacobian of the left-handed (s,θ,ϕ) coordinate system and I and J are the poloidal and toroidal currents. In Eq. (1), the prime denotes d/ds. The function $\beta_s(s,\theta,\phi)$ is given by $\beta_s = g_{s\phi}\mathbf{B}\cdot\boldsymbol{\nabla}\phi + g_{s\theta}\mathbf{B}\cdot\boldsymbol{\nabla}\theta$, where the metric tensor component $g_{s\phi} \equiv (\partial\mathbf{r}/\partial s)\cdot(\partial\mathbf{r}/\partial\phi)$. By taking the curl of the covariant representation of $\mathbf{B}$, Eq. (2), and crossing it with the first representation, Eq. (1), one finds that $\mathbf{j}\times\mathbf{B} = \left[F_T'(I' + \iota\, J')/\sqrt{g} + \mathbf{B}\cdot\boldsymbol{\nabla}\beta_s\right]\boldsymbol{\nabla} s$. If the equilibrium were exact, $\mathbf{j}\times\mathbf{B}$ would equal $\boldsymbol{\nabla} p = p'\boldsymbol{\nabla} s$ on each s-surface. That is, an approximate equilibrium solver, such as VMEC, fails to satisfy exact force balance by an amount $\delta f_s \equiv (\boldsymbol{\nabla} p - \mathbf{j}\times\mathbf{B})\cdot(\partial\mathbf{r}/\partial s)$, where

$$\delta f_s = p' - p_n' + \mathbf{B}\cdot\boldsymbol{\nabla}(\beta_p - \beta_s) \quad . \tag{3}$$

The nominal pressure profile $p_n(s)$ is defined by $p_n' = F_T'(I' + \iota J')/V'(s)$ where the s derivative of the volume inside a constant-s surface is $V'(s) \equiv \oint \sqrt{g} d\theta d\phi$, and the function β_p is defined by

$$\mathbf{B}\cdot\boldsymbol{\nabla}\beta_p = p_n'\,(1 - V'/\sqrt{g})\,. \tag{4}$$

Consequently, using VMEC output, the force imbalance $\delta\mathbf{f} = \delta f_s(s,\theta,\phi)\boldsymbol{\nabla} s$ can be calculated surface by surface.

It is natural to assume the force imbalance associated with the VMEC approximation is sufficiently small to treat the imbalance as a perturbation. Equilibrium perturbation theory is a well-studied subject in the context of ideal MHD stability and is reviewed in Ref. [10]. If the energy W is varied by displacing the magnetic surfaces to $\mathbf{r}(s,\theta,\phi) + \boldsymbol{\xi}(s,\theta,\phi)$ while holding $\iota(s)$ and $p(s)$ fixed, then $\delta W = \delta^1 W + \delta^2 W$. The first-order perturbed magnetic field is given by $\mathbf{B}_1 = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B})$. The variation that is linear in the displacement $\boldsymbol{\xi}$ is $\delta^1 W = \int(\boldsymbol{\nabla} p - \mathbf{j}\times\mathbf{B})\cdot\boldsymbol{\xi}\; d^3 r$. The part that is quadratic in $\boldsymbol{\xi}$ has the form $\delta^2 W = -\frac{1}{2}\int\boldsymbol{\xi}\cdot\mathcal{F}[\boldsymbol{\xi}] d^3 r$ where $\mathcal{F}$ is the Hermitian ideal MHD force operator acting on the displacement $\boldsymbol{\xi}$. Codes that examine the ideal MHD stability of a plasma assume that the equilibrium is exact, $\boldsymbol{\nabla} p = \mathbf{j}\times\mathbf{B}$, so that $\delta^1 W = 0$, and seek to minimize $\delta^2 W$. If a solution can be found that makes $\delta^2 W$ negative, then the plasma is unstable. Here we wish to minimize δW assuming $\delta^1 W$ is small, but non-zero, and that $\delta^2 W$ is positive, so there are no instabilities.

The ideal MHD stability code CAS3D [9] treats general plasma equilibria and extremizes δW by a Galerkin method. A non-dimensional representation of the normal displacement is used, $\xi^s \equiv \boldsymbol{\xi}\cdot\boldsymbol{\nabla} s = |\boldsymbol{\nabla} s|\xi_n$. The extremization leads to a system of linear equations for the Fourier harmonics of the scalar $\boldsymbol{\xi}$-components, in which the matrix is given by $\delta^2 W$ and the right-hand side by $\delta^1 W$. The assembly of matrix and right-hand side includes the discontinuities which are allowed in the resonant ξ^s harmonics at each rational surface $\iota(s_{mn}) = n/m$. The jump in a resonant ξ^s harmonic gives the surface current that flows on the $s = s_{mn}$ rational surface to keep a magnetic island from opening [11]. The magnetic field $\mathbf{B}_{\rm surf}$ that would be produced by this surface current is equal and opposite to the magnetic perturbation that is driving the formation of an island at $s = s_{mn}$ but produced by currents away from the resonant rational surface s_{mn}. Letting

$$\Delta_{mn} \equiv \left\| \frac{\partial}{\partial s}\left(\frac{\mathbf{B}_1\cdot\boldsymbol{\nabla} s}{\mathbf{B}\cdot\boldsymbol{\nabla}\phi}\right)\right\|_{mn} \tag{5}$$

be an (mn) Fourier harmonic, and $\|f\|$ be the jump of a quantity f across the rational surface, then the component of $\mathbf{B}_{\rm surf}$ which is normal, $\mathbf{n} = \boldsymbol{\nabla} s/|\boldsymbol{\nabla} s|$, to the resonant surface approximately is [6]

$$b^{\perp}_{mn} = |\mathbf{B}_{\rm surf}\cdot\mathbf{n}|_{mn} \approx \frac{1}{2}\frac{(F_T')^2}{|\sqrt{g}|B^2}\frac{\Delta_{mn}}{2\pi\, m}\oint B|\boldsymbol{\nabla} s|{\rm d}\theta{\rm d}\phi \quad . \tag{6}$$

Without an approximation, the Fourier harmonics $b^{\perp}_{mn}$ are computed by the BNORM code [12], which takes a general geometry into account. The width, $\mathcal{W}$, of the island that would be produced at the rational surface in the absence of the surface current is

$$\frac{\mathcal{W}}{a_{\rm minor}} = 2\sqrt{\left|\frac{2R}{m\,\iota'\sqrt{s}\,a_{\rm minor}}\frac{b^{\perp}_{mn}}{B}\right|} \quad . \tag{7}$$

with R and $a_{\rm minor}$ the major and minor radii of the plasma.

This procedure for studying magnetic islands was implemented in Ref. [11] using CAS3D to investigate the islands driven by a given displacement of the plasma edge for a W7-X variant. A similar concept has been developed for perturbed tokamaks in the IPEC code [13, 14]. For stellarator cases the basic idea was realized in the NSTAB code [15, 16], without giving an estimate for the island width from the discontinuous normal displacement.

If the shape of the outermost plasma surface is fixed, the drive of islands in stellarator equilibria by the natural resonances can be determined by extremizing δW for an approximate equilibrium, for which $\delta^1 W \neq 0$, with the

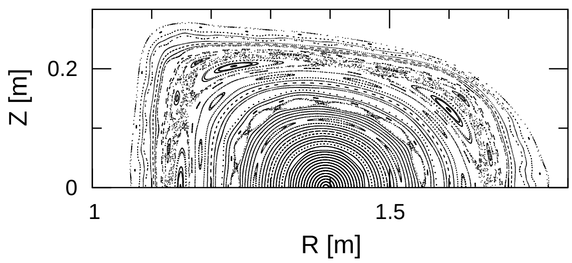

FIG. 1: PIES computation for an NCSX-type fixed-boundary equilibrium at $\langle\beta\rangle \approx 0.04$. Half of the triangular cross-section is shown.

boundary conditions that ξ^s vanish at the plasma edge, $s = 1$, and be regular at $s = 0$. Extremizing δW for an approximate equilibrium with general boundary conditions at the plasma edge, so that the plasma boundary may change, gives the general solution for which the jumps in ξ^s at the rational surfaces describe the drive for islands.

To illustrate this method, consider a fixed-boundary finite-plasma-β equilibrium representing the National Compact Stellarator Experiment (NCSX), which has been designed at Princeton Plasma Physics Laboratory [17, 18]. In Fig. 1 the island structure is shown which was computed by the PIES code in its fixed-boundary version for an NCSX-type case at $\beta \equiv 2\mu_0 p/B^2 = 0.04$. The same plasma boundary and the PIES rotational transform and pressure profile formed the input for the VMEC code, which was used in its fixed-boundary version, too. The pressure profile was slightly modified by narrow, differentiably connected, flat regions which were introduced at the principal natural resonances. The equation for the parallel current, $\mathbf{B} \cdot \boldsymbol{\nabla}(j_{||}/B) = -\boldsymbol{\nabla} \cdot \mathbf{j}_\perp$, gives an infinite $j_{||} = \mathbf{j} \cdot \mathbf{B}/B$ at rational surfaces unless either the resonant Fourier harmonic of the coordinate Jacobian or the pressure gradient p'_n is zero. On the natural rational surfaces, $\iota = n/m$, the resonant Fourier harmonic of the Jacobian is, in general, not zero, so the plasma pressure gradient must vanish.

According to Eqs. (5), (6) and (7), the square of the island width is proportional to the jump in the resonant harmonic of the flux derivative of the normal perturbed magnetic field at the respective resonant surface. Fig. 2 shows the dominant resonant harmonics of $\mathbf{B}_1 \cdot \boldsymbol{\nabla} s$ as calculated by the CAS3D code for the NCSX-type case of Fig. 1. At their respective resonant surfaces, the resonant harmonics are not smooth, as seen, for example, in the $(5, -3)$ harmonic (solid line) near $s \approx 0.5$. In Tab. I, the approximate $b^\perp_{mn}$ of Eq. (6) and the BNORM results are compared for the NCSX-type case considered here. The quality of the approximate $b^\perp_{mn}$ of Eq. (6) is found to be good. The island width data show a good agreement of PIES and CAS3D results. In this benchmark study, the single-processor time requirement of the CAS3D code-suite is smaller than that of PIES by a factor of about 4. In the CAS3D code-suite, the basic VMEC run was done in a radially highly resolved, and thus time-consuming way, taking $\approx 80\%$ of the total time. The CAS3D part proper takes only 8% of the total time, which is beneficial for an iterative process involving only the CAS3D part for determining a *healed* equilibrium without or with smaller islands.

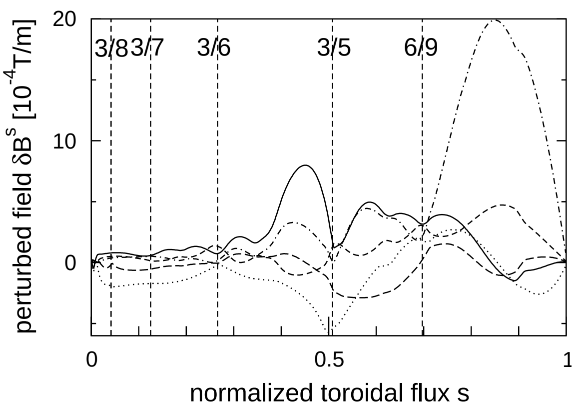

FIG. 2: Dominant resonant harmonics of $\mathbf{B}_1 \cdot \boldsymbol{\nabla} s$ versus normalized toroidal flux from a CAS3D calculation for the NCSX-type equilibrium of Fig. 1. Legend for harmonics: 3/5 solid; 3/6 dotted; 3/7 short-dashed; 3/8 long-dashed; 6/9 dot-dashed. The vertical dashed lines indicate the resonant surfaces.

A recent application of the perturbed-equilibrium method to a finite-β Wendelstein 7-X case (W7-X, [19]) complements the PIES–CAS3D benchmark [20]. The W7-X optimized stellarator was designed and is currently being built at the Institut für PlasmaPhysik, IPP. With its auxilliary coil system, W7-X will also provide for operation regimes in which the influence of low-order islands inside the plasma can be experimentally assessed. The magnetic topology of such a case was computationally studied in a $\langle\beta\rangle = 0.05$ W7-X equilibrium with the 5/6 *natural* island chain inside the plasma. The plasma boundary obtained from a free-boundary PIES calculation was analyzed with the CAS3D code retaining the plasma boundary found by PIES. With the CAS3D code-suite the widths of the flat pressure region and the island have been matched. Then the CAS3D island width exceeds the PIES result of $\mathcal{W} = 0.027$ m by 1 mm.

The accuracy with which the perturbed equilibrium approximates the true equilibrium has been checked [6] for the case of axisymmetry, where no islands can arise, and has also been benchmarked against a helically perturbed cylindrical equilibrium [21]. The island drive predicted by the CAS3D perturbed-equilibrium code has been successfully benchmarked with the IPEC code, which has a similar logic but is restricted to perturbations to axisymmetric equilibria [22].

TABLE I: Resonant harmonics of the normal component of the surface current field at resonant surfaces, $\iota = n/m$: $b^\perp_{mn}$ from the general-geometry BNORM code [12] compared to the approximation of Eq. (6). Island width, $\mathcal{W}$, from the PIES and CAS3D codes, normalized to the minor radius $a_{\rm minor} = 0.32$ m.

resonance	$b^\perp_{mn}$ (10^{-4}T)		$\mathcal{W}/a_{\rm minor}$	
$\iota = n/m$	BNORM	Eq. (6)	PIES	CAS3D
3/5	6.455	6.913	0.110	0.103
3/6	0.878	0.865	0.041	0.040
3/7	0.210	0.211	0.023	0.019
3/8	0.606	0.676	0.040	0.036
6/9	1.253	1.243	0.050	0.040

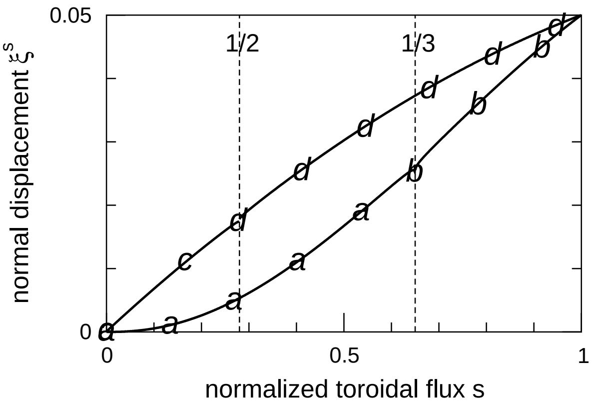

FIG. 3: A CAS3D computation with inhomogeneous problem and inhomogeneous boundary conditions finds the normal displacement that restores a *perfect* cylinder equilibrium from a *three-dimensionally distorted* cylindrical case. Labels: a for the $(3,-1)$ harmonic inside the $\iota = 1/3$ surface, b outside; c for the $(2,-1)$ harmonic inside the $\iota = 1/2$ surface, d outside.

If one chooses the external magnetic fields so the jumps in the resonant normal displacement harmonics vanish, then the drive for islands vanishes throughout the plasma. The result of an example CAS3D calculation is shown in Fig. 3. A cylindrical plasma was deformed by two helical perturbations with different helicities to constitute a general-geometry case with islands. An inhomogeneous boundary condition is used for ξ^s, so that the resonant harmonics become continuous at the respective resonances. As a result, the non-deformed plasma is restored to a very good approximation.

In summary, we have demonstrated that numerical tools originally developed for studying MHD stability also can be used as perturbed equilibrium codes to increase the accuracy of non-axisymmetric plasma equilibria and to assess the importance of magnetic islands in those equilibria. This approach promises a computationally efficient and theoretically insightful approach to both the design of stellarator plasmas with good magnetic surfaces, and the response of these plasmas to externally applied error fields.

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* Electronic address: `Carolin.Nuehrenberg@ipp.mpg.de`

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