Linear Tearing Mode Stability Equations for a Low Collisionality Toroidal Plasma

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Abstract

Tearing mode stability is normally analysed using MHD or two-fluid Braginskii plasma models. However for present, or future, large hot tokamaks like JET or ITER the collisionality is such as to place them in the banana regime. Here we develop a linear stability theory for the resonant layer physics appropriate to such a regime. The outcome is a set of 'fluid' equations whose coefficients encapsulate all neoclassical physics: the neoclassical Ohm's law, enhanced ion inertia, cross field transport of particles, heat and momentum all play a role. While earlier treatments have also addressed this type of neoclassical physics we differ in incorporating the more physically relevant 'semi-collisional fluid' regime previously considered in cylindrical geometry; semi-collisional effects tend to screen the resonant surface from the perturbed magnetic field, preventing reconnection. Furthermore we also include thermal physics, which may modify the results. While this electron description is of wide relevance and validity, the fluid treatment of the ions requires the ion banana orbit width to be less than the semi-collisional electron layer. This limits the application of the present theory to low magnetic shear – however this is highly relevant to the sawtooth instability - or to colder ions. The outcome of the calculation is a set of one-dimensional radial differential equations of rather high order. However, various simplifications that reduce the computational task of solving these are discussed. In the collisional regime, when the set reduces to a single second order differential equation, the theory extends previous work by Hahm et al (*Phys Fluids* ***31*** *3709 (1988)*) to include diamagnetic-type effects arising from plasma gradients, both in Ohm's law and the ion inertia term of the vorticity equation. The more relevant semi-collisional regime pertaining to JET or ITER, is described by a pair of second order differential equations, extending the cylindrical equations of Drake et al. (*Phys Fluids* ***26*** *2509 (1983)*) to toroidal geometry.

1. Introduction

Magnetic reconnection is believed to play a role in important tokamak phenomena such as sawteeth, neoclassical tearing modes (NTMs) and disruptions. It is normally analysed using resistive MHD or two-fluid Braginskii plasma models. Present large tokamaks, however, operate in a low collisionality regime where a kinetic model is appropriate. This was recognised in the development of neoclassical tearing mode theory, in both linear [1, 2] and non-linear [3, 4] situations, where the perturbed bootstrap current provides an instability source. A related theory of neoclassical bootstrap current driven, twisting parity ballooning modes was presented in Ref. 5. In these theories one can develop a systematic expansion procedure about a resonant surface, m = nq, where q is the safety factor, in which effective, fluid-like equations are derived but whose coefficients encapsulate kinetic effects, such as particle trapping: 'neoclassical fluid' equations. The linear theory presented in Refs. 2 and 5 was appropriate to a situation where these fluid-like equations were more collisional, in the sense that the collision frequencies $\nu_{e,i}$ exceeded the mode frequency, ω. Thus it omitted 'semi-collisional' electron effects occurring when $\omega\nu_e \sim k_{\|}^2 v_{the}^2$, with $k_{\|}$ a parallel wave-number and v_{the} the electron thermal velocity, previously considered in cylindrical geometry [6, 7, 8, 9]: in this regime there is a balance between parallel diffusive transport and the mode frequency. These semi-collisional effects tend to screen the resonant surface from the perturbed magnetic field, preventing reconnection and one finds that tearing mode instability requires large Δ' [8], where Δ' is the familiar tearing mode stability parameter.

Reference 8 has given the criterion for the validity of this semi-collisional theory as $\beta_e\left(L_s/L_n\right)^2 > 1$, where $\beta_e = 2\mu_0 n_e T_e/B^2$ is the electron beta, $L_s = Rq/\hat{s}$ (where $\hat{s} = (r/q)(dq/dr)$ is the magnetic shear) is the shear length and L_n is the density scale-length (R is the tokamak major radius, a the minor radius and r the radius of a given flux surface). Large tokamaks such as JET, and eventually ITER, operate in a regime where the collisional limit is not justified. Thus if we take typical ITER parameters [$n_e(r/a = 0.3) \sim 10^{20}\,m^{-3}$, $T_e(r/a = 0.3) \sim 22keV$, B = 5.7T, R/a = 2.9], assume q = 1 and that this lies near r/a ~ 0.3, and $L_n \sim a$, then $\beta_e\left(L_s/L_n\right)^2 \sim 0.12/\hat{s}^2$. Clearly for $\hat{s} << 1$, the semi-collisional regime is appropriate, a situation very relevant to sawtooth modelling.

Here we develop a theory that incorporates these semi-collisional effects into the neoclassical formalism. Furthermore, we include the effects of radial temperature gradients and thermal transport, absent from some earlier treatments (although electron neoclassical transport can be retained, its effect is negligible). The inclusion of thermal effects is of interest because there is evidence, e.g. from T-10 [10], that the radial electron temperature gradient, dT_e/dr, plays a role in the sawtooth phenomenon, with sawteeth being triggered when it exceeds a critical value. For the narrow semi-collisional resonant layers under discussion here, the question arises of whether to treat the ions as magnetised, $k_{\perp}\rho_i \leq 1$, or un-magnetised, $k_{\perp}\rho_i \gg 1$, where $k_{\perp}$ is a perpendicular wave-number and ρ_i is the ion Larmor radius. This has only been addressed within a cylindrical model [11] (although the case where the resonant layer is less than ρ_s, the ion Larmor radius at the sound speed, has been treated for the cold ion fluid model in toroidal geometry [12]). In this work we consider the magnetised case, which can be justified at low magnetic shear, $\hat{s} << 1$, a situation again relevant to the sawtooth instability, or for cold ions. The opposite case is extremely challenging, and will be deferred to later work, but suffice it to say that the effects of large ion orbits provide strong stabilisation [11].

In the following sections we develop a set of equations to describe tearing mode stability, namely coupled equations for the perturbed magnetic field, electrostatic potential and electron and ion densities, parallel flows, including the bootstrap current, and temperatures. Thus, Section 2 deals with the solution of the electron gyro-kinetic equation and the corresponding ion equation. Section 3 addresses Maxwell's equations and Ohm's law, while Section 4 develops the key vorticity equation. In Section 5 we consider some particular cases of interest particular. Firstly we address the collisional limit, $\omega\nu_e >> k_{||}^2 v_{the}^2$, obtaining the generalisation of the results of Ref. 2 to include the effects of temperature gradients. We also obtain the form of the equations for the toroidal version of the semi-collisional mode, appropriate when $\omega\nu_e \sim k_{||}^2 v_{the}^2$, generalising the cylindrical results of in Ref. 8 to toroidal geometry. An analytic solution of the stability problem posed by this latter set of equations will be discussed in a later paper. In the final section we discuss the stability problem presented by our general set of equations, their limitations, some possible simplifications and some plausible implications. Some details of the calculations concerning the vorticity equation, including the introduction of the notation of Glasser et al. [13] for toroidal geometry, appear in Appendix A. For convenience, Appendix B collects together some of the many symbols and notation introduced in the text.

2. The Gyro-kinetic Equations

(i) General Discussion

The gyro-kinetic equation for species j is [5, 14]

$$\left(v_{||}\mathbf{b}+\mathbf{v}_{dj}\right).\nabla g_j - i\omega g_j - C_j(g_j) = -i\frac{e_j}{T_j} f_{0j}(\omega-\omega_{*j}^T)\left[J_0(z_j)(\Phi - v_{||}A_{||}) + \frac{v_\perp}{k_\perp}\tilde{B}_{||}J_1(z_j)\right] \tag{1}$$

where $\mathbf{b}$ is a unit vector along the magnetic field, $v_{||}$ is the particle velocity along the magnetic field, Φ is the perturbed electrostatic potential, $A_{||}$ is the perturbed parallel component of the vector potential, $\hat{B}_{||}$ is the perturbed parallel magnetic field and we have written the perturbed distribution as

$$\delta f_j = -\frac{e_j\Phi}{T_j} f_{0j} + g_j e^{iL_j} \tag{2}$$

Here $L_j = \left|\mathbf{k}\times\mathbf{v}_\perp/\Omega_j\right|$, with $v_\perp$ the velocity perpendicular to the magnetic field, is the gyro-phase factor, $J_{0,\,1}$ are Bessel functions of argument $z_j = k_\perp v_\perp/\Omega_j$, f_{0j} are Maxwellian distributions and

$$\omega_{*j}^T = \omega_{*j}\left[1+\eta_j(u_j^2-\frac{3}{2})\right],\ u_j^2 = \frac{m_j v^2}{2T_j}\ ,\ \omega_{*j} = -n\frac{T_j}{e_j}\ \frac{d\ln n_j}{\partial\psi}$$

$$\mathbf{v}_{dj} = \frac{\mathbf{b}}{\Omega_j} \times \left(\frac{v_\perp^2}{2} \nabla \ln B + v_\parallel^2 \boldsymbol{\kappa} \right), \qquad \Omega_j = \frac{e_j B}{m_j}, \qquad \boldsymbol{\kappa} = \mathbf{b}.\nabla\mathbf{b} \tag{3}$$

with v the particle speed, Ω_j the cyclotron frequency of species j, $\boldsymbol{\kappa}$ is the curvature vector, n the toroidal mode number and all gradients are taken at constant (μ, v) with μ the magnetic moment, or (λ v) with $\lambda = 2\mu/v^2$ (thus $v_\parallel = \sigma v\sqrt{1-\lambda B}$, with $\sigma = \mathrm{sign}(v_\parallel)$). If θ is defined so that the safety factor, $q = \mathbf{B}.\nabla\varphi / \mathbf{B}.\nabla\theta$, is a flux function, then

$$\nabla_\parallel = \mathbf{b}.\nabla\theta \frac{\partial}{\partial\theta} + \mathbf{b}.\nabla\varphi \frac{\partial}{\partial\varphi} = \frac{I}{R^2 B} \left(\frac{\partial}{\partial\varphi} + \frac{1}{q}\frac{\partial}{\partial\theta} \right) \tag{4}$$

with $I = RB_\varphi$, so if we write g_j as

$$g_j(r,\theta,\varphi) = \hat{g}_j(r,\theta)e^{i(n\varphi - m\theta)} \tag{5}$$

then

$$\nabla_\parallel g_j = \frac{I}{R^2 Bq} \left[i(nq - m)\, \hat{g}_j + \frac{\partial \hat{g}_j}{\partial\theta} \right] e^{i(n\varphi - m\theta)} \tag{6}$$

where

$$nq - m \simeq nq'x \ , \ q' = \frac{dq}{d\psi} \tag{7}$$

with $x = \psi - \psi_s$, ψ_s being the resonant surface where m = nq, and prime denotes a derivative with respect to x.. It is convenient later to introduce

$$\hat{\nabla}_\parallel = \frac{I}{R^2 Bq} \frac{\partial}{\partial\theta} \tag{8}$$

Dropping carets we obtain

$$\frac{Iv_\parallel}{R^2 Bq} \left(\frac{\partial g_j}{\partial\theta} + inq'xg_j \right) + \mathbf{v}_{dj}.\nabla g_j - i\omega g_j - C_j(g_j)$$

$$= -\frac{ie_j}{T_j} f_{0j}\left(\omega - \omega_{*j}^T\right) \left[J_0(\Phi - v_\parallel A_\parallel) + J_1 \frac{v_\perp \tilde{B}_\parallel}{k_\perp} \right] \tag{9}$$

which we rewrite by expanding the Bessel functions for small z_j and by introducing h_j:

$$g_j = \frac{e_j}{T_j}\left(1 - \frac{\omega_{*j}^T}{\omega}\right)\Psi f_{0j} + h_j \tag{10}$$

where

$$A_{||} = \nabla_{||}\Psi / i\omega \tag{11}$$

Thus the parallel electric field is given by

$$E_{||} = -\frac{I}{R^2 Bq}\left(\frac{\partial}{\partial\theta} + inq'x\right)(\Phi - \Psi) \tag{12}$$

Then the fundamental kinetic equations are

$$\frac{Iv_{||}}{R^2 Bq}\left(\frac{\partial h_j}{\partial\theta} + inq'xh_j\right) + \mathbf{v}_{dj}.\nabla h_j - i\omega h_j - C_j(h_j)$$
$$= -\frac{ie_j}{T_j} f_{0j}\left(\omega - \omega_{*j}^T\right)\left(\Phi - \Psi + \frac{v_\perp{}^2\tilde{B}_{||}}{2\Omega_j}\right) + \frac{e_j}{T_j}\left(1 - \frac{\omega_{*j}^T}{\omega}\right) f_{0j}\mathbf{v}_{dj}.\nabla\Psi$$

(13)

We will solve these equations by introducing appropriate ordering schemes for electrons and ions. It is convenient to introduce an ordering parameter ε, where $\varepsilon = (\rho_e/\delta)^\lambda$ with δ the resonant layer width (where semi-collisional effects are manifest) and λ an exponent to be determined below, and $(m_e/m_i)^{1/2}$, defining $(m_e/m_i)^{1/2} = \varepsilon^\mu$. We can then exhibit the ordering of terms on the left-hand side of eqn. (13):

	ω_{bj}	$k_{\|\|}v_{thj}$	ω_{drj}	ω	ν_j	$k_r^2 D_{\perp j}^{neo}$	
j = electrons	1	$\varepsilon^{\lambda+1}$	ε^λ	$\varepsilon^{\lambda+2}$	ε^λ	$\varepsilon^{3\lambda}$	(14)
j = ions	1	$\varepsilon^{\lambda+1}$	$\varepsilon^{\lambda-\mu}$	$\varepsilon^{\lambda+2-\mu}$	ε^λ	$\varepsilon^{3\lambda-2\mu}$	

Here ω_{bj} is the bounce/transit frequency, $k_{||}v_{thj}$ the transit frequency over a wavelength $Rq/\delta q'$ associated with the resonant layer width, δ, ω_{drj} the radial magnetic drift frequency, respectively, for species j. (Although not explicit in eqn. (13), we have also included $k_r^2 D_{\perp j}^{neo}$, the neoclassical collisional radial diffusion rate associated with species j, which appears during the expansion of eqn. (13) and enters our final macroscopic equations.) This ordering automatically fulfils the semi-collisional condition, $\omega\nu_e \sim k_{||}^2 v_{the}^2$. However, there are a number of constraints on the parameters λ and μ that one may wish to impose:

- To develop a systematic and convenient ε-expansion for ions: $\lambda > \mu$
- To neglect ion sound: $k_{||} v_{thi} / \omega \le 1 \Rightarrow \mu \ge 1$
- To neglect cross-field ion neoclassical transport: $\nu_i {\rho_i}^2 / \omega\delta^2 \le 1 \Rightarrow \lambda \ge 1 + \mu/2$ (equality implies retention)

These criteria are displayed in Fig. 1

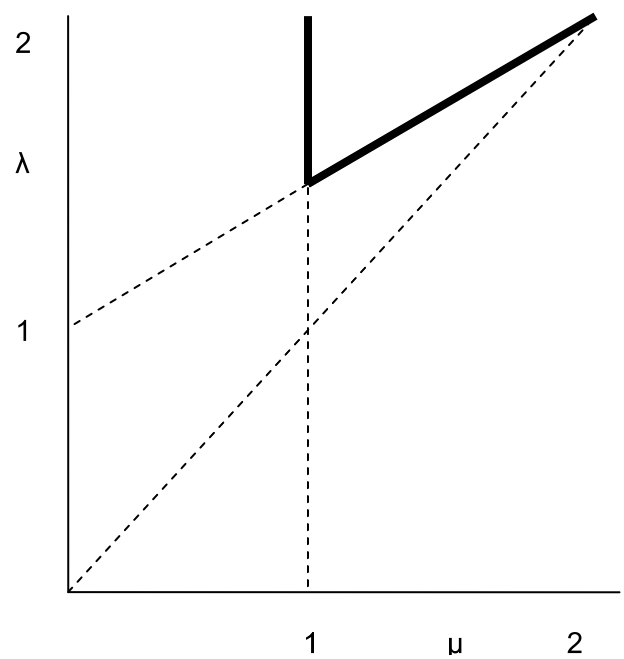

Fig. 1: The 'operating space' for choosing the exponents λ and μ; λ is involved in the exponents of the powers of $\varepsilon = (\rho_e / \delta)^\lambda$ *, representing the orders of the terms of the gyro-kinetic equations as shown in eqn. (14), while* $(m_e/m_i)^{1/2} = \varepsilon^\mu$.

Thus we could choose $\lambda = 2, \mu = 3/2;$ this allows us to ignore ion neoclassical transport and ion sound effects. Alternatively we can choose $\lambda = 7/4,\ \mu = 3/2$ which allows us to retain ion neoclassical transport. Having satisfied ourselves that there is a self-consistent ordering scheme for an expansion that allows us to ignore ion sound effects and retain ion neoclassical transport we will proceed with this by including the various physical effects at appropriate stages in the expansion, rather than employing the specific but complicated choice above, which would lead to many ‘empty orders’ in a formal expansion.

We will find that the solution of eqn. (13), order by order, mirrors that in standard neoclassical theory, with arbitrary functions being determined by collisional constraints arising from periodicity of the bounce/transit motion. For these constraints we use momentum conserving, pitch-angle scattering collision models. The lowest order solution for h_j is Maxwellian with perturbed densities, $\hat{n}_j$, and temperatures, $\hat{T}_j$, constant on a flux surface. Next order determines perturbed parallel flows from which one can compute the ion flow and bootstrap current driven by the perturbed gradients. Finally, solubility conditions in third order lead to ‘neoclassical fluid equations’ for $\hat{n}_j$ and $\hat{T}_j$ in response to the electromagnetic perturbations, which describe cross-field neoclassical transport, parallel collisional electron transport, neoclassical compressibility and the mode frequency, ω. However one can reasonably ignore the small electron cross-field transport to obtain closed algebraic expressions for the semi-collisional perturbed electron density and temperature.

(ii) Electron Solution

Dropping the e suffix on h_e for brevity and introducing the proton charge e, so that $e_j = - e$ for electrons, we have

$$\frac{Iv_{||}}{R^2Bq}\left(\frac{\partial h}{\partial\theta}+inq'xh\right)+\mathbf{v}_{de}.\nabla h-i\omega h-C_e(h)$$
$$=\frac{ie}{T_e}f_{0e}\left(\omega-\omega_{*e}^T\right)\left(\Phi-\Psi+\frac{v_\perp{}^2\tilde{B}_{||}}{2\Omega_e}\right)+\frac{e}{T_e}\left(1-\frac{\omega_{*e}^T}{\omega}\right)f_{0e}\mathbf{v}_{de}.\nabla\Psi \qquad (15)$$

We order the various terms in eqn. (15) such that in zeroth order

$$\frac{\partial h_0}{\partial\theta}=0 \;\Rightarrow\; h_0=h_0(v,\lambda,x,\sigma) \qquad (16)$$

and in first order

$$\frac{Iv_{||}}{R^2Bq}\frac{\partial h_1}{\partial\theta}=C_e(h_0)-v_{dre}\frac{\partial h_0}{\partial x}+\frac{e}{T_e}\left(1-\frac{\omega_{*e}^T}{\omega}\right)f_{0e}v_{dre}\frac{\partial\Psi}{\partial x} \qquad (17)$$

In eqn. (13)

$$v_{drj} = \mathbf{v}_{dj}.\nabla\psi \quad = \quad \frac{Iv_{||}}{R^2Bq}\,\frac{\partial}{\partial\theta}\left(\frac{Iv_{||}}{\Omega_j}\right) \tag{18}$$

and everything but the collision operator is annihilated by an orbit average. For passing particles we annihilate by applying the operator $\left\langle B(...)\,/v_{||}\right\rangle$, where

$$\langle ...\rangle \quad = \quad \oint\frac{(...)\,d\theta}{\mathbf{B}.\nabla\theta} \;\Big/\; \oint\frac{d\theta}{\mathbf{B}.\nabla\theta} = \oint(...)R^2 d\theta / \oint R^2 d\theta, \tag{19}$$

since

$$\frac{1}{\mathbf{B}.\nabla\theta} = \frac{q}{\mathbf{B}.\nabla\varphi} = \frac{qR^2}{I} \tag{20}$$

For trapped particles we integrate along the bounce orbit, summing over σ in the usual way [15] to obtain the constraint:

$$\left\langle \frac{B}{v_{||}}\; C_e\left(h_0\right)\right\rangle = 0 \tag{21}$$

which determines h_0, yielding

$$h_0 = \left[\frac{\hat{n}_e}{n_e} + \left(u^2 - \frac{3}{2}\right)\frac{\hat{T}_e}{T_e}\right] f_{0e} \tag{22}$$

It should be stressed that $\hat{n}_e$ and $\hat{T}_e$ (and later the same quantities for ions) only represent the contributions to perturbed density and temperature from h_e , whereas in calculating the full quantities, $\tilde{n}_e$ and $\tilde{T}_e$ one must recall we have represented f_e in terms of first g_e and then h_e. One can now integrate eqn. (17) to obtain

$$h_1 \quad = \quad \frac{Iv_{||}}{\Omega_e}\left[\frac{e}{T_e}\left(1 - \frac{\omega_{*e}^{T}}{\omega}\right)\Psi' f_{0e} - h_0'\right] \; + \; \bar{h}_1\left(v,\lambda,x,\sigma\right) \tag{23}$$

The next order equation is

$$v_{||}\;\hat{\nabla}_{||}h_2 \;+\; v_{||}\;\hat{\nabla}_{||}\left(\frac{Iv_{||}}{\Omega_e}\right)\frac{\partial h_1}{\partial x} \;+\; \frac{iv_{||}\,I\,nq'x}{qR^2B}\,h_0 \;=\; C_e\left(h_1\right) \tag{24}$$

Applying the operation used in eqn. (21), we obtain the constraint

$$\left\langle \frac{B}{v_{\|}}\, C_e\left(h_1\right)\right\rangle = \frac{iInq'x}{q} \quad \left\langle \frac{1}{R^2}\right\rangle h_0 \tag{25}$$

which determines $\overline{h}_1\left(v, \lambda, x, \sigma\right)$.

We introduce a momentum conserving, pitch angle scattering electron collision operator:

$$\begin{gathered}
C_e(h) = \nu_{ei}(v)\left[Lh + \frac{m_e}{T_e} v_{\|} u_{\|i} f_{0e}\right] + \nu_{ee}(v)\left[Lh + \frac{m_e}{T_e} v_{\|} u^{*}_{\|e} f_{0e}\right] \\
u_{\|i} = \left(1/n_i\right)\int d^3v\; v_{\|} h_i d^3v,\;\; u^{*}_{\|e} = \int d^3v\; \nu_{ee}(v) v_{\|} h_{1e} \Big/ \int d^3v \left(m_e v_{\|}^2 / T_e\right) \nu_{ee} f_{0e} \\
\nu_{ei} = \nu_0 / \left(m_e v^2 / 2T_e\right)^{3/2};\; \nu_{ee} = \nu_0 \phi\left(m_e v^2/2T_e\right) / \left(m_e v^2 / 2T_e\right)^{3/2} \\
\nu_0 = \frac{\sqrt{2}\pi n_e e^4 \ln\Lambda}{m_e^{1/2} T_e^{3/2}}
\end{gathered} \tag{26}$$

where the Lorentz collision operator, L, is given by

$$L \equiv \frac{2\, v_{\|}}{v^2 B}\, \frac{\partial}{\partial\lambda}\, \lambda v_{\|}\, \frac{\partial}{\partial\lambda} \tag{27}$$

and

$$\phi(x) = \left(1 - 1/2x\right)\eta(x) + \eta'(x),\;\; \eta(x) = \left(2/\sqrt{\pi}\right)\int_0^{\infty} e^{-t} t^{1/2} dt,\; \eta'(x) = d\eta/dx \tag{28}$$

Then we obtain an equation for $\overline{h}_1$:

$$\begin{aligned}
-\frac{IB}{\Omega_e}\left[\frac{e}{T_e}\left(1 - \frac{\omega_*^T}{\omega}\right)\Psi' f_{0e} - h_0'\right] &+ \frac{2}{v^2}\, \frac{\partial}{\partial\lambda}\, \lambda\left\langle v_{\|}\right\rangle \frac{\partial \overline{h}_1}{\partial\lambda} \\
&= \frac{iInq'x}{q\nu_{ei}(v)}\left\langle \frac{1}{R^2}\right\rangle h_0 - \frac{m_e}{T_e} f_{0e}\left[\frac{\nu_{ei}}{\nu_e}\left\langle B u_{\|i}\right\rangle + \frac{\nu_{ee}}{\nu_e}\left\langle B u^{*}_{\|e}\right\rangle\right]
\end{aligned} \tag{29}$$

with $\nu_e(v) = \nu_{ee}(v) + \nu_{ei}(v)$, so that

$$\begin{aligned}
h_1 = &\frac{I}{\Omega_e}\left[v_{\|} - \frac{B}{B_0}\hat{V}_{\|}\right]\left[\frac{e\Psi'}{T_e}\left(1 - \frac{\omega_{*e}^T}{\omega}\right) f_{0e} - h_0'\right] \\
&- i\frac{Inq'x}{q\nu_e}\left\langle \frac{1}{R^2}\right\rangle \frac{\hat{V}_{\|}}{B_0} h_0 + \frac{m_e}{T_e}\frac{\hat{V}_{\|}}{B_0} f_{0e}\left[\frac{\nu_{ei}}{\nu_e}\left\langle B u_{\|i}\right\rangle + \frac{\nu_{ee}}{\nu_e}\left\langle B u^{*}_{\|e}\right\rangle\right]
\end{aligned} \tag{30}$$

where

$$\hat{V}_{\|}(v,\lambda,\psi,\sigma) \;=\; \frac{\sigma v^2 B_0}{2}\int_{\lambda}^{\lambda_c}\frac{d\lambda'}{\left\langle\left|v_{\|}\ (\lambda')\right|\right\rangle} \tag{31}$$

in which B_0 is arbitrary (for later convenience we let $B_0 = \left\langle B^2\right\rangle^{1/2}$) and $\lambda_c = 1/B_{max}$.

We can use eqn. (30) to calculate $u_{\|e} = \int d^3v\ v_{\|}h_{1e}/n_e$, where $\int d^3v = \sum_{\sigma} B\int \pi v^3 dv d\lambda \big/ |v_{\|}|$, in order to help determine $j_{\|}$ later:

$$\begin{aligned}u_{\|e} =& \frac{B}{\left\langle B^2\right\rangle}\frac{f_c}{(1-0.37f_c)}\left(0.57+0.06f_c\right)\left\langle Bu_{\|i}\right\rangle + \frac{I}{\Omega_e}\frac{T_e}{m_e}\left[\frac{e\Psi'}{T_e}\left(1-\frac{\omega_{*e}}{\omega}(1+\eta_e)\right)-\frac{\hat{n}_e'}{n_e}-\frac{\hat{T}_e'}{T_e}\right]\\ &-\frac{I}{\Omega_e}\frac{T_e}{m_e}\frac{B^2}{\left\langle B^2\right\rangle}\frac{f_c}{(1-0.37f_c)}\left\{\left[\frac{e\Psi'}{T_e}\left(1-\frac{\omega_{*e}}{\omega}\right)-\frac{\hat{n}_e'}{n_e}\right](0.57+0.06f_c)-\left[\frac{e\Psi'}{T_e}\frac{\omega_{*e}}{\omega}\eta_e+\frac{\hat{T}_e'}{T_e}\right]\left(1.07-0.44f_c\right)\right\}\\ &-\frac{iInq'x}{q}\left\langle\frac{1}{R^2}\right\rangle\frac{B}{\left\langle B^2\right\rangle}\frac{f_c}{(1-0.37f_c)}\frac{T_e}{m_e}\tau_{ei}\left[(1.84-0.33f_c)\frac{\hat{n}_e}{n_e}+(4.46-1.26f_c)\frac{\hat{T}_e}{T_e}\right]\end{aligned} \tag{32}$$

Here the circulating fraction of particles is given by $f_c = \dfrac{3B_0^2}{4}\displaystyle\int_0^{\lambda_c}\frac{\lambda d\lambda}{\left\langle\sqrt{1-\lambda B}\right\rangle}$ and

$\tau_{ei} = 3\sqrt{\pi}\big/4\nu_0$ is the electron-ion momentum exchange time.

The equation for h_3 appears in next order:

$$\begin{aligned}& v_{\|}\hat{\nabla}_{\|}h_3 \;+\; v_{\|}\hat{\nabla}_{\|}\left(\frac{Iv_{\|}}{\Omega_e}\right)\frac{\partial h_2}{\partial\psi} \;+\; \frac{iInq'x}{qR^2B}v_{\|}h_1 \;-\; i\omega h_0\\ &= C_e\left(h_2\right) \;+\; \frac{ie}{T_e}\ f_{0e}\ \left(\omega\ -\ \omega_{*e}^{T}\right)\ \left(\Phi - \Psi + \frac{v_{\perp}{}^2\tilde{B}_{\|}}{2\Omega_e}\right)\end{aligned} \tag{33}$$

to which we apply the operator

$$\left\langle\int(...)\ d^3v\right\rangle \;=\; \sum_{\sigma}\left\langle B\int(...)\ \frac{\pi v^3 dv d\lambda}{|v_{\|}|}\right\rangle \tag{34}$$

that annihilates the first terms on both the left and the right. We evaluate the other terms obtaining a first equation for $\hat{n}_e$ and $\hat{T}_e$ (the details can be found in Ref. 16). For completeness we retain the small neoclassical cross-field transport terms. These arise from the annihilation of the h_2 term in eqn. (33), using eqn. (24) for h_2; it is this interaction of collisions and magnetic drifts that gives rise to neoclassical effects. This is a lengthy calculation and is recorded in Ref. 16, where it is performed for the simpler Lorentz collision model as an illustration. These terms are negligibly small and will henceforth be ignored in this paper. Furthermore we ignore small terms, $\sim (k_{\|}u_{\|i}/\omega)$, which is justified later by the calculation of $u_{\|i}$ in eqn. (49). The final result is:

$$
\begin{aligned}
& i\omega\frac{\hat{n}_e}{n_e}+\frac{ie}{T_e}\langle\Phi-\Psi\rangle(\omega-\omega_{*e})-i\left\langle\frac{\tilde{B}_{\|}}{B}\right\rangle\left[\omega-\omega_{*e}(1+\eta_e)\right] \\
& \quad -D_{\perp e}^{neo}\left[\frac{e\Psi''}{T_e}\left(1-\frac{\omega_*}{\omega}\left(1-\frac{\eta_e}{2}\right)\right)-\frac{\hat{n}''}{n_e}+\frac{\hat{T}''}{2T_e}\right] \\
& =\left(\frac{Inq'x}{q\left\langle B^2\right\rangle^{1/2}}\left\langle\frac{1}{R^2}\right\rangle\right)^2\frac{f_c}{(1-0.37f_c)}\frac{T_e}{m_e\nu_0}\left[(2.44-0.45f_c)\frac{\hat{n}_e}{n_e}+(5.94-1.69f_c)\frac{\hat{T}_e}{T_e}\right]
\end{aligned}
\tag{35}
$$

where , for the Lorentz collision model, $D_{\perp e}^{neo}=\frac{T_e I^2}{m_e\Omega_0^2\,\tau_{ei}}\left\langle\frac{B_0^2}{B^2}-f_c\right\rangle$ is the neoclassical cross-field particle diffusivity. A similar equation for the electron energy balance can be derived by applying the operation

$$
\left\langle\int(\ldots)\ \frac{m_e v^2}{2}d^3v\right\rangle \tag{36}
$$

to eqn. (33) to obtain:

$$
\begin{aligned}
& \frac{3}{2}i\omega\left(\frac{\hat{n}_e}{n_e}+\frac{\hat{T}_e}{T_e}\right)+\frac{3}{2}i\frac{e}{T_e}\langle\Phi-\Psi\rangle\left(\omega-\omega_{*e}(1+\eta_e)\right)-\frac{5}{2}i\left\langle\frac{\tilde{B}_{\|}}{B}\right\rangle\left[\omega-\omega_{*e}(1+2\eta_e)\right] \\
& \quad -\chi_{\perp e}^{neo}\left[\frac{e\Psi''}{T_e}\left(1-\frac{\omega_*}{\omega}\left(1+\frac{\eta_e}{2}\right)\right)-\frac{\hat{n}''}{n_e}-\frac{\hat{T}''}{2T_e}\right] \\
& =\left(\frac{Inq'x}{q\left\langle B^2\right\rangle^{1/2}}\left\langle\frac{1}{R^2}\right\rangle\right)^2\frac{f_c}{(1-0.37f_c)}\frac{T_e}{m_e\nu_0}\left((9.61-2.77f_c)\frac{\hat{n}_e}{n_e}+(17.15-5.47f_c)\frac{\hat{T}_e}{T_e}\right)
\end{aligned}
\tag{37}
$$

where $\chi_{\perp e}^{neo} \sim D_{\perp e}^{neo}$ is the neoclassical electron thermal conductivity. Ignoring the small neoclassical electron cross-field transport allows us to solve these equations algebraically to obtain explicit expressions for $\hat{n}_e$ and $\hat{T}_e$:

$$-D\frac{\hat{n}_e}{n_e} = (1+\lambda_4 s^2 - \lambda_2 s^2)\frac{e}{T_e}\left\langle \Phi - \Psi \right\rangle(1-\frac{\omega_{*e}}{\omega}) + \lambda_2 s^2 \frac{e}{T_e}\left\langle \Phi - \Psi \right\rangle\frac{\omega_{*e}}{\omega}\eta_e$$
$$-\left\langle \frac{\tilde{B}_{\|}}{B} \right\rangle(1-\frac{\omega_{*e}}{\omega})\left[1+\lambda_4 s^2 - \frac{5}{3}\lambda_2 s^2\right] + \left\langle \frac{\tilde{B}_{\|}}{B} \right\rangle\frac{\omega_{*e}}{\omega}\eta_e\left[1+\lambda_4 s^2 - \frac{10}{3}\lambda_2 s^2\right] \quad (38)$$

and

$$-D\frac{\hat{T}_e}{T_e} = (\lambda_1 s^2 - \lambda_2 s^2)\frac{e}{T_e}\left\langle \Phi - \Psi \right\rangle(1-\frac{\omega_{*e}}{\omega}) - (1+\lambda_1 s^2)\frac{e}{T_e}\left\langle \Phi - \Psi \right\rangle\frac{\omega_{*e}}{\omega}\eta_e$$
$$-\left\langle \frac{\tilde{B}_{\|}}{B} \right\rangle(1-\frac{\omega_{*e}}{\omega})\left[\frac{2}{3}+\frac{5}{3}\lambda_1 s^2 - \lambda_3 s^2\right] + \left\langle \frac{\tilde{B}_{\|}}{B} \right\rangle\eta_e\frac{\omega_{*e}}{\omega}\left[\frac{7}{3}+\frac{10}{3}\lambda_1 s^2 - \lambda_3 s^2\right] \quad (39)$$

where

$$D = \left\{1+(18.55-6.05f_c)s^2+(15.90-7.59f_c+0.63f_c^2)s^4\right\} \equiv 1+d_0 s^2 + d_1 s^4,$$
$$\lambda_1 = (2.44-0.45f_c), \quad \lambda_2 = (5.94-1.69f_c),$$
$$\lambda_3 = (6.41-1.85f_c), \quad \lambda_4 = (11.43-3.65f_c) \quad (40)$$

and the semi-collisional effects are represented through

$$s^2 \equiv \frac{i\left(\mathrm{Inq'x}\left\langle\frac{1}{R^2}\right\rangle\right)^2}{q^2\left\langle B^2\right\rangle}\left(\frac{T_e}{m_e\omega\nu_0}\right)\left(\frac{f_c}{1-0.37f_c}\right) \quad (41)$$

(ii) Ion Solution

We consider the 'collisional' case, $\nu_i > \omega$ We again find solutions analogous to the electron eqns. (22) and (23) for h_0 and h_1, where we again drop the suffices i on h_i. The equation for h_2 is

$$v_{\|} \hat{\nabla}_{\|} h_2 + v_{\|} \hat{\nabla}_{\|} \left(\frac{Iv_{\|}}{\Omega_i}\right) \frac{\partial h_1}{\partial \psi} = C_i\left(h_1\right) \tag{42}$$

which provides the constraint to determine $\overline{h}_1$

$$\left\langle \frac{B}{v_{\|}} C_i\left(h_1\right)\right\rangle = 0 \tag{43}$$

For the ions we also take a model pitch-angle scattering collision operator that conserves momentum [15], as in eqn. (26):

$$C_i(h_1) = \frac{2\nu_{ii}(v)v_{\|}}{Bv^2} \frac{\partial}{\partial\lambda}\lambda v_{\|}\frac{\partial}{\partial\lambda}h_1 + \nu_{ii}(v)v_{\|} f_{0i}\frac{m_i}{T_i} u^*_{\|i} \tag{44}$$

where

$$u^*_{\|i} = \frac{T_i}{m_i}\frac{\int d^3v\ \nu_{ii}(v)v_{\|}h_1}{\int d^3v\ \nu_{ii}(v)v_{\|}^2 f_{0i}} \tag{45}$$

$$\nu_{ii}(v) = \nu_i\phi(u)/u^3;\ \ \nu_i = \frac{\sqrt{2}\pi n_e e^4 \ln\Lambda}{m_i^{1/2}T_i^{3/2}}:\ \ u = m_i v^2 \big/ 2T_i$$

and the functional form of $\phi(u)$ can be found in eqn. (28). The constraint (43) yields

$$h_1 = \frac{I}{\Omega_i}\left(v_{\|} - \frac{B}{\left\langle B^2\right\rangle^{1/2}}\hat{V}_{\|}\right)\left[\frac{e}{T_i}\left(1-\frac{\omega^T_{*i}}{\omega}\right)\Psi' f_{0i} - h_0'\right] + \frac{\left\langle u^*_{\|i}B\right\rangle}{\left\langle B^2\right\rangle} f_{0i}\frac{m_i}{T_i}\hat{V}_{\|} \tag{46}$$

Solving self-consistently for $\left\langle u^*_{\|i}B\right\rangle$ we obtain, on using the result $\gamma = \int_0^\infty \nu_i t^{5/2} e^{-t} dt \big/ \int_0^\infty \nu_i t^{3/2} e^{-t} dt \simeq 1.33$ (so that $\gamma - 3/2 = 0.17$) arising from integrals over v of $\nu_{ii}(v)$ given in Ref. 15:

$$\left\langle u_{\|i}^{*} B\right\rangle = -\frac{IB}{\Omega_i}\,\frac{T_i}{m_i}\left\{\frac{e\Psi'}{T_i}\,\left[1-\frac{\omega_{*i}}{\omega}\left(1-0.17\eta_i\right)\right]+\frac{\hat{n}_i{}'}{n_i}-0.17\frac{\hat{T}_i'}{T_i}\right\} \tag{47}$$

Then

$$u_{\|i} = -\frac{I}{\Omega_i}\,\frac{T_i}{m_i}\left\{\frac{e\Psi'}{T_i}\,\left[1-\frac{\omega_{*i}}{\omega}\left(1+\eta_i\right)\right]+\frac{\hat{n}_i{}'}{n_i}+\frac{\hat{T}_i'}{T_i}\right\}\left(1-\frac{B^2}{\left\langle B^2\right\rangle}f_c\right)+\frac{\left\langle u_{\|i}^{*}B\right\rangle}{B}\frac{B^2}{\left\langle B^2\right\rangle}f_c \tag{48}$$

so that, using eqn. (47) to eliminate $\left\langle u_{\|i}^{*}B\right\rangle$,

$$u_{\|i} = -\frac{I}{\Omega_i}\,\frac{T_i}{m_i}\left\{\frac{e\Psi'}{T_i}\,\left[1-\frac{\omega_{*i}}{\omega}\left(1+\eta_i\right)\right]+\frac{\hat{n}_i{}'}{n_i}+\frac{\hat{T}_i'}{T_i}\right\}+1.17\frac{I}{\Omega_i}\,\frac{T_i}{m_i}\frac{B^2}{\left\langle B^2\right\rangle}f_c\left\{\frac{\hat{T}_i'}{T_i}-\frac{\omega_{*i}\eta_i}{\omega}\frac{e\Psi'}{T_i}\right\} \tag{49}$$

This result justifies the neglect of the term in $O(k_{\|}u_{\|i}/\omega)$ on the right-hand sides of eqns. (35) and (37). Equations (32) and (49) will allow us to calculate $j_{\|}$ later.

The equation for h_3 appears in next order:

$$\begin{aligned} & v_{\|}\hat{\nabla}_{\|}h_3 \;+\; v_{\|}\hat{\nabla}_{\|}\left(\frac{Iv_{\|}}{\Omega_i}\right)\frac{\partial h_2}{\partial\psi} \;+\; \frac{iInq'x}{R^2Bq}v_{\|}h_0 \;-\; i\omega h_0 \\ & = C_i\left(h_2\right) \;+\; \frac{ie}{T_i}\;f_{0i}\;\left(\omega\;-\;\omega_{*i}^{T}\right)\;\left(\Phi - \Psi + \frac{v_{\perp}{}^2\tilde{B}_{\|}}{2\Omega_i}\right) \end{aligned} \tag{50}$$

where we have taken the $z_i \to 0$ limit of the Bessel functions. We apply the operator (34) which annihilates the first terms on both the left and the right to obtain

$$\omega\hat{n}_i = \; n_i\left[\frac{e\left\langle\Phi\;-\;\Psi\right\rangle}{T_i}\left(\omega - \omega_{*i}\right)+\left\langle\frac{\tilde{B}_{\|}}{B}\right\rangle\left(\omega - \omega_{*i}\;\left(1+\eta_i\right)\right)\right] \tag{51}$$

Again a similar equation for the ion energy balance can be derived. More significantly, however, there is now a contribution from $C_i(h_2)$: this corresponds to ion neoclassical cross-field transport. Following Ref. 16 one obtains

$$\frac{3i}{2}\omega\left(\frac{\hat{n}_i}{n_i}+\frac{\hat{T}_i}{T_i}\right)=\frac{3i}{2}\frac{e\langle\Phi-\Psi\rangle}{T_i}\left[\omega-\omega_{*i}(1+\eta_i)\right]+\frac{5i}{2}\left\langle\frac{\tilde{B}_{\|}}{B}\right\rangle\left(\omega-\omega_{*i}\left(1+2\eta_i\right)\right)$$
$$+\frac{T_iI^2}{m_i\Omega_{i0}^2\tau_{ii}}\left\langle\frac{\langle B^2\rangle}{B^2}-f_c\right\rangle\left[\frac{\omega_{*i}}{\omega}\eta_i\frac{e\Phi''}{T_i}-\frac{\hat{T}_i''}{T_i}\right] \tag{52}$$

Equations (51) and (52) provide an explicit expression for $\hat{T}_i$:

$$\hat{T}_i-\frac{2i}{3}\frac{T_iI^2}{m_i\Omega_{i0}^2\tau_{ii}\omega}\left\langle\frac{\langle B^2\rangle}{B^2}-f_c\right\rangle\hat{T}_i''=-e\langle\Phi-\Psi\rangle\frac{\eta_i\omega_{*i}}{\omega}$$
$$-\frac{2i}{3}\frac{T_iI^2}{m_i\Omega_{i0}^2\tau_{ii}\omega}\left\langle\frac{\langle B^2\rangle}{B^2}-f_c\right\rangle\frac{\omega_{*i}}{\omega}\eta_ie\Phi''+\frac{2}{3}T_i\left\langle\frac{\tilde{B}_{\|}}{B}\right\rangle\left[\left(1-\frac{\omega_{*i}}{\omega}\right)-\frac{7}{2}\frac{\eta_i\omega_{*i}}{\omega}\right] \tag{53}$$

The equations for the ion quantities $\hat{n}_i$ and $\hat{T}_i$ are linearly algebraic if we ignore ion neoclassical transport, otherwise they involve a second order ordinary differential equation (ODE).

3. Maxwell's Equations

To obtain equations for the perturbed fields we utilise quasi-neutrality and the parallel and perpendicular Ampère's equations for Ψ and $\tilde{B}_{\|}$, using the perturbed charges and currents calculated from the gyro-kinetic solutions.

First, quasi-neutrality implies

$$\tilde{n}_e=\tilde{n}_i \tag{54}$$

where

$$\tilde{n}_j=-\frac{n_je_j}{T_j}\Phi+\frac{n_je_j}{T_j}\left(1-\frac{\omega_{*j}}{\omega}\right)\Psi+\hat{n}_j \tag{55}$$

Thus eqn. (54) relates Φ to Ψ and $\hat{n}_{i,e}$; the $\hat{n}_{i,e}$ are independent of θ and we will see below when considering Ampère's equations for Ψ, that Ψ is also independent of θ in leading order, implying the same is true for Φ.

Ampère's laws for $A_{\|}$ and $\tilde{B}_{\|}$ provide relationships to eliminate $\tilde{B}_{\|}$ and Ψ from our equations. To evaluate $\tilde{B}_{\|}$ we must calculate the perpendicular current arising from the first order in a Larmor radius expansion of the gyro-phase factor exp(iL_i) in eqn. (2) and use it in Ampère's law. We find [5, 14]

$$\frac{\tilde{B}_{\|}}{B} = -\frac{\mu_0 \tilde{p}}{B^2} \tag{56}$$

where we have used the definitions (10) and (21) of g_j and h_j which imply

$$\tilde{p} = \sum_j \tilde{p}_j = \hat{p} + \frac{n}{\omega} p_0{}' \Psi \,; \qquad \hat{p} = \sum_j \hat{p}_j \tag{57}$$

$\tilde{B}_{\|}$ can normally be neglected but will be seen to play a significant role in converting ∇B drifts to curvature drift in the vorticity equation discussed in the next section.

In the case of $A_{\|}$ we have

$$-\left|\nabla\psi\right|^2 \frac{\partial^2}{\partial x^2} A_{\|} = \mu_0 j_{\|} \tag{58}$$

where we must use the neoclassical current for $j_{\|}$. In lowest order, expressing $A_{\|}$ in terms of Ψ and writing $\Psi = \Psi^{(0)} + \Psi^{(1)} + \ldots$ where the expansion parameter is $\eta = \left(1/\beta_e\right)\left(L_s/L_n\right)^2 \left(\nu_e/\Omega_e\right)^2$, we have

$$\frac{\left|\nabla\psi\right|^2}{i\omega} \frac{I}{R^2 B q} \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial\theta} \Psi^{(0)} = 0 \quad \Rightarrow \quad \Psi^{(0)} = \Psi^{(0)}(x) \tag{59}$$

while in second order

$$-\frac{\left|\nabla\psi\right|^2}{i\omega} \frac{I}{R^2 B q} \frac{\partial^2}{\partial x^2}\left(\frac{\partial \Psi^{(1)}}{\partial\theta} + inq' x \Psi^{(0)}(x) \right) = \mu_0 j_{\|} \tag{60}$$

This imposes a solubility condition

$$\frac{nq' I}{\omega} \left\langle \frac{1}{R^2} \right\rangle \frac{d^2 (x\Psi^{(0)})}{dx^2} = -\mu_0 \left\langle \frac{j_{\|} B}{\left|\nabla\psi\right|^2} \right\rangle \tag{61}$$

Using eqns. (32) and (49) to evalulate the neoclassical parallel current we can calculate the above average of $j_{||}$. Writing $j_{||}$ as a sum of an inductive contribution driven by $E_{||}$ (see eqn. (12)) and the 'bootstrap' current driven by radial plasma gradients, we have

$$j_{||} = j_{||}^{\text{inductive}} + j_{||}^{\text{bootstrap}} \tag{62}$$

with

$$\left\langle \frac{j_{||}^{\text{inductive}} B}{|\nabla\psi|^2} \right\rangle = \frac{inq'xI}{q} \frac{\left\langle 1/R^2 \right\rangle \left\langle B^2 / |\nabla\psi|^2 \right\rangle}{\left\langle B^2 \right\rangle} \left(\frac{n_e e^2}{m_e \nu_0} \right) \frac{f_c}{(1-0.37 f_c)} \frac{T_e}{e} \left[(2.45 - 0.45 f_c) \frac{\hat{n}_e}{n_e} + (5.94 - 1.69 f_c) \frac{\hat{T}_e}{T_e} \right] \tag{63}$$

$$\left\langle \frac{j_{||}^{\text{bootstrap}} B}{|\nabla\psi|^2} \right\rangle = I\tilde{p}' \left[\left\langle \frac{1}{|\nabla\psi|^2} \right\rangle - \frac{\left\langle B^2 / |\nabla\psi|^2 \right\rangle}{\left\langle B^2 \right\rangle} \right] + I \frac{\left\langle B^2 / |\nabla\psi|^2 \right\rangle}{\left\langle B^2 \right\rangle} \frac{(1-f_c)}{(1-0.37 f_c)} \tilde{J}; \tag{64}$$

where

$$\tilde{J} = -\alpha_n \tilde{n}_e' (T_e + T_i) - \alpha_e n_e \tilde{T}_e' + \alpha_i n_i \tilde{T}_i';$$

with

$$\alpha_n = 1.06 - 0.06(1 - f_c), \; \alpha_e = 0.55 + 0.45(1 - f_c), \; \alpha_i = 0.183 - 1.25(1 - f_c) + 0.066(1 - f_c)^2$$

where we have introduced the total perturbations $\tilde{n}_{e,i}$ and $\tilde{T}_{e,i}$ in calculating the $j_{||}^{\text{bootstrap}}$ contribution, but retained $\hat{n}_e$ and $\hat{T}_e$ in the inductive term since these are manifestly proportional to $E_{||}$, as can be seen from eqns. (12), (38) and (39) when $\tilde{B}_{||}$ is ignored. (Note we have included the Pfirsch-Schlüter contribution in eqn. (64) so this is strictly the pressure gradient driven current, rather than just the bootstrap current.) Here the $\tilde{n}_j$ are defined in eqn. (55) and

$$\tilde{T}_j = \hat{T}_j + \frac{n}{\omega} T_{0j}{}' \Psi \tag{65}$$

It is convenient to write the total $j_{\|}$ in the form

$$\left\langle \frac{j_{\|}B}{|\nabla\psi|^2} \right\rangle = -\frac{inI\sigma_{\|}^{sc}\left(s^2\right)}{\mu_0 p_0'}\left(\frac{q'}{q}\right)^2\left\langle \frac{1}{R^2}\right\rangle^2 Lx\left(\Phi^{(0)}-\Psi^{(0)}\right)+\frac{Iq'}{q\mu_0 p_0'}\left\langle \frac{1}{R^2}\right\rangle H\tilde{p}'$$
$$-\frac{\left(1-f_c\right)}{\left(1-0.37f_c\right)}\left(\frac{Iq'}{q\mu_0 p_0'}\right)\left\langle \frac{1}{R^2}\right\rangle L\left[\alpha_n \tilde{n}_e'\left(T_e+T_i\right)+\alpha_e n\tilde{T}_e'-\alpha_i n\tilde{T}_i'\right] \tag{66}$$

where H is a flux-surface-averaged equilibrium quantity introduced in Ref. 13 and $L=\left(\mu_0 p_0' q/q'\left\langle 1/R^2\right\rangle\left\langle B^2\right\rangle\right)\left\langle B^2/|\nabla\psi|^2\right\rangle$; L was first introduced in Refs. 2 and 5 (these quantities are also defined in Appendix A). The neoclassical semi-collisional conductivity $\sigma_{\|}^{sc}$ is given by:

$$\sigma_{\|}^{sc}=\frac{\sigma_0}{D}\frac{f_c}{1-0.37f_c}\left\{\left(1-\omega_{*e}/\omega\right)\left(\sigma_n+d_1 s^2\right)-\left(\omega_{*e}\eta_e\sigma_T/\omega\right)\right\}$$

where

$$\sigma_n=2.45\text{-}0.45f_c,\ \sigma_T=5.94-1.69f_c \tag{67}$$

with $\sigma_0=\left(n_e e^2/m_e\nu_0\right)$. These three terms in eqn. (66) represent the parallel current driven by the electric field $E_{\|}$ (with neoclassical semi-collisional conductivity), Pfirsch-Schlüter and bootstrap current contributions, respectively. This expression still fails to reproduce the Spitzer resistivity nor, in the limit of a small fraction of trapped particles, the correct coefficient for the electron temperature gradient contribution to the bootstrap current as discussed in Ref. 16; however it provides a plausible prescription for investigating the bootstrap current at finite aspect ratio. (A more careful treatment using the Spitzer-Härm solution could be invoked [17]; we do not pursue this improvement further in this paper).

Combining eqns. (61) and (66) we obtain a neoclassical Ohm's law:

$$\frac{n}{\omega}\frac{d^2(x\Psi^{(0)})}{dx^2}=-\frac{inq'\sigma_{\|}^{sc}\left(s^2\right)}{p_0'q}\left\langle\frac{1}{R^2}\right\rangle Lx\left(\Phi^{(0)}-\Psi^{(0)}\right)+\frac{H\tilde{p}'}{p_0'}$$
$$-\frac{\left(1-f_c\right)}{\left(1-0.37f_c\right)}\frac{L}{p_0'}\left[\alpha_n\tilde{n}_e'\left(T_e+T_i\right)+\alpha_e n\tilde{T}_e'-\alpha_i n\tilde{T}_i'\right] \tag{68}$$

Equation (68) with eqns. (38) and (39) for $\hat{n}_e$ and $\hat{T}_e$ and eqns. (51) and (53) for $\hat{n}_i$ and $\hat{T}_i$ provides a relationship between $\Psi^{(0)}$ and Φ that includes the effects of neoclassical resistivity and the bootstrap current:

4. The Vorticity Equation

To close the system of equations we use the vorticity equation in the long wavelength limit of the ion finite Larmor radius in the Bessel functions [5]. In lowest order we will confirm $\Psi^{(0)}$ is flute-like and in first order we obtain an equation for $\Psi^{(1)}$; this can be solved, with constants of integration being determined by poloidal periodicity constraints. Finally in second order, the solubility condition on $\Psi^{(2)}$ provides a flux-surface-averaged equation for $\Psi^{(0)}$ after substituting for $\Psi^{(1)}$. However this also involves flux-surface averages of velocity moments of the magnetic drift term in the gyro-kinetic equations which can be evaluated by repeated integrations by parts in poloidal angle, use of the gyro-kinetic equations up to third order and noting conservation of momentum in ion-ion collisions. These manipulations give rise to terms that can be recognised as: (i) the enhanced neoclassical ion inertia which adds to that already present in the vorticity equation due to the usual ion polarisation drift; (ii) ion neoclassical cross-field viscosity, dominated by the perturbed ion temperature gradient; and (iii) a term arising from the parallel gradient of perturbed pressure. This equation will provide another second order ODE linking $\Phi^{(0)}$ and $\Psi^{(0)}$ that also involves $\hat{p}$ and $\hat{p}'$.

To obtain the vorticity equation we operate on the gyro-kinetic equations (1) with

$\sum_j e_j \int d^3v$ to obtain:

$$\frac{i}{\omega}\mathbf{B}.\nabla\left(\frac{j_{\|}}{B}\right) = -\sum_j \frac{e_j^2}{T_j}\int d^3v f_{0j}\left\{\left[1-\left(1-\frac{\omega_{*j}^T}{\omega}\right)J_0^2\right]\Phi - \left(1-\frac{\omega_{*j}^T}{\omega}\right)J_0 J_1 \frac{v_\perp}{k_\perp}\frac{\tilde{B}_{\|}}{B}\right\}$$
$$-i\sum_j e_j \int d^3v J_0 \frac{\mathbf{v}_{dj}.\nabla g_j}{\omega} + \frac{i}{\omega}\sum_j e_j \int d^3v \sigma \left|v_{\|}\right| \mathbf{b}.\nabla J_0 g_j \tag{69}$$

In the long wavelength limit, $z_j \ll 1$, and using Maxwell's equation (58) in terms of Ψ, this can be written as

$$\frac{B}{\omega^2}\nabla_{\|}\frac{|\nabla\psi|^2}{\mu_0 B}\frac{\partial^2}{\partial x^2}\nabla_{\|}\Psi = \sum_j \frac{e_j^2}{T_j}\int d^3v\, f_{oj}\left\{\frac{\omega_{*j}^T}{\omega}\Phi - \left(1-\frac{\omega_{*j}^T}{\omega}\right)\left[\frac{m_j^2 v_\perp^2}{2e_j^2 B^2}|\nabla\psi|^2\frac{d^2\Phi}{dx^2} + \frac{m_j v_\perp^2}{2e_j B}\tilde{B}_{\|}\right]\right\}$$
$$+i\sum_j e_j \int d^3v J_0 \frac{\mathbf{v}_{dj}.\nabla g_j}{\omega} - \frac{i}{\omega}\sum_j \frac{m_j}{e_j}\int d^3v \sigma \left|v_{\|}\right|_j \mathbf{b}.\nabla\left(\frac{m_j v_\perp^2 |\nabla\psi|^2}{2B^2}\right) g'' \tag{70}$$

This reduces to

$$\frac{I^2}{\omega^2 R^2 q^2}\left(\frac{\partial}{\partial\theta}+inq'x\right)\frac{|\nabla\psi|^2}{\mu_0 R^2 B^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial\theta}+inq'x\right)\Psi=-\frac{m_i n_i}{B^2}\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right]|\nabla\psi|^2\frac{d^2\Phi}{dx^2}$$

$$-\frac{np_0'}{\omega}\frac{\tilde{B}_{\|}}{B}+\frac{i}{\omega}\sum_j e_j\int d^3v\frac{Iv_{\|}}{R^2Bq}\frac{\partial}{\partial\theta}\left(\frac{Iv_{\|}}{\Omega_j}\right)\left[\frac{e_j}{T_j}\left(1-\frac{\omega_{*j}^T}{\omega}\right)\Psi' f_{0j}+h_j'\right]$$

$$-\sum_j e_j\int d^3v\left[\frac{e_j}{T_j}\left(1-\frac{\omega_{*j}^T}{\omega}\right)\Psi f_{0j}+h_j\right]\frac{\omega_{djn}}{\omega}-\frac{i}{\omega}\sum_j\frac{m_j}{e_j}\int d^3v\sigma\left|v_{\|}\right|_j \mathbf{b}.\nabla\left(\frac{m_j v_\perp^2|\nabla\psi|^2}{2B^2}\right)g'' \tag{71}$$

where the normal curvature term is given by

$$\omega_{djn}=\frac{\mathbf{b}}{\Omega_j}\times\left(\frac{v_\perp^2}{2}\nabla\ln B+v_{\|}^2\boldsymbol{\kappa}\right).\nabla(n\varphi-m\theta)$$

$$=-\frac{nB}{\Omega_j}\left[\frac{|\nabla\psi|^2}{R^2B^2}\left\{\frac{v_\perp^2}{2}\frac{\partial}{\partial\psi}\ln B+\frac{v_{\|}^2}{B^2}\frac{\partial}{\partial\psi}\left(\mu_0 p+\frac{B^2}{2}\right)\right\}+\left(\frac{v_\perp^2}{2}+v_{\|}^2\right)\frac{\nabla\psi.\nabla\theta}{R^2B^2}\frac{\partial}{\partial\theta}\ln B\right] \tag{72}$$

We solve this order by order: in lowest order we confirm the result

$$\frac{\partial^2}{\partial x^2}\frac{\partial\Psi^{(0)}}{\partial\theta}=0\Rightarrow\Psi^{(0)}=\Psi^{(0)}(x) \tag{73}$$

In first order, inserting $h_{0j}{}'$ on the right-hand side

$$\frac{I^2}{\omega^2R^2q^2}\frac{\partial}{\partial\theta}\left[\frac{|\nabla\psi|^2}{\mu_0R^2B^2}\frac{\partial^2}{\partial x^2}\left\{\frac{\partial}{\partial\theta}\Psi^{(1)}+inq'x\Psi^{(0)}\right\}\right]$$
$$=i\frac{I^2}{\omega R^2 q}\frac{\partial}{\partial\theta}\left(\frac{\tilde{p}'}{B^2}\right) \tag{74}$$

where we have used eqn. (57) so that a first integration yields

$$\frac{|\nabla\psi|^2}{\omega q\mu_0R^2B^2}\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial\theta}\Psi^{(1)}+inq'x\Psi^{(0)}\right]=i\frac{\tilde{p}'}{B^2}+F(x) \tag{75}$$

where F is an arbitrary function. Annihilating $\Psi^{(1)}$ on the left-hand side of eqn. (75), we obtain an equation for $F(x)$ so that

$$\frac{|\nabla\psi|^2}{\omega^2 q\,\mu_0 R^2 B^2}\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial\theta}\Psi^{(1)} + inq'x\Psi^{(0)}\right]$$

$$= \frac{i}{\omega}\left(\frac{1}{B^2} - \left\langle\frac{1}{|\nabla\psi|^2}\right\rangle\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle^{-1}\right)\tilde{p}' + \frac{inq'}{\mu_0\omega^2 q}\left\langle\frac{1}{R^2}\right\rangle\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle^{-1}\frac{d^2}{dx^2}x\Psi^{(0)}$$

(76)

Finally, in second order, we have

$$\frac{I^2}{\omega^2 R^2 q^2}\frac{\partial}{\partial\theta}\left[\frac{|\nabla\psi|^2}{\mu_0 R^2 B^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial\theta}\Psi^{(2)} + inq'x\Psi^{(1)}\right)\right] + \frac{inq'xI^2}{\omega^2 R^2 q^2}\frac{|\nabla\psi|^2}{\mu_0 R^2 B^2}\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial\theta}\Psi^{(1)} + inq'x\Psi^{(0)}\right]$$

$$= -m_i n_i\frac{|\nabla\psi|^2}{B^2}\left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\frac{d^2\Phi}{dx^2} - \frac{np_0'}{\omega}\frac{\tilde{B}_\parallel}{B}$$

$$+\frac{n}{\omega}\frac{np_0'}{\omega}\Psi^{(0)}\left[\frac{|\nabla\psi|^2}{R^2B^2}\frac{\partial}{\partial\psi}\left(\mu_0 p + B^2\right) + \frac{2\nabla\psi.\nabla\theta}{R^2B^2}\frac{\partial}{\partial\theta}\ln B\right]$$

$$+\frac{n}{\omega}\sum_j\int d^3v h_j\left\{\frac{m_j v_\perp^2}{2}\left(\frac{2|\nabla\psi|^2}{R^2B^2}\frac{\partial}{\partial\psi} + \frac{\nabla\psi.\nabla\theta}{R^2B^2}\frac{\partial}{\partial\theta}\right)\ln B + m_j v_\parallel^2\left[\frac{2|\nabla\psi|^2}{R^2B^2}\left\{\frac{1}{B^4}\frac{\partial}{\partial\psi}\left(\mu_0 p + \frac{B^2}{2}\right)\right\} + \frac{\nabla\psi.\nabla\theta}{R^2B^2}\frac{\partial}{\partial\theta}\ln B\right]\right\}$$

$$+i\frac{I^2}{\omega q R^2}\sum_j\int d^3v\frac{m_j v_\parallel}{B}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{B}\right)\left[\frac{e_j}{T_j}\left(1 - \frac{\omega_{*j}^T}{\omega}\right)\Psi^{(1)\prime}f_{0j} + h_j'\right] - \frac{i}{\omega}\sum_j\frac{m_j}{e_j}\int d^3v\sigma|v_\parallel|_j\,\mathbf{b}.\nabla\left(\frac{m_j v_\perp^2|\nabla\psi|^2}{2B^2}\right)g''$$

(77)

Applying the annihilator $\langle\ \rangle$ to eqn. (77) and using eqn. (76), we obtain

$$\frac{I^2}{\omega^2}\left[\frac{n^2q'^2}{\mu_0 q^2}\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle^{-1}\left\langle\frac{1}{R^2}\right\rangle^2 x\frac{d^2(x\Psi^{(0)})}{dx^2} + \frac{nq'x}{q}\omega\tilde{p}'\left(\left\langle\frac{1}{B^2R^2}\right\rangle - \left\langle\frac{1}{R^2}\right\rangle\left\langle\frac{1}{|\nabla\psi|^2}\right\rangle\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle^{-1}\right)\right]$$

$$= m_i n_i\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle\left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\frac{d^2\Phi}{dx^2} + \frac{np_0'}{\omega}\left\langle\frac{\tilde{B}_\parallel}{B}\right\rangle - \frac{n}{\omega}\frac{np_0'}{\omega}\left\langle\frac{|\nabla\psi|^2}{R^2B^2}\frac{\partial}{\partial\psi}\left(\mu_0 p + B^2\right) + \frac{2\nabla\psi.\nabla\theta}{R^2B^2}\frac{\partial}{\partial\theta}\ln B\right\rangle\Psi^{(0)}$$

$$-\frac{n}{\omega}\left\langle\sum_j\int d^3v h_j\left[\frac{m_j v_\perp^2}{2}\left(\frac{2|\nabla\psi|^2}{R^2B^2}\frac{\partial}{\partial\psi} + \frac{\nabla\psi.\nabla\theta}{R^2B^2}\frac{\partial}{\partial\theta}\right)\ln B + m_j v_\parallel^2\left[\frac{2|\nabla\psi|^2}{R^2B^2}\left\{\frac{1}{B^4}\frac{\partial}{\partial\psi}\left(\mu_0 p + \frac{B^2}{2}\right)\right\} + \frac{\nabla\psi.\nabla\theta}{R^2B^2}\frac{\partial}{\partial\theta}\ln B\right]\right]\right\rangle$$

$$- i\frac{I}{\omega}\left\langle\sum_j\int d^3v\,\frac{m_j v_\parallel}{B}\,\frac{I}{R^2 q}\,\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{B}\right)h_j'\right\rangle + \frac{inI^2}{\omega^2 q}p_0'\left\langle\frac{1}{R^2B^2}\frac{\partial}{\partial\theta}\Psi^{(1)\prime}\right\rangle$$

(78)

where we have noted that $\Phi = \Phi(x)$ and the last term vanishes for an up-down symmetric equilibrium. Substituting for $\tilde{B}_{\|}$ from eqn. (56) and evaluating the velocity integrals over h_{j0} in the normal curvature term, the right-hand side of eqn. (78) becomes

$$m_i n_i \left\langle \frac{|\nabla\psi|^2}{B^2} \right\rangle \left[1 - \frac{\omega_{*i}}{\omega}\left(1+\eta_i\right)\right] \frac{d^2\Phi}{dx^2} - \frac{2n}{\omega}\tilde{p}\left\langle \frac{|\nabla\psi|^2}{R^2B^2}\frac{\partial}{\partial\psi}\left(\mu_0 p + \frac{B^2}{2}\right) + \frac{\nabla\psi.\nabla\theta}{R^2B^2}\frac{\partial}{\partial\theta}\ln B \right\rangle$$

$$- i\frac{I}{\omega}\left\langle \sum_j \int d^3v \; \frac{m_j v_{\|}}{B}\;\frac{I}{R^2 q}\;\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{B}\right) h_j' \right\rangle + \frac{inI^2}{\omega^2 q} p_0' \left\langle \frac{1}{R^2B^2}\frac{\partial}{\partial\theta}\Psi^{(1)\prime} \right\rangle \tag{79}$$

where we have expressed $\hat{p}$ in terms of $\tilde{p}$. Using eqn. (76) the last term can be evaluated:

$$\frac{inI^2}{\omega^2 q\mu_0}\left\langle \frac{1}{R^2B^2}\frac{\partial}{\partial\theta}\Psi^{(1)\prime}\right\rangle p_0' = -\frac{np_0' I^2}{\omega}\tilde{p}\left[\left\langle \frac{1}{|\nabla\psi|^2 B^2}\right\rangle - \left\langle \frac{B^2}{|\nabla\psi|^2}\right\rangle^{-1}\left\langle \frac{1}{|\nabla\psi|^2}\right\rangle^2\right]$$

$$-p_0' \frac{n^2 q' I^2}{\omega^2 q}\left(x\Psi^{(0)}\right)' \left[\left\langle \frac{1}{R^2}\right\rangle\left\langle \frac{1}{|\nabla\psi|^2}\right\rangle\left\langle \frac{B^2}{|\nabla\psi|^2}\right\rangle^{-1} - \left\langle \frac{1}{R^2B^2}\right\rangle\right] \tag{80}$$

To evaluate the penultimate term, we make repeated use of the gyro-kinetic equations for h_{j0}, h_{j1}, h_{j2} and h_{j3}, with integrations by parts in θ and utilise the conservation of momentum in ion-ion collisions. Thus,

$$-i\frac{I}{\omega}\left\langle \sum_j e_j \int d^3v \; \frac{Iv_{\|}}{R^2Bq}\frac{\partial}{\partial\theta}\left(\frac{Iv_{\|}}{\Omega_j}\right) h_j'\right\rangle = i\frac{I}{\omega}\left\langle \sum_j \int d^3v \; \frac{Iv_{\|}m_j}{B}\frac{Iv_{\|}}{R^2Bq}\frac{\partial}{\partial\theta}h_j'\right\rangle \tag{81}$$

and using the gyro-kinetic equation (13) for h_j we have

$$\begin{aligned} & i\frac{I}{\omega}\left\langle \sum_j \int d^3v \; \frac{Iv_{\|}m_j}{B}\frac{Iv_{\|}}{R^2Bq}\frac{\partial}{\partial\theta}h_j'\right\rangle = \\ & i\frac{I}{\omega}\left\langle \sum_j \int d^3v \; \frac{Iv_{\|}m_j}{B}\left\{-\frac{Iv_{\|}}{R^2Bq}\frac{\partial}{\partial\theta}\left(\frac{Iv_{\|}}{B}\right)h_j'' + C_j(h_j') + i\omega h_j' - inq'\frac{Iv_{\|}}{R^2Bq}\left(xh_j\right)'\right\}\right\rangle \end{aligned} \tag{82}$$

since the right-hand side driving term in eqn. (13) is of even parity in $v_{\|}$.

We substitute the lowest order in h_j that gives a non-vanishing contribution to the various terms. For the last term this is h_{jo}, leading to a term related to the perturbed pressure $\tilde{p}_j$ while for the penultimate term, it is h_{j1}, leading to a term related to the perturbed parallel momentum, $m_j u_{\|j}$; the latter is clearly dominated by the heavier ions. The collisional term vanishes due to momentum conservation. The first term can clearly be recognised as related to the radial flux of toroidal momentum, Π:

$$\left\langle \frac{\Pi}{B} \right\rangle = \left\langle \sum_j \int d^3v \frac{m_j v_\|}{B} v_{drj} h_j \right\rangle \tag{83}$$

The contributions to this term from h_{j1} and h_{j2} vanish on integration by parts in θ and the first non-vanishing contribution is from h_{j3} and is also dominated by the ions. Thus expression (82) reduces to

$$-i\frac{I}{\omega}\left\langle \frac{\Pi}{B} \right\rangle'' - I m_i n_i \left\langle \frac{u_{\|i}}{B} \right\rangle' + \frac{nq' I^2}{\omega q}\left\langle \frac{1}{R^2B^2} \right\rangle \left(x\hat{p}\right)' \tag{84}$$

To calculate the term involving $\Pi\mathbf{(}h_{j3}\mathbf{)}$ we again integrate by parts in θ and use the gyro-kinetic equation for h_{j3} to obtain

$$-i\left\langle \frac{\Pi}{B} \right\rangle'' = i\frac{I}{2\omega}\left\langle \sum_j \int d^3v \; \frac{m_j v_\|}{B}\frac{I v_\|}{\Omega_j} \left\{ -\frac{I v_\|}{R^2 B q} \frac{\partial}{\partial\theta}\left(\frac{I v_\|}{\Omega_j}\right) h'''_{j2} + C_j\left(h''_{j2}\right) \right\} \right. \tag{85}$$

where the omitted terms are of higher order than those retained previously in eqn. (84). In a similar manner a further integration by parts on the first term and use of the gyro-kinetic equation yields

$$-\frac{i}{\omega}\left\langle \frac{\Pi}{B} \right\rangle'' = i\frac{I}{2\omega}\left\langle \sum_j \int d^3v \; \frac{m_j v_\|}{B} \left\{ C_j(h''_{j2}) - \frac{I v_\|}{3\Omega_j} C_j(h'''_{j1}) \right\} \right\rangle \tag{86}$$

This requires calculation of h_{j2}; such a calculation was carried out in Ref. 18, but a later more elegant treatment using the adjoint function to h_{j1} [19], identified a numerical error in Ref. 18. Both treatments exploited an approximate 'similarity' between the constraint equations on h_{j1} and h_{j2} and the integrals required for evaluating the quantity Π; this similarity requires a weak poloidal variation of $B(\theta)$. Although the accuracy of these calculations is not entirely clear, we propose to identify $\Pi \equiv \Pi_2$ where Π_2 is defined in eqn. (5) of Ref. 19. Using the evaluation of Π_2 given in eqn. (33) of Ref. 19, we find

$$\Pi = 0.1862 m_i n_i (2\hat{\varepsilon})^{3/2} \langle R \rangle^2 \frac{m_i T_i}{\tau_i e^3} \tilde{T}_i'' \tag{87}$$

where $\hat{\varepsilon} = (R_{max} - R_{min})/2\langle R \rangle$ on a flux surface.

Assembling all the contributions to eqn. (78) we have

$$\begin{aligned}
&\left(\frac{q'}{q}\right)^2 \frac{1}{\mu_0} \left\langle \frac{I}{R^2} \right\rangle^2 x \frac{d^2}{dx^2}\left(x\Psi^{(0)}\right) + x\left(\frac{q'}{q}\right)\left[\left\langle \frac{I^2}{R^2B^2} \right\rangle \left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle - \left\langle \frac{I^2}{R^2} \right\rangle \left\langle \frac{1}{|\nabla\psi|^2} \right\rangle\right] \frac{\omega}{n} \tilde{p}' \\
&+\mu_0 p_0{}' \left[\left\langle \frac{I^2}{|\nabla\psi|^2 B^2} \right\rangle \left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle - \left\langle \frac{I}{|\nabla\psi|^2} \right\rangle^2\right] \frac{\omega}{n}\tilde{p} + p_0{}' \frac{q'}{q}\left[\left\langle \frac{I^2}{R^2} \right\rangle \left\langle \frac{1}{|\nabla\psi|^2} \right\rangle - \left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \left\langle \frac{I^2}{R^2B^2} \right\rangle\right]\left(x\Psi^{(0)}\right)' \\
&+2\left\langle \frac{|\nabla\psi|^2}{R^2B^4} \frac{\partial}{\partial\psi}\left(\mu_0 p + \frac{B^2}{2}\right) + \frac{\nabla\psi.\nabla\theta}{R^2B^4} \frac{\partial}{\partial\theta}\left(\frac{B^2}{2}\right) \right\rangle \left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \frac{\omega}{n}\tilde{p} - \frac{q'}{q}\left\langle \frac{I^2}{R^2B^2} \right\rangle \left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \frac{\omega x}{n}(x\hat{p})' + \frac{i\omega}{n^2}\Pi''\left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \\
&\qquad = \frac{m_i n_i}{n^2}\left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \left[\left\langle \frac{|\nabla\psi|^2}{B^2} \right\rangle \omega\left(\omega - \omega_{*i}(1+\eta_i)\right) \frac{d^2\Phi}{dx^2} - \omega^2 \left\langle \frac{I u_{\|i}}{B} \right\rangle'\right]
\end{aligned} \tag{88}$$

Substituting for $u_{\|i}$ from eqn. (49) we can write the inertial term on the right-hand side of eqn. (88) as

$$\begin{aligned}
&\frac{m_i n_i}{n^2}\left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \left[\omega\left(\omega - \omega_{*i}(1+\eta_i)\right)\left(\left\langle \frac{|\nabla\psi|^2}{B^2} \right\rangle \frac{d^2\Phi}{dx^2} + \left\langle \frac{I^2}{B^2} \right\rangle \frac{d^2\Psi^{(0)}}{dx^2}\right) - 1.17 n\omega T_{i0}' \frac{I^2 f_c}{\langle B^2 \rangle} \frac{d^2\Psi^{(0)}}{dx^2}\right] \\
&+\frac{m_i n_i}{n^2}\left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \left[\omega^2 \left\langle \frac{I^2}{B^2} \right\rangle \frac{\hat{p}_i''}{e n_i} - 1.17\omega^2 \frac{I^2 f_c}{\langle B^2 \rangle} \frac{\hat{T}_i''}{e}\right]
\end{aligned} \tag{89}$$

Here one can recognise the neoclassical enhancement of the ion inertia when one sets $\Psi^{(0)} = \Phi$. The cross field momentum transport term, Π, in eqn. (88) is given by eqn. (87) in terms of $\hat{T}_i$. $\hat{T}_i$, which also appears in the inertial term, is itself given by eqn. (53).

It is interesting to separate out the convective parts of $\tilde{p}$ proportional to Ψ in eqn. (57). The terms in Ψ', Ψ, $\hat{p}'$ and $\hat{p}$ can then be combined and expressed in terms of the flux-surface-averaged quantities, E, F, and H defined by Glasser et al. [13, 20] and L defined in Ref. 5 (see Appendix A):

$$x\frac{d^2}{dx^2}\left(x\Psi^{(0)}\right)+D_I\Psi^{(0)}-\left(\omega x\hat{p}/n\right)'\left((L+H)/p_0'\right)+D_I\left(\omega\hat{p}/np_0'\right)$$

$$=\frac{\mu_0 m_i n_i}{n^2}\left(\frac{q}{Iq'\left\langle 1/R^2\right\rangle}\right)^2\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle\left[\omega\left(\omega-\omega_{*i}(1+\eta_i)\right)\left(\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle\frac{d^2\Phi}{dx^2}+\left\langle\frac{I^2}{B^2}\right\rangle\frac{d^2\Psi^{(0)}}{dx^2}\right)-1.17n\omega T_{i0}'\frac{I^2 f_c}{\left\langle B^2\right\rangle}\frac{d^2\Psi^{(0)}}{dx^2}\right]$$
$$+\frac{\mu_0 m_i n_i}{n^2}\left(\frac{q}{Iq'\left\langle 1/R^2\right\rangle}\right)^2\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle\left[\omega^2\left\langle\frac{I^2}{B^2}\right\rangle\frac{\hat{p}_i''}{en_i}-1.17\omega^2\frac{I^2 f_c}{\left\langle B^2\right\rangle}\frac{\hat{T}_i''}{e}\right]-\frac{i\omega\mu_0}{n^2}\Pi''\left(\frac{q}{Iq'\left\langle 1/R^2\right\rangle}\right)^2\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle$$

(90)

where $D_I = E+F+H$.

We can simplify eqn. (90) further if we ignore neoclassical ion thermal and transport and viscosity. Using relations (52) and (53) to determine $\hat{p}_i$ and $\hat{T}_i$ in the inertial term on the right-hand side of eqn. (90) (ignoring contributions from $\tilde{B}_{\|}$), we obtain:

$$x\left(d^2\left(x\Psi^{(0)}\right)/dx^2\right)+D_I\Psi^{(0)}-\left(\omega/np_0'\right)\left(x\hat{p}\right)'\left(L+H\right)+D_I\left(\omega\hat{p}/np_0'\right)=$$
$$\left(\mu_0 m_i n_i\omega^2/n^2\right)\left(q/Iq'\left\langle 1/R^2\right\rangle\right)^2\left\langle B^2/|\nabla\psi|^2\right\rangle\left\{\left\langle R^2\right\rangle\left(1-\omega_{*pi}/\omega\right)+\left(1.17f_c I^2/\left\langle B^2\right\rangle\right)\omega_{*pi}\eta_i/\omega\right\}\left(d^2\Phi^{(0)}/dx^2\right)$$

(91)

The contributions from $\hat{p}$, determined by eqns. (38), (39) and (52), have introduced effects from neoclassical ion compressibility through L; this equation also includes the effects of ion neoclassical inertia

5 Eigenvalue problem

The complete set of equations determining the stability of the system consists of the vorticity equation (90) with expression (87) for the perpendicular viscosity, Π and Ohm's law (68). However, these equations also involve the perturbed pressures, densities and temperatures given in eqns. (37), ignoring the neoclassical transport, (38) and (39) for electrons and eqns. (51), (52) and (53) for ions. Expression (56) provides the perturbed magnetic field, $\tilde{B}_{\|}$, but this is largely unimportant for normal values of β. Thus the general eigenvalue problem obtained by this method has the form of a tenth order system of ODE's, if we include cross-field neoclassical electron transport; ignoring this, as is very reasonable, it reduces to an eighth order system. If we also ignore ion neoclassical thermal transport and viscosity as in eqn. (91), then it simplifies to a fourth order system.

(a) The Collisional Case

However it is interesting to take the collisional limit, $s \to 0$, since this generalises Refs. 2 and 5 to include the effect of temperature gradients. Then the expressions for $\hat{n}_j$ and $\hat{T}_j$ are simpler and Fourier transforming allows one to reduce the Ohm's law and vorticity equations to a single second order ODE. This differs from the results in Refs. 2 and 5 by the substitution:

$$\omega - \omega_{*e} \to \omega - \omega_{*e}\left[1 + \left(5.9 - 1.7 f_c\right)\eta_e / \left(2.45 - 0.45 f_c\right)\right] \tag{92}$$

in Ohm's law (68) and the substitution

$$\left(1 - \omega_{*i}/\omega\right) \to \left\{\left(1 - \omega_{*pi}/\omega\right) + \left(1.17 f_c I^2 / \langle B \rangle^2 \langle R \rangle^2\right)\omega_{*pi}\eta_i/\omega\right\} \tag{93}$$

in the inertia term in the vorticity equation (91).

(b) The Semi-collisional Case

The semi-collisional regime is more appropriate to JET or ITER-like conditions than the collisional case discussed in the previous sub-section. This corresponds to assuming the semi-collisional quantity s ~ O(1), rather than s << 1, in the resonant layer. To simplify the analysis we ignore the curvature terms, D_I and H, although we retain the neoclassical effects represented by L and f_t. We proceed by analogy with the approach to analysing the semi-collisional regime adopted by Drake et al. [8], although our treatment is complicated by the presence of the bootstrap current in Ohm's law and neoclassical compressibility effects in the vorticity equation. With these assumptions we can reorganise the vorticity equation (91) and Ampère's law (68) with the neoclassical current as follows. On using Ampère's law, the vorticity equation becomes:

$$\begin{aligned} &\left[1 + \frac{1+\eta_i}{\hat{\omega}\tau} - 1.17 f_c \frac{I^2}{\langle R^2\rangle\langle B^2\rangle}\frac{\eta_i}{\hat{\omega}\tau}\right]\frac{d^2\Phi}{ds^2} = \\ &-iC\left\{s^2\hat{\sigma}(s^2)(\Phi - \Psi) + \frac{T_e}{e}\frac{\hat{\gamma}}{\hat{\beta}}s\left[\alpha_n(1+1/\tau)\frac{d}{ds}\left(\frac{\tilde{n}_e}{n_0}\right) + \alpha_e\frac{d}{ds}\left(\frac{\tilde{T}_e}{T_{e0}}\right) + \alpha_i\frac{d}{ds}\left(\frac{\tilde{T}_i}{T_{e0}}\right)\right] - \frac{T_e}{e}\frac{\hat{\mu}}{\hat{\beta}}\frac{d}{ds}\left(\frac{s\hat{p}}{p_{e0}}\right)\right\} \end{aligned} \tag{94}$$

while Ampère's law takes the form:

$$\frac{d^2}{ds^2}\left(s\Psi\right)=\hat{\omega}\left\{\hat{\beta}s\hat{\sigma}(s^2)\left(\Phi-\Psi\right)+\frac{T_e}{e}\hat{\gamma}\left[\alpha_n\left(1+1/\tau\right)\frac{d}{ds}\left(\frac{\tilde{n}_e}{n_0}\right)+\alpha_e\frac{d}{ds}\left(\frac{\tilde{T}_e}{T_{e0}}\right)+\alpha_i\frac{d}{ds}\left(\frac{\tilde{T}_i}{T_{e0}}\right)\right]\right\} \tag{95}$$

Here we have defined

$$\hat{\beta}=\mu_0 p_{e0}\left(\frac{d\left(\ell n\, n_{e0}\right)}{d\left(\ell n\, q\right)}\right)^2\frac{\left\langle B^2/\left|\nabla\psi\right|^2\right\rangle}{I^2\left\langle 1/R^2\right\rangle^2},\ \hat{\gamma}=\mu_0 p_{e0}\frac{\left(1-f_c\right)}{\left(1-0.37f_c\right)}\left(\frac{d\left(\ell n\, n_{e0}\right)}{d\left(\ell n\, q\right)}\right)\frac{\left\langle B^2/\left|\nabla\psi\right|^2\right\rangle}{\left\langle B^2\right\rangle\left\langle 1/R^2\right\rangle}$$

$$\hat{\mu}=\mu_0 p_{e0}\left(\frac{d\left(\ell n\, n_{e0}\right)}{d\left(\ell n\, q\right)}\right)\frac{\left\langle B^2/\left|\nabla\psi\right|^2\right\rangle}{\left\langle B^2\right\rangle\left\langle 1/R^2\right\rangle},\ C=\left(\frac{m_e}{m_i}\right)\left(\frac{\nu_0}{\omega_{*e}}\right)\frac{\left(1-0.37f_c\right)\left\langle B^2\right\rangle}{f_c\left\langle I^2/R^2\right\rangle\left\langle R^2\right\rangle\left\langle 1/R^2\right\rangle}\left(\frac{d\left(\ell n\, n_{e0}\right)}{d\left(\ell n\, q\right)}\right)^2$$

$$\hat{\sigma}(s^2)=\left\{\left(\hat{\omega}-1\right)\left(\sigma_n+d_1 s^2\right)-\eta_e\sigma_T\right\}/D\equiv\left(\sigma_0+\sigma_1 s^2\right)/D;$$

$$\hat{\omega}=\omega/\omega_{*e},\ \tau=T_{e0}/T_{i0} \tag{96}$$

Note that, for normal density profiles, both $\hat{\gamma}<0$ and $\hat{\mu}<0$. The expression for $\hat{\beta}$ reduces to that of Ref. 8 in the cylindrical limit, while $\hat{\gamma}$ and $\hat{\mu}$ arise entirely from toroidal neoclassical effects: $\hat{\gamma}$ from the bootstrap current and $\hat{\mu}$ from neoclassical compressibility effects. While C resembles the collisionality parameter in Ref. 8, it is reduced by a factor $\sim\left(B_\varphi^2/B_\theta^2\right)$ as a result of the neoclassical enhancement in ion inertia.

7. Discussion and Conclusions

We have derived a set of equations that describe the linear stability of tearing modes in the low collisionality regime appropriate to large tokamaks such as JET or ITER. Although these have the form of fluid-like equations for moments such as plasma density, temperature and current to feed into Maxwell's equations, they contain coefficients that encapsulate kinetic neoclassical effects, such as cross-field transport of particles, energy and momentum, the bootstrap current, neoclassical resistivity and neoclassical ion inertia and compressibility. The electron model corresponds to the semi-collisional regime in which parallel diffusive transport effects compete with the mode frequency ($\omega \sim k_{\|}^2 v_{the}^2 / \nu_e$). Thus, using the definition of the electron layer width, δ_{sc}, from the semi-collisional theory, assuming $\omega \sim \omega_{*e}$, i.e. $\omega_{*e} \sim \left(k_\theta \delta_{sc} \hat{s} / Rq\right)^2 v_{the}^2 / \nu_e$, one finds

$$\delta_{sc} \sim \left(\frac{\nu_e}{v_{the}} \frac{L_s^2}{L_n} \frac{r\rho_e}{nq}\right)^{1/2} \sim \left(\frac{a}{R}\right)^{3/4} \left(\frac{\nu_{*e}}{\hat{s}^2} \frac{Rq}{L_n} \frac{r\rho_e}{nq}\right)^{1/2} \tag{97}$$

where $L_s = Rq / \hat{s}$ is the magnetic shear length. However, the ion model we use assumes that $\left(k_r \rho_i\right)^2 \sim \left(\rho_i / \delta_{sc}\right)^2 < 1$, but this can only be justified at low magnetic shear, $\hat{s}$, or for a cold ion model. Using the definition (97) for the electron layer width, δ_{sc}, this implies

$$\hat{s} < \left(\frac{a}{R}\right)^{3/4} \left(\sqrt{\frac{m_e}{m_i}} \frac{Rq}{L_n} \frac{\nu_{*e}}{\rho_{*i}}\right)^{1/2}. \tag{98}$$

Nevertheless this is highly relevant for describing the resistive internal kink mode involved in the sawtooth phenomenon. A treatment for finite shear requires a kinetic, large orbit theory ($\rho_i > \delta_{sc}$), rather than the present fluid theory based on an ion Larmor radius expansion. This will be a major development, extending the cylindrical geometry theory of Cowley et al. [11] to the toroidal situation with finite ion banana orbits.

The full set of equations defining the eigenvalue problem was described at the beginning of Section 5: the vorticity equation (90) with expression (87) for the perpendicular viscosity, Π (or, ignoring ion thermal transport and perpendicular viscosity, eqn. (91)), and Ohm's law (68). The perturbed pressures, densities and temperatures appearing in these equations are given by eqns. (38) and (39) for electrons and eqns. (51) and (53) for ions (one should note the relations (55), (57) and (65) between various perturbed quantities); eqn. (56) provides the perturbed magnetic field, $\tilde{B}_{\|}$ (unimportant for typical values of plasma pressure).

This system of equations is comprised of ordinary differential equations in a local 'radial' co-ordinate about the mode resonant surface, m = nq(r). These can be of rather high order, but can be considerably simplified if we ignore the small electron neoclassical transport of density and temperature, i.e. $\omega > \varepsilon^{-3/2} q^2 \nu_e \rho_e^2$. This leads to algebraic expressions for $\hat{n}_e$ and $\hat{T}_e$. A further simplification, plausible but less convincing, is to ignore ion neoclassical transport of energy and momentum when one can also have algebraic solutions for $\hat{n}_i$ and $\hat{T}_i$. The radial component of Maxwell's equation provides an algebraic expression for $\tilde{B}_{\|}$; for normal values of plasma β this can be ignored. With all these assumptions the system reduces to a fourth order set of differential equations, consisting of a neoclassical Ohm's law and the vorticity equation; in general this will require numerical solution, with boundary conditions determined by matching to outer solutions and therefore involving the tearing mode stability parameter Δ'.

One can identify two other characteristic lengths besides the semi-collisional layer width: the resistive layer width,

$$\delta_\eta \sim \left(\frac{\eta}{\omega\mu_0}\right)^{1/2} \sim \left(\frac{L_n}{v_{the}\beta_e}\frac{\rho_e r}{nq}\right)^{1/2} \sim \left(\frac{a}{R}\right)^{3/4}\left(\frac{\nu_{*e}}{\beta_e}\frac{L_n}{Rq}\frac{\rho_e r}{nq}\right)^{1/2} \tag{99}$$

and the ion neoclassical transport length scale (we can safely ignore the electron neoclassical transport scale)

$$\delta_{\chi_i} \sim \left(\frac{R}{a}\right)^{3/4}\left(\frac{\nu_i\rho_i^2 q^2}{\omega}\right)^{1/2} \sim \left(\nu_{*i}\frac{L_n}{R}\frac{\rho_i r}{n}\right)^{1/2} \tag{100}$$

Clearly, depending on parameters, all these can compete: e.g. the semi-collisional layer, where the electron responses ($\hat{p}_e$, $\hat{n}_e$ and $\hat{T}_e$) have structure, can be broader or narrower than the resistive reconnection layer. One expects the collisional model to pertain if $\delta_\eta > \delta_{sc}$; in fact, using the estimates (97) and (99), the ratio

$$\left(\frac{\delta_{sc}}{\delta_\eta}\right)^2 \sim \frac{\beta_e}{2}\left(\frac{L_s}{L_n}\right)^2 \sim \frac{\beta_e}{2s^2}\left(\frac{Rq}{L_n}\right)^2 \tag{101}$$

(see also Ref. 8) is likely to be O(1) for a sawtooth with $\hat{s} \sim 0.1$, so one must indeed consider semi-collisional effects when modelling the sawtooth instability. Similarly one must consider the role of the ion responses ($\hat{p}_i$, $\hat{n}_i$ and $\hat{T}_i$) due to ion neoclassical transport.

We have generalised the existing theory of linear neoclassical tearing modes in the collisional regime [2] to include the effects of temperature gradients. It is worth recalling that Ref. 2 identified a strong reduction in growth rate of the Δ' driven tearing mode due to neoclassical resistivity and ion inertia effects, although the bootstrap current drive in the resonant layer would overwhelm the Glasser stabilisation effect [13] and lead to an unstable tearing parity 'interchange' mode, a linear analogue of finite island neoclassical tearing modes [3, 4]. In the more relevant semi-collisional regime we have derived a pair of second–order differential equations, extending the work of Drake et al. [8] (an analytic solution of these will appear in a later paper.) These equations are modified by the effects of the bootstrap current, neoclassical resistivity, the neoclassical ion inertia (which reduces the effective collisionality parameter C in Ref. 8 by a factor $\sim\left(B_\varphi^2/B_\theta^2\right)$), as well as a neoclassical compression of the pressure response. In future one can also consider the numerical solution of these equations in the context of the low shear sawtooth situation, as well as an extension to include finite ion orbit effects for m > 1 tearing modes. Of course the theory can be applied to resistive ballooning modes and it considerably extends the earlier treatment of Refs. 2 and 5 by including more physics: thermal effects and semi-collisional effects. These techniques may also be useful in analysing the finite island theory of neoclassical tearing modes.

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Appendix A: Some equilibrium relationships applied to the vorticity equation

Let us first introduce the flux-surface-averaged quantities E, F and H defined by Glasser et al. [13]

$$E = \frac{\mu_0 p_0' q}{q'^2 I \langle 1/R^2 \rangle}\left[\frac{Iq'}{\langle B^2 \rangle} - V''\right]\left\langle \frac{B^2}{|\nabla\psi|^2}\right\rangle$$

$$F = \left(\frac{\mu_0 p_0' q}{q' I \langle 1/R^2 \rangle}\right)^2 \left\{ I^2 \left[\left\langle \frac{B^2}{|\nabla\psi|^2}\right\rangle \left\langle \frac{1}{B^2|\nabla\psi|^2}\right\rangle - \left\langle \frac{1}{|\nabla\psi|^2}\right\rangle^2\right] + \left\langle \frac{1}{B^2}\right\rangle \left\langle \frac{B^2}{|\nabla\psi|^2}\right\rangle \right\}$$

$$H = \frac{\mu_0 p_0' q}{q' \langle 1/R^2 \rangle}\left[\left\langle \frac{1}{|\nabla\psi|^2}\right\rangle - \frac{\left\langle \frac{B^2}{|\nabla\psi|^2}\right\rangle}{\langle B^2 \rangle}\right] \tag{A.1}$$

where

$$\mathbf{B} = I\nabla\phi + \nabla\phi \times \nabla\psi \text{ , so that } \mathbf{B}.\nabla\theta = \frac{I}{R^2 q} \text{ and } \langle A \rangle \equiv \oint \frac{d\theta}{B.\nabla\theta} A \Big/ \oint \frac{d\theta}{B.\nabla\theta} .$$

It is also convenient to introduce the quantity L defined in Refs. 2 and 5:

$$L = \frac{\mu_0 p_0' q}{q' \langle 1/R^2 \rangle \langle B^2 \rangle}\left\langle \frac{B^2}{|\nabla\psi|^2}\right\rangle \tag{A.2}$$

We now consider separately the terms proportional to $\hat{p}, \hat{p}', \Psi$ and Ψ' in eqn. (88) when we recall eqn. (57):

$$\tilde{p} = \hat{p}' + \frac{n}{\omega} p_0{}' \Psi \tag{A.3}$$

First we evaluate the average curvature:

$$K \equiv \left\langle \frac{1}{B^2}\frac{\partial}{\partial\varphi}\left(\mu_0 p_0 + \frac{B^2}{2}\right) + \frac{\nabla\psi . \nabla\theta}{R^2 B^4}\frac{\partial}{\partial\theta}\left(\frac{B^2}{2}\right)\right\rangle$$

$$= \mu_0 p_0' \left\langle \frac{1}{B^2} \right\rangle + \frac{1}{2}\left\langle \frac{1}{B^2}\frac{\partial}{\partial\psi}B^2 \right\rangle + \frac{1}{2}\left\langle \frac{1}{R^2\ B^2}\frac{\partial}{\partial\theta}\ \left(\nabla\theta.\nabla\psi\right) \right\rangle \tag{A.4}$$

Now the Grad-Shafranov equation can be written (see eqn. (8) of Ref. 21, but in terms of our metric coefficients)

$$\frac{I}{R^2 q}\ \left[\frac{\partial}{\partial\psi}\left(\frac{q}{I}|\nabla\psi|^2\right) + \frac{q}{I}\frac{\partial}{\partial\theta}\ (\nabla\psi.\nabla\theta)\right] + \mu_0 p_0' + \frac{II'}{R^2} = 0 \tag{A.5}$$

so that eqn. (A.4) becomes, after some algebra,

$$K = \frac{\mu_0 p_0'}{2}\left\langle \frac{1}{B^2} \right\rangle + \frac{II'}{2}\left\langle \frac{1}{R^2\ B^2} \right\rangle - \frac{q'}{2q}\left\langle \frac{|\nabla\psi|^2}{R^2 B^2} \right\rangle + \frac{I'}{2I}\left\langle \frac{|\nabla\psi|^2}{R^2 B^2} \right\rangle - \frac{1}{2}\frac{\partial}{\partial\psi}\left(\frac{IV'}{q}\right) \tag{A.6}$$

Finally, introducing V'', where $V' = \left(q/I\right)\oint R^2 d\theta$, we have

$$K = \frac{1}{2}\ \left[\mu_0 p_0' \left\langle \frac{1}{B^2} \right\rangle + \frac{I^2 q'}{q}\left\langle \frac{1}{R^2\ B^2} \right\rangle - \frac{IV''}{q}\right] \tag{A.7}$$

Then gathering terms in Ψ, we have a contribution $I_1\Psi$ with

$$I_1 = (\mu_0 p_0')^2\left[\left\langle \frac{I^2}{|\nabla\psi|^2\,B^2} \right\rangle\left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle - \left\langle \frac{I}{|\nabla\psi|^2} \right\rangle^2\right] + \frac{\mu_0 p_o' q'}{q}\left[\left\langle \frac{I^2}{R^2} \right\rangle\left\langle \frac{1}{|\nabla\psi|^2} \right\rangle - \left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle\left\langle \frac{I^2}{R^2 B^2} \right\rangle\right]$$

$$+2\mu_0 p_0' \left\langle \frac{1}{B^2}\frac{\partial}{\partial\psi}\left(\mu_0 p_0 + \frac{B^2}{2}\right) + \frac{\nabla\psi.\nabla\theta}{R^2 B^4}\frac{\partial}{\partial\theta}\left(\frac{B^2}{2}\right)\right\rangle\left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \tag{A.8}$$

Using eqn. (A.8) and the definitions (A.1) we obtain

$$I_1 = \frac{q'^2}{V'^2}\left(E + F + H\right) \equiv \frac{q'^2}{V'^2}\left(D_I - \frac{1}{4}\right) \tag{A.9}$$

Turning to the terms proportional to Ψ', we find these trivially vanish. For the terms in $\hat{p}'$, we find they readily combine to yield $I_2\hat{p}'$, where

$$I_2 = \frac{q'^2}{V'^2}\left(L + H\right) \tag{A.10}$$

using the definition (A.2). Finally we consider the terms proportional to $\hat{p}$; this calculation is identical to that of the terms in Ψ, leading to a contribution:

$I_1\hat{p}$. As a result, finally, we find eqn. (90) can be written in the form of eqn. (91).

Appendix B: A summary of some symbols and notation employed

Here we collect some of the definitions and symbols employed in this paper.

- *Velocity space variables*:

  $v, \lambda, \sigma,$ and gyro-angle:

  $\sigma = v_{\|} / \left|v_{\|}\right|$: $\lambda = \frac{2\mu}{v^2}$; $v_{\|} = \sigma v (1-\lambda B)^{1/2}$;

  $u^2 = \frac{mv^2}{2T}$

- *Configuration space variables*:

  x, θ and φ, a non-orthogonal set in which field line trajectories are straight ($\varphi = q\theta$).

  $x = \left(\psi - \psi_0\right)$: the local flux variable. Prime is used to denote differential with respect to the poloidal flux, x ; thus

  $$h'_j = \frac{\partial h_j}{\partial x}; \quad \hat{n}'_j = \frac{d\hat{n}_j}{dx}; \quad \hat{T}'_j = \frac{d\hat{T}_j}{dx}; \quad \hat{p}_j = \frac{d\hat{p}_j}{dx}.$$

  s, a normalised radial variable representing semi-collisional effects, see eqn. (41).

  $$\hat{\nabla}_{\|} = \frac{I}{R^2 qB}\frac{\partial}{\partial\theta}.$$

  $$\left\langle X \right\rangle = \oint R^2 X d\theta / \oint R^2 d\theta.$$

- *Magnetic field variables*

  q is the safety factor; R the major radius, $I = RB_{\phi}$.

  $\hat{s}$ is the magnetic shear.

$$\hat{V}_{||} = \frac{\sigma v B_0}{2} \int_{\lambda}^{\lambda_c} \frac{d\lambda'}{\left\langle (1-\lambda' B)^{1/2} \right\rangle}$$

$$f_c = \frac{3}{4} B_0^2 \int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle (1-\lambda B)^{1/2} \right\rangle}; \qquad \lambda_c = \frac{1}{B_{max}}$$

  B_0 is an arbitrary constant magnetic field; a convenient choice is B_{max}, or $\left\langle B^2 \right\rangle^{1/2}$.

- *Perturbations* have the form $a(x) \exp i\left(n\varphi - m\theta - \omega t\right)$

$$\Psi = \frac{1}{i\omega} \hat{\nabla}_{||} A_{||} .$$

  $A_{||}$: longitudinal component of the perturbed vector potential.

  Φ : perturbed electrostatic potential.

  $\tilde{B}_{||}$: perturbed longitudinal magnetic field.

  $\tilde{p}$: total perturbed pressure.

  $\hat{n}_j$, $\hat{T}_j$: contributions to density and temperature from the h_{j0} part of the perturbed distribution functions.

  $u_{||j}$: longitudinal fluid velocity of species j.

  $u_{||j}^{*}$: weighted parallel flow for species j, as defined for electrons in eqn. (26) and ions in eqn. (45).

  Π : radial momentum flux.

- *Frequencies*

$$\omega_{*j}^{T} = \omega_{*j}\left[1+\eta_j(u_j^2-\frac{3}{2})\right], \quad \omega_{*j} = -n\frac{T_j}{e_j}\frac{d(\ell n\, n_j)}{\partial\psi},$$

$$\nu_{ei} = \nu_0/u^3,\ \nu_{ee} = \nu_0\phi(u)/u^3;\ \ \nu_0 = \frac{\sqrt{2}\pi n_e e^4 \ln\Lambda}{m_e^{1/2}T_e^{3/2}}.$$

$$\nu_{ii} = \nu_i\phi(u)/u^3;\ \ \nu_i = \frac{\sqrt{2}\pi n_e e^4 \ln\Lambda}{m_i^{1/2}T_i^{3/2}};\ \phi(u)=(1-\frac{1}{2u^2})\mathrm{erf}(u)+\frac{e^{-u^2}}{u\sqrt{\pi}},$$

so that $\tau_i = \frac{3}{4\sqrt{\pi}}\frac{1}{\nu_i}, \quad \tau_{ei} = \frac{3}{4\sqrt{\pi}}\frac{1}{\nu_0}.$

$$\Omega_j = \frac{e_j B}{m_j}; \quad e_e = -e, \quad e_i = +e\,.$$

- *Symbols used in obtaining vorticity equation*

E, F, H, D_I, L: symbols defined in Appendix A.

- *Equilibrium quantities*

Δ' - tearing mode stability parameter.

$\tau = T_e/T_i$.

L_n, L_T, L_p density, temperature and pressure scale-lengths; $\eta_j = d(\ell nT_j)/d(\ell n\, n_j) = L_{n_j}/L_{T_j}$.

$\hat{\beta}$, $\hat{\gamma}$ and C normalised pressure, bootstrap current and collisionality parameters of the semi-collisional theory, see eqn. (96).

- *Semi-collisional neoclassical current*

$\sigma_{\|}^{sc}, \hat{\sigma}, \sigma_n$ and σ_T semi-collisional conductivity eqn. (67), a normalised form in eqn. (96) and density gradient and temperature gradient contributions to eqn. (67).

D =1+d_0s^2+d_1s^4, denominator appearing in eqn. (40).

λ_j, j=1-4: coefficients of semi-collisionality effects defined in eqn. (40).

$\tilde{J}$: the bootstrap current factor in eqn. (64).

α_n , α_e and α_i: coefficients of density and electron and ion temperature gradient contributions to bootstrap current, eqn. (64).