Side conditioned axisymmetric equilibria with incompressible flows

G. N. Throumoulopoulos[1], H. Tasso[2], G. Poulipoulis[1]

[1] *University of Ioannina, Association Euratom - Hellenic Republic, Section of Theoretical Physics, GR 451 10 Ioannina, Greece*

[2] *Max-Planck-Institut für Plasmaphysik, Euratom Association, D-85748 Garching, Germany*

Abstract

Axisymmetric equilibria with incompressible flows of arbitrary direction are studied in the framework of magnetohydrodynamics under a variety of physically relevant side conditions consisting for example in that the plasma temperature or the magnetic field modulus are uniform on magnetic surfaces. To this end a set of pertinent non-linear ODEs are transformed to quasilinear ones and the respective initial value problem is solved numerically with appropriately determined initial values near the magnetic axis. Several equilibrium configurations are then constructed surface by surface. It turns out that in addition to the usual configurations with a magnetic axis, the non field aligned flow results to novel toroidal shell equilibria in which the plasma is confined within a couple of magnetic surfaces. In addition, the flow affects the elongation and triangularity of the magnetic surfaces and opens up the possibility of changing the magnetic field topology by creating double toroidal shell-like configurations.

I. Introduction

In a previous paper [1] the first two authors derived a generalized Grad-Shafranov equation governing the magnetohydrodynamic equilibrium states of an axisymmetric plasma with incompressible flows [Eq. (1) in Sec. II]. Owing to the flow, Eq. (1) contains five free surface functions, that is three surface functions in excess to those contained in the Grad-Shafranov equation. In particular, for flows non parallel to the magnetic field this equation contains an additional R^4-term, the coefficient of which depends on the density and the electrostatic potential and the variations of these quantities perpendicular to the magnetic surfaces. The conventional way of constructing analytic solutions to linearized versions of Eq. (1) is by assigning appropriately the free functions; accordingly, known solutions of the Grad-Shafranov equation, i.e the Solovév solution [2, 3] and the Hernegger-Maschke solution [4, 5] were extended in Refs. [6] and [7], respectively. For both extended solutions it turns out that the flow can change the magnetic field topology, thus resulting in a variety of new configurations of astrophysical and laboratory concern; specifically, in the case of the extended Solovév solution the configurations have an x-point in addition to the magnetic axis (see Fig. 11 in Ref. [6]) while in the case of the extended Hernegger-Maschke solution the flow gives rise to multitoroidal configurations [8] (Fig. 3 therein).

Instead of specifying the free surface functions of (1), it may be of physical or mathematical importance to introduce side conditions, e.g. isodynamicity: $B^2 = B^2(\psi)$, where ψ and B are the poloidal magnetic flux function and the magnetic field modulus, respectively. In the quasi-static case, viz. when the flow is neglected in the momentum equation but it is kept in Ohm's law, there is a unique configuration of this kind with circular magnetic surface cross-sections near the magnetic axis found in Ref. [9] and completely constructed in Ref. [10]. The same configuration persists in the case of incompressible axisymmetric flows parallel to the magnetic field [1]. In the case of non field aligned flows satisfying the side condition $P + B^2/2 = f(\psi)$, where P is the thermal pressure and f is an arbitrary smooth function of ψ, the magnetic surfaces near axis become elliptical with elongation perpendicular to the axis of symmetry [11].

The aim of the present study is to construct axisymmetric steady states with incompressible flows of arbitrary direction under a variety of side conditions including the aforementioned ones and to examine their generic characteristics particularly in connection with the impact of the flow on the magnetic topology. A preliminary investigation was conducted in Ref. [12]. Here, the construction is carried out numerically on the basis of a procedure suggested in Refs. [9], [10] and [1]. It consists of a reduction of the equilibrium problem to a set of ODEs for certain surface functions along with an integral relation determining the points of a magnetic surface cross section with the poloidal plane. The original set of ODEs, however, is non-linear. For this reason there is a difficulty in relation to the existence and uniqueness of solutions of the respective initial value problem. To resolve this problem we employ a couple of alternative transformations which reduce the ODEs to quasilinear ones. Then the problem is solved numerically. It turns out that the non field aligned flows give rise to a variety of novel equilibria and opens up the possibility of changing the magnetic field topology.

The side conditioned equilibrium equations are reviewed in Sec. II along with the solving procedure of Refs. [9] and [1]. Also, the generalized Grad-Shafranov equation is put in a form containing three surface functions in terms of the five original ones. This is helpful in obtaining subsequently a number of ODEs with equal number of unknown functions. In Sec. III the original ODEs of concern are mapped to quasilinear ones by a transformation which for the quasi-static case was employed in Ref. [10]. An alternative transformation is presented in Appendix B. There is, however, a peculiarity of the initial value problem when initial values are given on axis because the ODEs thereon become indefinite. To overcome this difficulty the derivatives of the unknown functions on axis are obtained in Appendix A by Mercier expansions around axis. Also, this method provides information on the shape of magnetic surfaces near axis. Thus, the problem becomes well posed and is solved then numerically. The various kinds of configurations associated with numerical solutions are presented in Sec. IV and are compared with the existing ones in the literature. Sec. V summarizes the study and the conclusions.

II. Side conditioned equilibrium equations

The MHD equilibrium states of an axisymmetric magnetized plasma with incompressible flows are determined by the generalized Grad-Shafranov equation [1],

$$(1-M^2)\Delta^\star\psi - \frac{1}{2}(M^2)'|\nabla\psi|^2 + \frac{1}{2}\left(\frac{X^2}{1-M^2}\right)' + R^2P_s' + \frac{R^4}{2}\left(\frac{\rho(\Phi')^2}{1-M^2}\right)' = 0, \tag{1}$$

along with the Bernoulli relation for the pressure

$$P = P_s(\psi) - \rho\left[\frac{v^2}{2} - \frac{R^2(\Phi')^2}{1-M^2}\right]. \tag{2}$$

Here, (z, R, ϕ) are cylindrical coordinates with z corresponding to the axis of symmetry; the function $\psi(R, z)$ labels the magnetic surfaces; $M(\psi)$ is the Mach function of the poloidal velocity with respect to the poloidal-magnetic-field Alfvén velocity; $\rho(\psi)$ and $\Phi(\psi)$ are the density and the electrostatic potential; $X(\psi)$ relates to the toroidal magnetic field; for vanishing flow the surface function $P_s(\psi)$ coincides with the pressure; v is the velocity modulus which can be expressed in terms of surface functions and R; $\Delta^\star = R^2\nabla\cdot(\nabla/R^2)$; and the prime denotes a derivative with respect to ψ. Derivation of (1) and (2) is provided in Ref. [1]. It should be noted, however, that (1) and (2) are slightly different in form from the respective equations (22) and (19) in [1] because of a different definition of P_s as was already mentioned in Refs. [6] and [7]. The surface quantities $M(\psi)$, $\Phi(\psi)$, $X(\psi)$, $\rho(\psi)$ and $P_s(\psi)$ are free functions for each choice of which (1) is fully determined and can be solved whence the boundary condition for ψ is given.

Eq. (1) can be simplified by the transformation [13, 14]

$$u(\psi) = \int_0^\psi \left[1 - M^2(g)\right]^{1/2} dg, \tag{3}$$

which reduces (1) to

$$\Delta^\star u + \frac{1}{2}\frac{d}{du}\left(\frac{X^2}{1-M^2}\right) + R^2\frac{dP_s}{du} + \frac{R^4}{2}\frac{d}{du}\left(\rho\frac{d\Phi}{du}\right)^2 = 0. \tag{4}$$

Also, (2) is put in the form

$$P = P_s(\psi) - \rho\left[\frac{v^2}{2} - R^2\left(\frac{d\Phi}{du}\right)^2\right]. \tag{5}$$

Note that no quadratic term as $|\nabla u|^2$ appears anymore in (4). The forms of (4) and (5) indicate one to introduce the new surface quantities

$$N(u) = \frac{X}{\sqrt{1-M^2}}, \tag{6}$$

$$L(u) = \sqrt{\rho}\frac{d\Phi}{du}, \tag{7}$$

which are helpful in reducing the number of the explicit free functions by one. In particular, (4) then contains three free surface functions $[N(u),\ P_s(u),\ L(u)]$ in place of the five original ones $[\rho(u),\ X(u),\ \Phi(u),\ M(u),\ P_s(u)]$.

Instead of specifying the free functions to determine (1), one can introduce side conditions as those mentioned in Sec. I. They can be expressed in terms of the thermal pressure P, the magnetic pressure $B^2/2$, the flow energy density $\rho v^2/2$ or combinations of them and consist in that these quantities remain uniform on magnetic surfaces. It is indeed plausible to assume isothermal magnetic surfaces in hot plasmas because of the huge heat conductivity parallel to $\mathbf{B}$ or try to eliminate neoclassical effects through an isodynamic condition [9]. Also, as was shown in Ref. [11], the condition $P+B^2/2 = f(\psi)$ is of mathematical importance because of a hidden symmetry. It is also relevant to magnetically confined plasmas, e.g. the relation $P + B^2/2 =$ const. is satisfied in a θ-pinch.

Such side conditions lead, in general, to an additional relation between $(\nabla u)^2, u$ and R as already accomplished in Ref. [1]. Specifically, (4) which contains a term with quartic R-dependence due to the assumed incompressibility, and (5) on account of the side conditions to be considered here can be put in the respective forms

$$|\nabla u|^2 = 2[i(u) + R^2 j(u) + R^4 k(u)] \tag{8}$$

$$\Delta^* u = -f(u) - R^2 g(u) - R^4 h(u) \tag{9}$$

where

$$i(u) = -\frac{N(u)^2}{2}, \tag{10}$$

$$f(u) = \frac{1}{2}\frac{dN(u)^2}{du} = -\frac{di(u)}{du}, \tag{11}$$

$$g(u) = \frac{dP_s(u)}{du}, \tag{12}$$

$$h(u) = \frac{1}{2}\frac{dL(u)^2}{du}. \tag{13}$$

The other coefficients $j(u)$ and $k(u)$ being side condition dependent are given in Table 1 for a variety of side conditions. Note that the expressions for $i(u)$,

$j(u)$, $k(u)$, $f(u)$, $g(u)$ and $h(u)$ in (10)-(13) and in Table 1 do not contain the density irrespective of side condition.

Equations (8) and (9) can be solved simultaneously by the following method suggested by Palumbo [9]. Employing R and u as independent coordinates instead of R and z and introducing the quantities $x = R^2$, $p = \partial u/\partial x$, $q = \partial u/\partial z$, $r = \partial^2 u/\partial x^2$ and $t = \partial^2 u/\partial z^2$, Eqs (8) and (9) are written in the respective forms

$$4xp^2 + q^2 = 2(i + xj + x^2 k), \tag{14}$$

$$4xr + t = -f - xg - x^2 h. \tag{15}$$

With the aid of (14) and $f + i' = 0$ which is identically satisfied [see (10) and (11)], Eq. (15) can be integrated with respect to x to yield

$$p = -\frac{1}{4}(g + j')x - \frac{1}{8}(h + k')x^2 + \frac{d(u)}{4}, \tag{16}$$

where $d(u)$ is a "constant of integration" surface quantity. Since z is a function of x and u, solutions of the equation

$$dz = -\frac{p}{q}dx + \frac{1}{q}du \tag{17}$$

exist provided that

$$\frac{\partial}{\partial u}\left(-\frac{p}{q}\right) = \frac{\partial}{\partial x}\left(\frac{1}{q}\right). \tag{18}$$

Using (14) and (16) for q^2 and p, (18) leads to five compatibility conditions in the form of ordinary differential equations (ODEs) [1]. These ODEs contain seven surface functions if the original five surface functions of (1) are employed [Eqs. (44)-(48) of Ref. [1]] or six surface functions if (4) here together with $M(u)$ and $L(u)$ as given by (6) and (7) are employed. Furthermore, complete solution of the problem requires determination of the function $z(x, u)$ which by (17) satisfies on each magnetic surface the equation

$$\left.\frac{\partial z}{\partial x}\right|_u = -\frac{p}{q} = \frac{\pm\frac{1}{4}[(g + j')x + \frac{1}{2}(h + k')x^2 - d]}{\left\{2(i + jx + kx^2) - \frac{x}{4}\left[(g + j')x + \frac{1}{2}(h + k')x^2 - d\right]^2\right\}^{1/2}}. \tag{19}$$

The solutions of (19) are in general hyperelliptic integrals [15], which are not related to known special functions unless they can be reduced to elliptic integrals. This occurs for field aligned flows ($\Phi' = 0$) implying, irrespective of side condition, $h = k = 0$. In this case, considered in Refs. [1, 16], the magnetic surfaces are identical in shape with those of the Palumbo solution (u contours of this solution are provided in Figures 1 and 2 of Ref. [10]); in particular, the magnetic surface cross sections near the magnetic axis are circular. Another case reducible to elliptic integrals is special non field aligned flows associated with the additional side condition $h + k' = 0$ which annihilates the coefficient of the largest powers of x in the denominator of (19). This condition is identically satisfied in the case of $P + B^2/2$ being uniform on a magnetic surface [12], a property related to a hidden MHD symmetry [11].

III. Transformed equations and numerical procedure

In the present study the problem will be solved numerically for generic non field aligned flows without imposing any side condition additional to any of those in Table 1. To this end, the set of the ODEs mentioned in Sec II should be solved on each magnetic surface and then the points of the magnetic surface cross section with the poloidal plane should be determined surface by surface on the basis of (19). There is, however, a difficulty because the ODEs are non-linear and therefore existence and uniqueness of solutions to the respective initial value problem is not guaranteed.

To overcome this difficulty the function u is mapped to a new function w by the transformation

$$\frac{du}{dw} = -\left(g + \frac{dj}{du}\right) \equiv \mathcal{F}(w), \qquad (20)$$

where $\mathcal{F}(w)$ is an arbitrary smooth function. Transformation (20) was employed in Ref. [10] for quasi-static isodynamic equilibria. Then, the method described in Sec. II can be employed along the same lines. Specifically, (8) and (9) lead to the following equations respective to (14) and (15):

$$|\nabla w|^2 = 4xp^2 + q^2 = \frac{2i}{\mathcal{F}^2} + x\frac{2j}{\mathcal{F}^2} + \frac{2k}{\mathcal{F}^2} \equiv \Theta(w) + xQ(w) + x^2\Xi(w), \qquad (21)$$

$$\Delta^\star w = 4xr + t = \frac{1}{2}\frac{d\Theta}{dw} + x\left(1 + \frac{1}{2}\frac{dQ}{dw}\right) - x^2\tilde{H}. \qquad (22)$$

Here, $p \equiv \partial w/\partial x$ ($x = R^2$), $q \equiv \partial w/\partial z$, $r \equiv \partial^2 w/\partial x^2$ and $q \equiv \partial^2 w/\partial z^2$; to make further consideration convenient we have introduced the new surface functions $\Theta(w) \equiv 2i/\mathcal{F}^2$, $Q(w) \equiv 2j/\mathcal{F}^2$, $\Xi(w) \equiv 2k/\mathcal{F}^2$ and

$$\tilde{H}(w) \equiv \frac{h}{\mathcal{F}} + \frac{2k}{\mathcal{F}^3}\frac{d\mathcal{F}}{dw}.$$

Integration of (22) with respect to x yields

$$p = \frac{1}{4}\left(\tilde{H}x^2 + x - Y\right), \tag{23}$$

where $Y(w)$ is the "integration constant"; the relation for z respective to (19) reads

$$\left.\frac{\partial z}{\partial x}\right|_w = -\frac{p}{q} = \frac{\pm\frac{1}{4}[Hx^2 + x - Y]}{\left[\Theta + Qx + x^2\Xi - \frac{1}{4}x\left(Hx^2 + x - Y\right)^2\right]^{1/2}}, \tag{24}$$

where

$$H(w) \equiv -\frac{1}{2}\left(\tilde{H} + \frac{1}{2}\frac{d\Xi}{dw}\right). \tag{25}$$

The case of field aligned flows is recovered for $\Xi = H = 0$ and the special case of non-parallel flows associated with the additional side condition $h + k' = 0$ corresponds to $H = 0$. Furthermore, the compatibility relation (18) leads to the following set of five first-order ODEs for the functions $\Theta(w)$, $Q(w)$, $\Xi(w)$, $H(w)$ and $Y(w)$:

$$4Q + 2\Theta Y' - Y(Y + \Theta') = 0, \tag{26}$$

$$8\Xi + Y(4 - Q') + 2QY' + \Theta' = 0, \tag{27}$$

$$-3 - 2\Theta H' + Q' + 2\Xi Y' + H(6Y + \Theta' - Y\Xi') = 0, \tag{28}$$

$$-2QH' + H(Q' - 8) + \Xi' = 0, \tag{29}$$

$$-5H^2 - 2\Xi H' + H\Xi' = 0. \tag{30}$$

Note that (26)-(30) are *quasi-linear*, viz. the derivatives appear linearly, and therefore Picard' s theorem guarantees existence and uniqueness of the respective initial value problem. Initial values near the magnetic axis can be obtained on the basis of Mercier expansions around axis because of indefiniteness of some of (26)-(30) thereon when they are put in solved forms.

Therefore, the derivatives of the flux functions involved on axis are needed. Details are given in Appendix A. Also, this method provides information on the magnetic field on axis and the shape of the magnetic surfaces near axis. Once initial values are established the problem is well posed and can be solved numerically. It is recalled that the solving procedure consists of the following two steps: a) solve the initial value problem to obtain uniquely the functions $\Theta(w)$, $Q(w)$, $\Xi(w)$, $H(w)$ and $Y(w)$ and b) for any given w integrate (24) to obtain the z-coordinates of the points of each magnetic surface cross section with the poloidal plane. Accordingly, we have developed a programme in Mathematica 5.1. As shown in Appendix A for up-down symmetric configurations to be considered here there are three free parameters (R_0, Ξ_0 and H_0) associated with the radial distance of the magnetic axis and the functions Ξ and H thereon. Also, the problem can be solved by another transformation, alternative to (20), which is presented in Appendix B.

IV. Configurations in connection with numerical solutions

For the solutions to be presented here we have chosen, without loss of generality, $R_0 = 1$ and $w_0 = 0$. Depending on the sign of Ξ_0 there are two kinds of configurations:

1. Toroidal configurations for $\Xi_0 < 0$ having a single magnetic axis [located on $(z = 0, R = R_0)$ where $w = w_0 = 0$]. These configurations have magnetic field topology similar to those of Refs. [10] and [11]. The functions $\Theta(w)$, $Q(w)$, $\Xi(w)$, $H(w)$ and $Y(w)$ for such a solution are given in Fig. 1. Respective w-contours are shown in Fig. 2. In this and subsequent figures to follow a set of w-contours of the quasi-static (Palumbo) solution or another reference equilibrium are also given for comparison. In general for $\Xi_0 < 0$ the magnetic surfaces are more elongated horizontally (parallel to the mid-plane $z = 0$) up to the magnetic axis as compared with the quasi-static ones.

2. Toroidal shells for $\Xi_0 > 0$ in which the plasma is contained within two toroidal surfaces. A solution of this kind is given in Fig. 3. Although mathematically the solution has an extremum on $w = 0$, the physically acceptable part of the solution is restricted to non positive values of $\Theta(w)$ (see Appendix A), thus resulting in a toroidal vertically elongated shell. In particular, w-contours for the solution of Fig. 3 are shown in Fig. 4.

The various kinds of configurations can also be classified in terms of the parameters Ξ_0 and H_0 as follows.

IV1. $\Xi_0 = 0$ *and* $H_0 \neq 0$

Let us first note that $\Xi(w) = 0$ implies parallel flows (or the quasi-static equilibrium) because then it follows from (30) that $H(w) = 0$. This should not be confused with the case of $\Xi_0 = 0$ (on axis) in this subsection which involves non parallel flows when $H_0 \neq 0$. Indeed, Ξ_1 then remains finite (see Apendix A):

$$\Xi_1(\Xi_0 = 0, H_0) = \frac{(R_0 + 2H_0R_0^3)\,(2 + 9H_0R_0^2 - |2 - H_0R_0^2|)}{2R_0^4\,(1 + 2H_0R_0^2)^2}.$$

As in the case of parallel flows, however, the magnetic surfaces near axis have circular cross sections in accordance with (36). For $H_0 > 0$ the surfaces far from axis are less parallel elongated than those of the quasi-static equilibrium (Fig. 5). For $H < 0$ the triangularity can change drastically. As an example, a configuration with inverse triangularity is shown in Fig. 6.

IV2. $\Xi_0 \neq 0$ *and* $H_0 = 0$

Equilibria for $\Xi_0 < 0$ and $H_0 = 0$ have been constructed in Ref. [11] within the framework of the constrained Pohlmeyer-Lund-Regge model (w-curves are provided in Figures 1 and 2 therein). Unlike in Ref. [11], however, no restriction on the elongation of the magnetic surfaces (parallel to the mid-plane $z = 0$) was found here. As a matter of fact such a configuration with very elongated surfaces is presented in Fig. 7 (in purple). Thus, the limitation on the elongation reported in [11] may be due to the particular method of solution in that paper. For $\Xi_0 > 0$ the equilibrium becomes a toroidal shell with magnetic surfaces elongated perpendicular to the mid-plane $z = 0$. A configuration of this kind is shown in Fig. 8.

IV3. $\Xi_0 \neq 0$ *and* $H_0 \neq 0$

In this generic case there is a variety of configurations in connection with the four combinations of signs of Ξ_0 and H_0. It is particularly interesting to examine whether there are configurations with two magnetic axes. This requires two positive roots of the second order-polynomial in the numerator of

(24) and four positive roots of the fourth-order polynomial in the denominator appropriately located with respect to the roots of the numerator. (Note that for a quasi-static equilibrium ($H = \Xi = 0$) only configurations with a single magnetic axis are possible.) For this reason we first examined this requirement by applying the Sturm theorem and Descartes rule [17]. The former determines the exact number of real roots of a polynomial with real coefficients; the latter determines the maximum number of positive roots of such a polynomial. It turns out that the requirement is compatible with the Sturm theorem and Descartes rule. Then, by inspection we found that the requirement can be fulfilled for $\Xi_0 > 0$ and $H_0 < 0$. An equilibrium of this kind shown in Fig. 9 consists of a toroidal shell reaching the axis of symmetry, similar to those reported in subsection IV2, and a second thin shell-like configuration located farther from the axis of symmetry. The distance between the two configurations decreases as $|\Xi_0/H_0|$ takes larger values. As can be seen in Fig. 9, however, the magnetic surfaces of the shell-like configuration do not close. Closeness does not improve either by varying Ξ_0 and H_0 or by constructing the magnetic surfaces by using a different numerical method in FORTRAN. Therefore, the existence of double toroidal shell configurations with closed magnetic surfaces remains an open question.

For the other three combinations of signs of H_0 and Ξ_0 one obtains configurations similar to those presented in subsection IV2. Specifically, single-magnetic-axis equilibria correspond to $\Xi_0 < 0$ irrespective of the sign of H_0. Such a configuration is presented in Fig. 2 (in yellow). As in the case of $H_0 = 0$ there is no limitation on the elongation of the magnetic surfaces parallel to the mid-plane $z = 0$ as $|\Xi_0|$ increases. A configuration with very elongated magnetic surfaces is presented in Fig. 7 (in yellow). The parallel elongation far from axis, however, decreases as H_0 takes larger positive values (Fig. 7). In addition, as shown in Fig. 10, for negative values of H_0 the triangularity of the magnetic surfaces far from axis is affected drastically. Finally, single-toroidal-shell equilibria are derived for $\Xi_0 > 0$ and $H_0 > 0$. A configuration of this kind is shown in Fig. 4 (in yellow).

V. Summary and Conclusions

We have studied axisymmetric equilibria with incompressible flows under side conditions of physical relevance by a procedure introduced in Refs. [9], [10] and [1]. This procedure reduces the problem to a set of ODEs for certain surface functions and an integral relation determining the points of the cross section of a magnetic surface with the poloidal plane. Because of the nonlinearity of the original ODEs, we have employed two alternative transformations [(20) or (51)] mapping the original ODEs to quasilinear ones [Eqs. (26)-(30) or (39)-(43)] and containing equal number of unknown surface functions; thus, existence and uniqueness of the respective initial value problem is guaranteed. Also, because of indefiniteness of the ODEs on the magnetic axis, initial values are determined in terms of the derivatives of the unknown surface functions near axis. These derivatives are obtained by Mercier expansions around axis in conjunction with a l'Hospital-like procedure. The equilibrium problem then has been solved numerically surface by surface with two free parameters (Ξ_0 and H_0) associated with the non-field aligned flows. The flow results in the following novel kinds of up-down symmetric equilibria:

1. Configurations with circular magnetic surfaces near axis, for $\Xi_0 = 0$ and $H_0 \neq 0$, as the quasi-static ones corresponding to $\Xi_0 = H_0 = 0$. However, the flow affects the shape of the surfaces far from axis.

2. Configurations with magnetic surfaces elongated parallel to the mid-plane $z = 0$ for $\Xi_0 < 0$. The special case of equilibria of this kind constructed by a different method in Ref. [11] are recovered for $H_0 = 0$. Unlike in Ref. [11], however, no restriction on the elongation has been found in the present study.

3. Toroidal shells for $\Xi_0 > 0$ and $H_0 \geq 0$ in which the plasma is confined in the interior of two nested magnetic surfaces. The magnetic surfaces are elongated perpendicular to the mid-plane $z = 0$ compared with the quasi-static ones.

4. Equilibria consisting of a toroidal shell reaching the axis of symmetry and a second shell-like configuration for $\Xi_0 > 0$ and $H_0 < 0$. Thus, the flow opens up the possibility of changing the magnetic field topology.

Also, the shape of the magnetic surfaces far from axis is affected by the value and sign of H_0; specifically: a) they become less elongated parallel to the mid-plane $z = 0$ as H_0 takes larger positive values and b) the triangularity of those surfaces is affected drastically for negative values of H_0.

It is emphasized that the above reported conclusions hold irrespective of the particular condition of Table 1 except for $P + B^2/2$ being uniform on surfaces which corresponds to $H_0 = 0$. The properties of particular equilibria in connection with profiles of the (original) physical quantities, i.e. pressure, magnetic field, velocity etc, deserves further investigation. Also, in view of the tough and in general unsolved stability problem of steady states with flow, the stability of the equilibria constructed here remains an open question.

Appendix A. Characteristics near magnetic axis and initial values for ODEs (26)-(30)

Firstly, it is instructive to analyze possible solutions of (21) and (22) near the magnetic axis on the basis of Mercier expansions as follows. Employing a Cartesian system (x, y) centered on the magnetic axis (R_0, z_0), i.e. $R = R_0+x$ and $z = z_0 + y$, we expand the w surfaces in x and y around the magnetic axis up to second order:

$$w - w(0) = \frac{a}{2}x^2 + \frac{b}{2}y^2 + cxy + \text{higher orders.} \tag{31}$$

Also, we expand the flux functions contained in (21) and (22) up to first-order in $w - w(0)$, i.e.

$$\Theta(w) = \Theta(0)+\Theta'(0)\,[w - w(0)]\,, \quad Q(u) = Q(0)+Q'(0)\,[w - w(0)] \quad \text{etc}, \tag{32}$$

and R^2 and R^4 up to second order in x and y. On the basis of the zeroth-, first- and second-order equations thus obtained from (21) and (22), one can derive the following relations for certain of the surface quantities on axis and for the coefficients a, b and c of $w - w(0)$ in (31):

$$\Theta(0) = R_0^4\Xi(0), \tag{33}$$

$$Q(0) = -2R_0^2\Xi(0), \tag{34}$$

$$c = 0, \tag{35}$$

$$\frac{a}{b} = \frac{\left[R_0 + 2H(0)R_0^3\right]^2}{\left[R_0 + 2H(0)R_0^3\right]^2 - 4\Xi(0)}. \tag{36}$$

Henceforth to simplify notation, any surface quantity and its derivative with respect to w on axis will be denoted by the subscripts 0 and 1, respectively, e.g, $\Xi(0) \to \Xi_0$ and $\Xi'(0) \to \Xi_1$. On account of (10) and $\Theta = 2i/\mathcal{F}^2$ [see (21)], Eq. (33) implies that $\Xi_0 \leq 0$ and therefore it follows from (36) that in general the magnetic surfaces near axis are elongated parallel to the mid-plane $z = 0$. In particular, for $\Xi_0 = 0$ they become circular irrespective of the value of H_0. This is an extension of the quasi-static configuration for which $\Xi_0 = H_0 = 0$. Also, unlike the quasi-static equilibrium, the toroidal magnetic field on axis (given below in the u space) does not vanish:

$$(B_\phi)_0 = \frac{1}{R_0}\frac{N_0 - R_0^2 M_0 L_0}{\sqrt{1 - M_0^2}}. \tag{37}$$

In the present study we will consider equilibria symmetric with respect to the mid-plane $z = 0$ and therefore a potential magnetic axis is located on $(z = 0, R = R_0)$. The turning points of the w-curves on the poloidal cross section where $\partial z/\partial x|_w$ vanishes correspond to the roots of the numerator of (24). Also, the intersection points of the w-curves with the mid-plane $z = 0$ where $\partial w/\partial x|_w$ tends to infinity are associated with the roots of the polynomial in the denominator of (24). Thus, the requirement that the abscissa of the magnetic axis is R_0 is fulfilled if one of the roots of the numerator of (24) is equal to R_0^2. This leads to the following relation for surface quantities on axis in addition to (33) and (34):

$$Y_0 = R_0^2\left(1 + H_0 R_0^2\right). \tag{38}$$

Therefore, R_0 and two out of the five quantities Θ_0, Q_0, Ξ_0, H_0 and Y_0 are free. To make comparison with the existing solutions [9, 10, 11] convenient we choose Ξ_0, H_0 (and R_0) as the free parameters. The initial value problem, however, has the peculiarity that when (26)-(30) are written in solved forms:

$$\frac{d\mathcal{A}_i}{dw} = f_i(\Theta, Q, \Xi, H, Y) \quad (i = 1, 2, \ldots 5),$$

where $\mathcal{A}_1 = \Theta$, $\mathcal{A}_2 = Q$ etc, certain of the functions f_i become indefinite on axis (of the form zero over zero). For this reason the derivatives Θ_1, Q_1 etc are needed. To this end, we consider the following algebraic equations, resulting from (26)-(30) to zeroth-order in $w - w_0$ on the basis of the expansions (32):

$$4Q_0 - Y_0^2 + 2Y_1 1\Theta_0 - Y_0\Theta_1 = 0, \tag{39}$$

$$4Y_0 - Q_1 Y_0 + 2Q_0 Y_1 + \Theta_1 + 8\Xi_0 = 0, \tag{40}$$

$$-3 + Q_1 + 6H_0 Y_0 - 2H_1\Theta_0 + H_0\Theta_1 + 2Y_1\Xi_0 - Y_0\Xi_1 = 0, \tag{41}$$

$$-8H_0 - 2H_1Q_0 + H_0Q_1 + \Xi_1 = 0, \tag{42}$$

$$-5H_0^2 - 2H_1\Xi_0 + H_0\Xi_1 = 0. \tag{43}$$

There is, however, an additional obstacle to obtain from these equations all the five derivatives on axis when the constraints (33), (34) and (38) are imposed explicitly (directly), because (39)-(43) then become dependent [viz. the determinant of the coefficients of the set (39)-(43) vanishes]. To overcome this difficulty we have employed a l'Hospital-like procedure consisting of the following steps:

1. Solve the set of (39)-(43) when (33), (34) and (38) are used explicitly to calculate four out of the five first order derivatives, e.g. Θ_1^e, Q_1^e, Ξ_1^e and Y_1^e, in terms of Ξ_0, H_0 R_0 and H_1. Here, the superscript "e" denotes explicit treatment of the constraints.

2. Solve the five algebraic equations without using the constraints to calculate all five derivatives $\Theta_1^i, \ldots H_1^i$, in terms of Θ_0, Q_0, Y_0, Ξ_0, H_0 and R_0 with the subscript "i" indicating implicit (indirect) treatment of the constraints.

3. Since the expression for H_1^i becomes indefinite upon imposing (33), (34) and (38) explicitly we introduce in the constraints (33) and (34), which cause the indefiniteness, respective derivative terms via a small parameter ϵ:

$$\Theta_0 = \Xi_0 R_0^4 + \epsilon\Theta_1, \quad Q_0 = -2\Xi_0 R_0^2 + \epsilon Q_1. \tag{44}$$

Inserting (38) and (44) in the expression for H_1^i and then taking the limit of $\epsilon \to 0$ we obtain H_1^i in terms of Θ_1 and Q_1:

$$H_1^i = -\frac{2H_0^2 \left(Q_1 + 4H_0Q_1R_0^2 + 2H_0\Theta_1\right)}{\left(1 + 2H_0R_0^2\right)^2 \left(Q_1R_0^2 + \Theta_1\right)}. \tag{45}$$

4. The derivatives Θ_1 and Q_1 can then be calculated in terms of Ξ_0, H_0 and R_0 by solving the equations

$$\Theta_1 = \Theta_1^e(\Xi_0, H_0, R_0; H_1^i) \text{ and } Q_1 = Q_1^e(\Xi_0, H_0, R_0; H_1^i) \tag{46}$$

after substituting H_1^i in (46) from (45). Subsequently, $H_1(\Xi_0, H_0, R_0)$ can be calculated from (45) and $\Xi_1(\Xi_0, H_0, R_0)$ together with $Y_1(\Xi_0, H_0, R_0)$ can be obtained from the expressions for Ξ_1^i and Y_1^i.

The expressions for the five derivatives in terms of Ξ_0, H_0 and R_0 and certain intermediate quantities in the above treatment being lengthy will not be given here explicitly. For $H_0 = 0$ it turns out that $H_1 = 0$ and therefore $H(w) \equiv 0$. This is the special cases of non parallel flows associated with the side condition $P + B^2/2 = f(w)$ which was studied in Ref. [11]. In this case (30) is satisfied identically and one has to solve the four ODEs (26)-(29) for the functions $\Theta(w)$, $Q(w)$, $\Xi(w)$, and $Y(w)$. The derivatives on axis then become

$$\Xi_1 = 0, \tag{47}$$

$$\Theta_1 = -\frac{R_0^4}{(R_0^2 - 6\Xi_0)} - 8\Xi_0, \tag{48}$$

$$Q_1 = \frac{3(R_0^2 - 4\Xi_0)}{R_0^2 - 6\Xi_0}, \tag{49}$$

$$Y_1 = -\frac{3}{R_0^2 - 6\Xi_0}. \tag{50}$$

The quasi-static equilibrium is recovered for $\Xi_0 = 0$ and therefore $\Xi(w) \equiv 0$ on account of (47). In this case (29) is also satisfied identically and the problem reduces to the three ODEs (26)-(28) for $\Theta(w)$, $Q(w)$ and $Y(w)$. These are identical with Eqs. (17) in Ref. [10] for the respective functions $P(\lambda)$, $Q(\lambda)$ and $X(\lambda)$ which were derived by geometric considerations. Also, for $H_0 = \Xi_0 = 0$ the derivatives Θ_1, Q_1, and Y_1 calculated from (48)-(50) agree with P_1, Q_1, and X_1 in Ref. [10].

On the basis of the above procedure we have obtained the derivatives on axis by symbolic computations in Mathematica 5.1. After that, the initial value problem is well posed and can be solved numerically.

Appendix B. Alternative transformation

A transformation alternative to (20) which on the basis of the Palumbo procedure leads to a set of five quasi-linear ODEs with equal number of unknown functions is

$$\frac{du}{dw} = -\frac{1}{2}\left(h + \frac{dk}{du}\right) \equiv \mathcal{F}(w). \tag{51}$$

Eq. (21) then remains identical in form and the equation respective to (22) reads

$$\Delta^\star w = 4xr + t = \frac{1}{2}\frac{d\Theta}{dw} - x\tilde{V}(w) + x^2\left(2 + \frac{1}{2}\frac{d\Xi}{dw}\right), \tag{52}$$

where

$$\tilde{V}(w) \equiv \frac{g}{\mathcal{F}} + \frac{2j}{\mathcal{F}^3}\frac{d\mathcal{F}}{dw}. \tag{53}$$

The integral of (53) with respect to x corresponding to (23) is

$$p = \frac{1}{4}\left(x^2 - Vx - Y\right), \tag{54}$$

where

$$V(w) \equiv \tilde{V}(w) + \frac{1}{2}\frac{dQ}{dw} \tag{55}$$

and $Y(w)$ is the "integration constant". The relation for z is put in the form

$$\left.\frac{\partial z}{\partial x}\right|_w = -\frac{p}{q} = \frac{\pm\frac{1}{4}[x^2 - Vx - Y]}{\left[\Theta + Qx + x^2\Xi - \frac{1}{4}x\left(x^2 - Vx - Y\right)^2\right]^{1/2}}. \tag{56}$$

It is convenient to write the numerator of (56) in terms of its roots as

$$x^2 - V(w)x - Y(w) = (x - S(w))\,(x - X(w)), \tag{57}$$

thus replacing V and Y by the functions S and X ($V = S + X$ and $Y = -SX$). The compatibility condition (18) then furnishes the following quasi-linear ODEs for $\Theta(w)$, $Q(w)$, $\Xi(w)$, $S(w)$ and $X(w)$:

$$4Q - S^2X^2 - 2X\Theta S' - S\left(-2\Theta X' + \Theta' X\right) = 0, \tag{58}$$

$$\begin{aligned} 4S^2X + 8\Xi - 2QXS' + 2\Theta\left(S' + X'\right) & \\ +S\left[X\left(4X + Q'\right) - 2QX' - \Theta'\right] - X\Theta' &= 0, \end{aligned} \tag{59}$$

$$\begin{aligned} -3S^2 - X\left(3X + Q' + 2\Xi S'\right) + 2Q\left(S' + X'\right) + \Theta' & \\ +S\left[-Q' - 2\Xi X' + X\left(\Xi' - 12\right)\right] &= 0, \end{aligned} \tag{60}$$

$$Q' + 2\Xi S' + 2\Xi\left(S' + X'\right) - (S + X)\left(\Xi' - 8\right) = 0, \tag{61}$$

$$\Xi' - 5 = 0. \tag{62}$$

Eq. (62) is decoupled from the other equations and has the solution

$$\Xi(w) = \Xi_0 + 5(w - w_0). \tag{63}$$

In this respect transformation (51) is advantageous over (20). The spacial cases of non-parallel flows with $H = 0$ and of parallel flows ($H = \Xi = 0$), however, can not be recovered conveniently because the coefficient of the x^2-term in the numerator of (56) is unity. Also, unlike in the case of the transformation (20) for $H = 0$ for which one of the ODEs [Eq. (30)] is identically satisfied, in the present case for $V = 0$ the problem becomes over-determined because one has to solve the four ODEs (58)-(61) for three functions [$Q(w)$, $\Theta(w)$ and $S(w)$]. Information on the configuration characteristics near magnetic axis can be obtained by considering (21) and (38) on the basis of expansions around axis [Eqs. (31) and (32)]. In particular, relations (33) and (34) for surface functions on axis remain valid, while (38) is replaced, in connection with (57), by $S_0 = R_0^2$ (or $X_0 = R_0^2$). Also, as in the case of (26)-(30), Eqs. (58)-(62) become indefinite on axis when they are put in solved forms and therefore the derivatives Θ_1, Q_1, S_1 and X_1 on axis are required for the initial value problem. These can be found by a procedure similar to that described in Appendix A.

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Table captions

Table 1: The coefficients $j(u)$ and $k(u)$ for several side conditions involving the quantities P, B^2 and ρv^2. Note that the condition $P + \rho v^2/2 = f(u)$ not included in the Table can be satisfied only for parallel flows [$d\Phi/du = 0$] because of the explicit R-dependence of the last term in (5).

Figure captions

Fig. 1: Numerical solutions of Eqs. (26)-(30) for $\Xi_0 = -0.5$ and $H_0 = 0.5$ associated with a configuration with a magnetic axis. In the graphs only the parts of the solutions satisfying the physical requirement $\Theta(w) \leq 0$ are given.

Fig. 2: Two sets of w-contours the one (in yellow) for $\Xi_0 = -0.5$ and $H_0 = 0.5$ associated with the profiles of Fig. 1 and the other (in purple) for $\Xi_0 = 0$ and $H_0 = 0.5$. Both equilibria have a magnetic axis on which $w = 0$.

Fig. 3: Numerical solutions of Eqs. (26)-(30) for $\Xi_0 = 0.1$ and $H_0 = 0.5$ associated with a toroidal shell. The physically acceptable parts of the solutions correspond to non positive values of $\Theta(w)$.

Fig. 4: A set of w-contours associated with a toroidal shell for $\Xi_0 = 0.1$ and $H_0 = 0.5$ (in yellow) associated with the profiles of Fig. 3. The innermost and outermost magnetic surfaces of the shell correspond to the first and second root of $\Theta(w)$ (see Fig. 3). The second set of w-contours (in purple) corresponds to an equilibrium with a magnetic axis ($\Xi_0 = 0$ and $H_0 = 0.5$).

Fig. 5: w-curves for $\Xi_0 = 0$ and $H_0 = 0.5$ (in yellow). The curves near axis are circular as the quasi-static ones shown in purple but far from axis they become less elongated parallel to the mid-plane $z = 0$.

Fig. 6: A configuration with inverse triangularity (in yellow) for $\Xi_0 = 0$ and $H_0 = -0.4$. The w-curves in purple correspond to the quasi-static configuration.

Fig. 7: Two configurations with a magnetic axis and very elongated magnetic surfaces parallel to the mid-plane $z = 0$, the one for $\Xi_0 = -10$ and $H_0 = 0$ (in purple) and the other for $\Xi_0 = -10$ and $H_0 = 1$ (in yellow). The respective values of the eccentricity [$e = \sqrt{1 - a/b}$ with a/b as given by (36)] are 0.988 and 0.9.

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shell and a smaller toroidal shell-like configuration.

Fig. 10: Impact of $H_0 < 0$ on the triangularity of an equilibrium with a magnetic axis ($\Xi_0 = -0.1$): a) a quasi-static like configuration ($H_0 = -0.09$); b) an eye-like configuration ($H_0 = -0.3$); and c) a configuration with inverse triangularity ($H_0 = -0.4$).

List of Tables

Side condition	$j(u)$	$k(u)$
$P = P(u)$	$\frac{2\left[LMN + (1-M^2)(P_s - P)\right]}{M^2(1-M^2)}$	$\frac{L^2(1-2M^2)}{M^2(1-M^2)}$
$B^2 = B^2(u)$	$B^2 + \frac{2LMN}{1-M^2}$	$-\frac{L^2M^2}{1-M^2}$
$\rho v^2 = f(u)$	$\frac{1}{M^2}\left(\frac{2LMN}{1-M^2} + f\right)$	$-\frac{L^2}{M^2(1-M^2)}$
$P + B^2/2 = f(u)$	$\frac{2(f - P_s)}{1-M^2}$	$-\frac{L^2}{1-M^2}$
$B^2 + \rho v^2 = f(u)$	$\frac{4LMN + f(1-M^2)}{1-M^4}$	$-\frac{L^2}{1-M^2}$
$P + B^2/2 + \rho v^2/2 = f(u)$	$\frac{2LMN + (1-M^2)(f-P_s)}{1-M^2}$	$\frac{L^2(M^2-2)}{1-M^2}$

Table 1: The coefficients $j(u)$ and $k(u)$ for several side conditions involving the quantities P, B^2 and ρv^2. Note that the condition $P + \rho v^2/2 = f(u)$ not included in the Table can be satisfied only for parallel flows $[d\Phi/du = 0]$ because of the explicit R-dependence of the last term in (5).

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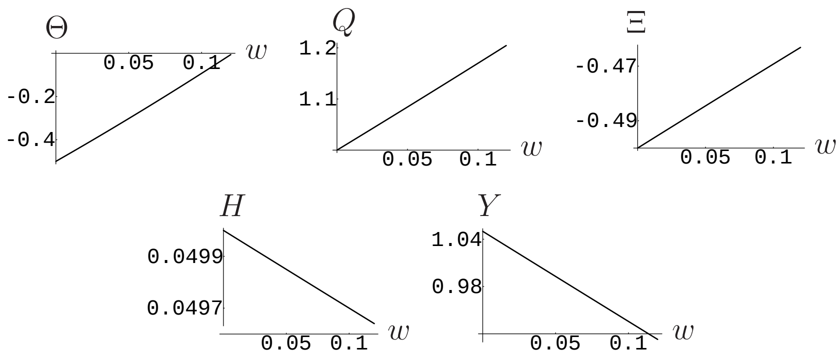

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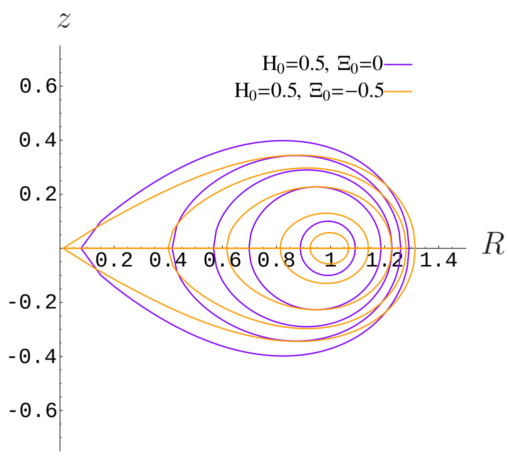

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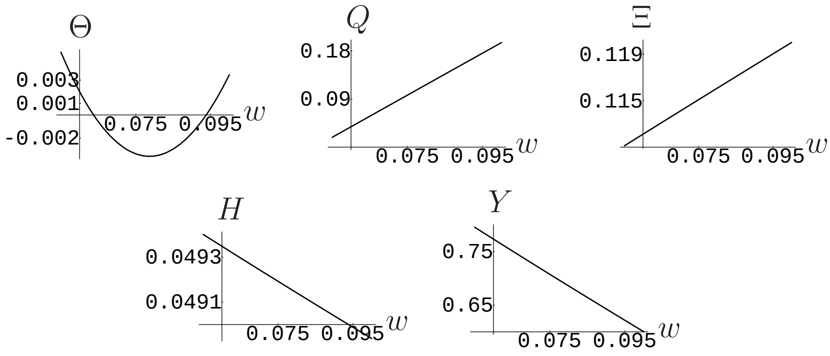

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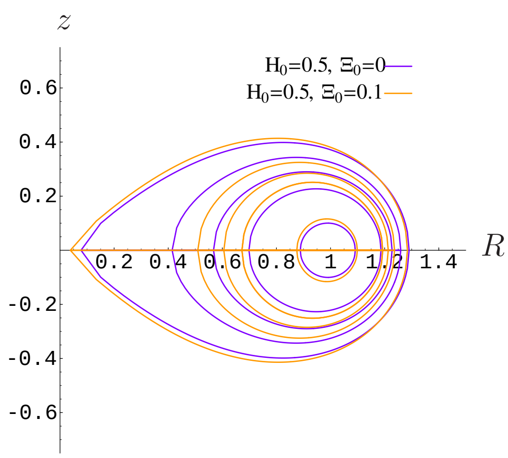

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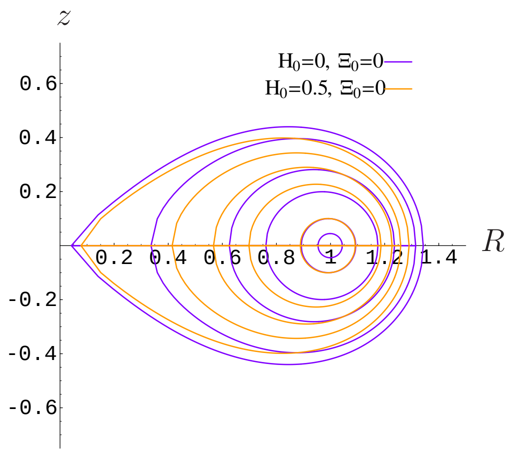

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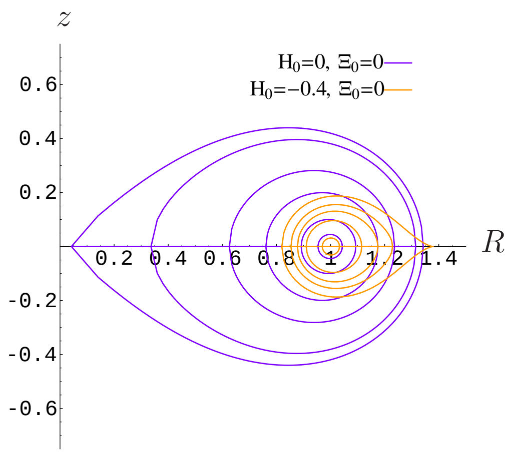

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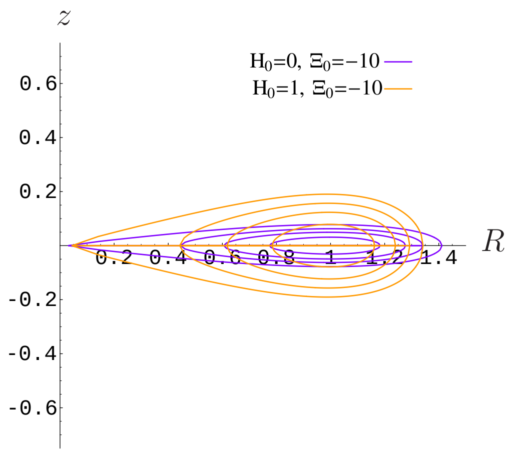

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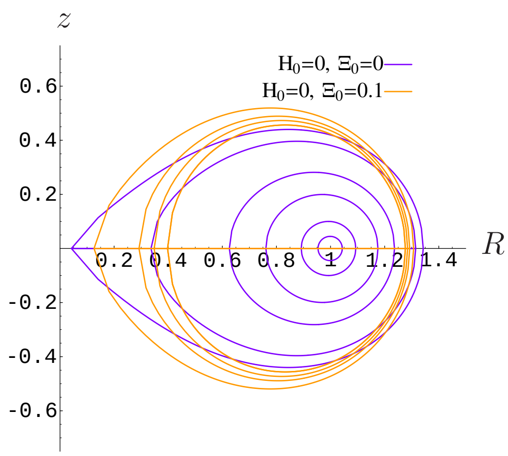

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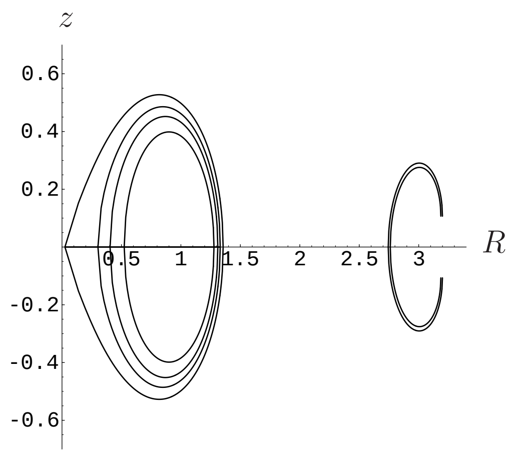

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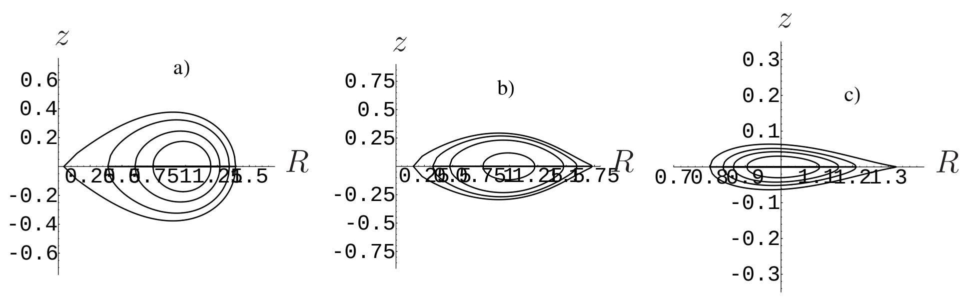

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