# On rapid plasma rotation

P. Helander

*Max-Planck-Institut für Plasmaphysik, 17491 Greifswald, Germany*

The conditions under which rapid plasma rotation may occur in a general three-dimensional magnetic field with flux surfaces, such as that of a stellarator, are investigated. Rotation velocities comparable to the ion thermal speed are found to be attainable only in magnetic fields whose strength $B$ depends on the arc length $l$ along the field in approximately the same way for all field lines on each flux surface $\psi$, i.e., $B \simeq f(\psi, l)$. Moreover, it is shown that the rotation must be in the direction of the vector $\nabla\psi \times \nabla B$.

PACS numbers: 52.55.-s, 52.55.Hc, 52.25.Dg

Magnetic confinement schemes aim at insulating the plasma from the surroundings by means of a strong magnetic field. In the limit of infinite field strength, $B \to \infty$, the transport across magnetic flux surfaces (if such exist) vanishes, while that within flux surfaces remains finite. This is true regardless of whether the cross-field transport is due to collisions or turbulence. In the former case the diffusivity scales as $B^{-2}$ (in most regimes), while turbulent transport can have various dependencies, including $B^{-1}$ (Bohm) and $B^{-2}$ (gyro-Bohm, Rechester-Rosenbluth etc.) In any case, the cross-field transport disappears in the limit $B \to \infty$ since the gyro-radius $\rho$ and cross-field drifts then vanish. Magnetic fusion experiments approximate this limit in the sense that the transport across flux surfaces is many orders of magnitude slower than that within them.

Transport theory for magnetized plasmas usually relies on an expansion in the smallness of $\delta = \rho/L \ll 1$, where $L$ is the macroscopic length scale. In zeroth order transport occurs only within, and not across, flux surfaces, as noted above, and one usually proceeds quickly to first order, where a drift- or gyro-kinetic equation is derived and solved for the kind of transport under consideration, be it collisional or turbulent. However, already the zeroth-order kinetic equation implies some fundamental constraints on the spatial dependence of plasma parameters. For instance, the temperature must be constant on each flux surface because heat conduction eliminates any temperature variation within flux surfaces. For a similar reason, the rotation is constrained to be purely toroidal in an axisymmetric tokamak to lowest order in $\delta$. This is because parallel viscosity eliminates any poloidal rotation substantially greater than the diamagnetic speed (which vanishes in the limit $\delta \to 0$). The only rotation comparable to the ion thermal speed that is allowed is in the toroidal direction.

These results are well known in the tokamak, but it appears that the consequences have not been worked out in the case of a general, three-dimensional field, such as that of a stellarator. This is the aim of the present paper, where we find that rapid rotation can only occur in a certain class of magnetic fields. If the magnetic field is written (locally) as

$$\mathbf{B} = \nabla\psi \times \nabla\alpha, \tag{1}$$

then it turns out that plasma rotation comparable to the ion thermal speed can only occur if the field strength in lowest order only depends on $\psi$ and the arc length $l$ along the field, i.e., if

$$B \simeq f(\psi, l) \tag{2}$$

for some function $f$. Interestingly, such fields have attracted attention because of their favorable confinement properties [1]. They are an important subclass of "omnigenous" magnetic fields, which are fields where the time-averaged cross-field drift vanishes for all particle orbits [2, 3]. Quasi-axisymmetric [4] and quasi-helically symmetric [5, 6] fields are examples of fields having the property (2), but the latter is a weaker condition than quasisymmetry. We also find that the the rotation velocity vector must point in the direction $\nabla\psi \times \nabla B$, so that the streamlines coincide with lines of constant magnetic field strength. Since these contraints follow already in zeroth order, they are independent of the cross-field transport and hold in all collisionality regimes (except extremely low ones, it turns out).

We start from the ion kinetic equation

$$\frac{\partial f}{\partial t} + (\mathbf{V} + \mathbf{v}) \cdot \nabla f + \frac{e}{m}\left(\mathbf{E}' + \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{V}}{\partial t} - (\mathbf{V} + \mathbf{v}) \cdot \nabla \mathbf{V}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f) + S, \tag{3}$$

where $\mathbf{v} = \mathbf{u} - \mathbf{V}(\mathbf{r}, t)$ is the velocity vector measured relative to velocity field $\mathbf{V}$, $\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B}$ the electric field in the moving frame, $e$ and $m$ the ion charge and mass, respectively, $C$ the collision operator and $S$ represents any sources present in the plasma. Although $\mathbf{V}$ is in principle arbitrary, we shall choose it to be equal to the lowest-order mean ion velocity. As in MHD, the electic field is ordered to be so large that the $E \times B$ velocity is comparable to the thermal speed, $E \sim v_T B$, while the collsion frequency is taken to be comparable to the transit frequency, $v_T/L$, in order to allow for all conventional collisionality regimes. The dependent variables $f = f_0 + f_1 + \ldots$, $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \ldots$ and, unconventionally, also the magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \ldots$ are expanded in the smallness of $\delta = v_T/\Omega L \ll 1$, where $\Omega = eB/m$. In order to study equilibrium (rather than the approach to it), the time derivatives of zeroth-order quantities are assumed to be small, $\partial f_0/\partial t \ll (v_T/L) f_0$, whilst higher-order quantities may vary more rapidly (to allow for turbulence, for instance). The electric field is thus electrostatic in lowest order, $\mathbf{E}_0 = -\nabla\Phi_0$.

The largest terms in Eq. (3) are of order $\Omega f$, and the others of order $\delta\Omega f = v_T f/L$ or smaller. In lowest order, then, our kinetic equation becomes simply

$$\frac{e}{m}\left(\mathbf{E}_0' + \mathbf{v} \times \mathbf{B}_0\right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0,$$

and can only hold for all $\mathbf{v}$ if

$$(\mathbf{v} \times \mathbf{B_0}) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{4}$$

and $\mathbf{E}_0' = 0$, so that

$$\mathbf{V}_\perp = \mathbf{V} - \mathbf{V} \cdot \mathbf{bb} = \frac{\mathbf{B_0} \times \nabla \Phi_0}{B_0^2} \tag{5}$$

and $\mathbf{b} \cdot \nabla \Phi_0 = 0$, where $\mathbf{b} = \mathbf{B}_0 / B_0$. We shall assume that the magnetic field at least approximately (i.e., in lowest order) possesses flux surfaces, which we label by $\psi$, so that $\Phi_0 = \Phi_0(\psi, t)$.

A drift kinetic equation can now be derived in the conventional way by averaging over the gyro-angle [7, 8, 9, 10], and there is no need to repeat this well-know calculation here. If the velocity space coordinates are chosen to be $w = mv^2/2$ and $\mu = mv_\perp^2/2B_0$, the result is in lowest order

$$\frac{\partial f_0}{\partial t} + (v_\| \mathbf{b} + \mathbf{V}) \cdot \nabla f_0 + \dot{w} \frac{\partial f_0}{\partial w} + \dot{\mu} \frac{\partial f_0}{\partial \mu} = \frac{\partial f_0}{\partial t} + \bar{\Lambda}(f_0) = C(f_0), \tag{6}$$

where $\dot{\mu} = 0$ and

$$\dot{w} = e\tilde{E}_\| v_\| - m v_\| \mathbf{V} \cdot \nabla \mathbf{V} \cdot \mathbf{b} - m v_\|^2 \mathbf{b} \cdot \nabla \mathbf{V} \cdot \mathbf{b} + \mu B_0 \mathbf{V} \cdot \nabla \ln B_0,$$

with $\tilde{E}_\| = \mathbf{b} \cdot \mathbf{E}_1' = \mathbf{b} \cdot (\mathbf{E}_1 + \mathbf{V} \times \mathbf{B}_1)$.

In equilibrium, the source term balances transport losses and is therefore also relatively small, usually of order $\delta$ or $\delta^2$ as mentioned in the introduction. The solutions to the resulting equilibrium equation are found from a familiar H-theorem argument [9]. Multiplying the equation by $\ln f_0$ and integrating over velocity space gives

$$\nabla \cdot \mathbf{G} = -\int \ln f_0 C(f_0)\ 2\pi v_\perp dv_\perp dv_\|, \tag{7}$$

where

$$\mathbf{G} = -\int (\mathbf{V} + v_\| \mathbf{b}) f_0 (\ln f_0 - 1)\ 2\pi v_\perp dv_\perp dv_\|$$

is the entropy flux. The left-hand side of Eq. (7) is annihilated by a flux-surface average, defined as the volume average between two neighboring flux surfaces, and it follows that $f_0$ must be a Maxwellian, whose density $n$ and temperature $T$ may vary over each flux surface. Substituting this Maxwellian into Eq. (6) without the time-derivative gives an equation which can only be satisfied if the following relations are satisfied [9, 10]:

$$\mathbf{b} \cdot \nabla \ln n - \frac{e\tilde{E}_\|}{T} + \frac{m}{T} \mathbf{V} \cdot \nabla \mathbf{V} \cdot \mathbf{b} = 0,$$

$$\mathbf{b} \cdot \nabla T = 0,$$

$$\mathbf{V} \cdot \nabla \left( \ln n - \frac{3}{2} \ln T \right) = 0,$$

$$\nabla \cdot (n\mathbf{V}) = 0,$$

$$\mathbf{b} \cdot \nabla \mathbf{V} \cdot \mathbf{b} - \frac{1}{3} \nabla \cdot \mathbf{V} = 0.$$

The first of these equations relates $\tilde{E}_{\|}$ to the density variation on each flux surface and will be of no concern to us. The second equation implies that irrational flux surfaces (and, by continuity, also rational ones) are isothermal. Since $\mathbf{V} \cdot \nabla \psi = 0$, the third equation thus implies

$$\mathbf{V} \cdot \nabla n = 0.$$

This reduces the fourth equation to an incompressibility condition,

$$\nabla \cdot \mathbf{V} = 0, \tag{8}$$

and the fifth one to

$$\mathbf{b} \cdot \nabla \mathbf{V} \cdot \mathbf{b} = 0. \tag{9}$$

We now recall Eq. (5) and note that

$$0 = \nabla \times (\mathbf{V} \times \mathbf{B}_0) = \mathbf{B}_0 \cdot \nabla \mathbf{V} - \mathbf{V} \cdot \nabla \mathbf{B}_0,$$

which combined with Eq. (9) leads to

$$\mathbf{V} \cdot \nabla \mathbf{B}_0 \cdot \mathbf{B}_0 = 0.$$

Since $(\nabla \mathbf{B}_0) \cdot \mathbf{B}_0 = B_0 \nabla B_0$ we thus conclude that

$$\mathbf{V} \cdot \nabla B_0 = 0.$$

In other words, the streamlines of the flow are given by the intersection between flux surfaces and surfaces of constant $B_0$. This means that the velocity field can be written as

$$\mathbf{V}(\mathbf{r}) = g(\mathbf{r}) \nabla \psi \times \nabla B_0$$

for some function $g(\mathbf{r})$ of the spatial coordinates $\mathbf{r}$. The parallel component of the flow is thus

$$V_{\|} \mathbf{b} = g(\mathbf{r}) \nabla \psi \times \nabla B_0 - \frac{d\Phi_0}{d\psi} \frac{\mathbf{b} \times \nabla \psi}{B_0}.$$

Taking the scalar product of this equation with $\mathbf{b} \times \nabla\psi$ gives an expression for $g$,

$$g\mathbf{b} \cdot \nabla B_0 + \frac{1}{B_0}\frac{d\Phi_0}{d\psi} = 0,$$

and thus enables us to write down an explicit expression for the lowest-order flow velocity,

$$\mathbf{V} = -\frac{d\Phi_0}{d\psi}\frac{\nabla\psi \times \nabla B_0}{\mathbf{B}_0 \cdot \nabla B_0}. \tag{10}$$

If $\mathbf{B}_0$ is written in Clebsch coordinates (1), then $\mathbf{V}$ becomes

$$\mathbf{V} = \frac{\nabla\Phi_0 \times \nabla B_0}{(\nabla\psi \times \nabla B_0) \cdot \nabla\alpha}.$$

The requirement (8) that this flow field should be incompressible now implies a constraint on the spatial variation of the magnetic field strength,

$$(\nabla\psi \times \nabla B_0) \cdot \nabla(\mathbf{B}_0 \cdot \nabla B_0) = 0. \tag{11}$$

If $B_0$ is expressed in coordinates $(\psi, \alpha, l)$, where $l$ is the arc length along $\mathbf{B}_0$ then it follows from Eq. (11) that

$$(\nabla\psi \times \nabla B_0) \cdot \nabla\dot{B}_0 = 0,$$

where $\dot{B}_0 = \partial B_0/\partial l$. This relation already indicates our desired result: the parallel variation of $B$ should not change when moving along a curve of constant $B$. We note that

$$\frac{\partial B_0}{\partial\alpha}\frac{\partial\dot{B}_0}{\partial l} - \frac{\partial B_0}{\partial l}\frac{\partial\dot{B}_0}{\partial\alpha} = 0, \tag{12}$$

and it follows that $\dot{B}_0$ must be expressible as a function of $\psi$ and $B_0$, i.e., $\dot{B}_0 = \dot{B}_0(\psi, B_0)$, at least locally. This implies, in turn, that $B_0$ must satisfy the requirement (2). To see this formally, we note that Eq. (12) can be written as

$$\frac{\partial \ln \dot{B}_0}{\partial l} = \frac{\partial}{\partial l}\ln\left(\frac{\partial B_0}{\partial\alpha}\right),$$

and integrated once, to yield

$$\frac{\partial B_0}{\partial l} = F(\psi, \alpha)\frac{\partial B_0}{\partial\alpha},$$

with $F$ an arbitrary function. The general solution is

$$B_0 = B_0(\psi, l'),$$

where $l' = l - l_0(\psi, \alpha)$ is an arc length coordinate with a different origin from $l$, constructed to that the origins on different field lines lie on mod-B surfaces. The function $l_0$ is a function of field line related to $F$ by $F\partial l_0/\partial\alpha = -1$. We conclude that rotation at a speed comparable to the thermal speed is only possible if the magnetic field strength in lowest order only depends on $\psi$ and the arc length $l'$. The converse is also true: the flow field (10) satisfies the conditions (8)-(9) if $B_0$ is independent of $\alpha$, and our theorem can thus be stated in the following way. *The lowest-order drift kinetic equation admits solutions where the mean flow velocity is comparable to the thermal speed if, and only if, the magnetic field is approximately satisfies the requirement (2).*

Another way of stating this result is that a sufficiently large radial electric field is only possible in magnetic fields with the property (2). "Sufficiently large" in this context refers to fields that are strong enough to produce flow velocities comparable to $v_T$ (sonic rotation), and it is worth noting that this may occur for fields that are in fact much smaller than $E \sim v_T B$ (though formally of this order, in the sense of the gyroradius ordering assumed). The result (10) can be written as

$$\frac{\mathbf{V}}{E/B_0} = \frac{\mathbf{n} \times \nabla B_0}{\mathbf{b} \cdot \nabla B_0}, \tag{13}$$

where $\mathbf{n} = \nabla\psi/|\nabla\psi|$ is the unit vector normal to the flux surfaces and $E = -\mathbf{n} \cdot \nabla\Phi_0$ is the electric field. The point is that the right-hand side of (13) can be relatively large [but not infinite in fields with the property (2)], in which case the parallel component of the velocity (10) is significantly larger than the perpendicular one. In tokamaks, for instance, $|\mathbf{n} \times \nabla B_0|/(\mathbf{b} \cdot \nabla B_0) \sim q/\epsilon \gg 1$, where $q$ is the safety factor and $\epsilon$ the inverse aspect ratio. As is well known, sonic rotation thus occurs already for radial electric fields of order $E \sim \epsilon v_T B/q$. In a stellarator, a similar estimate tends to hold approximately, but the details depend of course on the specific magnetic configuration. Importantly, sonic rotation can occur at roughly the same electric field as when the poloidal $E \times B$ drift cancels the poloidal component of $v_\|$ for a thermal ion. This "resonance" condition is thought to strongly affect neoclassical transport [11].

The result that the eletric field cannot be large unless the magnetic field satisfies (2) suggests a paradox in the low-density limit, since any electric field strength is possible in vacuum. The resolution lies in our ordering of the collision frequency, $\nu_i \sim v_T/L$. This is the standard neoclassical ordering, and is usually followed by a subsidiary ordering where the collsion frequency is taken to be smaller or larger than the transit frequency, but usually not as small as $\nu \sim \delta v_T/L$. At extremely low densities, this latter case must be allowed, in which case the lowest-order drift kinetic equation becomes $\bar{\Lambda}(f_0) = 0$ and does not constrain $f_0$ to be Maxwellian or $B_0$ to satisfy Eq. (2).

Before concluding, we remark that the condition (2) is satisfied by quasi-helically symmetric fields. By definition, the latter satisfy [6]

$$B = B(\psi, m\theta - n\varphi) = B(\psi, \vartheta),$$

where $(m, n)$ are integers and $(\psi, \theta, \varphi)$ are Boozer coordinates, so that

$$\mathbf{B} = \beta(\psi, \theta, \varphi)\nabla\psi + I(\psi)\nabla\theta + J(\psi)\nabla\varphi = \nabla\psi \times \nabla(\theta - \iota\varphi).$$

It follows that

$$\mathbf{B} \cdot \nabla B = \frac{m\iota - n}{\iota I + J} B^2 \frac{\partial B}{\partial \vartheta}$$

is a function only of $\psi$ and $m\theta - n\varphi$, and hence Eq. (11) is satisfied. As one would expect from their similarity to tokamaks, quasi-symmetric stellarator plasmas are thus in principle capable of rapid rotation. We note, however, that in general the presence of the centrifugal force modifies the usual force balance relation, so that $\mathbf{j} \times \mathbf{B} = \nabla p$ does not hold and Boozer coordinates, as usually defined, do not exist.

In conclusion, we have considered the question of plasma rotation in general three-dimensional magnetic confinement systems, and found that sonic rotation is only possible in certain magnetic fields, and can only occur in the direction of constant magnetic field strength. In the special case of a tokamak, plasma rotation must therefore be purely toroidal in lowest order, as is well known both theoretically and experimentally. (Although the poloidal rotation in experiments has been reported to exceed its neoclassical prediction, it is still far smaller than the toroidal rotation [12, 13].) In stellarators, the radial electric field and rotation velocity are set by the condition of ambipolar cross-field transport, and is usually fairly slow in experiments, $V \ll v_T$. It is often the case that neoclassical transport dominates, and the magnitude and direction of the rotation then depend on the collisionality and heating channel. Faster rotation is only possible if the magnetic equilibrium satisfies the condition (2), and the flow is then given by Eq. (13). These conditions are approximate in the sense that they only need to be satisfied to lowest order in gyroradius, but are independent of the cross-field transport. They therefore hold in all (conventional) collisionality regimes, and also in the presence of gyro-kinetic turbulence.

**Acknowledgment**

It is a pleasure to acknoledge useful discussions with C. Beidler, P. Catto, H. Maaßberg, J. Nührenberg, and Yu. Turkin.

## References


[1] A.A. Skovoroda and V.D. Shafranov, Plasma Phys. Rep. **21**, 886 (1995).

[2] L.S. Hall and B. McNamara, Phys. Fluids **18**, 552 (1975).

[3] J.R. Cary and S. Shasharina, Phys. Plasmas **4**, 3323 (1997).

[4] J. Nührenberg, W. Lotz, S. Gori, in *Theory of Fusion Plasmas* (Editrice Compositori, Bologna, 1994), p. 3.

[5] J. Nührenberg and R. Zille, Phys. Lett. A **129**, 113 (1988).

[6] A.H. Boozer, Plasma Phys. Contr. Fusion **37**, A103 (1995).

[7] R.D. Hazeltine and A.A. Ware, Plasma Phys. Contr. Fusion **20**, 673 (1978).

[8] I.B. Bernstein and P.J. Catto, Phys. Fluids **28**, 1342 (1985).

[9] F.L. Hinton and S.K. Wong, Phys. Fluids **28**, 3082 (1985).

[10] R. Hazeltine and F.L. Hinton, Phys. Plasmas **12**, 102506 (2005).

[11] H. Maaßberg, R. Burhenn, U. Gasparino, G. Khner, H. Ringler, and K.S. Dyabilin Phys. Fluids B **5**, 3627 (1993).

[12] K. Crombé, Y. Andrew, M. Brix, C. Giroud, S. Hacquin, N. C. Hawkes, A. Murari, Phys. Rev. Lett. **95**, 155003 (2005).

[13] W.M. Solomon, K.H. Burrell, R. Andre, L.R. Baylor, R. Budny, P. Gohil, R.J. Groebner, C.T. Holcomb, W.A. Houlberg, and M.R. Wade, Phys. Plasmas **13**, 056116 (2006).