Many-particle approach to the gyrokinetic theory

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ABSTRACT

A systematic first-principle approach to the many-particle formulation of the gyrokinetic theory is suggested. The gyrokinetic many-particle Hamiltonian is derived using the Lie transform technique. The generalized gyrokinetic equation is obtained following the Born-Bogoliubov-Green-Kirkwood-Yvon (BBGKY) approach. The microscopic expression for the self-consistent potential and the polarization density is obtained. It is shown that new terms appear in the gyrokinetic polarization which can not be derived in the conventional approach. An expression for the collision term is obtained in the Landau approximation.

1 Introduction

The gyrokinetic theory is a basic framework to describe microinstabilities, turbulence, and resultant anomalous transport observed in magnetically confined plasmas. Basic equations for the gyrokinetic theory are the gyrokinetic Vlasov equation and the equations for the electromagnetic self-consistent fields. Operating with the self-consistent field, the gyrokinetic theory is, in fact, a many-particle theory. However, the derivation schemes Refs. [1, 2, 3, 4, 5] based on the Lie transform, consider rather a one-particle motion of the guiding center in a gyro-dependent self-consistent field (although Refs. [4, 5] contain the potentiality to incorporate particle collisions within the Klimontovich approach Ref. [6] to the statistical mechanics). Formulated on the basis of the rigorous theory of the guiding center one-particle motion Ref. [7], the gyrokinetic theory remains semi-phenomenological in the treatment of many-particle effects.

In this paper, we suggest a systematic first-principle approach to the many-particle formulation of the gyrokinetic theory. We start with a system of N charged particles in a strong external magnetic field. Using the Lie transform technique (see Ref. [8]), we separate the fast gyro-motion from the dynamics of the system in a way which is analogous to the conventional derivation of the one-particle gyrokinetic theory. The many-particle Hamiltonian function resulting from the Lie transform is used to derive the generalized gyrokinetic equation

with the collision term in accordance with the well-established Born-Bogoliubov-Green-Kirkwood-Yvon (BBGKY) methodology (see Ref. [9]). Finally, the microscopic expression for the self-consistent potential and the polarization density is obtained. It is shown that new terms appear in the gyrokinetic polarization which can not be derived in the conventional approach. An expression for the collision term is obtained in the Landau approximation.

The paper is organized as follows. The many-particle Hamiltonian function is derived in Sec. 2. The generalized gyrokinetic equation is derived in Sec. 3. The main conclusions and an outlook of the paper are given in Sec. 4. In Appendix A, we discuss how to include the magnetic interaction in our theory.

2 The gyrokinetic many-particle Hamiltonian

2.1 N-particle Lagrange form.

Consider a system of N charged particles in a strong magnetic field. It is well known (see Ref. [7]) that the free motion of such particles can be approximately described by the guiding center Lagrange form:

$$\gamma_j = e_j \mathbf{A}_j^* d\mathbf{R}_j + \frac{e_j \mu_j}{m_j} d\theta_j - \left(\frac{m_j v_{\parallel j}^2}{2} + \mu_j B \right) dt + \mathcal{O}(\epsilon_B^2). \quad (1)$$

This description involves a perturbative computation with respect to the small parameter $\epsilon_B = \rho_j / L_B$ where ρ_j is the gyro-radius of the j th particle and L_B is a characteristic length of the external magnetic field (see Ref. [7] for a detailed discussion). The parameters in Eq. (1) are the position of the guiding center $\mathbf{R}_j$, the parallel velocity $v_{\parallel j}$, the magnetic moment μ_j and the gyro-phase θ_j . The external magnetic field is defined as the curl of the magnetic potential $\mathbf{B} = \nabla \times \mathbf{A}$, the extended magnetic potential is given by the expression $\mathbf{A}_j^* = \mathbf{A} + (m_j v_{\parallel j} / e_j) \mathbf{b}$ where $\mathbf{b} = \mathbf{B} / B$, $B = |\mathbf{B}|$. The one-particle Poisson brackets resulting from Eq. (1) have the form:

$$\{R_\tau, R_\nu\} = -\frac{\varepsilon_{\tau\nu\lambda} b_\lambda}{e B_\parallel^*}; \quad \{\mathbf{R}, v_\parallel\} = \frac{\mathbf{b}^*}{m}; \quad \{\mu, \theta\} = -\frac{e}{m}, \quad (2)$$

where the latin index which distinguishes between the particles has been skipped, the Greek indexes correspond to the Cartesian coordinates of the particle, $\varepsilon_{\tau\nu\lambda}$ is the Levi-Civita symbol, $\mathbf{B}^* = \nabla \times \mathbf{A}^*$, $B_\parallel^* = \mathbf{b} \cdot \mathbf{B}^*$ and $\mathbf{b}^* = \mathbf{B}^* / B_\parallel^*$.

Note that the particle's magnetic moment μ_j is an invariant of the free motion. Its invariance is destroyed if we take into account the binary Coulomb interaction between the particles. The reason is that the Coulomb potential:

$$V_{kj} = \frac{e_k e_j}{r(\theta_k, \theta_j)}, \quad r = |\mathbf{R}_k + \rho_k(\theta_k) - \mathbf{R}_j - \rho_j(\theta_j)| \quad (3)$$

depends on the gyro-phases of the interacting particles. Here, e_k is the charge

of the k th particle and $\rho_k(\theta_k)$ is its gyroradius:

$$\rho(\theta_k) = \sqrt{\frac{2m_k\mu_k}{e_k^2 B}} (\mathbf{e}_1 \cos \theta_k - \mathbf{e}_2 \sin \theta_k) \ , \quad \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{b}. \quad (4)$$

Finally, the system of N interacting particles in a strong magnetic field is described with accuracy to $\mathcal{O}(\epsilon_B^2)$ by the Poincaré-Cartan form $\gamma = \hat{\gamma} - \mathcal{H}dt$ with the symplectic part:

$$\hat{\gamma}(z_1, \dots, z_N) = \sum_{j=1}^N \left(e_j \mathbf{A}^*(z_j) d\mathbf{R}_j + \frac{m_j \mu_j}{e_j} d\theta_j \right) \ , \quad (5)$$

and the N -particle Hamiltonian:

$$\begin{aligned} \mathcal{H}(z_1, \dots, z_N) &= \sum_{j=1}^N \left[\frac{m_j v_{\parallel j}^2}{2} + \mu_j B(\mathbf{R}_j) \right] + \\ &\quad \frac{1}{2} \sum_{k,j=1}^N V_{kj}(\mathbf{R}_k, \mu_k, \theta_k; \mathbf{R}_j, \mu_j, \theta_j). \end{aligned} \quad (6)$$

Here, $z_j = (\mathbf{R}_j, v_{\parallel j}, \mu_j, \theta_j)$ is the phase-space coordinate of the j th particle. Note that we always imply $k \neq j$ when the potential V_{kj} is used.

2.2 Lie transform

The gyro-dependency of the binary potential in Eq. (6) represents a substantial complication in the computations because one has to resolve the gyro-motion which is much faster than the motion of the guiding centers. To overcome this complication, one can use the fact that the ratio of the potential energy to the kinetic energy is small for typical plasma parameters:

$$\delta = \left(\frac{1}{2} \sum_{k,j=1}^N V_{kj} \right) / \sum_{j=1}^N \left[\frac{m_j v_{\parallel j}^2}{2} + \mu_j B(\mathbf{R}_j) \right] \ll 1. \quad (7)$$

Thus, a perturbation theory can be constructed which separates the fast gyro-motion from the drift component. Following the derivation scheme of the conventional gyrokinetic theory Ref. [1], we apply a Lie transform technique [note that the canonical Lie transform Ref. [8] is sufficient in our case because the symplectic part Eq. (5) does not contain any gyro-dependency]. An obvious difference from the conventional derivation is that now the Lie transform acts in $6N$ -dimensional phase space whereas in the conventional case it has acted only in the 6-dimensional phase space. Besides that, it is not enough to apply just a single Lie transform to separate all the gyro-dependency because the Hamiltonian function Eq. (6) depends on N gyro-phases. Consequently, a composition

of N transformations is required:

$$\hat{T} = \prod_{k=1}^N \hat{T}_k, \quad \hat{T}_k = \exp \left(- \int_0^\delta \hat{\mathcal{L}}[S_k(\delta')] d\delta' \right), \quad \hat{\mathcal{L}}[S_k] = \{S_k, \cdot\}. \quad (8)$$

Note that N -particle Poisson brackets generated by the symplectic form Eq. (5) are used in Eq. (8). A particular canonical Lie transform $\hat{T}_k$ is defined by a single generating function S_k and is responsible for the separation of a single gyro-phase θ_k from the particle dynamics. The composition of the transformations Eq. (8) starts at θ_N and proceeds consequently to θ_1 separating a single gyro-phase on each step. A particular transform $\hat{T}_k$ acts on the N -particle system which depends only on k gyro-phases $(\theta_1, \theta_2, \dots, \theta_k)$ because the rest of the gyro-phases has already been separated from the dynamics on the previous steps of the composition Eq. (8).

2.3 First order

Applying the apparatus of the perturbative canonical Lie transform (reviewed in Refs. [8, 10]), one obtains in first order of the perturbation for the $(N - k)$ th stage of the transformation the following equation:

$$\sum_{j=1}^N \hat{D}_{0j} S_k^{(1)} = \bar{H}_k^{(1)} - \bar{H}_{k+1}^{(1)}, \quad \hat{D}_{0j} = \frac{\partial}{\partial t} + \{\cdot, h_0(z_j)\} \approx \omega_{cj} \frac{\partial}{\partial \theta_j}. \quad (9)$$

Here, $\bar{H}_k^{(1)}$ is the first-order (in δ) N -particle Hamiltonian function resulting from the $(N - k)$ th stage of the transformation. The Hamiltonian function $\bar{H}_{k+1}^{(1)}$ results from the previous $(N - k - 1)$ th stage of the transformation. The operator $\hat{D}_{0j}$ corresponds to the Lagrange time derivative taken along the unperturbed trajectory of the j th particle [here, the one-particle Hamiltonian $h_0(z) = mv_{\parallel}^2/2 + \mu B$]. The approximation of this operator as $\omega_{cj} \partial/\partial \theta_j$ implies that the drift time-scale is neglected in the evolution equation (9) for the generating function $S_k^{(1)}$.

Applying the operation $\langle \dots \rangle_k$ (average over the gyro-phase θ_k) to Eq. (9) and setting $\langle S_k \rangle_k = 0$, one obtains the following recurrence relation for the Lie-transformed Hamiltonian:

$$\bar{H}_k^{(1)} = \langle \bar{H}_{k+1}^{(1)} \rangle_k = \oint \frac{d\theta_k}{2\pi} \bar{H}_{k+1}^{(1)}, \quad k < N, \quad (10)$$

where for the first stage of the transformation sequence the Hamiltonian is as follows:

$$\bar{H}_N^{(1)} = \langle H^{(1)} \rangle_N, \quad H^{(1)} = \frac{1}{2} \sum_{nj} V_{nj}, \quad V_{nj} = V_{nj}(\theta_n, \theta_j). \quad (11)$$

Finally, combining Eqs. (11) and (10), one can write the resulting first-order gyrokinetic correction to the N -particle Hamiltonian in the form:

$$\bar{H}^{(1)} = \frac{1}{2} \sum_{nj} \langle V_{nj} \rangle_{nj}, \quad \langle V_{nj} \rangle_{nj} = \oint \frac{d\theta_n}{2\pi} \oint \frac{d\theta_j}{2\pi} V_{nj}. \quad (12)$$

2.4 Second order

In the second order, the equation for the generating function on the $(N - k)$ th stage of the transformation can be written in the form (see Ref. [10]):

$$\sum_{j=1}^N \hat{D}_{0j} S_k^{(2)} = 2 \left[\bar{H}_k^{(2)} - \bar{H}_{k+1}^{(2)} \right] - \left\{ S_k^{(1)}, \bar{H}_k^{(1)} + \bar{H}_{k+1}^{(1)} \right\}. \quad (13)$$

Applying the operation of the gyro-average $\langle \dots \rangle_k$ to Eq. (13) and assuming $\langle S_k^{(2)} \rangle_k = 0$, one obtains the following recurrence relation for the second-order gyrokinetic correction to the N -particle Hamiltonian:

$$\bar{H}_k^{(2)} = \left\langle \bar{H}_{k+1}^{(2)} + \frac{1}{2} \left\{ S_k^{(1)}, \tilde{H}_k^{(1)} \right\} \right\rangle_k, \quad \tilde{H}_k^{(1)} = \bar{H}_{k+1}^{(1)} - \bar{H}_k^{(1)}, \quad k < N, \quad (14)$$

where we have used the relations $\langle \bar{H}_{k+1}^{(1)} \rangle_k = \bar{H}_k^{(1)}$ and $\langle \{ \bar{H}_k^{(1)}, S_k^{(1)} \} \rangle_k = 0$.

Note that for the first stage of the transformation sequence, the Hamiltonian is as follows:

$$\bar{H}_N^{(2)} = \frac{1}{2} \left\langle H^{(2)} + \left\{ S_N^{(1)}, \tilde{H}_N^{(1)} \right\} \right\rangle_N, \quad H^{(2)} = 0. \quad (15)$$

Hence, the second-order gyrokinetic correction to the N -particle Hamiltonian is given by the expression:

$$\bar{H}^{(2)} = \frac{1}{2} \left\langle \sum_{k=1}^N \left\{ S_k^{(1)}, \tilde{H}_k^{(1)} \right\} \right\rangle_{1\dots N}, \quad (16)$$

and the N -particle gyrokinetic Hamiltonian can be written in the form:

$$\bar{H}(z_1, \dots, z_N) = \sum_{j=1}^N h_0(z_j) + \frac{1}{2} \sum_{k,j=1}^N \langle V_{kj} \rangle_{kj} + \frac{1}{2} \left\langle \sum_{k=1}^N \left\{ S_k^{(1)}, \tilde{H}_k^{(1)} \right\} \right\rangle_{1\dots N} \quad (17)$$

2.5 The generating function

Consider in detail the k th stage of the Lie transform. Let us introduce the set $L_1 = \{\theta_{k+1}, \dots, \theta_N\}$ of the gyro-phases which are already transformed out [on the current step k of the composition of the transforms Eq. (8)] and the set $L_2 = \{\theta_1, \dots, \theta_{k-1}\}$ of the gyro-phases which have to be transformed out on the following steps. Recall that the current step of the transform handles the

gyro-phase θ_k so that we can introduce the set of all gyro-phases still present in the system $\bar{L}_2 = L_2 \cup \{k\}$.

Using these notations, we can rewrite Eq. (9) in the form:

$$\sum_{j \in \bar{L}_2} \omega_{cj} \frac{\partial S_k^{(1)}}{\partial \theta_j} = -\tilde{H}_k^{(1)}, \quad \tilde{H}_k^{(1)} = \bar{H}_{k+1}^{(1)} - \bar{H}_k^{(1)}. \quad (18)$$

Taking into account that $\langle V_{mn} \rangle_k = V_{mn}$ if both $m, n \neq k$, one can obtain the relation:

$$\tilde{H}_k^{(1)} = \sum_{j \in L_1} (\langle V_{kj} \rangle_j - \langle V_{kj} \rangle_{kj}) + \sum_{j \in L_2} (V_{kj} - \langle V_{kj} \rangle_k). \quad (19)$$

One can see that Eq. (18) contains the coupling between the gyro-motion of different particles. To separate this effect, we write the generating function in the form:

$$S_k^{(1)}(\theta_1, \dots, \theta_k) = \langle S_k^{(1)} \rangle_{L_2}(\theta_k) + \{S_k^{(1)}\}_{L_2}(\theta_1, \dots, \theta_k) \quad (20)$$

where $\langle \dots \rangle_{L_2}$ denotes an average over the gyro-phases $(\theta_1, \dots, \theta_{k-1})$. In the following, the function $\langle S_k^{(1)} \rangle_{L_2}$ will be called the *one-phase part* of the generating function and the function $\{S_k^{(1)}\}_{L_2}$ will be called the *many-phase part* of the generating function.

Starting from Eq. (18) and taking Eq. (19) into account, one can write the evolution equations for both the one-phase part and the many-phase part of the generating function as follows:

$$\omega_{ck} \frac{\partial \langle S_k^{(1)} \rangle_{L_2}}{\partial \theta_k} = -\langle \tilde{H}_k^{(1)} \rangle_{L_2}, \quad \langle \tilde{H}_k^{(1)} \rangle_{L_2} = \sum_{j \in L_1 \cup L_2} (\langle V_{kj} \rangle_j - \langle V_{kj} \rangle_{kj}) \quad (21)$$

$$\sum_{j \in \bar{L}_2} \omega_{cj} \frac{\partial \{S_k^{(1)}\}_{L_2}}{\partial \theta_j} = -\{\tilde{H}_k^{(1)}\}_{L_2} \quad (22)$$

$$\{\tilde{H}_k^{(1)}\}_{L_2} = \left[\sum_{j \in L_2} (V_{kj} - \langle V_{kj} \rangle_j) - \sum_{j \in L_2} (\langle V_{kj} \rangle_k - \langle V_{kj} \rangle_{kj}) \right]. \quad (23)$$

The equation for the one-phase part of the generating function Eq. (21) has an analog in the conventional gyrokinetic theory and can easily be solved:

$$\langle S_k^{(1)} \rangle_{L_2} = -\frac{1}{\omega_{ck}} \int_0^{\theta_k} d\theta_k \sum_{j \in L_1 \cup L_2} (\langle V_{kj} \rangle_j - \langle V_{kj} \rangle_{kj}). \quad (24)$$

2.6 The final form of the gyrokinetic Hamiltonian

Implementing the splitting of the generating function into one-phase and many-phase parts, one can rewrite the Poisson brackets in Eq. (16) in the following

form:

$$\begin{aligned} \left\langle \left\{ S_k^{(1)}, \tilde{H}_k^{(1)} \right\} \right\rangle_{1\dots N} &= \left\langle \left\{ \langle S_k^{(1)} \rangle_{L_2}, \langle \tilde{H}_k^{(1)} \rangle_{L_2} \right\} \right\rangle_{1\dots N} \\ &+ \left\langle \left\{ \{S_k^{(1)}\}_{L_2}, \{\tilde{H}_k^{(1)}\}_{L_2} \right\} \right\rangle_{1\dots N} \end{aligned} \quad (25)$$

where the relations $\langle \{\tilde{H}_k^{(1)}\}_{L_2} \rangle_{L_2} = \langle \{S_k^{(1)}\}_{L_2} \rangle_{L_2} = 0$ have been used.

Substituting the expression Eq. (24) for the one-phase part of the generating function $\langle S_k \rangle_{L_2}$ into Eq. (25), one can obtain the following:

$$\begin{aligned} &\left\langle \left\{ \langle S_k^{(1)} \rangle_{L_2}, \langle \tilde{H}_k^{(1)} \rangle_{L_2} \right\} \right\rangle_k \approx \\ &- \left\langle \frac{1}{2} \sum_{k,j,n=1}^N \frac{1}{B_k} \frac{\partial}{\partial \mu_k} \left(\langle V_{kj} \rangle_j - \langle V_{kj} \rangle_{kj} \right) \left(\langle V_{kn} \rangle_n - \langle V_{kn} \rangle_{kn} \right) \right\rangle_k, \end{aligned} \quad (26)$$

where the notation $B_k = B(\mathbf{R}_k)$ has been used. The drift time scales have been neglected in Eq. (26) which means that only the $\{\mu, \theta\}$ -components of the Poisson brackets have been retained. In all aspects, the derivation of Eq. (26) is analogous to the corresponding computation in the conventional gyrokinetic theory (see Ref. [10] for a detailed discussion).

Finally, one can write the expression for the generalized gyrokinetic Hamiltonian function [supplemented by Eqs. (22) and (23)] which includes many-particle effects rigorously:

$$\begin{aligned} \bar{H}(z_1, \dots, z_N) &= \sum_{j=1}^N h_0(z_j) + \frac{1}{2} \sum_{k,j=1}^N \langle V_{kj} \rangle_{kj} + \frac{1}{2} \left\langle \sum_{k=1}^N \left\{ \langle S_k^{(1)} \rangle_{L_2}, \langle \tilde{H}_k^{(1)} \rangle_{L_2} \right\} \right\rangle_{1\dots N} \\ &- \frac{1}{2} \left\langle \sum_{k,j,n=1}^N \frac{1}{B_k} \frac{\partial}{\partial \mu_k} \left(\langle V_{kj} \rangle_j - \langle V_{kj} \rangle_{kj} \right) \left(\langle V_{kn} \rangle_n - \langle V_{kn} \rangle_{kn} \right) \right\rangle_k \end{aligned} \quad (27)$$

In this section we have derived the N -particle gyrokinetic Hamiltonian which will be used in the following to construct the gyrokinetic equation using the BBGKY approach.

3 The gyrokinetic equation

3.1 Derivation starting from the Liouville equation

Following the conventional procedure for the derivation of the BBGKY hierarchy with a given N -particle Hamiltonian, we start from the Liouville equation (see Ref. [9], p. 118):

$$\left(\frac{\partial}{\partial t} + i\hat{L} \right) \rho(t) = -\varepsilon \{ \rho(t) - \rho_{\text{rel}}(t) \}, \quad -i\hat{L} = \{ \bar{H}, \cdot \}, \quad \varepsilon \rightarrow +0, \quad (28)$$

where the infinitesimal right-hand-side selects retarded solutions (this reflects time irreversibility of the statistical mechanics). The function $\rho(t)$ is the “true”

N -particle distribution function, the function $\rho_{\text{rel}}(t)$ is the so-called “relevant” distribution function which results from the relaxation processes (a time asymptote of the “true” distribution function, see Ref. [9] for a detailed discussion).

As usual, the next step in the derivation is to define the s -particle distribution function:

$$f_s(z_1, \dots, z_s) = \frac{1}{(N-s)!} \int \rho(z_1, \dots, z_N; t) dz_{s+1} \dots dz_N, \quad (29)$$

which can be expressed through the one particle distribution function $f(z)$ and the s -particle correlation function $g_s(z_1, \dots, z_s)$ according to the expression:

$$f_s(z_1, z_2, \dots, z_s) = f(z_1)f(z_2) \dots f(z_s) + g_s(z_1, z_2, \dots, z_s) \quad (30)$$

Substituting the expression for the s -particle distribution function Eq. (29) in the Liouville equation (28), one obtains an infinite set of equations (the BBGKY set, see Refs. [9] for a detailed derivation). One can truncate this set of equations using the weakness of the interaction between the particles [i.e. smallness of the parameter δ , see Eq.(7)]. Since the N -particle gyrokinetic Hamiltonian Eq. (27) has been derived with accuracy up to the second power in the binary potential, we apply the same ordering here as well. Then, the equation for a one-particle distribution function resulting from the Liouville equation (28) takes the form:

$$\begin{aligned} \frac{\partial f(z_1)}{\partial t} + \left\{ f(z_1), h_0(z_1) \right\} + \left\{ f(z_1), \int dz_2 f(z_2) \langle V_{12} \rangle \right\} + \int dz_2 \left\{ g_2(z_1, z_2), \langle V_{12} \rangle \right\} + \\ + \left\{ f(z_1), \frac{1}{(N-1)!} \int dz_2 \dots dz_N f(z_2) \dots f(z_N) \bar{H}^{(2)} \right\} = \mathcal{O}(\|V_{12}\|^3), \end{aligned} \quad (31)$$

where $\|V_{12}\|$ is some measure for the binary potential and the two-particle correlation function $g_2(z_1, z_2)$ results from the equation:

$$\begin{aligned} \frac{\partial g_2(z_1, z_2)}{\partial t} + \left\{ g_2(z_1, z_2), h_0(z_1) + h_0(z_2) \right\} + \left\{ f(z_1)f(z_2), \langle V_{12} \rangle \right\} + \\ + \mathcal{O}(\|V_{12}\|^2) = -\varepsilon g_2(z_1, z_2). \end{aligned} \quad (32)$$

One can see that the first and second terms in the gyrokinetic equation (31) correspond to the free evolution of the kinetic system, the third term describes the self-consistent field, the fourth term corresponds to the collisions (in the Landau approximation) and the fifth term (the “polarization term”) corresponds to the gyrokinetic polarization (see discussion in the next paragraph). Note that in the second order of the perturbation theory, the correlations due to collisions (expressed through g_2) do not mix with the correlations due to the gyrokinetic polarization [expressed through $\bar{H}^{(2)}$]. However, in the higher orders of the perturbation theory, the gyrokinetic polarization will directly influence the collision term.

3.2 Gyrokinetic polarization

The gyrokinetic polarization results from the second order correction to the gyrokinetic Hamiltonian which is given, according to Eq. (27), by the expression:

$$\begin{aligned} \bar{H}^{(2)} = & \frac{1}{2} \left\langle \sum_{k=1}^N \left\{ \{S_k^{(1)}\}_{L_2}, \{\tilde{H}_k^{(1)}\}_{L_2} \right\} \right\rangle_{1\dots N} - \\ & - \frac{1}{2} \left\langle \sum_{k,j,n=1}^N \frac{1}{B_k} \frac{\partial}{\partial \mu_k} \left(\langle V_{kj} \rangle_j - \langle V_{kj} \rangle_{kj} \right) \left(\langle V_{kn} \rangle_n - \langle V_{kn} \rangle_{kn} \right) \right\rangle_k \end{aligned} \quad (33)$$

Assuming identity of the particles, one can rewrite the integral over the phase space in the “polarization term” of the kinetic equation as follows:

$$\begin{aligned} \frac{1}{(N-1)!} \int dz_2 \dots dz_N f(z_2) \dots f(z_N) \bar{H}^{(2)}(z_1, z_2, \dots, z_N) = \\ = \chi_1(z_1) + \chi_2(z_1) + \mathcal{I}_1^{pol}(z_1) + \mathcal{I}_2^{pol}(z_1) + \xi(z_1), \end{aligned} \quad (34)$$

where the “polarization potentials” are given by the expressions:

$$\chi_1(z_1) = - \left\langle \int dz_2 f(z_2) \frac{1}{2B_1} \frac{\partial}{\partial \mu_1} [\langle V_{12} \rangle_2 - \langle V_{12} \rangle_{12}]^2 \right\rangle_1 \quad (35)$$

$$\chi_2(z_1) = - \int dz_2 f(z_2) \frac{1}{2B_2} \frac{\partial}{\partial \mu_2} [\langle V_{12} \rangle_1 - \langle V_{12} \rangle_{12}]^2 \quad (36)$$

$$\begin{aligned} \mathcal{I}_1^{pol}(z_1) = & - \left\langle \int dz_2 dz_3 f(z_2) f(z_3) \times \right. \\ & \left. \frac{1}{2B_1} \frac{\partial}{\partial \mu_1} \left(\langle V_{12} \rangle_2 - \langle V_{12} \rangle_{12} \right) \left(\langle V_{13} \rangle_3 - \langle V_{13} \rangle_{13} \right) \right\rangle_1 \end{aligned} \quad (37)$$

$$\begin{aligned} \mathcal{I}_2^{pol}(z_1) = & - \int dz_2 dz_3 f(z_2) f(z_3) \times \\ & \frac{1}{B_2} \frac{\partial}{\partial \mu_2} \left(\langle V_{12} \rangle_1 - \langle V_{12} \rangle_{12} \right) \left(\langle V_{23} \rangle_3 - \langle V_{23} \rangle_{23} \right) \end{aligned} \quad (38)$$

$$\begin{aligned} \xi(z_1) = & \frac{1}{(N-1)!} \int dz_2 \dots dz_N f(z_2) \dots f(z_N) \times \\ & \frac{1}{2} \left\langle \sum_{k=1}^N \left\{ \{S_k^{(1)}\}_{L_2}, \{\tilde{H}_k^{(1)}\}_{L_2} \right\} \right\rangle_1 \end{aligned} \quad (39)$$

Following the common definition of the self-consistent potential, we introduce the quantity:

$$e\phi_{(1)} = \int dz_2 f(z_2) V_{12}, \quad (40)$$

where e is the charge of the particle and the function $f(z_2)$ is the one-particle distribution function, see Eq. (31). We will see that Eq. (40) represents only the lowest-order part of the self-consistent field potential which appears to be an asymptotic series with respect to the powers of the binary potential $\|V_{12}\|$.

Using the definition Eq. (40), one can show that the term $\mathcal{I}_1^{pol}$ corresponds to the common second order correction in the gyrokinetic Hamiltonian (see Ref. [1]):

$$\mathcal{I}_1^{pol} = -\frac{e^2}{2B(\mathbf{R}_1)} \frac{\partial}{\partial \mu_1} \langle \tilde{\phi}_{(1)}^2(z_1) \rangle, \quad \tilde{\phi}_{(1)} = \phi_{(1)} - \langle \phi_{(1)} \rangle \quad (41)$$

and the term $\mathcal{I}_2^{pol}$ corresponds to the conventional polarization density (see Ref. [1]):

$$\mathcal{I}_2^{pol} = \int dz_2 \frac{e}{B(\mathbf{R}_2)} \frac{\partial f(z_2)}{\partial \mu_2} \tilde{\phi}_{(1)}(z_2) \langle V_{12} \rangle_1 \quad (42)$$

The relation $\oint d\theta_3 \langle V_{12} \rangle_{12} (\langle V_{23} \rangle_3 - \langle V_{23} \rangle_{23}) = 0$ has been applied to derive Eq. (42).

Summarizing, we can write the generalized electrostatic gyrokinetic equation in the following form:

$$\begin{aligned} \frac{\partial f(z_1)}{\partial t} + \left\{ f(z_1), h_0(z_1) \right\} + \left\{ f(z_1), e\langle \phi(z_1) \rangle \right\} + \left\{ f(z_1), -\frac{e^2}{2B(\mathbf{R}_1)} \frac{\partial}{\partial \mu_1} \langle \tilde{\phi}_{(1)}^2(z_1) \rangle \right\} \\ + \left\{ f(z_1), \chi_1(z_1) + \chi_2(z_1) \right\} + \left\{ f(z_1), \xi(z_1) \right\} = J(z_1), \end{aligned} \quad (43)$$

where $J(z_1)$ is the collision integral which will be discussed in the next section. The electromagnetic gyrokinetic equation is discussed in Appendix A.

We define the electrostatic self-consistent potential as follows:

$$e\phi(z_1) = \int dz_2 \left[f(z_2) + \frac{e}{B(\mathbf{R}_2)} \frac{\partial f(z_2)}{\partial \mu_2} \tilde{\phi}_{(1)}(z_2) \right] V_{12} \quad (44)$$

The conventional gyrokinetic Poisson law can easily be derived from this definition. One has to act with Laplacian Δ on Eq. (44) and take into account that for the case of the Coulomb potential $\Delta V_{12} = \Delta(e^2/|\mathbf{x}_1 - \mathbf{x}_2|) = -4\pi e^2 \delta(\mathbf{x}_1 - \mathbf{x}_2)$ (where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the coordinates of the particles). Note that the nonlinear gyrokinetic corrections $\mathcal{I}_1^{pol}$ and $\mathcal{I}_2^{pol}$ are defined with $\phi_{(1)}$ instead of ϕ (in accordance with the order of the perturbation theory used). In the final form of the Poisson law, however, one can neglect this difference.

Besides familiar gyrokinetic quantities, Eq. (43) contains several new terms such as the second-order corrections χ_1 and χ_2 which are proportional to $\int f(z_2) V_{12}^2 dz_2$. It is clear that these quantities can not be obtained in the conventional approach which operates solely with the self-consistent potential $\phi \propto \int f(z_2) V_{12} dz_2$. Physically, the potentials χ_1 and χ_2 can be understood as a self-action of a particle induced by another particle.

The second-order correction ξ is absent in the conventional theory, too. This potential appears due to interaction between the cyclotron motion of different particles and couples arbitrarily many particles together. Closer consideration of this quantity shows a kind of hierarchy of clusters of coupled particles. In

fact, Eq. (39) can be rewritten in the form:

$$\xi(z_1) = \sum_{k=2}^N \frac{(N-k)!}{(N-1)!} \int \frac{1}{2} \left\langle \left\{ \{S_k^{(1)}\}_{L_2}, \{\tilde{H}_k^{(1)}\}_{L_2} \right\} \right\rangle_1 \prod_{i=2}^k f(z_i) dz_i \quad (45)$$

One sees that the quantity ξ is represented by a series of powers of the one-particle distribution function (note that this representation has a formal analogy to the *cluster expansion* used in the statistical physics). At the same time, ξ has only the second order in the binary potential $\|V_{12}\|$. The different terms in Eq. (45) correspond to the closed “clusters” with different number of coupled particles. For example, the term with $k = 2$ couples two particles with the gyro-phases θ_1 and θ_2 ; the term with $k = 3$ couples three particles with the gyro-phases θ_1 , θ_2 and θ_3 ; the term with $k = N$ corresponds to a cluster which contains all N particles.

A more detailed study of the properties of the new potentials χ_1 , χ_2 and ξ resulting from the microscopic description of the gyrokinetic polarization will be done in future.

3.3 Collision integral

The collision integral resulting from the solution of Eq. (32) can be written in a compact form as follows (see Ref. [9]):

$$J_1(z_1, t) = \int dz_2 iL'_{12} \left[\int_0^\infty d\tau e^{-\varepsilon\tau} e^{-i\tau L_{12}^{(0)}} iL'_{12} \right] f(z_1, t) f(z_2, t) \quad (46)$$

with the Liouville operators defined according to the expressions:

$$-iL_{12}^{(0)} = \left\{ h_0(z_1) + h_0(z_2), \cdot \right\}, \quad -iL'_{12} = \left\{ \langle V_{12} \rangle, \cdot \right\}, \quad (47)$$

where $h_0(z) = mv_{\parallel}^2/2 + \mu B(\mathbf{R})$ is the free one-particle Hamiltonian function. The collision integral Eq. (46) is accurate up to the second order in the binary potential (Landau level of the perturbation theory truncation). Note that non-Markov effects have been neglected in this expression. The quantity ε is small and corresponds to the decay of statistical correlations.

Substituting the explicit expressions for the Poisson brackets Eq. (2) into the definition of the Liouville operators Eq. (47), one can write the following:

$$-iL'_{12} = \frac{\partial \langle V_{12} \rangle}{\partial \mathbf{R}_1} \cdot \left[\left(\frac{\mathbf{b}_1^*}{m_1} \frac{\partial}{\partial v_{\parallel 1}} - \frac{\mathbf{b}_2^*}{m_2} \frac{\partial}{\partial v_{\parallel 2}} \right) + \left(\frac{\mathbf{b}_1^*}{qB_{\parallel 1}^*} \times \frac{\partial}{\partial \mathbf{R}_1} - \frac{\mathbf{b}_2^*}{qB_{\parallel 2}^*} \times \frac{\partial}{\partial \mathbf{R}_2} \right) \right] \quad (48)$$

The Fourier transform of the gyro-averaged potential is given by the expression:

$$\langle V_{12}(r) \rangle = \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{R}_{12}} J_0(k_{\perp} \rho_1) J_0(k_{\perp} \rho_2) V_{12}(k), \quad \mathbf{R}_{12} = \mathbf{R}_1 - \mathbf{R}_2, \quad (49)$$

where r corresponds to the distance between the particles and $V_{12}(k) = 4\pi e_1 e_2 / k^2$ for the case of Coulomb interaction (note that, in principle, one can use other types of potentials, too). Finally, neglecting the coordinate dependency coming from the external magnetic field, one can write the gradient of the binary potential in the following form:

$$\frac{\partial \langle V_{12} \rangle}{\partial \mathbf{R}_1} \approx \int \frac{d^3 k}{(2\pi)^3} i \mathbf{k} e^{i \mathbf{k} \cdot \mathbf{R}_{12}} J_0(k_\perp \rho_1) J_0(k_\perp \rho_2) V_{12}(k). \quad (50)$$

Recall that the evolution operator $\exp \left[-i\tau L_{12}^{(0)} \right]$ describes the free motion of the guiding center (which in general is not a straight line), so that

$$\exp \left[-i\tau L_{12}^{(0)} \right] \mathbf{R}_{12} = \mathbf{R}_{12} - \int_0^\tau \mathbf{v}_{12}(t') dt' \quad (51)$$

$$\mathbf{v}_{12} = \{ \mathbf{R}_1, h_0(z_1) \} - \{ \mathbf{R}_2, h_0(z_2) \} \quad (52)$$

Thus, substituting Eqs. (48), (50) and (51) into the collision integral Eq. (46), one can obtain the gyrokinetic collision integral in the following form:

$$\begin{aligned} J(z_1, t) = & \frac{b_{1\mu}^*}{m_1} \frac{\partial}{\partial v_{\parallel 1}} \int d\mathbf{R}_2 2\pi B_{\parallel}^* d\mu_2 dv_{\parallel 2} \tilde{A}_{\mu\nu}^{(12)} \\ & \left(\frac{b_{1\nu}^*}{m_1} \frac{\partial}{\partial v_{\parallel 1}} - \frac{b_{2\nu}^*}{m_2} \frac{\partial}{\partial v_{\parallel 2}} \right) f(z_1, t) f(z_2, t) + \\ & \left(\frac{\mathbf{b}_1^*}{qB_{\parallel 1}^*} \times \frac{\partial}{\partial \mathbf{R}_1} \right)_\mu \int d\mathbf{R}_2 2\pi B_{\parallel}^* d\mu_2 dv_{\parallel 2} \tilde{A}_{\mu\nu}^{(12)} \\ & \left(\frac{\mathbf{b}_1^*}{qB_{\parallel 1}^*} \times \frac{\partial}{\partial \mathbf{R}_1} - \frac{\mathbf{b}_2^*}{qB_{\parallel 2}^*} \times \frac{\partial}{\partial \mathbf{R}_2} \right)_\nu f(z_1, t) f(z_2, t), \end{aligned} \quad (53)$$

where the factor $\tilde{A}_{\mu\nu}^{(12)}$ is given by the expression:

$$\begin{aligned} \tilde{A}_{\mu\nu}^{(12)}(\mathbf{R}_1, \mathbf{R}_2) = & - \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \exp \left[i(\mathbf{k}_1 + \mathbf{k}_2) \cdot (\mathbf{R}_1 - \mathbf{R}_2) \right] \\ & J_0(k_{1\perp} \rho_1) J_0(k_{1\perp} \rho_2) J_0(k_{2\perp} \rho_1) J_0(k_{2\perp} \rho_2) V_{12}(k_1) V_{12}(k_2) k_{1\mu} k_{2\nu} \\ & \int_0^\infty d\tau e^{-\varepsilon\tau} \exp \left[-i \mathbf{k}_2 \cdot \int_0^\tau \mathbf{v}_{12}(t') dt' \right] \end{aligned} \quad (54)$$

In the derivation of Eq. (53), we have neglected the action of the “free” Liouville operator $L_{12}^{(0)}$ on the coordinate dependency coming from the external magnetic field as well as on the distribution functions $f(z_1)$ and $f(z_2)$ (it implies that the variations of the distribution functions during a single small-pitch-angle scattering event are neglected).

Expanding the ambient magnetic field in Eq. (54) in the Taylor series around $\mathbf{R} = \mathbf{R}_1$ and keeping only the first term in the expansion (i.e. assuming that the Debye length is larger than the gyroradius $\lambda_D > \rho_L$), one can simplify the collision integral to the following form:

$$\begin{aligned}
J(z_1, t) = & \frac{b_{1\mu}^*}{m_1} \frac{\partial}{\partial v_{\parallel 1}} \int 2\pi B_{\parallel}^* d\mu_2 dv_{\parallel 2} A_{\mu\nu}^{(12)} \\
& \left(\frac{b_{1\nu}^*}{m_1} \frac{\partial}{\partial v_{\parallel 1}} - \frac{b_{2\nu}^*}{m_2} \frac{\partial}{\partial v_{\parallel 2}} \right) f(z_1, t) f(z_2, t) + \\
& \left(\frac{\mathbf{b}_1^*}{qB_{\parallel 1}^*} \times \frac{\partial}{\partial \mathbf{R}} \right)_{\mu} \int 2\pi B_{\parallel}^* d\mu_2 dv_{\parallel 2} A_{\mu\nu}^{(12)} \\
& \left[f(z_2, t) \left(\frac{\mathbf{b}_1^*}{qB_{\parallel 1}^*} \times \frac{\partial}{\partial \mathbf{R}} \right)_{\nu} f(z_1, t) - f(z_1, t) \left(\frac{\mathbf{b}_2^*}{qB_{\parallel 2}^*} \times \frac{\partial}{\partial \mathbf{R}} \right)_{\nu} f(z_2, t) \right]
\end{aligned} \tag{55}$$

where $z_1 = (\mathbf{R}, v_{\parallel 1}, \mu_1)$, $z_2 = (\mathbf{R}, v_{\parallel 2}, \mu_2)$ and the factor $A_{\mu\nu}^{(12)}$ is given by the expression:

$$\begin{aligned}
A_{\mu\nu}^{(12)} = & \int \frac{d^3 k}{(2\pi)^3} J_0^2(k_{\perp} \rho_1) J_0^2(k_{\perp} \rho_2) V_{12}^2(k) k_{\mu} k_{\nu} \times \\
& \int_0^{\infty} d\tau e^{-\varepsilon \tau} \exp \left[-i \mathbf{k} \cdot \int_0^{\tau} \mathbf{v}_{12}(t') dt' \right]
\end{aligned} \tag{56}$$

To benchmark our result with already existing approaches consider a straight homogeneous ambient magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$. Here, the first term in Eq. (55) (the parallel-velocity-scattering part of the collision integral) can be written in the form:

$$\begin{aligned}
J(z) = & \frac{\omega_p^4}{n^2} \int \frac{d^3 k}{(2\pi)^3} \int d^3 v' k_z \frac{\partial}{\partial v_z} \frac{J_0^2(k_{\perp} \rho) J_0^2(k_{\perp} \rho') \pi \delta[k_z(v_z - v'_z)]}{(k_z^2 + k_{\perp}^2)^2} \\
& \times \left[f(z') k_z \frac{\partial}{\partial v_z} f(z) - f(z) k_z \frac{\partial}{\partial v'_z} f(z') \right]
\end{aligned} \tag{57}$$

In this expression, we have used $d^3 v' = 2\pi B_0 d\mu' dv'_{\parallel}$, the plasma frequency $\omega_p^2 = 4\pi n e^2 / m$ and the Fourier transform of the Coulomb potential:

$$V_{12}(k) = \int V_{12}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3 r = \frac{4\pi e_1 e_2}{k^2}, \quad k^2 = k_z^2 + k_{\perp}^2 \tag{58}$$

We compare the collision integral Eq. (57) with the collision integral derived by N. Rostoker in Ref. [11] [in practice, we compare our result with the collision integral Eq. (B1) of Ref. [12] which is more compact than the integral published in Ref. [11]]:

$$St\{f\} = \frac{\omega_p^4}{n^2} \int \frac{d^3 k}{(2\pi)^3} \sum_{l, l'} \int d^3 v' \left(k_z \frac{\partial}{\partial v_z} + \frac{l\omega_c}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} \right) \tag{59}$$

$$\frac{J_l^2(k_\perp \rho) J_{l'}^2(k_\perp \rho') \pi \delta[k_z(v_z - v'_z) + \omega_c(l - l')]}{(k_z^2 + k_\perp^2)^2 |\epsilon(k_z v_z + l \omega_c, \mathbf{k})|} \left[f(z') \left(k_z \frac{\partial}{\partial v_z} + \frac{l \omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) f(z) - f(z) \left(k_z \frac{\partial}{\partial v'_z} + \frac{l' \omega_c}{v'_\perp} \frac{\partial}{\partial v'_\perp} \right) f(z') \right].$$

One can see that our result coincides with the $l = l' = 0$ term of Eq. (59) which describes the scattering of the parallel velocity component. Note that Refs. [11, 12] derive the collision integral in the Lenard-Balescu approximation whereas we stay within the Landau approximation. This explains the absence of the dielectric function $\epsilon(\omega, \mathbf{k})$ in the denominator of our collision integral (contrary to the integral of Rostoker). Absence of the higher-order Bessel functions $l \neq 0$ in our expression may be attributed to the absence of the gyro-dependent component in the gyrokinetic dynamics and to the approximations employed to solve the equation for the generating functions S_k [see Eq. (9)].

Summarizing, we have obtained a gyrokinetic collision integral Eq. (55) which depends explicitly on the external magnetic field and is anisotropic (clear separation on the parallel and the perpendicular directions in the differential operators entering the collision integral). The complex form of a guiding center trajectory including all possible particle drifts in an inhomogeneous magnetic field is explicitly reflected in the gyrokinetic collision integral through the integral over time in Eq. (56) (which is equivalent to an integral along the particle trajectory). Also, the gyrokinetic shielding effect enters explicitly the gyrokinetic collision integral through the Bessel functions which appear in Eq. (56). The relation of the collision integral derived to the result of Rostoker, Ref. [11], has been established. At this point, we would like to stress that our result is not limited to the straight homogeneous magnetic field (contrary to Refs. [11, 12]) which may be an important improvement. Thus, in Ref. [13] it was indicated that one is faced with *singularities* when using the Rostoker-type collision operator. It was suggested that particle drifts are to be incorporated in the collision operator to get rid of these singularities.

Note that Eq. (55) may be of interest in the physics of a non-neutral (pure-electron) plasma cooled in a strong magnetic field to the liquid or crystal state (by means of cyclotron radiation). In this case, collisions are responsible for the equipartition of the energy between the degrees of freedom (see Ref. [14]). Another possible area of application is production of antihydrogen introducing antiprotons into a positron plasma and applying a strong magnetic field so that the antiprotons and positrons form so-called guiding center atoms (see Ref. [15]) and recombine (the recombination rate will depend on the collision process). In fusion science, a direct effect of the magnetic field on the collision process becomes important if the Debye length is comparable with the gyroradius $\lambda_D \sim \rho_L$. This condition may be satisfied for electron-ion collisions in a tokamak reactor. Further analysis of the collision integral Eq. (55) and its applications will be done in future.

4 Conclusions

In this paper, we have developed a first-principle approach to the gyrokinetic theory which allows a rigorous treatment of many-particle effects such as gyrokinetic polarization and collisions. Microscopic expressions for the self-consistent field and the gyrokinetic polarization have been derived. It is shown that all effects known from the conventional approach to the gyrokinetic theory are still present in our theory.

Several new terms appear in the gyrokinetic polarization (the “polarization potentials” χ_1 , χ_2 and ξ) which could not be derived in the conventional approach. The potentials χ_1 and χ_2 can be interpreted as a manifestation of a “self-action” of a gyrokinetic particle induced by another particle. The potential ξ corresponds to interaction between the cyclotron degrees of freedom of different particles.

We have derived an expression for the gyrokinetic collision integral in the Landau approximation. This integral depends explicitly on the external magnetic field and is anisotropic. The complex form of a guiding center trajectory including all possible particle drifts in an inhomogeneous magnetic field and the gyrokinetic shielding effect enter explicitly the gyrokinetic collision term. The relation of the collision integral derived to the result of Rostoker, Ref. [11], has been established.

The next step in the theory developed here is further analysis of the gyrokinetic collision integral Eq. (55) along with a detailed study of the properties of the new potentials χ_1 , χ_2 and ξ resulting from the microscopic description of the gyrokinetic polarization effect.

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A Electromagnetic corrections

In the main text, we have considered the many-particle gyrokinetic theory based on the electrostatic interaction between the particles. The microscopic expression for the self-consistent electrostatic potential has been derived. Meanwhile, nothing has been said about the self-consistent magnetic potential. In this appendix, we address the natural question about the place of the self-consistent magnetic potential in our theory. We show approximately how one can derive the expression for the self-consistent magnetic field and gyrokinetic Ampere’s law starting from the microscopic picture. We do not intend to perform a detailed and rigorous study of associated effects here.

To derive the microscopic expression for the self-consistent magnetic field potential, one needs to take into account the motion of the particles. In the

main text, we have considered the quantity $\phi'_{(kj)} = V_{kj}/e_j$ (with $V_{kj} = e_k e_j / r_{kj}$ according to the Coulomb law) as an electrostatic potential created by the k th particle in the position of the j th particle. Strictly speaking, it is true only in the rest frame of the k th particle. In the laboratory frame (where all calculation have been performed) one has to take into account two additional effects: the Lorentz transformation of the electrostatic potential $\phi'_{(kj)}$ and the Lorentz contraction of the distance between the particles r_{kj} (appearing in the Coulomb law).

The Lorentz transformation of the electrostatic potential from the rest frame of the k th particle to the laboratory frame $\phi'_{(kj)} \rightarrow \{\phi_{(kj)}, A_{\parallel(kj)}\}$ can be written as follows:

$$\phi_{(kj)} = \frac{\phi'_{(kj)}}{\sqrt{1 - v_{\parallel k}^2/c^2}}, \quad A_{\parallel(kj)} = \frac{v_{\parallel k}}{c} \frac{\phi'_{(kj)}}{\sqrt{1 - v_{\parallel k}^2/c^2}}, \quad (60)$$

Note that $A'_{\parallel(kj)} = 0$ in the rest frame of the k th particle. In Eq. (60), we have neglected the perpendicular motion (drifts) of the gyrocenters (the drifts are usually much slower than the parallel motion).

In the rest frame of the k th particle, the potential $\phi'_{(kj)} = e_k / r_{kj}^{(k)}$. Here, $r_{kj}^{(k)}$ is the distance between the particles which is measured in the rest frame of the k th particle. In the computations, however, the distance $r_{kj}^{(lab)}$ measured in the laboratory frame has to be used. To express $r_{kj}^{(k)}$ through $r_{kj}^{(lab)}$ let us consider the “rod” which connects the particles k and j . The rest frame of this “rod” moves with respect to the laboratory frame with the speed $\Delta v_{\parallel} = (v_{\parallel k} - v_{\parallel j}) [1 + \mathcal{O}(v_{\parallel}^2/c^2)]$ where the second term takes into account the relativistic correction to the Galilean definition of the relative velocity of the particles k and j (we neglect this correction here). Thus, the natural length of the “rod” at rest is $r_{kj}^{(0)} = r_{kj}^{(lab)} / \sqrt{1 - \Delta v_{\parallel}^2/c^2}$ where the square root takes into account the Lorentz contraction. Finally, the length of the “rod” in the rest frame of the k th particle (which we need to substitute in the Coulomb law) is given by the expression:

$$r_{kj}^{(k)} = r_{kj}^{(0)} \sqrt{1 - \frac{v_{\parallel j}^2}{c^2}} = r_{kj}^{(lab)} \sqrt{\frac{1 - v_{\parallel j}^2/c^2}{1 - \Delta v_{\parallel}^2/c^2}} \quad (61)$$

Here, we have used the fact that the speed of the rest frame of the k th particle measured in the rest frame of the “rod” is $v_{\parallel j} [1 + \mathcal{O}(v_{\parallel}^2/c^2)]$.

Substituting Eq. (61) in Eq. (60), one can write the electrostatic and magnetic potentials created by the k th particle in the position of the j th particle and measured in the laboratory frame as follows:

$$\phi_{(kj)} = \frac{e_k}{r_{kj}^{(lab)}} \sqrt{\frac{1 - \Delta v_{\parallel}^2/c^2}{(1 - v_{\parallel j}^2/c^2)(1 - v_{\parallel k}^2/c^2)}} = \left(1 + \frac{v_{\parallel k} v_{\parallel j}}{c^2}\right) V_{kj} + \mathcal{O}\left(\frac{v_{\parallel}^4}{c^4}\right)$$

$$A_{\parallel(kj)} = \frac{v_{\parallel k}}{c} \frac{e_k}{r_{kj}^{(lab)}} \sqrt{\frac{1 - \Delta v_{\parallel}^2/c^2}{(1 - v_{\parallel j}^2/c^2)(1 - v_{\parallel k}^2/c^2)}} = \frac{v_{\parallel k}}{c} V_{kj} + \mathcal{O}\left(\frac{v_{\parallel}^3}{c^3}\right) \quad (63)$$

The total parallel magnetic potential acting on the j th particle in the laboratory frame is given by the expression:

$$e_j A_{\parallel}(z_j) = \frac{1}{c} \sum_{k=1}^N v_{\parallel k} V_{kj} \quad (64)$$

The total potential energy of the system in the laboratory frame is as follows:

$$W_{\text{pot}} = \frac{1}{2} \sum_{k,j=1}^N \left(1 + \frac{v_{\parallel k} v_{\parallel j}}{c^2}\right) V_{kj}, \quad (65)$$

where the familiar factor 1/2 takes care that each pair of particles will be counted just once. Finally, recalling that the canonical momentum has the form $\mathcal{P} = \mathbf{p} - (e/c)\mathbf{A}$, we find the expression for the N -particle Hamiltonian function:

$$\begin{aligned} \mathcal{H}(z_1, \dots, z_N) &= \sum_{j=1}^N \left[\frac{1}{2m_j} \left(m_j v_{\parallel j} - \frac{1}{c^2} \sum_{k=1}^N v_{\parallel k} V_{kj} \right)^2 + \mu_j B(\mathbf{R}_j) \right] \\ &+ \frac{1}{2} \sum_{k,j=1}^N \left(1 + \frac{v_{\parallel k} v_{\parallel j}}{c^2}\right) V_{kj}. \end{aligned} \quad (66)$$

In the first order, one can write the gyro-dependent Hamiltonian as follows:

$$\mathcal{H}^{(1)} = \frac{1}{2} \sum_{k,j=1}^N \left[1 - \frac{v_{\parallel k} v_{\parallel j}}{c^2} \right] V_{kj}. \quad (67)$$

The second order Hamiltonian function is not zero in the electromagnetic case (contrary to the electrostatic case) and is given by the expression:

$$\mathcal{H}^{(2)} = \frac{1}{2} \sum_{k,n,j=1}^N \frac{v_{\parallel k} v_{\parallel n}}{c^2} \frac{V_{kj} V_{nj}}{m_j c^2} \quad (68)$$

Applying the Lie transform technique, one can easily obtain the electromagnetic first- and second-order many-particle gyrokinetic corrections:

$$\bar{H}^{(1)} = \frac{1}{2} \sum_{k,j=1}^N \left[1 - \underbrace{\frac{v_{\parallel k} v_{\parallel j}}{c^2}}_{new} \right] \langle V_{kj} \rangle_{kj} \quad (69)$$

$$\bar{H}^{(2)} = \frac{1}{2} \left\langle \sum_{k=1}^N \left\{ S_k^{(1)}, \bar{H}_k^{(1)} \right\} \right\rangle_{1\dots N} + \underbrace{\frac{1}{2} \sum_{k,n,j=1}^N \frac{v_{\parallel k} v_{\parallel n}}{c^2} \frac{\langle V_{kj} V_{nj} \rangle_{knj}}{m_j c^2}}_{new} \quad (70)$$

Finally, substituting the Hamiltonian function in the BBGKY hierarchy, one can write the electromagnetic generalized gyrokinetic equation:

$$\frac{\partial f(z_1)}{\partial t} + \left\{ f(z_1), h_0(z_1) \right\} + \left\{ f(z_1), e\langle \phi(z_1) \rangle - \frac{v_{\parallel}}{c} \langle A_{\parallel}(z_1) \rangle \right\} + \dots \quad (71)$$

where the self-consistent magnetic potential is defined as follows:

$$eA_{\parallel}(z_1) = \int dz_2 \frac{v_{\parallel}^2}{c} f(z_2) V_{12} - \int dz_2 f(z_2) \frac{eA_{\parallel(1)}(z_2)}{m_2 c^2} V_{12} \quad (72)$$

$$eA_{\parallel(1)}(z_2) = \int dz_3 \frac{v_{\parallel}^3}{c} f(z_3) V_{23} \quad (73)$$

In this appendix, we are interesting only in a rough benchmark of our approach against the conventional picture (we want to derive Ampere's law). Thus, only the last term in Eq. (70) has been taken into account (assuming $j \neq 1$, $k \neq n$). Note that effect of the gyrokinetic polarization [the first term in Eq. (70)] will lead to a slight modification of the definition of the self-consistent magnetic potential.

To derive Ampere's law, we act with Laplacian Δ on both sides of Eq. (72). Recall that $\Delta(1/r) = -4\pi\delta(\mathbf{r})$ so that $\Delta V_{12} = -4\pi e^2 \delta(\mathbf{x}_1 - \mathbf{x}_2)$. Assuming $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_2 = \mathbf{R} + \boldsymbol{\rho}$, one can obtain the expression:

$$\begin{aligned} -\Delta A_{\parallel}(\mathbf{x}) &= \frac{4\pi}{c} \int dz e v_{\parallel} f(z) \delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x}) - \\ &\quad \frac{4\pi e^2}{m c^2} \int dz A_{\parallel}(z) f(z) \delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x}). \end{aligned} \quad (74)$$

Here, the last term can be rewritten as follows:

$$\frac{4\pi e^2}{m c^2} \int dz A_{\parallel}(\mathbf{R} + \boldsymbol{\rho}) f(z) \delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x}) = \frac{4\pi n(\mathbf{x}) e^2}{m c^2} A_{\parallel}(\mathbf{x}) = \frac{\omega_p^2}{c^2} A_{\parallel}(\mathbf{x}) \quad (75)$$

where the relations $n(\mathbf{x}) \approx \int f(\mathbf{x} - \boldsymbol{\rho}, v_{\parallel}, \mu) B_{\parallel}^* d\mu dv_{\parallel} d\theta$ and $\omega_p = \sqrt{4\pi n e^2 / m}$ have been used. Finally, we derive Ampere's law which includes the familiar skin terms:

$$\left(\frac{\omega_p^2}{c^2} - \Delta \right) A_{\parallel}(\mathbf{x}) = \frac{4\pi}{c} \int dz e v_{\parallel} f(z) \delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x}) \quad (76)$$

Recall that effect of the gyrokinetic polarization has been skipped in the derivation.

Summarizing, we have found a way of including the magnetic interaction into the many-particle gyrokinetic theory (it appears to be the lowest-order relativistic correction to the electrostatic case). We have derived a microscopic expression for the self-consistent magnetic potential. As a benchmark for our theory, we have derived the conventional gyrokinetic Ampere's law which is a direct consequence of the microscopic expression for the self-consistent magnetic potential. The effect of the microscopic magnetic interaction on the gyrokinetic polarization and the gyrokinetic collision integral will be studied in details in future.

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