Contributions to the Control of Hybrid and Switched Linear Systems

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Abstract

Hybrid linear systems are a class of hybrid systems where the continuous time evolution is governed by a set of first order linear ordinary differential equations and the jump dynamics are described by a set of first order linear difference equations. Switched linear systems are a subclass of hybrid linear systems with a continuous evolution of the system states. Due to the large number of physical applications, control of hybrid and switched linear systems has received considerable attention over the past years. This dissertation provides novel contributions to the control of such systems and extends some existing results in this area. This work focuses on different problems regarding stability and stabilization of switched linear systems and optimal control of hybrid linear systems.

The major part of this thesis concerns stability and stabilization of switched linear systems. The results are primarily presented in terms of conditions for stability of autonomous switched systems. Then, stabilization methods aim at designing a local state feedback for each mode of a controlled switched system to satisfy the stability criteria for the closed loop system modes. Parts of these results rely on the concept of common left eigenvectors and left eigenstructure assignment. To this end, several techniques for eigenstructure assignment in the context of linear systems are developed. Afterwards, these techniques are employed for characterizing exponential stability and for stabilization of a class of switched linear systems with state dependent switching and certain restrictions on the switching manifolds. In addition, they are used for quadratic stabilization of a class of controlled switched linear systems with arbitrary switching signals where the open loop constituent matrices share an invariant subspace to which a common quadratic Lyapunov function can be associated. Another stability approach makes use of the Kalman-Yakubovic-Popov lemma to demonstrate quadratic stability of a class of switched linear systems. This class is characterized by arbitrary switching between two modes, where the difference of the constitute matrices is of rank $m \geq 1$, and to which a symmetric transfer function matrix can be associated. These results extend existing results in quadratic stability of rank-1 difference switched linear systems.

The thesis also addresses problems in linear quadratic control of hybrid linear systems. In these problems, cost functions are quadratic, the final time and the number and sequence of switches are given. Constraints are specified by linear (in-)equalities in the state space. Switching between different dynamics may either occur at fixed or free time instances. A parameterization method with respect to initial and end points of each interval in a generalized time domain is employed. Numerical solutions for such problems are suggested.

Zusammenfassung

Hybride lineare Systeme sind eine Klasse von hybriden Systemen, bei denen die kontinuierliche Zeitentwicklung durch einen Satz von linearen Differentialgleichungen erster Ordnung, und die Sprungdynamik durch einen Satz von linearen Differenzengleichungen erster Ordnung beschrieben werden. Geschaltete lineare Systeme sind eine Unterklasse von hybriden linearen Systemen mit einer stetigen Zeitentwicklung der Systemzustände. Aufgrund der großen Anzahl physikalischer Anwendungen erfuhr die Regelung von hybriden und geschalteten linearen Systemen in den letzten Jahren beträchtliche Aufmerksamkeit. Diese Dissertation stellt neue Beiträge zur Regelung solcher Systeme vor und erweitert einige vorhandene Ergebnisse in diesem Bereich. Diese Arbeit konzentriert sich auf verschiedene Aspekte der Stabilität und Stabilisierung geschalteter linearer Systeme sowie der optimalen Regelung hybrider linearer Systeme.

Der größte Teil dieser Arbeit betrifft die Stabilität und Stabilisierung von geschalteten linearen Systemen. Die Darstellung der Ergebnisse bezieht sich in erster Linie auf Bedingungen für die Stabilität autonomer geschalteter linearer Systeme. Stabilisierungsverfahren zielen dann auf den Entwurf einer lokalen Zustandsrückführung für jeden Modus eines geregelten geschalteten linearen Systems, um die Stabilitätskriterien für die Modi des geschlossenen Regelkreises zu befriedigen. Teile dieser Ergebnisse beruhen auf dem Konzept gemeinsamer linker Eigenvektoren und der Zuweisung linker Eigenstruktur. Zu diesem Zweck werden mehrere Verfahren zur Zuweisung der Eigenstruktur von linearen Systemen entwickelt. Anschließend werden diese Techniken zur Charakterisierung exponentieller Stabilität und zur Stabilisierung einer Klasse von geschalteten linearen Systemen mit zustandsabhängigem Schalten unter bestimmten Einschränkungen bezüglich der Schaltmannigfaltigkeiten eingesetzt. Darüber hinaus werden sie zur quadratischen Stabilisierung einer Klasse von beliebig schaltenden linearen Systemen eingesetzt, in der die Dynamikmatrizen des offenen Kreises einen invarianten Untervektorraum gemeinsam haben, der mit einer gemeinsamen quadratischen Lyapunov-Funktion assoziiert werden kann. Ein weiterer Stabilitätsansatz nutzt das Kalman-Yakubovic-Popov-Lemma, um quadratische Stabilität einer Klasse von geschalteten linearen Systemen zu zeigen. Diese Klasse wird von beliebigem Schalten zwischen zwei Modi gekennzeichnet, für welche die Differenz der Dynamikmatrizen von Rang $m \geq 1$ ist, und mit denen eine symmetrische Übertragungsfunktionsmatrix assoziiert werden kann. Diese Ergebnisse erweitern vorhandene Ergebnisse zur quadratischen Stabilität geschalteter linearer Systeme, bei denen die Differenz der Dynamikmatrizen Rang 1 aufweist.

Die Dissertation befasst sich auch mit der linear quadratischen Regelung von hybriden linearen Systemen. Hierbei sind die Kostenfunktionen quadratisch, der Zeithorizont und die Anzahl und Reihenfolge des Schaltens gegeben. Einschränkungen sind durch lineare (Un-) Gleichungen im Zustandsraum gegeben. Die Zeitpunkte des Schaltens zwischen unterschiedlichen Dynamiken können entweder vorgegeben oder frei sein. Ein Parametrierungsverfahren in Bezug auf Ausgangs- und Endpunkte der einzelnen Intervalle eines generalisierten Zeitbereichs wird verwendet. Numerische Lösungen werden für solche Aufgaben vorgeschlagen.

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Publications

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- N. Bajcinca, D. Flockerzi, and Y. Kouhi. On a geometrical approach to quadratic Lyapunov stability and robustness. In *Proc. of the 52th Conference on Decision and Control*, pages 1–6, 2013
- Y. Kouhi, N. Bajcinca, J. Raisch, and R. Shorten. A new stability result for switched linear systems. In *Proc. of the European Control Conference*, pages 2152–2156, 2013a
- Y. Kouhi, N. Bajcinca, and R. G. Sanfelice. Suboptimality bounds for linear quadratic problems in hybrid linear systems. In *Proc. of the European Control Conference*, pages 2663–2668, 2013b
- Y. Kouhi and N. Bajcinca. On the left eigenstructure assignment and state feedback design. In *Proc. of the American Control Conference*, pages 4326–4327, 2011a
- Y. Kouhi and N. Bajcinca. Nonsmooth control design for stabilizing switched linear systems by left eigenstructure assignment. In *Proc. of the IFAC World Congress*, pages 380–385, 2011b
- Y. Kouhi and N. Bajcinca. Robust control of switched linear systems. In *Proc. of the 50th IEEE Conference on Decision and Control and European Control Conference*, pages 4735–4740, 2011c

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Chapter 1

Introduction and literature review

1.1 Hybrid and switched linear systems

After over two decades of research, many problems concerning control of hybrid and switched systems still remain unsolved. These problems are important as many physical systems have hybrid features. Examples can be found in mechanical, chemical, biological, network systems, etc. Hybrid systems combine multiple dynamics. Typically, such systems involve switches between different flows, between flows and jumps, or only switches between different discrete dynamics. The switching rule is often governed by an external signal or characterized by state space constraints. The external signal can be caused by different sources. The most common type of this signal depends on time or states. When the dynamics of a hybrid system is linear, we call the system a hybrid linear system.

Control of hybrid and switched linear systems are particularly interesting from two points of view. First, these systems inherit some properties of standard linear systems. Second, the switching nature of these systems gives rise to nonlinear behaviors. Thus, the control approaches developed for these systems borrow concepts from both linear and nonlinear control theory. Numerous examples of these systems can be found in physical systems. To gain more intuition about the models of switched and hybrid linear systems, we now illustrate some applications.

(i) **Boost converter:** Each electrical circuit may have switching nature due to existence of electrical switches, diodes, transistors, or capacitors; see Julius and Van der Schaft (2002), Heemels et al. (2009), Sira-Ramirez (1989), and Schiffer et al. (2012). The following example has been selected from Heemels et al. (2009).

An established example in power systems which behaves as a hybrid system is a DC-DC boost converter. Power converters are widely used in variable speed DC motor drives, computer power supplies, cell phones, and cameras (Heemels et al., 2009). The boost converter can generate an output voltage which is greater than the input voltage. The circuit of a DC-DC converter is depicted in Figure 1.1.1. This model consists of a load R, a capacitor C, an inductor L, a flyback diode D, and a switch S. The input voltage is denoted by $u := E$ and the voltage of two sides of the capacitor C is denoted by $v_c(t)$. R_c and R_L are the series resistors for the capacitor and the inductor, respectively. The switching period T_s for the switch S is given. The duty cycle $\alpha(t) \in [0,1]$, that is, the ratio of the activation duration of on mode for switch S to the period T_s, is considered as the external input. When the switch S is open, depending on the voltage of two sides of the flyback diode D, this diode may function in on or off modes. Now, assuming $x(t) = [i_L(t) \;\; v_c(t)]^\top$ as the state variables of the boost converter, depending on the status of the diode D and the switch S, three operation modes for this system can be investigated. This model can be written in the form of

$$\dot{x}(t) = A_{\sigma(t,x)}x(t) + B_{\sigma(t,x)}u \qquad \sigma(t,x) \in \mathcal{L} := \{1,2,3\}, \tag{1.1}$$

where A_i and B_i for $i \in \mathcal{L}$ are specified as follows:

1) In the first mode the switch S is closed and the entire current i_L passes through the switch S, thus $i_L > 0$ or $[1 \;\; 0]\; x > 0$. In this case, the matrix A_1 and the vector B_1 are given by

$$A_1 = \begin{bmatrix} \frac{-R_L}{L} & 0 \\ 0 & \frac{-1}{(R+R_c)C} \end{bmatrix}, \; B_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}. \tag{1.2}$$

The system works in this mode until time reaches to $t = (n-1+\alpha)T_s$, where $n \in \mathbb{N}$. Then the system switches to mode 2).

2) In this mode the switch S is open and the flyback diode is on. Then, $i_L > 0$ or $[1 \;\; 0]x > 0$, and the data are computed to be

$$A_2 = \begin{bmatrix} \frac{-1}{L}\left(R_L + \frac{R_cR}{R+R_c}\right) & \frac{-1}{L}\frac{R}{R+R_c} \\ \frac{R}{C(R+R_c)} & \frac{-1}{C(R+R_c)} \end{bmatrix}, \; B_2 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}. \tag{1.3}$$

The system can have two transitions from this mode. If $i_L = 0$ or $[1 \;\; 0]x = 0$ holds, then the system switches to mode 3). Otherwise, after spending time up to $t = nT_s$ the system switches to mode 1).

3) In this mode the switch S is open and the flyback diode is off. It implies that $i_L = 0$ or $[1 \;\; 0]x = 0$, and

$$A_3 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{(R+R_c)C} \end{bmatrix}, \; B_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.4}$$

The system stays in this mode until the time instance $t = nT_s$. Afterwards, the system switches to mode 1).

As will be explained in Section 1.2, the model of the boost converter (1.1) is in the form of a controlled switched linear system. The switching rules depend on both the states and time.

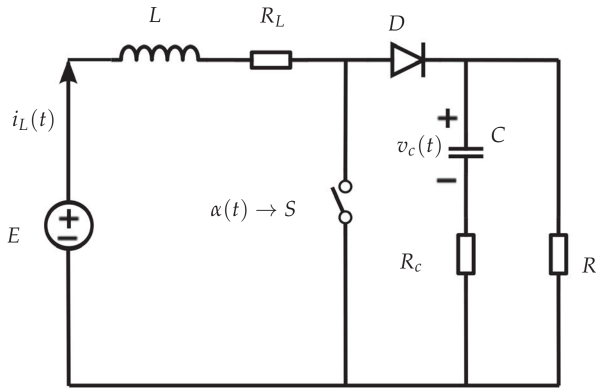

Figure 1.1.1: A DC-DC boost converter (Heemels et al., 2009).

(ii) **Mobile multi-agent systems with switching topology:** This example has been selected from Olfati-Saber and Murray (2004). In multi-agent systems, each agent exchanges information with some other agents called its "neighbors". This communication defines a network topology. The network topology is often represented by a directed graph $G = (V, E, A)$, where $V = \{v_1, \ldots, v_n\}$ denotes the set of nodes associated to the n agents, $E \subseteq V \times V$ is the set of edges which corresponds to the communication between agents, and $A = [a_{ij}]$ with non-negative entries is the $n \times n$ adjacency matrix (Ren and Beard, 2007). The network may have a switching topology perhaps due to the agents' changing positions or existence of obstacles between the agents. In this case, one can associate a directed graph $G_k \subseteq \Gamma$ corresponding to each new topology, where Γ is a set of finite collections of digraphs with n nodes and k belongs to an index set $\mathcal{L} := \{1, \ldots, \ell\} \subset \mathbb{N}$ with $\ell := |\Gamma|$. Then, the states of the network evolve with different dynamics, which can be expressed by a switched linear system in the form of

$$\dot{x} = -L_{\sigma(t,x)} x, \tag{1.5}$$

where $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathcal{L}$ is a switching signal which specifies the active network topology. The matrix $L_k = [l_{k,ij}]_{n \times n}$ with $k \in \mathcal{L}$ is called the Laplacian of the graph G_k, which is defined by

$$l_{k,ij} = \begin{cases} \displaystyle\sum_{p=1,\ p \neq i}^{n} a_{k,ip} & j = i, \\ -a_{k,ij}, & j \neq i. \end{cases}$$

Note that for two different agents $i, j \in \{1, \ldots, n\}$ which are not neighbors in G_k, we have $a_{k,ij} = 0$ for each $k \in \mathcal{L}$. The aim of the consensus algorithm consists in characterizing a communication routine such that all states exponentially converge to the same variable, namely

$$x_1(\infty) = x_2(\infty) = \cdots = x_n(\infty) = \frac{1}{n} \sum_{i=1}^{n} x_i(0). \tag{1.6}$$

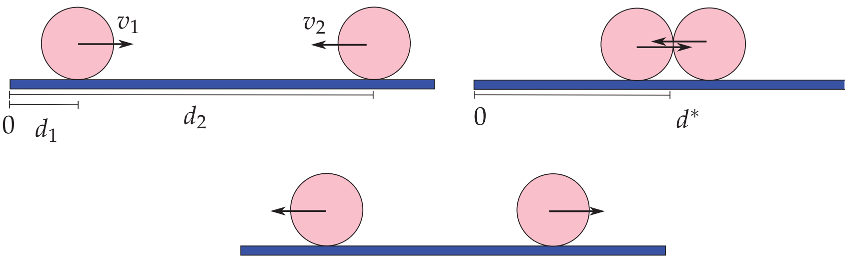

Figure 1.1.2: Schematic of the moving objects before, during, and after collision (Goebel et al., 2009).

Now, note that (1.5) defines a switched linear system, where switching signal depends on time and state variables.

(iii) **Collisions:** The system dynamics of two moving particles may change instantly when a collision between them occurs. More precisely, the collision may lead to sudden changes on the velocities of the mobile objects. Examples of such collisions are billiard balls and bouncing balls (Goebel et al. (2009), Lygeros (2004) and Brogliato (1999)). We now illustrate the following example taken from Goebel et al. (2009).

Consider two particles which are moving towards each other with constant speeds v_1 and v_2; see Figure 1.1.2. For simplicity, let's assume that the diameters of both particles are zero. The states of this system can be considered as $x = [d^\top \ v^\top]^\top$, where $d = [d_1 \ d_2]^\top$ and $v = [v_1 \ v_2]^\top$ represent the positions and the velocity of the two particles, respectively. The system dynamics are specified by Newton's second law as

$$\dot{x} = \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ v_1 \\ v_2 \end{bmatrix} := Ax. \tag{1.7}$$

Note that the system dynamics evolves in accordance with (1.7) whenever $x \in C := \{x : d_1 \leq d_2\}$. A collision occurs when the positions of the particles are the same and $v_1 \geq v_2$. In a collision time instance, positions of the particles remain the same, that is, $d^+ = d$, while the velocities can be derived by the momentum equation

$$m_1 v_1^+ + m_2 v_2^+ = m_1 v_1 + m_2 v_2, \tag{1.8}$$

and the energy dissipation equation

$$v_1^+ - v_2^+ = -\rho(v_1 - v_2), \tag{1.9}$$

where m_1 and m_2 are the masses of the particles and the constant $0 < \rho < 1$ is called restitution coefficient. Thus, the model of this system for the points

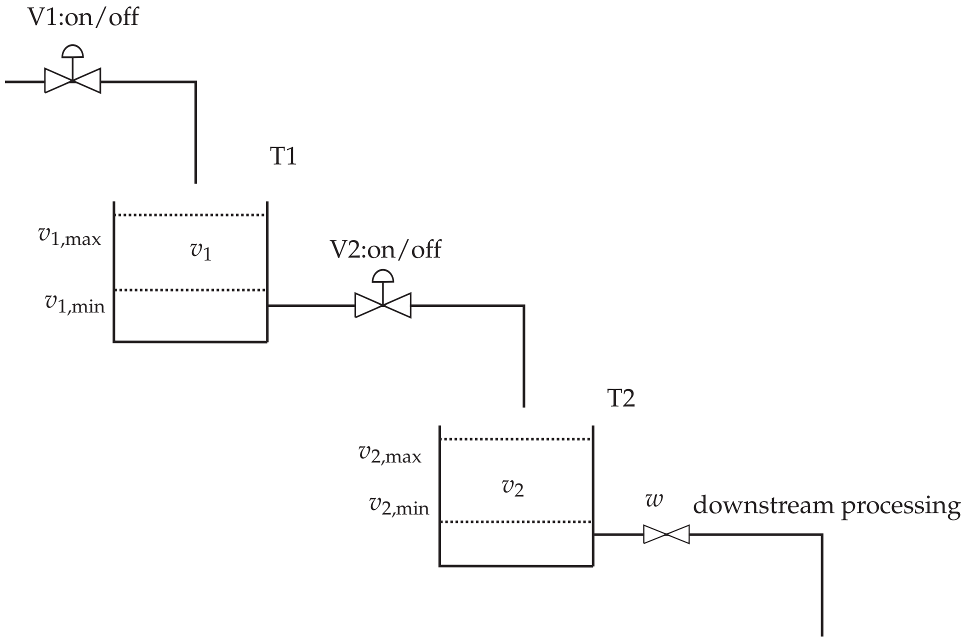

Figure 1.1.3: Schematic of the chemical process plant (Simeonova et al., 2006).

belonging to the collision set specified by $D = \{x : d_1 = d_2,\ v_1 \geq v_2\}$ reads

$$\begin{bmatrix} d_1^+ \\ d_2^+ \\ v_1^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{m_1 - m_2\rho}{m_1+m_2} & \frac{m_2(1+\rho)}{m_1+m_2} & 0 & 0 \\ \frac{m_1(1+\rho)}{m_1+m_2} & \frac{m_2 - m_1\rho}{m_1+m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ v_1 \\ v_2 \end{bmatrix} := Gx. \tag{1.10}$$

Consequently, we observe that the model of this system can be represented in the form of

$$\mathcal{H} : \begin{cases} \dot{x} &= A\,x \qquad x \in C, \\ x^+ &= G\,x \qquad x \in D. \end{cases} \tag{1.11}$$

We talk extensively about stability and optimal control of hybrid systems of this form in Chapter 6.

(iv) **Chemical process systems:** Many chemical process plants can include hybrid or switching effects. For example, opening and closing of valves in a plant may suddenly change inputs, outputs, or states of the system. Different scenarios of interconnection of tank systems are introduced in the literature; see Raisch et al. (1999) or Simeonova et al. (2006). The following example has been selected from Simeonova et al. (2006).

The chemical plant depicted in Figure 1.1.3 is a simplified version of the benchmark chemical plant (Simeonova, 2008). The tank T1 is a batch chemical reactor which is followed by the buffer tank T2. This system operates automatically and includes four successive operation modes: filling with the raw material, production by the chemical reaction, discharging (*i.e.*, harvesting of the final product), cleaning and waiting for the next operation. Although the control reacts in a discontinuous way, the final product must be delivered in a continuous way to the downstream processing stage. To this end, there is an intermediate buffer tank T2 between the batch reactor and the downstream processing plant, which is discontinuously fed from the reactor, but continuously withdrawn. The states of the system are v_1, the volume of T1, and v_2, the volume of T2. Four operation modes for this system can be investigated as follows:

1) This mode refers to the condition that the valve V1 is closed (off) and V2 is considered to be open. The initial volume of V1 equals $v_{1,\max}$ and the initial volume of T2 equals $v_{2,\min}$. The material in T1 is discharging with the output flow rate $r[m^3/h]$, and T2 is being fed with the input flow rate $r[m^3/h]$, and is discharging with the output flow rate $w[m^3/h]$. Defining $x = [v_1\ v_2]^\top$, the dynamics can be written as follows:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} r \\ w \\ q \end{bmatrix} := B_1 u. \tag{1.12}$$

This mode lasts until the volume of the tank T1 reaches to a minimum value $v_{1,\min}$. Then, the plant switches to mode 2).

2) This mode refers to the condition that the valves V1 and V2 are both closed (off). Thus, V1 is in standby. The material in T2 is discharging with the output flow rate $w[m^3/h]$. Then, the dynamics is written as follows:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} r \\ w \\ q \end{bmatrix} := B_2 u. \tag{1.13}$$

Mode 2) lasts for a certain amount of time $p_2[h]$, decided by the operator. Then, the plant operates in mode 3).

3) This mode refers to the case that the valve V1 is open (on), and V2 is closed (off). Then, T1 is filled with the raw material with the flow rate $q[m^3/h]$, and T2 is discharging with the output flow rate $w[m^3/h]$. The dynamics then reads

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} r \\ w \\ q \end{bmatrix} := B_3 u. \tag{1.14}$$

This mode lasts until the volume of T1 reaches the maximum value $v_{1,\max}$. Then, the plant switches to mode 4).

4) This mode refers to the case that both V1 and V2 are closed (off). Only T2 is discharging with the output flow rate $w[m^3/h]$. Then, the system dynamics can be written as follows:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} r \\ w \\ q \end{bmatrix} := B_4 u. \tag{1.15}$$

Mode 4) lasts for certain amount of time $p_4[h]$. Then, the plant goes again to mode 1), and the cycle continues.

Now, consider that the dynamics of the overall system can be represented as a switched linear system of the form

$$\dot{x} = B_{\sigma(t,x)} u \qquad \sigma(t,x) \in \mathcal{L} := \{1,2,3,4\}, \tag{1.16}$$

where $\sigma(t,x) : \mathbb{R} \times \mathbb{R}^2 \to \mathcal{L}$ represents the switching signal which depends on time and the states of the system.

1.2 Model of hybrid and switched linear systems

As the previous physical applications suggest, a hybrid linear system can be represented by the following general model:

$$\mathcal{H} : \begin{cases} \dot{x} & = A_{\sigma(t,x)}\ x + B_{\sigma(t,x)}\ u \qquad x \in C, \\ x^+ & = G_{\sigma(t,x)}\ x + H_{\sigma(t,x)}\ v \qquad x \in D, \end{cases} \tag{1.17}$$

where

$$\sigma(t,x) \in \mathcal{L} := \{1,\ldots,\ell\}, \tag{1.18}$$

indicates the switching signal between different dynamics (A_i, B_i, G_i, H_i) for $i \in \mathcal{L}$. In this model, $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ denote the *flow* and *jump* sets, respectively. Note that from the model (1.17), different models of switched linear systems can be derived by restricting the parameters, as will be stated in the following.

(i) If $D = \emptyset$ and $C = \mathbb{R}^n$, then (1.17) is converted to a continuous switched linear system. If $\sigma(t,x) = \sigma(t)$, that is, the switching signal is only a function of time, then a switched linear system with arbitrary time-dependent switching signal results. If $\sigma(t,x) = \sigma(x)$, that is, the switching signal is only a function of the states, then a switched linear system with state dependent switching results. If the control input is not identically equal to zero, that is, $B_i \neq 0$ for some $i \in \mathcal{L}$, then we refer to the system as a controlled switched linear system. Otherwise, we refer to the system simply as a switched linear system. Note that throughout this thesis, we do not consider the switching signal as a control input.

(ii) If $C = \emptyset$ and $D = \mathbb{R}^n$, then we will arrive at a discrete time controlled switched linear system. Again, if the switching signal σ is only a function of time, a discrete time controlled switched linear system with arbitrary switching signal results, and if it is only a function of the states, a discrete time controlled switched linear system is obtained. If the control input is identically zero for all modes, then we simply refer to the system as a discrete switched linear system.

From the above statements, we can argue that switched linear systems form a subclass of hybrid linear systems. In this dissertation we discuss different problems associated with stability of continuous switched linear systems and optimal control of hybrid linear systems. Then, depending on the class of switched and hybrid systems, we study the following solutions of these systems.

a) For a (controlled) switched linear system with arbitrary switching signal, we assume a finite number of switching within a finite time interval occurs. Therefore, solutions of such systems are unique for each initial condition and are always absolutely continuous, implying that such solutions are continuous and differentiable almost everywhere. Moreover, these solutions follow the direction of vector field almost everywhere. Thus, for switched linear systems with arbitrary switching signals we consider the Caratheodory solutions. For illustration of Caratheodory solutions, see Appendix A.6.2.1 and Cortes (2008).

b) Solutions of a (controlled) switched linear system with state dependent switching are always absolutely continuous. However, in this case a trajectory of the switched system can slide on switching manifolds, and thus may not follow the direction of vector field almost everywhere. For this reason, for switched linear systems with state space constraints, we consider the Filippov solutions. For the definition of the Filippov solutions, see Appendix A.6.2.2, Filippov (1988), and Cortes (2008). It is worth mentioning that Filippov solutions may not be unique for switched linear systems with state dependent switching; see Cortes (2008).

c) Solutions of a hybrid linear system can exhibit jumps. Therefore, such solutions are absolutely continuous within flow evolutions, while they are not necessarily continuous during jumps; see Goebel et al. (2012). Thus, for a hybrid linear system due to the discontinuity of its solutions and the different nature of flow and jump dynamics, we investigate the notion of generalized time domain for the domain of state variables in the hybrid system (1.17). The generalized time domain consists of two elements as a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The first element in the product indicates the actual time, whereas the second ingredient in the product counts the jump events. For system modelling and general solutions in a generalized time domain, see Goebel et al. (2012). Similar to item b), solutions of a hybrid linear system may not necessarily be unique.

In this dissertation, we tackle three different problems regarding control of hybrid and switched linear systems: stability and stabilization of switched linear systems, and optimal control of hybrid linear systems. We now briefly elaborate these problems and point out some of the previous results related to these topics. For different stability and stabilization problems in the contexts of discrete time linear time varying and switched linear systems one can refer to, for example, Lin and Antsaklis (2009); Ahmadi et al. (2014); Daafouz et al. (2002); Barabanov (2005); Wirth (2005b) and the references therein.

1.3 Stability of switched linear systems

The stability problem for switched linear systems is perhaps the most well studied topic in the context of switched linear systems. The primary motivation for studying the stability problem for a switched linear system perhaps stems from the fact that Hurwitz stability of constituent matrices is not sufficient for stability of the associated switched linear system; see Example 3.2.1 and the references therein. On the other hand, it has also been found that even if all matrices of a switched linear system are unstable, one might be able to construct a routine of switching, such that some or all trajectories of the switched linear system exponentially converge to zero (Wicks et al., 1998). Research on the stability problem has given rise to many notions such as common quadratic Lyapunov function (Shorten and Narendra, 2003; Wulff, 2005), single Lyapunov function (De Schutter and Heemels, 2004), multiple Lyapunov functions (Branicky, 1998), converse Lyapunov theorem (Wirth, 2005a), average dwell time (Hespanha and Morse, 1999), common piecewise linear Lyapunov functions (Molchanov and Pyatnitskiy, 1989), copositive Lyapunov functions for a class of rank-1 difference positive switched systems with arbitrary number of subsystems (Fornasini and Valcher, 2010), etc. In the following, we briefly represent those approaches which are most closely related to the results in this thesis.

1.3.1 Common Quadratic Lyapunov function (CQLF)

The existence of a common quadratic Lyapunov function guarantees exponential stability of a switched linear system both under arbitrary and state dependent switching; see, *e.g.*, Griggs et al. (2010). Moreover, it is known that quadratic stability exhibit robust properties in terms of perturbations of constituent matrices and discretization of switched linear systems; see Rossi et al. (2011) and Shorten et al. (2007). The quadratic stability problem for a switched system with arbitrary switching signal can be stated as follows:

Problem 1.3.1. Consider the switched linear system

$$\dot{x} = A_{\sigma(t)}x(t) \qquad \sigma(t) \in \mathcal{L} := \{1, \dots, \ell\}, \tag{1.19}$$

where all matrices A_i for $i \in \mathcal{L}$ are Hurwitz and σ is the switching signal between the A_i's. Find conditions on the real matrices $A_i \in \mathcal{A} := \{A_1, \dots, A_\ell\}$ such that the switched system (1.19) is quadratically stable, that is, there exists a common quadratic Lyapunov function $V(x) = x^\top Px$ with $P = P^\top > 0$ such that

$$A_i^\top P + PA_i < 0 \qquad \forall i \in \{1, \dots, \ell\}. \quad \square \tag{1.20}$$

Note that (1.20) is a linear matrix inequality (LMI) and a common Lyapunov solution P can be computed numerically. It is known that a set of Hurwitz matrices similar to

1) set of upper (lower) triangular matrices (Shorten and Narendra, 1998),
2) set of pairwise commuting matrices (Narendra and Balakrishnan, 1994),
3) set of matrices which belong to solvable Lie Algebraic conditions (Liberzon et al., 1999; Agrachev and Liberzon, 2001),
4) set of real diagonalizable matrices which any pair of them share at least $(n-1)$ right eigenvectors (Shorten and Cairbre, 2001),

are classes of matrices that a common quadratic Lyapunov function for the associated switched linear system exists. Now, we point out two other important results which are related to the stability concepts introduced in Chapters 4 and 5 of this thesis.

1) **Rank-1 difference switched systems:** One of the well-known results in the context of quadratic stability is expressed by the next theorem.

   Theorem 1.3.1. *(See Shorten and Narendra (2003)) Consider the switched linear system*

   $$\dot{x} = \left(A - \sigma(t)\frac{bc}{d}\right)x \qquad \sigma(t) \in \{0, 1\}, \tag{1.21}$$

   where $A \in \mathbb{R}^{n\times n}$ is Hurwitz, $b \in \mathbb{R}^n$, $c^\top \in \mathbb{R}^n$, and $d > 0$ is a scalar. Then, the following statements are equivalent:

   a) The switched system (1.21) *is quadratically stable,*
   b) The transfer function $g(s) = c(sI - A)^{-1}b + d$ is strictly positive real,
   c) The matrix $A\left(A - \frac{bc}{d}\right)$ has no real negative eigenvalue. □

   The proof of this theorem uses the Kalman-Yakubovic-Popov lemma proposed for strictly positive real systems to guarantee the existence of a CQLF (Shorten and Narendra, 2003). This result has been proven and extended to the class of complex matrices separately by Laffey (2009) and King and Nathanson (2006).

2) **Two dimensional switched systems:** For a pair of two dimensional matrices, existence of a CQLF is expressed by the following result:

   Theorem 1.3.2. *(See Shorten and Narendra (2002)) Suppose in Problem 1.3.1 $\ell = 2$, and $A_1, A_2 \in \mathbb{R}^{2\times 2}$ are Hurwitz. Then the switched system* (1.19) *is quadratically stable if and only if the matrices A_1A_2 and $A_1A_2^{-1}$ do not have any real negative eigenvalue.* □

Stability of second order switched systems including two subsystems under singular perturbations has been considered by El Hachemi et al. (2011).

Despite much effort similar results for matrices with higher dimensions have only been derived recently (Kouhi et al., 2013a).

1.4 Stabilization of switched linear systems

The stabilization problem for switched linear systems has been studied from diverse points of view. Some research considers stabilization of the switched system (1.19) with Hurwitz constituent matrices, by restricting the rate of the switching signal. More precisely, when all matrix constituents are Hurwitz, it is shown that if the switching events are sufficiently far apart in time, then the switched linear system will be stable. This concept is well-known as stability with average dwell time. Finding a reasonable upper bound for the average dwell time has been investigated in Hespanha and Morse (1999), Solo (1994), Geromel and Colaneri (2006), and Chesi et al. (2012). Some other research assumes that the matrices A_i for all $i \in \mathcal{L}$ in (1.19) are not necessarily stable, and they look for a switching law that makes the switched system exponentially stable (Wicks et al., 1998; Feron, 1996). In this dissertation, however, we follow a standard approach for quadratic stabilization of a controlled switched linear system using local state feedback for each mode, similar to Cheng (2004) and De Schutter and Heemels (2004). This problem for a controlled switched system with arbitrary switching signal can be stated as follows:

Problem 1.4.1. Consider the controlled switched system

$$\dot{x} = A_{\sigma(t)}\, x(t) + B_{\sigma(t)}\, u(t) \qquad \sigma(t) \in \mathcal{L} := \{1, \ldots, \ell\}. \tag{1.22}$$

Find a local state feedback $u = K_{\sigma(t)}x$ such that the resulting closed loop switched system

$$\dot{x} = \left(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}\right) x, \tag{1.23}$$

is quadratically stable, that is, there exists a common quadratic Lyapunov function $V(x) = x^\top P x$ with $P = P^\top > 0$ such that

$$(A_i + B_iK_i)^\top P + P(A_i + B_iK_i) < 0 \qquad \forall i \in \mathcal{L}. \quad \square \tag{1.24}$$

Note that (1.24) is not an LMI due to the existence of the coefficient PB_iK_i in the inequality, where both P and K_i are unknown parameters. Nevertheless, one can convert this inequality to the LMI form as follows; see De Schutter and Heemels (2004). First by pre-multiplying by P^{-1} and post-multiplying by P^{-1}, we get

$$(P^{-1}A_i^\top + P^{-1}K_i^\top B_i^\top) + (A_iP^{-1} + B_iK_iP^{-1}) < 0 \qquad \forall i \in \mathcal{L}. \tag{1.25}$$

Thus with the assumptions $X := P^{-1}$ and $Y_i := K_iP^{-1}$, we can rewrite (1.25) as

$$XA_i^\top + A_iX + Y_i^\top B_i^\top + B_iY_i < 0 \qquad \forall i \in \mathcal{L}. \tag{1.26}$$

Now, it is evident that (1.26) is an LMI and can be solved numerically. However, numerical methods do not provide any insight into the feasibility of such LMI's. For this reason, finding sufficient conditions on A_i and B_i for all $i \in \mathcal{L}$ under which the solvability of these LMI's is guaranteed, is a crucial problem. Several articles dealing with this problem have been published, *e.g.*, Cheng (2004); Sun and Zheng (2001). For example, in Cheng (2004) a necessary and sufficient condition for stabilization of second order controlled switched linear systems with single input has been explored.

In this dissertation, we consider stabilization of a class of controlled switched linear systems where all open loop matrices A_i's, $i \in \mathcal{L}$, share a right invariant subspace with appropriate dimension to which a common quadratic Lyapunov function can be associated. Satisfying some rank conditions with respect to B_i's, for this class we show that quadratic stabilization can be achieved if sufficient number of input channels are available. Moreover, we demonstrate that this approach can be employed for stabilization of a class of controlled switched systems, where their open loop matrices have invariant subspaces which have sufficiently small distances, and a certain form of Riccati inequalities for the matrices associated with these invariant subspaces hold.

The concept of common controlled invariant subspaces has been previously considered in the context of linear parameter varying and switched linear systems, often for stability of a system in a subspace of the state space or for mode decoupling; see, *e.g.*, Balas et al. (2003), Yurtseven et al. (2010), Haimovich and Braslavsky (2010), and Blanchini (1999). Furthermore, it has also been used for determining a sequence of stabilizable controlled switched linear systems (Sun and Zheng, 2001). For these purposes, algorithms have been developed to compute a largest common controlled invariant subspace of a switched system for a subspace of $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and $u \in \mathbb{U}$. The largest common invariant subspace for such a switched linear system can be computed by an iterative algorithm using fixed point theory; see, *e.g.*, Tsatsomeros (2001) and Julius and Van der Schaft (2002).

1.5 Optimal control of hybrid linear systems

Optimal control problems for hybrid systems are recognized as challenging mathematical problems. In such problems, one often associates several cost values each of which corresponds to a part of a trajectory evolving with a given dynamics. Then, the objective is to minimize the sum of these costs over a fixed or free time interval, and fixed or free switching time instances. For such optimal control problems, the maximum principle for hybrid systems holds; see Sussmann (1999); Caines et al. (2006); Azhmyakov et al. (2007); Liberzon (2011); Passenberg et al. (2011). In one of the early works by Sussmann (1999), a general condition for the maximum principle for hybrid systems including fixed switching sequences and free switching time instances has been proposed. In Caines et al. (2006) the concept is generalized to a setting with guard conditions. Typically, such algorithms extend the classical maximum principle by additional requirements on the switching manifolds known as transversality conditions. Despite the sound theory, computational difficulties arise even in a setting with quadratic costs for hybrid linear systems (Riedinger et al., 1999; Azhmyakov et al., 2009; Xu and Antsaklis, 2004).

Linear quadratic regulator (LQR) problems for hybrid and switched linear systems, similar to optimal control problems for the general form of hybrid systems, may have fixed or free switching times, and fixed or free switching sequences; see, for instance, Problem 6.5.1 as a possible scenario. Several results dealing with these problems exist in the literature (Xu and Antsaklis, 2004; Azhmyakov et al., 2009). For instance, in Xu and Antsaklis (2004), an efficient numerical algorithm for the LQR problem with free switching times is obtained by introducing a parameterization in terms of switching times.

Some other research provides numerical solutions for optimal control of systems which are closely related to hybrid linear systems. For example, in the context of piecewise linear systems, Rantzer and Johansson (2000) suggests lower and upper bounds to the optimal cost by formulating a semi-definite and a convex programming problem. Analytic expressions for sub-optimal solutions to an LQR problem for LTI systems with state space and input constraints have been reported in Johansen et al. (2002).

1.6 Organization of the thesis

In Chapter 2, we study different approaches concerning left eigenstructure assignment for multi-input systems, pole placement for single input systems, and partial pole placement for single- and multi-input systems. This chapter can be viewed as the basis for the upcoming results on the stability and stabilization of switched linear systems in Chapter 3 and Chapter 4. Moreover, this chapter extends the result by Kouhi and Bajcinca (2011a) on left eigenstructure assignment.

In Chapter 3, we study exponential stability and stabilization of switched linear systems with state dependent switching signals. In these problem settings, we assume certain restrictions on the geometry of switching manifolds. This chapter extends parts of the results by Kouhi and Bajcinca (2011b) on exponential stability and stabilization of switched linear systems. Our techniques use the concept of common left eigenvectors and left eigenstructure assignment introduced in Chapter 2.

In Chapter 4, we apply the concept of common eigenvectors and left eigenstructure assignment for stability and stabilization of switched linear systems under arbitrary switching. Our approach for stabilization investigates a local state feedback design for each subsystem of a controlled switched system. The main result in this chapter discusses stabilization of a controlled switched system whose open loop constituent matrices share an invariant subspace to which a common quadratic Lyapunov function can be associated. We then use the left eigenstructure assignment technique for imposing a number of desired left eigenvectors that are perpendicular to the common invariant subspace, thereby guaranteeing quadratic stability of the closed loop matrices of subsystems. Moreover, we discuss robust stability of a switched linear system when its Hurwitz matrices share $(n-1)$ real left eigenvectors. This chapter extends the result by Kouhi and Bajcinca (2011c) on quadratic stability and stabilization of switched linear systems.

In Chapter 5, we extend the results by Shorten and Narendra (2003) on quadratic stability, and Shorten et al. (2009) concerning weak quadratic stability of switching systems with two modes and rank-1 difference matrices. We show that their results can be extended to switched systems with rank $m \geq 1$ difference matrices, provided that a symmetric transfer function matrix can be associated with the pair of matrices. Moreover, we use this approach for computing a set of control inputs which stabilize a class of controlled switched linear systems. Parts of these results have been published in Kouhi et al. (2013a) and Kouhi et al. (2014).

In Chapter 6, we briefly study the stability problem for hybrid linear systems. We show that their stability is equivalent to stability of switched linear systems, if bilinear transformations for converting discrete evolutions to continuous dynamics are used. The main results of this chapter, however, are related to the optimal control of hybrid linear systems. We investigate two problem configurations. In the first scenario, the class of hybrid linear systems is specified by a single flow and a single jump dynamics, and state space constraints are represented by polyhedral sets. For this system, we consider an optimal control problem with a fixed sequence of switching occurring at fixed time instances. Our solution algorithm determines upper and lower bounds for the overall cost function of this problem. In the next scenario, we consider an optimal control problem for hybrid linear systems including multiple flow dynamics with fixed sequence of switching, such that switches can take place at free time instances. This result solves a problem related to maximum principle for hybrid linear systems. Parts of the results in this chapter have been published by Kouhi et al. (2013b).

In Chapter 7, we provide the conclusions.

In Appendix A, some necessary definitions and existing concepts are given to make this dissertation necessarily self-contained.

Chapter 2

Left eigenstructure assignment

2.1 Introduction

Eigenstructure assignment for multi-input systems and pole placement for single-input systems using static state feedback are old topics in the control systems theory. From a classical point of view, pole placement for a single input system is often used to assign some or all eigenvalues of a linear system to stabilize the closed loop system, and/or to improve the rate of convergence of system solutions (Kailath, 1980; Saad, 1986). For multi-input systems, due to more freedom on control inputs, eigenstructure assignment has given rise to different problem formulations. For example, right eigenstructure assignment is used to shape the solutions of a linear system (Wonham, 1967; Liu and Patton, 1998), and left eigenstructure is used for disturbance attenuation (Choi, 1998a,b). In this chapter, however, our motivation is not to provide new results in this sense, nor to be involved fundamentally in different eigenstructure assignment methods for LTI systems. Although some of the techniques introduced in this chapter can be used for control of a linear system, some others essentially may not make sense for this purpose. In fact, the current chapter should be viewed as an introduction for the results of the upcoming chapters on stabilization of controlled switched linear systems.

Our main interest in this chapter is how to exploit the potential of control inputs for assigning as many desired left eigenvectors as possible using state feedback, such that the closed loop system is stable. It should be mentioned that this aim is different from conventional left eigenstructure assignment proposed in the literature, including Choi (1998a,b) and the references therein.

2.2 Left eigenvector assignment

First, we present left eigenstructure assignment for single input systems. Then, in the next part we extend this to multi-input systems.

2.2.1 Single-input systems

Consider a single input controllable LTI system given by the state space representation

$$\dot{x} = Ax + bu, \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the state vector and the control input, respectively. Suppose we aim at designing a state feedback $u = kx$, where k is a row vector, for assigning a desired left eigenvector $w \in \mathbb{C}^n$ and its corresponding eigenvalue $\lambda_1 \in \mathbb{C}_{<0}$ to the closed loop system, where $\mathbb{C}_{<0}$ denotes the open left half plane. Then, by a simple computation we have

$$w^*(A + bk) = \lambda_1 w^* \implies k = -\frac{w^*(A - \lambda_1 I)}{w^* b} \qquad \text{for } w^* b \neq 0. \tag{2.2}$$

The left eigenvector w appears nonlinearly in the closed loop system matrix $A_{\text{cl}} = A + bk$ as

$$A_{\text{cl}} = \left(I - \frac{bw^*}{w^* b}\right) A + \lambda_1 \frac{bw^*}{w^* b}. \tag{2.3}$$

Note that the matrix $I - bw^*/w^*b$ in (2.3) has one eigenvalue equal to 0 with corresponding left eigenvector $c_1 := w$, and $n-1$ eigenvalues equal to 1 with corresponding left eigenvectors c_i such that $c_i^\top b = 0$ for $i \in \{2, \dots, n\}$. Now, a natural problem arising from the gain (2.2) and the closed loop description (2.3) is to explore the proper selection of w and λ_1 such that $k^\top \in \mathbb{R}^n$ and A_{cl} is a Hurwitz matrix (Kouhi and Bajcinca, 2011a).

Lemma 2.2.1. *(Simon and Mitter, 1968) The characteristic polynomial of the closed loop system* (2.3) *is given by*

$$\begin{aligned} det(\lambda I - A_{cl}) &= (\lambda - \lambda_1) \frac{w^*\, adj(\lambda I - A) b}{w^* b} \\ &= (\lambda - \lambda_1)(\lambda^{n-1} + \beta_1 \lambda^{n-2} + \cdots + \beta_{n-1}), \end{aligned} \tag{2.4}$$

where the coefficients β_i for $i \in \{1, \dots, n-1\}$ are in the form of

$$\beta_i = \frac{w^*(a_i I + a_{i-1} A + a_{i-2} A^2 + \cdots + A^i) b}{w^* b}, \tag{2.5}$$

and a_i for $i \in \{1, 2, \dots, n\}$ are the coefficients of the characteristic polynomial of A, i.e., $p(\lambda) = det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$.

Proof: The characteristic polynomial of the closed loop system matrix A_{cl} is computed as follows:

$$\begin{aligned} \det(\lambda I - A - bk) &= \det(\lambda I - A)\det[I - (\lambda I - A)^{-1} bk] \\ &= p(\lambda)\ \det\left[I - (\lambda I - A)^{-1} bk\right] \\ &= p(\lambda)\left[1 - k(\lambda I - A)^{-1} b\right] \\ &= p(\lambda)\left[1 + \frac{w^*(A - \lambda_1 I)(\lambda I - A)^{-1} b}{w^* b}\right] \\ &= p(\lambda)\left[\frac{w^*(\lambda I - A - \lambda_1 I + A)(\lambda I - A)^{-1} b}{w^* b}\right] \\ &= (\lambda - \lambda_1)\ \frac{w^* \text{adj}(\lambda I - A) b}{w^* b}. \end{aligned}$$

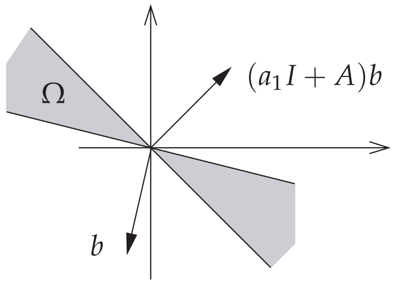

Figure 2.2.1: The interior of the colored region, denoted by Ω, is the proper region for selection of a desired left eigenvector w, when $n = 2$.

Note that in deriving the last line we used the fact

$$p(\lambda)(\lambda I - A)^{-1} = \det(\lambda I - A).(\lambda I - A)^{-1} = \operatorname{adj}(\lambda I - A);$$

see also Appendix A.2 and Kailath (1980). On the other hand, the Resolvent formula (Kailath, 1980) for the adjoint matrix is given by

$$\begin{aligned}\operatorname{adj}(\lambda I - A) &= A^{n-1} + (\lambda + a_1)A^{n-2} + \cdots + \\ &\qquad (\lambda^{n-1} + a_1\lambda^{n-2} + \cdots + a_{n-1})I \\ &= \lambda^{n-1}I + \lambda^{n-2}(a_1 I + A) + \cdots + (a_{n-1}I + \cdots + a_1 A^{n-2} + A^{n-1}).\end{aligned} \tag{2.6}$$

Pre-multiplying equation (2.6) by w^* and post-multiplying by b, and dividing it by w^*b leads to the formulation (2.5). □

Now, consider that deriving a simple statement for stability of A_{cl} from the polynomial (2.4) based on the Routh-Hurwitz stability criteria is not straightforward. Therefore, we will study this problem in more detail in the sequel. However, in the cases when $n = 2$ or $n = 3$, and λ_1 and consequently w are real, we can get particularly simple illustrative results. For instance, with $n = 2$, $\lambda_1 \in \mathbb{R}_{<0}$, and $w \in \mathbb{R}^2$, stability of A_{cl} is equivalent to have

$$\beta_1 = \frac{w^\top (a_1 I + A)b}{w^\top b} > 0. \tag{2.7}$$

This inequality holds if the inner-products of w with b, and w with $(a_1 I + A)b$ share the same sign. Hence, the set of all desired left eigenvectors w that makes $\beta_1 > 0$ can be specified geometrically, as illustrated by the shaded area Ω in Figure 2.2.1. If $n = 3$, $\lambda_1 \in \mathbb{R}_{<0}$, and $w \in \mathbb{R}^3$, then in addition to (2.7), we require

$$\beta_2 = \frac{w^\top (a_2 I + a_1 A + A^2)b}{w^\top b} > 0. \tag{2.8}$$

Again, the two expressions $\beta_1 > 0$ and $\beta_2 > 0$ with respect to the parameter w characterize a region in the three dimensional state space. This region can be computed geometrically (Kouhi and Bajcinca, 2011b).

2.2.1.1 Closed loop stability

To solve the problem of selecting a suitable w which stabilizes the closed loop matrix A_{cl} for the general case, we use the linear transformation $z = Tx$. Then, the closed loop system $\dot{x} = (A + bk)x$ is converted to the form

$$\dot{z} = (A_c + b_c k_c)z, \quad A_c = T^{-1}AT, \; b_c = T^{-1}b, \text{ and } k_c = kT. \tag{2.9}$$

Consider now the particular case where A_c and b_c are in controllable canonical form

$$A_c = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & & & . \\ . & . & . & . & & & . \\ . & . & . & . & & & . \\ 0 & 0 & 0 & . & . & . & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & . & . & . & -a_1 \end{bmatrix}, \; b_c = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix}. \tag{2.10}$$

Note that for any controllable pair (A, b) this transformation matrix T exists, and is given by

$$T = \Phi_c(A, b)\Phi_c^{-1}(A_c, b_c), \tag{2.11}$$

where $\Phi_c(A, b)$ and $\Phi_c(A_c, b_c)$ are the controllability matrices of the original and of the transformed systems, respectively. Now, referring to (2.4), the closed loop characteristic polynomial for the closed loop matrix $A_{c,\text{cl}} = A_c + b_c k_c$ reads

$$\det(\lambda I - A_{c,\text{cl}}) = (\lambda - \lambda_1)\frac{w_c^* \text{adj}(\lambda I - A_c)b_c}{w_c^* b_c},$$

where $w_c = [w_{c,1} \; \ldots \; w_{c,n}]^\top$ is the left eigenvector for the matrix $A_{c,\text{cl}}$ corresponding to the eigenvalue λ_1. On the other hand, one can observe that the closed loop matrix $A_{c,\text{cl}}$ is also in canonical form

$$A_{c,\text{cl}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ \frac{\lambda_1 w_{c,1}^*}{w_{c,n}^*} & \frac{-w_{c,1}^* + \lambda_1 w_{c,2}^*}{w_{c,n}^*} & \cdots & \frac{-w_{c,(n-1)}^* + \lambda_1 w_{c,n}^*}{w_{c,n}^*} \end{bmatrix}.$$

As b_c in (2.10) has only one non-zero element, for computing $w_c^* \text{adj}(\lambda I - A_c)b_c$ we only require to compute the n'th column of $\text{adj}(\lambda I - A_c)$. Note that $w_c^* b_c = w_{c,n}^* \neq 0$ indicates that $w_{c,n}^* \neq 0$. Thus, the characteristic polynomial of $A_{c,\text{cl}}$ equals

$$\begin{aligned} &\det(\lambda I - A_{c,\text{cl}}) = \\ &= \lambda^n + \frac{1}{w_{c,n}^*}\left(w_{c,(n-1)}^* - \lambda_1 w_{c,n}^*\right)\lambda^{n-1} + \cdots - \frac{1}{w_{c,n}^*}\lambda_1 w_{c,1}^* \\ &= \frac{1}{w_{c,n}^*}(\lambda - \lambda_1)\left(w_{c,n}^* \lambda^{n-1} + w_{c,(n-1)}^* \lambda^{n-2} + \cdots + w_{c,2}^* \lambda + w_{c,1}^*\right). \end{aligned} \tag{2.12}$$

The last equation reveals the fact that the characteristic polynomial of $A_{c,\text{cl}}$ does not depend on the variables $a_1, a_2, \ldots, a_n$, but it is determined by the parameters $w_{c,1}, \ldots, w_{c,n}$, and λ_1. Therefore, a left eigenvector w_c which stabilizes $A_{c,\text{cl}}$ can be characterized by the coefficients of the following stable polynomial

$$h_c(\lambda) = w^*_{c,n}\lambda^{n-1} + w^*_{c,(n-1)}\lambda^{n-1} + \cdots + w^*_{c,2}\lambda + w^*_{c,1}. \tag{2.13}$$

On the other hand, from the transformation $A_{c,\text{cl}} = T^{-1}A_{\text{cl}}T$, it is simple to show that $w = (T^*)^{-1}\, w_c$. Thus, after finding an appropriate w_c from (2.13), we can compute a vector w which stabilizes $A_{\text{cl}} = A + bk$ (Kouhi and Bajcinca, 2011a). In the case, when λ_1 is not real, for having a real feedback gain k the eigenvalues of the closed loop matrix A_{cl} must come in complex conjugate pairs. This implies that $h_c(\lambda)$ must have the form of $h_c(\lambda) = w^*_{c,n}(\lambda - \lambda^*_1)h_{c,1}(\lambda)$, where $h_{c,1}(\lambda)$ is a polynomial with real coefficients.

When λ_1 and w_c are restricted to be real, several contributions in the literature are available that specify simple sufficient conditions for stability of polynomials in the form of (2.13); for example, the results by Craven and Csordas (1998) and Dimitrov and Peña (2005). Now, we stress one of these results in the next lemma.

Lemma 2.2.2. *(See Dimitrov and Peña (2005)). Let γ be the unique real root of the polynomial $\gamma^3 - 5\gamma^2 + 4\gamma - 1 = 0$, namely $\gamma = 4.0796$. If the coefficients of the polynomial $h_c(\lambda)$ defined by* (2.13) *are all positive real and the following conditions hold*

$$w_{c,i}w_{c,i+1} \geq \gamma w_{c,i-1}w_{c,i+2} \qquad \forall i \in \{2, \ldots, n-2\}, \tag{2.14}$$

then $h_c(\lambda)$ is Hurwitz. □

For other results on stability of polynomials, see Katkova and Vishnyakova (2008) and the references therein.

2.2.1.2 Pole placement

In this section, we show that (2.11) and (2.12) can be additionally used for pole placement; see Kouhi and Bajcinca (2011a). To this end, consider a controllable linear system (2.1), and let $\{\lambda_1, \ldots, \lambda_n\}$ be a set of numbers including complex conjugate pairs. We wish to design a state feedback law $u = kx$, with $k^\top \in \mathbb{R}^n$, which assigns the desired eigenvalues λ_i for all $i \in \{1, \ldots, n\}$ to the closed loop matrix $A_{\text{cl}} = A + bk$. We perform the following algorithm for achieving this goal.

a) Obtain an appropriate canonical left eigenvector w_c corresponding to λ_1 from the following relationship

$$\frac{1}{w^*_{c,n}}\left(w^*_{c,n}\lambda^{n-1} + w^*_{c,(n-2)}\lambda^{n-1} + \cdots + w^*_{c,2}\lambda + w^*_{c,1}\right) = (\lambda - \lambda_2)\ldots(\lambda - \lambda_n). \tag{2.15}$$

b) Obtain the transformation matrix T defined in accordance with (2.11).

c) Obtain the desired feedback gain from

$$k = -\frac{w_c^* T^{-1}(A - \lambda_1 I)}{w_c^* b_c}. \tag{2.16}$$

Note that (2.16) results by inserting the equation $w = (T^*)^{-1}\, w_c$ into (2.2).

2.2.1.3 Single shift eigenvalue

Consider again the (not necessarily controllable) linear system (2.1). Suppose the aim is to design a state feedback $u = kx$ that only shifts the real eigenvalue μ_1 of the open loop matrix A and replaces it by a real desired number λ_1, while the other eigenvalues of A remain unchanged. To this end, we choose the open loop left eigenvector w corresponding to the eigenvalue μ_1, that is, $w^\top A = \mu_1 w^\top$, to be the desired left eigenvector for A_{cl} corresponding to the desired eigenvalue λ_1. Then, referring to the computation (2.2), the formula for the appropriate feedback gain reads

$$k = -\frac{w^\top (A - \lambda_1 I)}{w^\top b} = -\frac{(\mu_1 - \lambda_1) w^\top}{w^\top b} \qquad \text{for } w^\top b \neq 0. \tag{2.17}$$

The closed loop characteristic polynomial then will be computed by (2.4). On the other hand, by following the same procedure on derivation of (2.4), one can argue that the open loop characteristic polynomial $p(\lambda)$ equals

$$p(\lambda) = (\lambda - \mu_1)\frac{w^\top \text{adj}(\lambda I - A) b}{w^\top b}. \tag{2.18}$$

Comparing (2.4) and (2.18), we conclude that except one eigenvalue, all other eigenvalues of the closed loop and open loop matrices are the same.

The single shift eigenvalue technique has been introduced by Simon and Mitter (1968).

2.2.1.4 Partial pole placement

Consider now the linear system (2.1), with controllability of the system to be discussed in the sequel. We want to study the problem of designing a state feedback $u = kx$ that only shifts the eigenvalues $\mu_1, \ldots, \mu_m$ of the open loop matrix A, and replaces them by the desired numbers $\lambda_1, \ldots, \lambda_m$ for some $1 < m \leq n$, while keeping the other eigenvalues of A, namely $\mu_{m+1}, \ldots, \mu_n$, unchanged. To this end, we follow an algorithm introduced by Saad (1986). Let us denote the real Schur decomposition of $A^\top$ as

$$A^\top = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} L_1^\top & \star \\ 0 & L_2^\top \end{bmatrix} \begin{bmatrix} Q_1^\top \\ Q_2^\top \end{bmatrix}, \tag{2.19}$$

where columns of $T = [Q_1\ Q_2] \in \mathbb{R}^{n\times n}$ consist of orthonormal vectors, $L_1 \in \mathbb{R}^{m\times m}$, and $L_2 \in \mathbb{R}^{(n-m)\times(n-m)}$; see (Stewart, 2001) and Appendix A.2.9. As T is orthonormal, that is, $T^{-1} = T^\top$, (2.19) reveals that eigenvalues of L_1 are equal to m eigenvalues of A, and eigenvalues of L_2 are the same as other $n-m$ eigenvalues of A. Without loss of generality, let us assume that the set of eigenvalues of L_1 and L_2 are given by $\{\mu_1, \ldots, \mu_m\}$ and $\{\mu_{m+1}, \ldots, \mu_n\}$, respectively. Note that (2.19) implies the following relationships hold:

$$Q_1^\top A = L_1 Q_1^\top \quad \text{and} \quad Q_2^\top A Q_2 = L_2. \tag{2.20}$$

Further, introducing the parametrization $k = \eta Q_1^\top$, where $\eta^\top \in \mathbb{R}^m$, we use the transformation $x = Tz$ and (2.20) to get the similarity condition

$$A_{\text{cl}} := A + bk \sim T^\top A_{\text{cl}} T = \begin{bmatrix} L_1 + Q_1^\top b\eta & 0 \\ \star & L_2 \end{bmatrix}. \tag{2.21}$$

This is an indication that the set of eigenvalues $\{\mu_{m+1}, \ldots, \mu_n\}$ in A_{cl} remain unchanged, as the eigenvalues of L_2 are $\mu_{m+1}, \ldots, \mu_n$. However, the eigenvalues $\mu_1, \ldots, \mu_m$ can be changed by means of the parameter η in the term $L_1 + Q_1^\top b\eta$. Now, we can employ our pole placement technique elaborated in Section 2.2.1.2 in the upper left block for assigning the desired eigenvalues $\lambda_1, \ldots, \lambda_m$. To this end, define $L_{c,1}$ and b_{c,L_1} to be the canonical controllable form of L_1 and $Q_1^\top b$, respectively. Then, we adopt the formula (2.16) for computing η as

$$\eta = -\frac{w_{c,L_1}^* T_{L_1}^{-1}(L_1 - \lambda_1 I)}{w_{c,L_1}^* b_{c,L_1}}, \tag{2.22}$$

where $T_{L_1} = \Phi_c(L_1, Q_1^\top b)\Phi_c^{-1}(L_{c,1}, b_{c,L_1})$, and w_{c,L_1} is the left eigenvector of the matrix $L_{c,1}$ corresponding to the eigenvalue λ_1. Note that η in (2.22) exists if the pair $(L_1, Q_1^\top b)$ is controllable, or the controllability matrix $\Phi_c(L_1, Q_1^\top b)$ has full rank. In fact, by the definition of the controllability matrix, we have

$$\begin{aligned}\Phi_c(L_1, Q_1^\top b) &= [Q_1^\top b \;\; L_1 Q_1^\top b \;\; \ldots \;\; (L_1)^{m-1} Q_1^\top b] \\ &= Q_1^\top [b \; Ab \; \ldots \; A^{m-1} b].\end{aligned} \tag{2.23}$$

Therefore, if

$$\text{rank} \; \left(Q_1^\top [b \; Ab \; \ldots \; A^{m-1} b]\right) = m,$$

then partial pole placement is possible.

2.2.2 Multi-input systems

In this section, we generalize the results of single input systems to multi-input systems in the form of

$$\dot{x} = Ax + Bu, \tag{2.24}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are again the state and the control input vectors, respectively. Let $w_i \in \mathbb{C}^n$ for $i \in \{1, \ldots, m\}$ be m linearly independent vectors including complex conjugate pairs, and let $\lambda_i \in \mathbb{C}_{<0}$ for $i \in \{1, \ldots, m\}$ be numbers coming in complex conjugate pairs. Define $W = [w_1 \ldots w_m]$ and $\Lambda = \text{diag}([\lambda_1, \ldots, \lambda_m])$. Suppose we aim at computing a state feedback $u = Kx$ with $K \in \mathbb{R}^{m \times n}$, such that the closed loop matrix $A_{\text{cl}} = A + BK$ has m assigned left eigenvectors given by the columns of W corresponding to m desired eigenvalues given by the diagonal entries of Λ. It is simple to show that such feedback gain has the form

$$K = -(W^*B)^{-1}(W^*A - \Lambda W^*) \qquad \text{for } \det(W^*B) \neq 0. \tag{2.25}$$

This design leads to the closed loop system matrix

$$A_{\text{cl}} = \left(I - B(W^*B)^{-1}W^*\right) A + B(W^*B)^{-1}\Lambda W^*. \tag{2.26}$$

Note that the columns of $-(W^*B)^{-1}$ and the rows of $W^*A - \Lambda W^*$ consist of conjugate pairs of complex vectors and thus their multiplication, K defined by (2.25), is a real matrix. The matrix $I - B(W^*B)^{-1}W^*$ in (2.26) has m eigenvalues equal to 0 with corresponding left eigenvectors $c_i := w_i$ for $i \in \{1, \ldots, m\}$, and $(n-m)$ eigenvalues equal to 1 with corresponding left eigenvectors c_i such that $c_i^\top B = 0$ for $i \in \{m+1, \ldots, n\}$ (Kouhi and Bajcinca, 2011a). Now, we explore under which conditions the closed loop description A_{cl} given by (2.26) is Hurwitz.

Lemma 2.2.3. *(See Kouhi and Bajcinca (2011a)). The characteristic polynomial of the closed loop system matrix* (2.26) *is given by*

$$\det(\lambda I - A_{cl}) = \det(\lambda I - A)^{1-m} \ \det(\lambda I - \Lambda)\frac{W^* \operatorname{adj}(\lambda I - A)B}{\det(W^*B)}. \tag{2.27}$$

Proof: Defining $p(\lambda) = \det(\lambda I - A)$, the characteristic polynomial of the closed loop system is computed as follows:

$$\begin{aligned}
\det(\lambda I - A - BK) &= \det(\lambda I - A)\det[I - (\lambda I - A)^{-1}BK] \\
&= p(\lambda) \det\left[I - (\lambda I - A)^{-1}BK\right] \\
&= p(\lambda) \det\left[I - K(\lambda I - A)^{-1}B\right] \\
&= p(\lambda) \det\left[I + (W^*B)^{-1}(W^*A - \Lambda W^*)(\lambda I - A)^{-1}B\right] \\
&= p(\lambda) \det\left((W^*B)^{-1}\left[W^*B + (W^*A - \Lambda W^*)(\lambda I - A)^{-1}B\right]\right) \\
&= p(\lambda) \det\left((W^*B)^{-1}\right) \det\left([W^*(\lambda I - A) + W^*A - \Lambda W^*](\lambda I - A)^{-1}B\right) \\
&= p(\lambda) \ \frac{\det(\lambda I - \Lambda).\det\left(W^*(\lambda I - A)^{-1}B\right)}{\det(W^*B)} \\
&= p(\lambda)^{1-m} \det(\lambda I - \Lambda)\frac{\det\left(W^*\operatorname{adj}(\lambda I - A)B\right)}{\det(W^*B)},
\end{aligned}$$

where in deriving the third line of the proof, we used the general identity

$$\det(I_n - XY) = \det\ (I_m - YX),$$

for any two matrices $X \in \mathbb{C}^{n\times m}$ and $Y \in \mathbb{C}^{m\times n}$; see Appendix A.2.3. □

Therefore, for closed loop stability, we propose the following corollary.

Corollary 2.2.1. *Assume $\lambda_i \in \mathbb{C}_{<0}$ for all $i \in \{1, \ldots, m\}$, $\det(W^*B) \neq 0$, and (A, B) is controllable. The closed loop matrix* (2.26) *is Hurwitz if and only if $\det(W^* \operatorname{adj}(\lambda I - A)B)$ has no zero in the closed right half plane.*

2.2.2.1 Closed loop stability having $(n-1)$ inputs

Although designing a convenient W which stabilizes the closed loop system matrix (2.26) for a general m is not straightforward, we introduce a method for the case $m = n-1$ and $W \in \mathbb{R}^{n\times(n-1)}$, based on the results by Kouhi and Bajcinca (2011c). Suppose the controller gain $K \in \mathbb{R}^{(n-1)\times n}$ in the form of (2.25) imposes a pre-specified set of $(n-1)$ linearly independent common real left eigenvectors given by the columns of $W = [w_1 \dots w_{n-1}]$ and the corresponding negative real eigenvalues given by the diagonal entries of $\varLambda = \operatorname{diag}([\lambda_1, \dots, \lambda_{n-1}])$ to the closed loop system matrix $A_{\mathrm{cl}} = A + BK$. For the closed loop stability, we need to ensure that the last eigenvalue λ_n is negative real. Note that λ_n must be real because a real matrix cannot have only one complex eigenvalue with non-zero imaginary part. Now, let the characteristic polynomials of the open and closed loop system (2.24) be denoted by $p(\lambda)$ and $q(\lambda)$, respectively, that is,

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n, \tag{2.28}$$

$$q(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \dots + \alpha_n. \tag{2.29}$$

It is a known fact that $-\alpha_1$ in (2.29) is equal to the sum of the closed loop eigenvalues, and simultaneously is equal to the trace of A_{cl}, that is,

$$\operatorname{tr}(A_{\mathrm{cl}}) = \sum_{i=1}^{n} \lambda_i = -\alpha_1;$$

see Appendix A.2.6 and Kailath (1980). Hence, λ_n can now be computed as

$$\lambda_n = -\alpha_1 - \sum_{i=1}^{n-1} \lambda_i = \operatorname{tr}(A_{\mathrm{cl}}) - \operatorname{tr}(\varLambda)\,. \tag{2.30}$$

Further, we compute $\operatorname{tr}(A_{\mathrm{cl}})$ from (2.26)

$$\begin{aligned}\operatorname{tr}(A_{\mathrm{cl}}) &= \operatorname{tr}(\left(I - B(W^\top B)^{-1}W^\top\right) A + B(W^\top B)^{-1}\varLambda W^\top) \\ &= \operatorname{tr}(A) - \operatorname{tr}\left(B(W^\top B)^{-1}W^\top A\right) + \operatorname{tr}\left(B(W^\top B)^{-1}\varLambda W^\top\right).\end{aligned}$$

Using the identity $\operatorname{tr}(EF) = \operatorname{tr}(FE)$ for general matrices E and F, as well as the fact that $\operatorname{tr}(A) = -a_1 = -\operatorname{tr}\left(a_1/(n-1)\ I_{n-1}\right) = -\operatorname{tr}\left(a_1/(n-1)\ (W^\top B)^{-1}W^\top B\right)$, we have

$$\begin{aligned}\operatorname{tr}(A_{\mathrm{cl}}) &= -a_1 - \operatorname{tr}\left(W^\top AB(W^\top B)^{-1}\right) + \operatorname{tr}\left(W^\top B(W^\top B)^{-1}\varLambda\right) \\ &= -\operatorname{tr}\left(W^\top\left(a_1/(n-1)\ I_n + A\right) B(W^\top B)^{-1}\right) + \operatorname{tr}(\varLambda).\end{aligned}$$

Consequently, recalling (2.30), we can derive a simple expression for λ_n

$$\lambda_n = -\operatorname{tr}\left(W^\top(a_1/(n-1)\ I_n + A)B(W^\top B)^{-1}\right). \tag{2.31}$$

In the sequel we provide an algorithm for designing a matrix W which satisfies $\lambda_n < 0$. To this end, let $\theta \in \mathbb{R}^n$ be a vector belonging to $\ker(B^\top)$, that is,

$$\theta^\top B = 0, \tag{2.32}$$

and consider the parametrization

$$W^\top = \mathcal{W}^\top + \mu\theta^\top, \tag{2.33}$$

where $\mathcal{W}$ is an arbitrary matrix in $\mathbb{R}^{n\times(n-1)}$ and $\mu \in \mathbb{R}^{n-1}$ is an unknown vector which must be determined. From the definition of θ, it is obvious that

$$(W^\top B)^{-1} = (\mathcal{W}^\top B)^{-1}.$$

Defining the matrix $X \in \mathbb{R}^{n\times(n-1)}$ and the vector $Y \in \mathbb{R}^{n-1}$ as

$$\begin{aligned} X &= \Big(a_1/(n-1)\ I_n + A\Big)B(\mathcal{W}^\top B)^{-1}, \\ Y^\top &= \theta^\top\Big(a_1/(n-1)\ I_n + A\Big)B(\mathcal{W}^\top B)^{-1}, \end{aligned} \tag{2.34}$$

and substituting $W^\top$ from (2.33) into (2.31), we get

$$\begin{aligned} \lambda_n &= -\mathrm{tr}\left((\mathcal{W}^\top + \mu\theta^\top)\,(a_1/(n-1)\ I_n + A)\,B(\mathcal{W}^\top B)^{-1}\right) \\ &= -\mathrm{tr}\,(\mathcal{W}^\top X) - \mathrm{tr}\,(\mu Y^\top). \end{aligned}$$

As θ, μ, and Y are vectors, we can write

$$\mathrm{tr}(\mu Y^\top) = \mathrm{tr}(Y^\top\mu) = Y^\top\mu,$$

which leads to

$$\lambda_n = -\mathrm{tr}\,(\mathcal{W}^\top X) - Y^\top\mu. \tag{2.35}$$

Thus, the last equation indicates that the requirement $\lambda_n < 0$ is equivalent to have

$$Y^\top\mu > -\mathrm{tr}\,(\mathcal{W}^\top X). \tag{2.36}$$

Note that if we fix $\mathcal{W}$, then X and Y will be known variables. Then, (2.36) represents an inequality with μ as an unknown variable. This inequality indicates that the inner-product of the known vector Y and the unknown vector μ must be greater than the known scalar $-\mathrm{tr}\,(\mathcal{W}^\top X)$. The set of vectors μ fulfilling such characteristics indeed identifies a geometric cone in the space $\mathbb{R}^{n-1}$. Now, sweeping over all possible $\mathcal{W} \in \mathbb{R}^{n\times(n-1)}$ would yield the region $\Omega \subseteq \mathbb{R}^{n\times(n-1)}$ of all desired matrices W.

In the following example, we explain how we can compute an appropriate W based on the proposed algorithm.

Example 2.2.1. Consider the controlled linear system (2.24) with

$$A = \begin{bmatrix} 3 & -3 & -7 \\ 0 & -4 & 0 \\ 1 & 3 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}. \tag{2.37}$$

A has eigenvalues at $2, -4, -4$, and thus is not Hurwitz. Suppose we want to determine a state feedback gain $K \in \mathbb{R}^{2\times 3}$ which assigns two eigenvalues at $-2, -3$, and makes the closed loop matrix $A_{\text{cl}} = A + BK$ Hurwitz. To this end, we assume that the left eigenvectors corresponding to these two eigenvalues are columns of the matrix $W = [w_1\ w_2]$, where $w_1, w_2 \in \mathbb{R}^2$ are unknown. Now, we use the parameterization (2.33). The vector θ coming from (2.32) and an arbitrary vector $\mathcal{W}$ are given by:

$$\mathcal{W}^\top = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \ \theta = [-2\ \ 0\ \ 1]^\top.$$

Then, from (2.34) the matrix X and the vector Y are computed as

$$X = \begin{bmatrix} -3 & 0.333 \\ -1 & 1 \\ 3 & -4 \end{bmatrix}, \ Y = \begin{bmatrix} 9 & -4.667 \end{bmatrix}^\top.$$

As $-\mathrm{tr}(\mathcal{W}^\top X) = 6.6667$, we can select $\mu^\top = [2\ \ 1]$ to satisfy the condition of inequality (2.36). This leads to

$$W^\top = \mathcal{W}^\top + \mu\theta^\top = \begin{bmatrix} -3 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix},$$

$$K = \begin{bmatrix} 8 & -7 & -9 \\ -6.667 & 4 & 8 \end{bmatrix}.$$

The closed loop matrix then equals

$$A_{\mathrm{cl}} = \begin{bmatrix} 4.333 & -6 & -8 \\ 8 & -11 & -9 \\ 3.667 & -3 & -7 \end{bmatrix},$$

which has the eigenvalues $-2, -3$, and -8.667.

2.3 Left eigenstructure assignment

In Section 2.2.2.1 we have discovered that finding a set of m desired left eigenvectors which can stabilize the closed loop system matrix (2.26) under the feedback gain (2.25), in general, is challenging. Therefore, we leave out this problem and slightly change our strategy. In the new scheme, we reduce the number of desired left eigenvectors to $m-1$ and increase the number of desired eigenvalues to n. Hence, we expect to establish both left eigenvectors assignment and closed loop stability.

Consider again the linear system (2.24) with controllability to be discussed in the sequel. Imagine that the controller gain K is supposed to impose a prespecified set of $m-1$ linearly independent left eigenvectors with complex conjugate pairs given by the columns of $W = [w_1 \dots w_{m-1}]$ corresponding to $m-1$ eigenvalues coming in complex conjugate pairs given by the diagonal elements of $\Lambda = \mathrm{diag}([\lambda_1, \dots, \lambda_{m-1}])$, and to assign all other eigenvalues to be equal to $n-m+1$ numbers $\lambda_m, \dots, \lambda_n$ with complex conjugate pairs. As we intend to accomplish left eigenvectors assignment and complete pole placement tasks, we combine the concepts elaborated in Section 2.2.2.1 and Section 2.2.1.4. To this end, let $B = [b_1 \dots b_j \dots b_m]$ consists of columns b_i corresponding to the channel u_i in the control input $u^\top = [u_1 \dots u_j \dots u_m]$ for all $i \in \{1, \dots, m\}$. We withdraw an input channel b_j from B, and define the new matrices $\bar{B} \in \mathbb{R}^{n\times(m-1)}$ and $\bar{u} \in \mathbb{R}^{m-1}$ as

$$\bar{B} = [b_1 \dots b_{j-1}\ b_{j+1}\ \dots b_m], \quad \bar{u} = [u_1 \dots u_{j-1}\ u_{j+1} \dots u_m]^\top. \tag{2.38}$$

We then reformulate (2.24) in the form of:

$$\dot{x} = Ax + \bar{B}\bar{u} + b_j u_j. \tag{2.39}$$

We now consider the structure $\bar{u} = \bar{K}x$ and $u_j = k_j x$ for the control inputs. Specifically, we use the state feedback gain $\bar{K} \in \mathbb{R}^{(m-1)\times n}$ for assigning $m-1$ desired left eigenvectors given by the columns of W and their corresponding eigenvalues $\lambda_1, \dots, \lambda_{m-1}$, and use k_j for assigning the remaining eigenvalues $\lambda_m, \dots, \lambda_n$. Thus, if $\det(W^*\bar{B}) \neq 0$, then referring to (2.25) the controller gain $\bar{K}$ can be computed as

$$\bar{K} = -(W^*\bar{B})^{-1}\left(W^*(A + b_j k_j) - \Lambda W^*\right), \tag{2.40}$$

yielding to the closed loop matrix

$$A_{\mathrm{cl}} = \left[(I - \bar{B}(W^*\bar{B})^{-1}W^*)A + \bar{B}(W^*\bar{B})^{-1}\Lambda W^*\right] + \left(I - \bar{B}(W^*\bar{B})^{-1}W^*\right) b_j k_j. \tag{2.41}$$

Defining

$$\begin{aligned} \hat{A} &= \left[(I - \bar{B}(W^*\bar{B})^{-1}W^*)A + \bar{B}(W^*\bar{B})^{-1}\Lambda W^*\right], \\ \hat{b} &= (I - \bar{B}(W^*\bar{B})^{-1}W^*)b_j, \end{aligned} \tag{2.42}$$

the closed loop matrix in (2.41) can be described by

$$A_{\mathrm{cl}} = \hat{A} + \hat{b}k_j. \tag{2.43}$$

Now, the problem of determining k_j can be solved by the partial pole placement algorithm for single input systems presented in Section 2.2.1.4. The real Schur decomposition of $\hat{A}^\top$ defines the transformation matrix $T = [Q_1\ Q_2]$, where $Q_1 \in \mathbb{R}^{n\times(n-m+1)}$ and $Q_2 \in \mathbb{R}^{n\times(m-1)}$, as well as matrices $\hat{L}_1 \in \mathbb{R}^{(n-m+1)\times(n-m+1)}$ and $\hat{L}_2 \in \mathbb{R}^{(m-1)\times(m-1)}$, satisfying: $Q_1^\top \hat{A} = \hat{L}_1 Q_1^\top$ and $Q_2^\top \hat{A} Q_2 = \hat{L}_2$. We assume in this description $\hat{L}_2$ has the eigenvalues $\lambda_1, \dots, \lambda_{m-1}$. The appropriate feedback gain k_j then has the structure

$$k_j = \eta Q_1^\top, \tag{2.44}$$

where $\eta^\top \in \mathbb{R}^{(n-m+1)}$ is an unknown parameter. Let $(\hat{L}_{c,1}, \hat{b}_{c,\hat{L}_1})$ be the controllable canonical form of the pair $(\hat{L}_1, Q_1^\top \hat{b})$. Then, computation of η can be achieved by adopting (2.22). This leads to

$$\eta = -\frac{w^*_{c,\hat{L}_1} T^{-1}_{\hat{L}_1}(\hat{L}_1 - \lambda_1 I)}{w^*_{c,\hat{L}_1}\hat{b}_{c,\hat{L}_1}}, \tag{2.45}$$

where $T_{\hat{L}_1} = \Phi_c(\hat{L}_1, Q_1^\top\hat{b})\Phi_c^{-1}(\hat{L}_{c,1}, \hat{b}_{c,\hat{L}_1})$ and $w_{c,\hat{L}_1}$ refers to the left eigenvector of $\hat{L}_{c,1}$ corresponding to the eigenvalue λ_1. This is of course possible if $\Phi_c(\hat{L}_1, Q_1^\top\hat{b})$ has full rank, or

$$\operatorname{rank}\left(Q_1^\top[\hat{b}\ \hat{A}\hat{b}\ \dots\ \hat{A}^{n-m}\hat{b}]\right) = n - m + 1.$$

For notational simplicity, we define the matrix

$$\Pi = I - \bar{B}(W^*\bar{B})^{-1}W^*. \tag{2.46}$$

Obviously $\Pi^k = \Pi$ for all $k \in \mathbb{N}$ and $W^*\Pi = 0$. Therefore, recalling $\hat{A}$ and $\hat{b}$ from (2.42), we have

$$\hat{A}^k\hat{b} = \Pi(A\Pi)^k b_j \qquad \forall\, k \in \{1, \dots, n-m\},$$

and

$$\begin{aligned} Q_1^\top[\hat{b}\ \hat{A}\hat{b}\ \dots\ \hat{A}^{n-m}\hat{b}] &= Q_1^\top[\Pi b_j\ \Pi(A\Pi)b_j\ \dots\ \Pi(A\Pi)^{n-m}b_j] \\ &= Q_1^\top \Pi \Phi_c(A\Pi, b_j) \begin{bmatrix} I_{n-m+1} \\ 0 \end{bmatrix}. \end{aligned} \tag{2.47}$$

In (2.47), $\Phi_c(A\Pi, b_j)$ is the controllability matrix of the pair $(A\Pi, b_j)$. Now, for simplicity assume that the matrix ΠA has a full set of (left) eigenvectors. We can therefore choose a set of $(n-m+1)$ right eigenvectors of ΠA, collected in the columns of the matrix V, which are orthogonal to the left eigenvectors given by the columns of W, that is, $W^*V = 0$. It is simple to check that the columns of V are also right eigenvectors of $\hat{A}$. Note that the columns of Q_1 and V form bases for the left and right $\hat{A}$ invariant subspaces corresponding to the same eigenvalues, respectively. This implies that $\text{rank}(Q_1^\top V) = n-m+1$. Furthermore, having the identities $Q_1^\top \Pi V = Q_1^\top V$ and $\text{rank}(Q_1^\top V) = n-m+1$, we can argue that *rank* $(Q_1^\top \Pi) = n - m + 1$. Therefore, we reach to the following statement.

Lemma 2.3.1. *Left eigenstructure assignment can be achieved whenever ΠA has a full set of eigenvectors, det$(W^*\bar{B}) \neq 0$, and the pair $(A\Pi, b_j)$ is controllable, that is,*

$$\textit{rank}\ \Phi_c(A\Pi, b_j) = n. \quad \square \tag{2.48}$$

After designing the vector k_j, the controller gain $\bar{K}$ can be designed in accordance with (2.40), leading to the design of a proper controller gain K.

As a result, the algorithm for left eigenstructure assignment reads as follows:

1) extract the channels b_j and $\bar{B}$, such that ΠA has full set of eigenvectors, the condition (2.48), and $\det(W^*\bar{B}) \neq 0$ are fulfilled.

2) compute the matrix $\hat{A}$ and the vector $\hat{b}$ from (2.42).

3) derive the transformation matrix $T = [Q_1\ Q_2]$, $\hat{L}_1$, and $\hat{L}_2$ from the real Schur decomposition of $\hat{A}^\top$.

4) compute k_j from (2.44) and (2.45).

5) compute $\bar{K}$ from (2.40), and re-order the inputs $\bar{u} = \bar{K}x$ and $u_j = k_jx$ in (2.39) to find the control $u = Kx$.

Now, we give an example to illustrate this algorithm.

Example 2.3.1. Consider the multi-input system (2.24) with

$$A = \begin{bmatrix} 3 & -3 & -7 \\ 0 & -4 & 0 \\ 1 & 3 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}. \tag{2.49}$$

The eigenvalues of A are located at $2, -4, -4$. Suppose we want to assign the desired left eigenvector $W = [-1 \;\; 1 \;\; 1]^\top$ and its eigenvalue at $\Lambda := \lambda_1 = -2$, as well as the desired eigenvalues $\lambda_2 = -5$ and $\lambda_3 = -6$ to the closed loop matrix. We choose the first control input corresponding to the first column of B, namely $\bar{B} = [1 \;\; 1 \;\; 2]^\top$, for assigning W and its corresponding eigenvalue $\Lambda = -2$ as $W^\top \bar{B} \neq 0$. Then, the second input channel corresponding to the second column of B, namely $b_2 = [1 \;\; 0 \;\; 2]^\top$, is responsible for partial pole placement. As rank $\Phi_c(A\Pi, b_2) = 3$ and ΠA has full set of eigenvectors, Lemma 2.3.1 implies that left eigenstructure assignment is possible. Based on this knowledge, we compute the matrix $\hat{A}$ and the vector $\hat{b}$ as

$$\hat{A} = \begin{bmatrix} 5 & -5 & -9 \\ 2 & -6 & -2 \\ 5 & -1 & -9 \end{bmatrix}, \; \hat{b} = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix}.$$

Now, real Schur factorization of $\hat{A}^\top$ gives us

$$Q_1^\top = \begin{bmatrix} 0.4082 & 0.8165 & -0.4082 \\ 0.7071 & 0 & 0.7071 \end{bmatrix}, \; Q_2^\top = [-0.5774 \;\; 0.5774 \;\; 0.5774],$$

and the matrices

$$\hat{L}_1 = \begin{bmatrix} -4 & 0 \\ 4.6188 & -4 \end{bmatrix}, \; \hat{L}_2 = -2.$$

One can now compute the feedback gain $k_2 = [3 \;\; 5 \;\; -2]$ through equations (2.44) and (2.45). Finally, we can use (2.40) to compute $\bar{K}$ with $\Lambda = -2$, yielding the result $\bar{K} = [0.5 \;\; -4.5 \;\; -1]$ and

$$A_{\mathrm{cl}} = \begin{bmatrix} 6.5 & -2.5 & -10 \\ 0.5 & -8.5 & -1 \\ 8 & 4 & -11 \end{bmatrix}.$$

2.4 Conclusions

In this chapter, we have discussed:

a) pole placement and partial pole placement for linear single input systems,

b) left eigenvector assignment for single- and multi-input systems,

c) left eigenstructure assignment for multi-input systems in the sense that all eigenvalues and a number of left eigenvectors are assigned to the closed loop systems.

We also spoke about stability of closed loop systems, and explored the required controllability conditions for accomplishing the tasks in items (a-c). In the next chapters, we utilize the benefits of these techniques for control of switched linear systems.

Chapter 3

Switched linear systems with state dependent switching

3.1 Introduction

Switched linear systems are a class of differential equations

$$\dot{x} = f(x) := A_{\sigma(t,x)}x \qquad \sigma(t,x) \in \mathcal{L} = \{1, \ldots, \ell\}, \tag{3.1}$$

which feature discontinuous right hand side. Although in this equation the vector field consists of a set of linear functions, the discontinuity on f makes the study of such systems interesting. One of the challenges associated with switched linear systems is indeed stability. It is known that Hurwitz stability of A_i for $i \in \mathcal{L}$ is not sufficient for stability of the switched system (3.1), since nonlinear behaviors such as sliding mode (Khalil, 2002) or fast switching phenomenon may cause instability. Therefore, for gaining better insight into this problem, one can restrict the switching rules to two scenarios, namely to state dependent switching and time dependent switching behaviors. In the former problem setting the switching rule is specified by some switching manifolds in the state space, while in the latter problem, in the most general case, the switched system is allowed to arbitrarily switch between different modes over time.

In this chapter, we study a class of switched linear systems whose switching law is governed by a given manifold in the state space. This class of systems can be viewed as a particular class of hybrid linear systems with multiple linear vector fields, and state dependent switching. Thus, due to the existence of different directions of the vector field in different regions of the state space, Filippov solutions (see Appendix A.6.2.2 and Cortes (2008); Filippov (1988)) are particularly of interest. We intend to study under which conditions on switching matrices and switching constraints, Filippov solutions exponentially converge to zero. Our approach in this chapter relies on the concept of common left eigenvectors. The basic idea in our approach stems from the fact that if all matrices in a switched system share a number of real left eigenvectors corresponding to negative real eigenvalues, then Filippov solutions of the switched system converge exponentially to an invariant subspace which is orthogonal to the common left eigenvectors. Hence, if the switching manifold has a restrictive topology in the sense that it is disjoint with that invariant subspace, then after passing some certain amount of time,

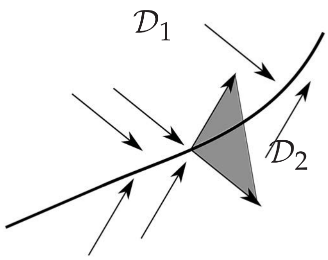

Figure 3.2.1: Sliding mode on the switching surface.

no switching is expected to happen. Thus, the overall switched system is reduced to a single linear LTI system after lasting finite time. Consequently, stability of the switched system is guaranteed by Hurwitz stability of the subsystem matrices. We then combine this concept and the left eigenstructure assignment techniques introduced in Chapter 2 for stabilization of controlled switched linear systems.

3.2 Exponential stability of switched linear systems

In this section, we consider the stability problem for switched linear systems with state dependent switching laws. Our stability result is based on the concept of common left eigenvectors. We first express our stability theory, and then discuss our stabilization approach in the next section.

Consider the switched linear system

$$\dot{x} = A_{\sigma(x)}\, x(t) \qquad \sigma(x) \in \mathcal{L} := \{1, \ldots, \ell\}, \tag{3.2}$$

where $\sigma : \mathbb{R}^n \to \mathcal{L}$ represents the switching signal between the constituent matrices $\mathcal{A} = \{A_1, \ldots, A_\ell\}$. Further, assume that switching to a new dynamics can occur when a trajectory of the switched system hits a given manifold (not necessarily consisting of one part) in the state space defined by

$$\mathcal{M} = \{x \in \mathbb{R}^n : M(x) = 0\}. \tag{3.3}$$

Our goal is to study exponential stability of such switched linear systems. We say the switched linear system is exponentially stable if there exist numbers $\alpha \geq 1$ and $\beta > 0$ such that $\|x(t)\| \leq \alpha e^{-\beta t}\|x_0\|$ for any $x_0 := x(0) \in \mathbb{R}^n$; see, *e.g.* , Shorten et al. (2007). Note that even when all matrices A_i for $i \in \mathcal{L}$ are Hurwitz, the switched system (3.2) might be unstable. Figure 3.2.1 illustrates one possibility how instability may arise due to the directions of the vector fields on the switching manifold in the boundaries of $\mathcal{D}_1$ and $\mathcal{D}_2$; see also Cortes (2008). In words, if the vector fields given by $f_1 := A_1 x$ and $f_2 := A_2 x$ are pointing towards different regions in the neighborhood of the switching manifold, then the Filippov set valued map, constructed by the convex hull of the vector fields around the switching manifold, includes the switching manifold itself. This implies that the trajectory slides on the surface of this manifold. Now, suppose that the switching manifold is stretched to infinity, and the vector fields are still pointing outwards with

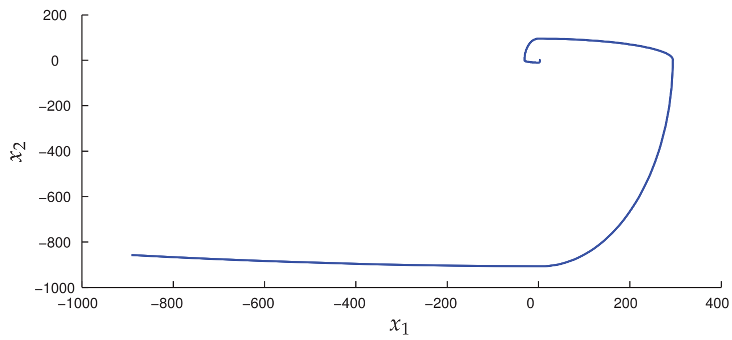

Figure 3.2.2: A trajectory of the switched system in Example 3.2.1.

respect to the switching manifold. Then, evidently some trajectories of the switched system move along the manifold and diverge to infinity. Thus, the system becomes unstable.

Sliding on the switching manifold is not the only reason that may cause instability of the switched system defined by (3.2). There are some other possibilities regarding this issue. For instance, it is evident that existence of complex eigenvalues with non-zero imaginary parts in linear systems leads to trajectories that "circle" around the origin. Then, one can piece together several such systems, and construct some switching rule which provides an unstable switched linear system even if the constituent systems are stable. Here, we present such an example taken from De Schutter and Heemels (2004); Branicky (1998).

Example 3.2.1. (See Branicky (1998)) Consider a switched linear system under state dependent switching in the form of

$$\Sigma : \quad \begin{cases} \dot{x} = A_1 x & x_1 x_2 \leq 0, \\ \dot{x} = A_2 x & x_1 x_2 > 0, \end{cases}$$

where $x = [x_1 \ x_2]^\top$ and

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}.$$

A_1 and A_2 have eigenvalues at $-1 \mp 31.62i$, and thus both are Hurwitz. Furthermore, on the switching manifold the vector fields of the two subsystems point in the same direction, implying that no sliding mode behavior takes place. However, this switched system is not stable. A trajectory of this system emerging from the initial condition $x_0 = [1 \quad 1]^\top$ is depicted in Figure 3.2.2. The picture reveals that the trajectory goes beyond all bounds.

For such reasons stability of switched linear systems has been considered as a non-trivial problem for many years. In this chapter, we aim at establishing a new approach to this problem based on the concept of common left eigenvectors. Our basic idea is inspired by the following simple fact. As a special case, suppose all matrices in $\mathcal{A} = \{A_1, \ldots, A_\ell\}$ are Hurwitz and share a set of m real left eigenvectors given by the columns of the matrix $W = [w_1 \ldots w_m]$. We will show that all solutions of such switched systems converge exponentially to an invariant set defined by

$$\mathcal{X}_{n-m} = \{x \in \mathbb{R}^n : W^\top x = 0\}. \tag{3.4}$$

Motivated by this fact, we consider the case where the switching manifold has a restricted topology, that is, the invariant subspace $\mathcal{X}_{n-m}$ and the switching manifold are disjoint, as illustrated by Figure 3.2.3. Then, we can intuitively deduce that the switched system (3.2) is exponentially stable. This argument is stated and proved by the following theorem.

Theorem 3.2.1. *Consider the switched system* (3.2) *with the switching manifold defined by* (3.3). *Assume that the matrices A_i for all $i \in \mathcal{L}$ are Hurwitz. Furthermore, suppose the sets $\mathcal{X}_{n-m}$ and $\mathcal{M}$ defined by* (3.4) *and* (3.3), *respectively, are disjoint in the sense that $(\mathcal{X}_{n-m} + \epsilon\mathbb{B}) \cap \mathcal{M} = \emptyset$, for some $\epsilon > 0$ and with $\mathbb{B}$ being a closed unit ball. Then, if all matrices A_i for $i \in \mathcal{L}$ share a set of real left eigenvectors given by the columns of W, the system* (3.2) *and* (3.3) *is exponentially stable.*

Proof: Note that the following property

$$x \in \mathcal{X}_{n-m} \Rightarrow W^\top \dot{x} = W^\top A_i x = \Lambda_i W^\top x = 0 \qquad \forall i \in \mathcal{L}$$

indicates that the set $\mathcal{X}_{n-m}$ is invariant for the switched system (3.2). Now, we demonstrate that starting from each initial condition x_0, the trajectory of the switching system (3.2) reaches to the interior of the set $(\mathcal{X}_{n-m} + \epsilon\mathbb{B})$ in finite time. To this end, we need to show that the distance between the trajectory and the invariant set decreases over time. Let's denote the distance between each point x and the invariant set $\mathcal{X}_{n-m}$ by $\mathrm{dist}(x, \mathcal{X}_{n-m})$. As $\mathcal{X}_{n-m}$ represents a linear subspace of the state space, the projection of a point x onto $\mathcal{X}_{n-m}$, denoted by $\mathrm{Proj}_{\mathcal{X}_{n-m}}(x)$, equals

$$\mathrm{Proj}_{\mathcal{X}_{n-m}}(x) = x - W(W^\top W)^{-1}(W^\top x); \tag{3.5}$$

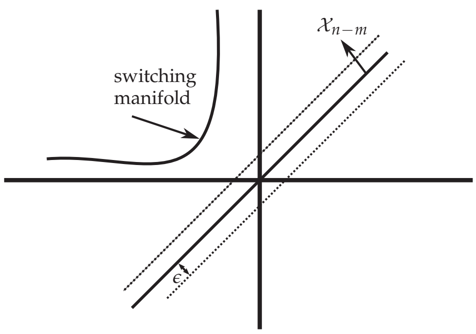

Figure 3.2.3: The geometry of the invariant subspace $\mathcal{X}_{n-m}$ and the switching manifold.

see Appendix A.8.1 and Boyd and Vandenberghe (2004). Notice the distance from a point x to the subspace $\mathcal{X}_{n-m}$ is equal to the norm of the difference between the vector x and its projection onto the subspace $\mathcal{X}_{n-m}$. Thus, this distance is bounded by:

$$\begin{aligned}\operatorname{dist}(x,\mathcal{X}_{n-m}) &= \left\| x - \operatorname{Proj}_{\mathcal{X}_{n-m}}(x) \right\| = \left\| W(W^\top W)^{-1}(W^\top x) \right\| \\ &= (x^\top W(W^\top W)^{-1}W^\top x)^{1/2} \leq \lambda_{\max}\left((W^\top W)^{-1}\right)^{1/2} \|W^\top x\|. \end{aligned} \tag{3.6}$$

We show that $\|W^\top x\|$ is always decreasing over time. Consider a time instance where mode i is active. Since the columns of W consist of the set of left eigenvectors of A_i, it follows that $W^\top A_i = \varLambda_i W^\top$, where the diagonal elements of $\varLambda_i = \operatorname{diag}([\lambda_{i1},\ldots,\lambda_{im}])$ are the corresponding eigenvalues of A_i. Then, one can write

$$\frac{d}{dt}\left[\ \|W^\top x\|^2\ \right] = \frac{d}{dt}[x^\top W W^\top x] = x^\top A_i^\top W W^\top x + x^\top W W^\top A_i x = 2x^\top W \varLambda_i W^\top x.$$

Now, defining $\mathbf{M} = \{1,\ldots,m\}$, we have

$$\frac{d}{dt}[x^\top W W^\top x] = \ 2x^\top W\varLambda_i W^\top x \leq 2\left(\max_{\substack{j\in\mathbf{M}\\ i\in\mathcal{L}}}\lambda_{ij}\right) x^\top W W^\top x.$$

Let's take $\lambda_0 = \max_{\substack{j\in\mathbf{M}\\ i\in\mathcal{L}}}\lambda_{ij}$, then the last inequality indicates

$$\|W^\top x\|^2 \leq\ e^{2\lambda_0 t}\|W^\top x_0\|^2 \Rightarrow \|W^\top x\| \leq\ e^{\lambda_0 t}\|W^\top x_0\|.$$

Consequently, the following bound for $\operatorname{dist}(x,\mathcal{X}_{n-m})$ in (3.6) is attained

$$\operatorname{dist}(x,\mathcal{X}_{n-m}) \leq \lambda_{\max}\left((W^\top W)^{-1}\right)^{1/2} e^{\lambda_0 t}\ \|W^\top x_0\|. \tag{3.7}$$

As the inequality (3.7) is valid for all modes $i \in \mathcal{L}$, it is also valid for the points belonging to the switching manifold, where the vector fields in these points are obtained by constructing the convex hull of the vector fields around them. Therefore, the trajectory reaches the interior of the set $\mathcal{X}_{n-m} + \epsilon\mathbb{B}$ in finite time, say t_1. On the other hand, as the vector field of the switched linear system is globally Lipschitz, $x(t)$ cannot have finite escape time. Now, as $(\mathcal{X}_{n-m} + \epsilon\mathbb{B}) \cap \mathcal{M} = \emptyset$, no switching takes places after t_1, and consequently the switched system (3.2) reduces to a linear system $\dot{x} = A_i x$, for some $A_i \in \mathcal{A}$. By assumption each subsystem is exponentially stable, and its convergence rate to the origin is not slower than $-\lambda_0$. Hence, we can argue that the overall switched system is exponentially stable. □

In some situations, Theorem 3.2.1 can be used for testing stability of switched linear systems for which no common quadratic Lyapunov function (see Problem 1.3.1 for a definition) exists, as illustrated by the next example.

Example 3.2.2. (See also Kouhi and Bajcinca (2011b)) It is well known that no common quadratic Lyapunov function for the switched system (3.2) exists, if there exist positive definite matrices $R_i = R_i^\top$, $i \in \mathcal{L}$, satisfying

$$\sum_{i=1}^{\ell}(A_i^\top R_i + R_i A_i) > 0; \tag{3.8}$$

see Boyd et al. (1994). Consider now an example adopted from De Schutter and Heemels (2004) with matrices A_i and R_i, $i \in \{1,2\}$, as follows:

$$A_1 = \begin{bmatrix} -0.001 & 0 & 0 \\ 0.3 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.01 & 0 & 0 \\ 0.05 & -1 & -10 \\ 0 & 0.1 & -1 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 0.02 & 0.01 & -0.1 \\ 0.01 & 0.299 & 0.704 \\ -0.1 & 0.704 & 2.470 \end{bmatrix}, \; R_2 = \begin{bmatrix} 0.08 & 0.025 & 0 \\ 0.025 & 0.212 & -0.553 \\ 0 & -0.553 & 1.971 \end{bmatrix}.$$

The matrices A_1, A_2, $R_1 > 0$, and $R_2 > 0$ satisfy inequality (3.8), as the eigenvalues of $\sum_{i=1}^{\ell}(A_i^\top R_i + R_i A_i)$ equal $0.002,\ 0.2735,\ 0.7768$. Thus, no statement about the stability of the switched system (3.2) based on the concept of a common quadratic Lyapunov function is possible. However, by defining $x = [x_1\ x_2\ x_3]^\top$ and the switching manifold as

$$\mathcal{M} = \{x \in \mathbb{R}^3 : x_2^2 + x_3^2 - a x_1^2 = -1,\ 0 < a < 10,\ a \in \mathbb{N}\},$$

the switched system (3.2) satisfies the following conditions:

(i) $w = [1\ \ 0\ \ 0]^\top$ is a left eigenvector for both A_1 and A_2,

(ii) A_1 and A_2 are Hurwitz,

(iii) the switching manifold $\mathcal{M}$ and the space $\mathcal{X}_2 + \epsilon\mathbb{B}$ do not intersect with $\epsilon = 0.02$ and

$$\mathcal{X}_2 = \{x \in \mathbb{R}^3 : w^\top x = 0\}.$$

Then, referring to Theorem 3.2.1, the switched system defined by (3.2) and (3.3) is exponentially stable.

Example 3.2.3. (See Kouhi and Bajcinca (2011a)) Consider the case that $n = 2$ in (3.2). Assume all matrices are Hurwitz and share a real left eigenvector w corresponding to the eigenvalues λ_{1i} for all $i \in \mathcal{L}$. Then, exponential stability of the switched system (3.2) is equivalent to quadratic stability of this system (for the definition of quadratic stability, see Problem 1.3.1) and no restriction on the switching manifold is required. To clarify this fact, it is evident that A_i's share a real right eigenvector v corresponding to the eigenvalue λ_{2i} for $i \in \mathcal{L}$, satisfying $w^\top v = 0$. Now, employing common quadratic Lyapunov function as $V(x) = x^\top P x$ with $P = ww^\top + \epsilon vv^\top$ and a scalar $\epsilon > 0$, we have

$$\frac{1}{2}\left(A_i^\top P + PA_i\right) = \lambda_{1i} ww^\top + \frac{1}{2}\epsilon(A_i^\top vv^\top + vv^\top A_i). \tag{3.9}$$

Without loss of generality, assume w and v are normal. This implies that the transformation matrix $T = [w\ \ v]$ is orthonormal, *i.e.*, $T^\top = T^{-1}$. As eigenvalues of a matrix are not affected by exploiting a linear transformation, the matrix $A_i^\top P + PA_i$ is negative definite if and only if $T^{-1}(A_i^\top P + PA_i)T$ is negative definite. On the other hand, we have

$$\frac{1}{2}T^{-1}(A_i^\top P + PA_i)T = \begin{bmatrix} \lambda_{1i} & \frac{1}{2}\epsilon v^\top A_i w \\ \frac{1}{2}\epsilon v^\top A_i w & \lambda_{2i}\epsilon \end{bmatrix}. \tag{3.10}$$

Having in mind that

$$\text{trace}\left(\frac{1}{2}T^{-1}(A_i^\top P + PA_i)T\right) = \lambda_{1i} + \epsilon\lambda_{2i} < 0,$$

the matrix $\frac{1}{2}T^{-1}(A_i^\top P+PA_i)T$ is negative definite if only if if its determinant is positive, or

$$\det\left(\frac{1}{2}T^{-1}(A_i^\top P + PA_i)T\right) = \epsilon\lambda_{1i}\lambda_{2i} - \frac{1}{4}\epsilon^2(v^\top A_i w)^2 > 0. \tag{3.11}$$

Notice $\lambda_{1i}\lambda_{2i} = \det(A_i)$; see Appendix A.2.6. Thus, for (3.11) to hold we require

$$\epsilon < \min\frac{4\det(A_i)}{(v^\top A_i w)^2} \qquad \forall i \in \mathcal{L}. \tag{3.12}$$

We generalize the result of Example 3.2.3 to the class of switched systems including $(n-1)$ common real left eigenvectors in the next chapter.

3.3 Exponential stabilization of controlled switched systems

In this section, we combine the left eigenstructure assignment techniques presented in Chapter 2 and the proposed idea on stability of switched linear systems in this chapter, for stabilization of controlled switched linear systems. For this purpose, we use local state feedbacks to impose common left eigenvectors to all constituent matrices of a switched linear system. Analogous to the stability approach, the key requirement here is that the switching manifold and the desired invariant subspace constructed by the desired common left eigenvectors are disjoint. First, we describe the algorithm for single input controlled switched systems. Afterwards, we discuss the problem of multi-input controlled switched linear systems.

3.3.1 Stabilization of single-input controlled switched systems

Consider a single input controlled switched linear system with the switching manifold $\mathcal{M}$ defined by (3.3), and the system dynamics given by

$$\dot{x} = A_{\sigma(x)}x(t) + b_{\sigma(x)}u(t), \tag{3.13}$$

where $A_{\sigma(x)} \in \mathbb{R}^{n\times n}$, $b_{\sigma(x)} \in \mathbb{R}^n$, and

$$A_{\sigma(x)} \in \mathcal{A} = \{A_1, ..., A_\ell\},\ b_{\sigma(x)} \in \mathcal{B} = \{b_1, ..., b_\ell\}. \tag{3.14}$$

To stabilize the switched system (3.13), we use local state feedback design $u = k_{\sigma(x)}x$ for $\sigma(x) \in \mathcal{L}$. The following corollary, which is a direct consequence of Theorem 3.2.1, is convenient for this purpose.

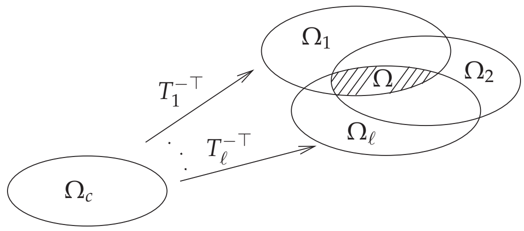

Figure 3.3.1: Geometrical construction of the common left eigenvector.

Corollary 3.3.1. *See Kouhi and Bajcinca (2011b). Assume local feedback gains k_i for all $i \in \mathcal{L}$ exist, such that the following conditions hold:*

(i) $A_{cl,i} = A_i + b_i k_i$ *are Hurwitz, and there exist a real vector* w *and numbers* $\lambda_{i1} \in \mathbb{R}_{<0}$ *such that* $w^\top A_{cl,i} = \lambda_{i1} w^\top$ *hold for all* $i \in \mathcal{L}$,

(ii) $(\mathcal{X}_{n-1} + \epsilon\mathbb{B}) \cap \mathcal{M} = \emptyset$ *for some* $\epsilon > 0$, *where* $\mathcal{X}_{n-1} = \{x \in \mathbb{R}^n : \; w^\top x = 0\}$.

Then, the switched system (3.13) *is exponentially stabilizable.* □

Therefore, the objective of control design consists in imposing an appropriately selected common left eigenvector w and its corresponding eigenvalues λ_{i1} to the closed loop matrices $A_{\mathrm{cl},i} = A_i + b_i k_i$ for all $i \in \mathcal{L}$. To this end, one can adopt the formula (2.2)

$$k_i = -\frac{w^\top (A_i - \lambda_{i1} I)}{w^\top b_i} \qquad \text{for } w^\top b_i \neq 0. \tag{3.15}$$

Constructing a proper common left eigenvector w_i which fulfills the first condition of the above corollary can be done, in principle, by following the procedure stated in Chapter 2.2.1.1. Consider a polynomial in the form of (2.13) whose coefficients are elements of $w_{c,i}$ instead of w_c. Let Ω_c be the whole set of appropriate left eigenvectors $w_{c,i}$ which makes the polynomial to be stable. With slight abuse of notation, let's define the set Ω_i as $\Omega_i := (T_i^\top)^{-1}\Omega_c$, where $T_i = \Phi_{ci}(A_i, b_i)\Phi_{ci}^{-1}(A_{ci}, b_{ci})$ for each $i \in \mathcal{L}$. The desired set of a common left eigenvector w, denoted by Ω, consists of all vectors that can ensure stability of all $A_{\mathrm{cl},i}$'s simultaneously for $i \in \mathcal{L}$. Indeed, this set is specified by intersection of the sets Ω_i, that is, $\Omega = \cap_{i=1}^{\ell}\Omega_i$; see Figure 3.3.1. Furthermore, in order to guarantee exponential stability of the switched system (3.13), due to the second condition of Corollary 3.3.1, one has to pick an eigenvector w from Ω that additionally guarantees the disjunction condition $(\mathcal{X}_{n-1} + \epsilon\mathbb{B}) \cap \mathcal{M} = \emptyset$. By illustration of two examples, let us explain how we can determine such a left eigenvector.

Example 3.3.1. Consider the controlled switched system (3.13) with the data

$$A_1 = \begin{bmatrix} 3 & -3 & -7 \\ 0 & -4 & 0 \\ 1 & 3 & -5 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 3 & -5 \\ 2 & -6 & -2 \\ 7 & 1 & -11 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \; b_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

A_2 is Hurwitz but A_1 has an eigenvalue at 2, and thus is not Hurwitz (A_1 and A_2 are adopted from Tsatsomeros (2001)). Defining $x = [x_1 \; x_2 \; x_3]^\top$, suppose the switching manifold is given by

$$\mathcal{M} = \{x \in \mathbb{R}^3 : -x_1 + x_2 + x_3^2 = -2a, \; a \in \mathbb{N}\}.$$

Assume we are interested in designing a control $u = k_{\sigma(x)}x$ for $\sigma(x) \in \{1, 2\}$, such that the resulting switched system is exponentially stable. Using the transformation $z = T_i x$, with $T_i = \Phi_{ci}(A_i, b_i)\Phi_{ci}^{-1}(A_{ci}, b_{ci})$ for $i \in \mathcal{L} := \{1, 2\}$, given by

$$T_1 = \begin{bmatrix} -72 & -8 & 1 \\ -8 & 2 & 1 \\ -32 & 6 & 2 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -4 & 7 & 1 \\ -8 & -2 & 0 \\ 20 & 17 & 2 \end{bmatrix},$$

the matrices A_i for $i \in \mathcal{L}$ are converted to control canonical forms; see Chapter 2.2.1.1. Now, let's denote a desired left eigenvector of the controllable canonical form of the i^{th} mode by $w_{ci} = [w_{ci1}\ w_{ci2}\ w_{ci3}]^\top$ for each $i \in \mathcal{L}$. For each $i \in \{1, 2\}$, the polynomial

$$h_i(\lambda) = w_{ci3}\lambda^2 + w_{ci2}\lambda + w_{ci1}$$

is Hurwitz when $w_{c,ij} > 0$ for $j \in \{1, 2, 3\}$. On the other hand, recall from Chapter 2.2.1.1 that the relationship between the left eigenvectors w and $w_{c,i}$ corresponding to the closed loop matrices $A_{\text{cl},i}$ and $A_{\text{c,cl},i}$ (controllable canonical form of $A_{\text{cl},i}$), respectively, is given by $w = T_i^{-\top} w_{ci}$ for each $i \in \mathcal{L}$. Therefore, for computing an appropriate common left eigenvector that stabilizes this switched system, we need to satisfy the following criteria:

$$\begin{aligned} &T_1^{-\top} w_{c1} - T_2^{-\top} w_{c2} = 0, \\ &w_{c,ij} > 0 \qquad \forall\, i \in \{1, 2\},\ j \in \{1, 2, 3\}. \end{aligned}$$

Define the vector $w = [w_1\ w_2\ w_3]^\top$. To meet the condition $(\mathcal{X}_2 + \epsilon\mathbb{B} \cap \mathcal{M}) = \emptyset$ in accordance with item (ii) of Corollary 3.3.1, we consider the following restrictions on the elements of w: $w_1 + w_2 = 0$, $-2w_2 + w_3 \le 0$, $w_2 \ge 0$, and $w_3 \ge 0$. Having these inequalities in hand, it turns out that the disjoint condition is satisfied. Now, the problem associated with finding w can be approached by optimization technique. Let us define the vectors $w_c^\top = [w_{c1}^\top\ w_{c2}^\top]$, $\psi_1 = [0\ \ -2\ 1]$, $\psi_2 = [0\ \ -1\ 0]$, $\psi_3 = [0\ \ 0\ \ -1]$, $\nu = \gamma[1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0]^\top$, and the matrices

$$R = \begin{bmatrix} I_3 & -I_3 \\ 1\ 1\ 0 & 0\ 0\ 0 \end{bmatrix} \begin{bmatrix} (T_1^\top)^{-1} & 0 \\ 0 & (T_2^\top)^{-1} \end{bmatrix}, \quad S = \begin{bmatrix} & -I_6 & \\ \psi_1(T_1^\top)^{-1} & 0\ 0\ 0 \\ \psi_2(T_1^\top)^{-1} & 0\ 0\ 0 \\ \psi_3(T_1^\top)^{-1} & 0\ 0\ 0 \end{bmatrix}.$$

Then, we introduce the following static optimization problem

$$\text{minimize } \|w_c\|^2$$

$$\text{subject to} \begin{cases} Rw_c = 0, \\ Sw_c \le -\nu, \end{cases}$$

for some small $\gamma > 0$. Note that the variable ν is added to the problem to rule out computation of the zero solution. Now, the optimization problem can be solved by "fmincon" command in the MATLAB optimization toolbox. With $\gamma = 0.01$, the results of this optimization problem are

$$w_{c1} = \begin{bmatrix} 0.01 \\ 0.076 \\ 0.0135 \end{bmatrix}, \quad w_{c2} = \begin{bmatrix} 0.1213 \\ 0.0832 \\ 0.01 \end{bmatrix}, \quad w = \begin{bmatrix} -0.0035 \\ 0.0035 \\ 0.0068 \end{bmatrix}.$$

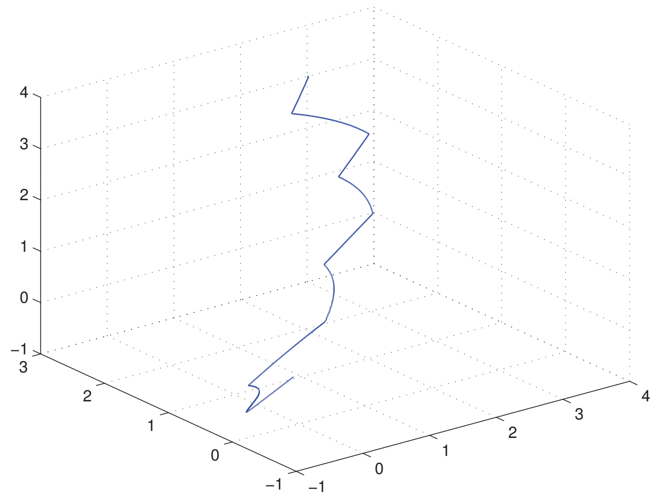

Figure 3.3.2: A trajectory of the switched system in Example 3.3.1.

We choose the corresponding eigenvalue to w to be $\lambda_1 = -2$. Now, (3.15) gives us

$$k_1 = [0.8077 \ -1.7615 \ -0.3308], \ k_2 = [-4.3854 \ \ 1.8021 \ \ 5.0312],$$

leading to the closed loop matrices

$$A_{\text{cl1}} = \begin{bmatrix} 3.8077 & -4.7615 & -7.3308 \\ 0.8077 & -5.7615 & -0.3308 \\ 2.6154 & -0.5231 & -5.6615 \end{bmatrix}, \ A_{\text{cl2}} = \begin{bmatrix} -3.3854 & 4.8021 & 0.0312 \\ 2 & -6 & -2 \\ -1.7708 & 4.6042 & -0.9375 \end{bmatrix}.$$

Figure 3.3.2 depicts the trajectory of the system emerging from the point $x_0 = [3\ 3\ 3]^\top$ by the assumption that initially the first mode is active. It can be seen that the trajectory eventually converges to zero.

Example 3.3.2. Consider the controlled switched system (3.13) with the data

$$A_1 = \begin{bmatrix} 3 & -3 & -1 & 1 \\ 0 & -4 & 0 & 1 \\ 1 & 3 & -5 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 3 & -5 & 2 \\ 2 & -6 & -2 & 0 \\ 7 & 1 & -1 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \ b_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

None of the matrices A_1 and A_2 are Hurwitz. Defining $x = [x_1\ x_2\ x_3\ x_4]^\top$, suppose the switching manifold is given by

$$\mathcal{M} = \{x \in \mathbb{R}^4 : -x_1 + x_2 + \tfrac{1}{2}x_3^2 = -a, \ x_4 = 0, \ a \in \mathbb{N}\}.$$

Assume we are interested in designing the control input in the form of $u = k_{\sigma(x)}x$ for $\sigma(x) \in \{1, 2\}$, such that the resulting switched system becomes exponentially stable. By using the transformations $z = T_i x$, where $T_i = \Phi_{ci}(A_i, b_i)\Phi_{ci}^{-1}(A_{ci}, b_{ci})$ for $i \in \mathcal{L} := \{1, 2\}$, given by

$$T_1 = \begin{bmatrix} 1 & -6 & 4 & 1 \\ -1 & -14 & 2 & 1 \\ 2 & -34 & 6 & 2 \\ -20 & 6 & 2 & 0 \end{bmatrix}, \ T_2 = \begin{bmatrix} -34 & 15 & 9 & 1 \\ -38 & -10 & 2 & 0 \\ 80 & 64 & 8 & 0 \\ 356 & 49 & 7 & 1 \end{bmatrix},$$

the matrices A_i for $i \in \{1,2\}$ are converted to controllable canonical forms; see Chapter 2.2.1.1. Now, let's pick a desired left eigenvector of the controllable canonical form with the following entries: $w_{ci} = [w_{ci1}\ w_{ci2}\ w_{ci3}\ w_{ci4}]^\top$ for each $i \in \mathcal{L}$. For having the polynomial

$$h_i(\lambda) = w_{ci4}\lambda^3 + w_{ci3}\lambda^2 + w_{ci2}\lambda + w_{ci1}$$

to be Hurwitz, we employ Lemma 2.2.2. Sufficient conditions for stability of this polynomial are stated by

$$\begin{aligned} w_{c,ij} > 0 \qquad & \forall i \in \{1,2\},\ j \in \{1,2,3,4\}, \\ w_{c,i2}w_{c,i3} \geq 4.0796\ w_{c,i1}w_{c,i4} \qquad & \forall i \in \{1,2\}. \end{aligned}$$

On the other hand, $w = T_i^{-\top} w_{ci}$ for each $i \in \mathcal{L}$ is a common left eigenvector of the two systems, thus

$$T_1^{-\top} w_{c1} - T_2^{-\top} w_{c2} = 0.$$

Define $w = [w_1\ w_2\ w_3\ w_4]^\top$. To satisfy the condition $(\mathcal{X}_2 + \epsilon\mathbb{B} \cap \mathcal{M}) = \emptyset$ in accordance with item (ii) of Corollary 3.3.1, we impose the following restrictions: $w_1 + w_2 = 0$, $w_2 + w_3 \leq 0$, $w_2 - w_3 \leq 0$, and $w_2 \geq 0$. With these inequalities it turns out that the disjoint condition is satisfied. For solving the problem for w, we again employ the optimization technique. Let us define the vectors $w_c^\top = [w_{c1}^\top\ w_{c2}^\top]$, $\psi_1 = [0\ 1\ 1\ 0]$, $\psi_2 = [0\ 1\ -1\ 0]$, $\psi_3 = [0\ -1\ 0\ 0]$, $\nu = \gamma[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0]^\top$, and the matrices

$$R = \begin{bmatrix} I_4 & -I_4 \\ 1\ 1\ 0\ 0 & 0\ 0\ 0\ 0 \end{bmatrix} \begin{bmatrix} (T_1^\top)^{-1} & 0 \\ 0 & (T_2^\top)^{-1} \end{bmatrix}, \quad S = \begin{bmatrix} -I_8 & \\ \psi_1 (T_1^\top)^{-1} & 0\ 0\ 0\ 0 \\ \psi_2 (T_1^\top)^{-1} & 0\ 0\ 0\ 0 \\ \psi_3 (T_1^\top)^{-1} & 0\ 0\ 0\ 0 \end{bmatrix}.$$

The following static optimization problem now includes all the constraints

$$\text{minimize } \|w_c\|^2$$

$$\text{subject to} \begin{cases} Rw_c = 0, \\ Sw_c \leq -\nu, \\ -w_{c,12}w_{c,13} + 4.0796\ w_{c,11}w_{c,14} \leq 0, \\ -w_{c,22}w_{c,23} + 4.0796\ w_{c,21}w_{c,24} \leq 0. \end{cases}$$

Note that the vector ν is entered into the problem for excluding computation of zero solution. Now, the "fmincon" command in the MATLAB optimization toolbox can solve this optimization problem. The parameters of this optimization problem with $\gamma = 0.01$ are computed to be

$$w_{c1} = \begin{bmatrix} 0.0061 \\ 0.0061 \\ 0.0467 \\ 0.0061 \end{bmatrix}, \quad w_{c2} = \begin{bmatrix} 0.7548 \\ 0.5810 \\ 0.1236 \\ 0.0141 \end{bmatrix}, \quad w = \begin{bmatrix} 0.0129 \\ -0.0129 \\ 0.0031 \\ 0.0013 \end{bmatrix}.$$

One can then choose the eigenvalue associated to w to be $\lambda_1 = -2$, and then use (3.15) to compute the feedback gains

$$k_1 = [-11.1744\ 0.3837\ 3.5930\ -0.9186],\ k_2 = [-2.5202\ -6.4899\ 2.3283\ -2.2172].$$

Consequently, the closed loop matrices equal

$$A_{\text{cl1}} = \begin{bmatrix} -8.1744 & -2.6163 & 2.5930 & 0.0814 \\ -11.1744 & -3.6163 & 3.5930 & 0.0814 \\ -21.3488 & 3.7674 & 2.1860 & -0.8372 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$A_{\text{cl2}} = \begin{bmatrix} -1.5202 & -3.4899 & -2.6717 & -0.2172 \\ 2 & -6 & -2 & 0 \\ 7 & 1 & -1 & 1 \\ -1.5202 & -7.4899 & 4.3283 & -2.2172 \end{bmatrix}.$$

Note that A_{cl1} and A_{cl2} are both Hurwitz, but no common quadratic Lyapunov function can be associated with them. This can be checked by using an appropriate LMI software.

3.3.2 Stabilization of multi-input controlled switched systems

Now, consider a multi-input controlled switched linear system with the switching manifold $\mathcal{M}$ defined by (3.3), and the system dynamics given by

$$\dot{x} = A_{\sigma(x)}x(t) + B_{\sigma(x)}u(t), \tag{3.16}$$

where $A_{\sigma(x)} \in \mathbb{R}^{n\times n}$, $B_{\sigma(x)} \in \mathbb{R}^{n\times m}$, and

$$A_{\sigma(x)} \in \mathcal{A} = \{A_1, ..., A_\ell\},\ B_{\sigma(x)} \in \mathcal{B} = \{B_1, ..., B_\ell\}. \tag{3.17}$$

The objective of the control design is to exponentially stabilize (3.16) by means of local state feedbacks $u = K_{\sigma(x)}x$, where $K_{\sigma(x)} \in \mathbb{R}^{m\times n}$ corresponds to the open loop system $(A_{\sigma(x)}, B_{\sigma(x)})$ for $\sigma(x) \in \mathcal{L}$. For this target, we can again refer to the following corollary which is a direct consequence of Theorem 3.2.1.

Corollary 3.3.2. *Let K_i for all $i \in \mathcal{L}$ be local state feedback gains such that*

(i) $A_{cl,i} = A_i + B_iK_i$ are Hurwitz, and there exist a matrix $W \in \mathbb{R}^{n\times(m-1)}$ and a diagonal matrix $\Lambda = diag([\lambda_1, \dots, \lambda_{m-1}])$ with real entries satisfying: $W^\top A_{cl,i} = W^\top \Lambda$,

(ii) $(\mathcal{X}_{n-m+1} + \epsilon\mathbb{B}) \cap \mathcal{M} = \emptyset$ for some $\epsilon > 0$, where $\mathcal{X}_{n-m+1} = \{x \in \mathbb{R}^n :\ W^\top x = 0\}$.

Then, the switched system (3.16) *is exponentially stabilizable.* □

We employ the algorithm described in Chapter 2.3 for achieving the criteria of Corollary 3.3.2. Namely, we use $(m-1)$ inputs for assigning an appropriate set of left eigenvectors given by the columns of $W = [w_1 \dots w_{m-1}]$ and their corresponding eigenvalues given by the diagonal entries of $\Lambda = \text{diag}([\lambda_1, \dots, \lambda_{m-1}])$. In addition, we use a single input for assigning the remaining eigenvalues $\lambda_m, \dots, \lambda_n$. In summary this algorithm works as follows:

(i) a set of linearly independent desired left eigenvectors given by the columns of $W = [w_1 \dots w_{m-1}]$ are selected in a manner such that $(\mathcal{X}_{n-m+1} + \epsilon\mathbb{B}) \cap \mathcal{M} = \emptyset$.

(ii) each K_i for $i \in \mathcal{L}$ splits into two parts, a matrix $\bar{K}_i$ and a vector k_{ij} corresponding to the channels $\bar{B}_i = [b_{i1} \dots b_{ij-1}\ b_{i(j+1)} \dots b_{im}]$ and b_{ij}, respectively. The separation is carried out in a way such that $\Pi_i A_i$ has a full set of eigenvectors and the controllability matrices $\Phi_c(A_i\Pi_i, b_{ij})$ have rank n, where $\Pi_i = I - \bar{B}_i(W^\top \bar{B}_i)^{-1}W^\top$ and $\det(W^\top \bar{B}_i) \neq 0$ for all $i \in \mathcal{L}$. Note that the index j is not fixed and can be different for each i. Then, we compute the matrix $\hat{A}_i$ and the vector $\hat{b}_i$ adopted from (2.42) as

$$\hat{A}_i = \left[(I - \bar{B}_i(W^\top \bar{B}_i)^{-1}W^\top)A_i + \bar{B}_i(W^\top \bar{B}_i)^{-1}\Lambda W^\top\right],$$
$$\hat{b}_i = (I - \bar{B}_i(W^\top \bar{B}_i)^{-1}W^\top)b_{ij}. \tag{3.18}$$

(iii) the transformation matrices $T_i = [Q_{i1}\ Q_{i2}]$ are acquired from the real Schur decomposition of $\hat{A}_i^\top$ for all $i \in \mathcal{L}$.

(iv) k_{ij} is computed by adopting (2.44) and (2.45) for each mode $i \in \mathcal{L}$.

(v) $\bar{K}_i$ is computed by adopting (2.40) for mode i, and the inputs $\bar{u} = \bar{K}_i x$ and $u_j = k_{ij}x$ in (2.39) are re-arranged to find the control $u = K_i x$ for each $i \in \mathcal{L}$.

Now, we provide an example for this algorithm.

Example 3.3.3. Consider a controlled switched system with matrices A_1 and A_2 as those in Example 3.3.1 with the switching manifold given by

$$\mathcal{M} = \{x \in \mathbb{R}^3 : -x_1 + x_2 + x_3 = 2a - 1,\ a \in \mathbb{Z}\}.$$

The matrices B_1 and B_2 are

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}. \tag{3.19}$$

For stabilization of this switched system, we follow the steps of the above algorithm. By looking at the equation of the switching manifold $\mathcal{M}$, we decide $W = [-1\ 1\ 1]^\top$ to be the desired common left eigenvector. As $(\mathcal{X}_2 + \epsilon\mathbb{B}) \cap \mathcal{M} = \emptyset$ and $\det(W^\top B_i) \neq 0$ for all $i \in \mathcal{L}$ and a sufficiently small $\epsilon > 0$, the requirement of item (i) has already been fulfilled. For both closed loop matrices $A_{\text{cl}1}$ and $A_{\text{cl}2}$, we choose the closed loop eigenvalues corresponding to W to be $\lambda_1 = -2$, and the other eigenvalues to be $\lambda_2 = -5$ and $\lambda_3 = -6$. As all the data for the first subsystem is identical to those in Example 2.3.1, we can follow the entire procedure in this example for finding the appropriate controllers. This gives us $k_{12} = [3\ \ 5\ -2]$ and $\bar{K}_1 = [0.5\ -4.5\ -1]$.

In the second subsystem, we optionally take the first input for partial pole placement, whereas the second input is used for assigning the left eigenvector W and its corresponding eigenvalue λ_2. According to this separation, let us define the vectors

$b_{21} = [1\ 1\ 1]^\top$ and $\bar{B}_2 = [2\ 0\ 1]^\top$. Define also $\varPi_2$ similar to $\varPi$ in (2.46) by signifying the equation with index "2". As rank $\Phi_c(A_2\varPi_2, b_{21}) = 3$ and $\varPi_2 A_2$ has a full set of eigenvectors, Lemma 2.3.1 implies that left eigenstructure assignment is possible. Based on this knowledge, we compute the matrix $\hat{A}_2$ and the vector $\hat{b}_2$ in accordance with (2.42)

$$\hat{A}_2 = \begin{bmatrix} -5 & 9 & 1 \\ -4 & 0 & 4 \\ 1 & 7 & -5 \end{bmatrix},\ \hat{b}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Now, computing the real Schur factorization of $\hat{A}_2^\top$ defines the matrices Q_{21}, Q_{22}, $\hat{L}_{21}$, and $\hat{L}_{22}$ with appropriate dimensions, satisfying $Q_{21}^\top \hat{A}_2 = \hat{L}_{21} Q_{21}^\top$ and $Q_{22}^\top \hat{A}_2 Q_{22} = \hat{L}_{22}$. These matrices are numerically computed as

$$Q_{21}^\top = \begin{bmatrix} 0.4082 & 0.8165 & -0.4082 \\ 0.7071 & 0 & 0.7071 \end{bmatrix},\ \hat{L}_{21} = \begin{bmatrix} -4 & 0 \\ 9.2376 & -4 \end{bmatrix},$$
$$Q_{22}^\top = [-0.5774\ \ 0.5774\ \ 0.5774], \quad \hat{L}_{22} = -2.$$

Now, we use (2.44) and (2.45) to compute the appropriate state feedback for this mode, leading to the gain $k_{22} = [-0.9167\ \ -1.5833\ \ 0.6667]$. Finally, as $W^\top \bar{B}_2 \neq 0$, we employ (2.40) to obtain $\bar{K}_2$, yielding the gain $\bar{K}_2 = [-6.9167\ \ 4.4167\ \ 6.6667]$. Consequently, the closed loop matrices of the both subsystems equal

$$A_{\text{cl1}} = \begin{bmatrix} 6.5 & -2.5 & -10 \\ 0.5 & -8.5 & -1 \\ 8 & 4 & -11 \end{bmatrix},\ A_{\text{cl2}} = \begin{bmatrix} -7.75 & 4.25 & 3 \\ -4.9167 & -1.5833 & 4.6667 \\ -0.8333 & 3.8333 & -3.6667 \end{bmatrix}.$$

3.4 Conclusions

In this chapter, we proposed sufficient conditions for stabilization of a class of single- and multi-input controlled switched linear systems under state dependent switching rules. In this class, the switching signal is specified by a given manifold in the state space. Our proposed approach investigates local state feedback design for each subsystem. The left eigenvectors and eigenstructure assignment techniques for LTI systems are the key tools for the design. On this basis, we showed that if all closed loop matrices of a controlled switched linear system share a number of left eigenvectors, then all solutions of the switched system converge to a known invariant subspace. The idea then lies in the appropriate selection of the desired common left eigenvectors to ensure Hurwitz stability of the closed loop matrices, and to avoid the situation that the common invariant subspace and the switching manifold intersect. Apparently, the structure of the switching manifold enforces a significant restriction in this technique. Thus, the problem of selecting the desired common eigenvectors may include a lot of constraints especially when the switching manifold has a complicated topology.

Chapter 4

Switched linear systems with arbitrary switching signals

4.1 Introduction

Despite many interesting results, the stability problem for switched linear systems with arbitrary time-dependent switching signals still remains a challenging research topic. Similar to stability theory for nonlinear systems, Lyapunov's second theorem is the essential tool for dealing with this problem. It turns out that if there exists a positive definite function which decreases in time along the system solutions, then the switched linear system is exponentially stable (Shorten et al., 2007). Such a function is called a common Lyapunov function for the switched system. When the common Lyapunov function is quadratic and satisfies separately the Lyapunov function properties for all modes, it is called a common quadratic Lyapunov function (CQLF), and the switched system is called quadratically stable. It is known that quadratic stability of a switched system implies its exponential stability, and we will also give a short proof for this fact in this chapter. As a matter of fact, referring to the converse Lyapunov theorem, exponential stability for a switched linear system is equivalent to the existence of a common Lyapunov function for such system; see Shorten et al. (2007) and the references therein.

In this chapter, we study quadratic stability and stabilization (see Problems 1.3.1, and 1.4.1) of several classes of switched linear systems. We show that if all constituent matrices of a switched linear system are individually Hurwitz and share an $(n-m)$ dimensional right invariant subspace to which a common quadratic Lyapunov function can be associated, and if a set of m common left eigenvectors perpendicular to this invariant subspace exists, then the switched linear system is quadratically stable. One direct consequence of this result is quadratic stability of switched systems whose constituent matrices have $(n-1)$ common right eigenvectors (Shorten and Cairbre, 2001), or $(n-1)$ common real left eigenvectors (Kouhi and Bajcinca, 2011b). Moreover, we discuss robust stability of switched linear systems, particularly for systems with $(n-1)$ common real left eigenvectors.

In addition, we propose the notion of block similarity for controlled switched linear systems, and demonstrate that the quadratic stability problem for block similar matrices is equivalent. This result may particularly be applicable for controlled switched systems where the closed loop matrices can be transformed to block triangular form. Then, we identify one such class, namely the class of controlled switched systems where the open loop matrices share an invariant subspace with appropriate dimension to which a common quadratic Lyapunov function can be associated. For this class, we utilize the left eigenvectors assignment approach to design local feedbacks which make the controlled switched systems quadratically stable. This result extends parts of the results by Kouhi and Bajcinca (2011c), where $(n-1)$ inputs are assumed for stabilization of a controlled switched linear system. We develop further this approach for stabilization of a controlled switched system whose open loop matrices own $(n-m)$ dimensional invariant subspaces which are sufficiently close, and all matrices associated with the invariant subspaces satisfy a certain form of Riccati inequalities instead of Lyapunov inequalities.

4.2 Quadratic stability of switched linear systems

We study quadratic stability of switched linear systems under arbitrary time-dependent switching signals. Our switched linear system is defined as

$$\dot{x} = A_{\sigma(t)}x(t) \quad A_{\sigma(t)} \in \mathcal{A} := \{A_1, \ldots, A_\ell\}, \tag{4.1}$$

where $x(t_0) = x_0 \in \mathbb{R}^n$ and $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{L} := \{1, \ldots, \ell\}$ is a piecewise constant function referred to as the switching signal between different modes, that is, $\dot{x} = A_i x(t)$ for $i \in \mathcal{L}$. Note that the only restriction on the switching is that a finite number of switching occurs within each finite time interval. Given an initial condition x_0, the Caratheodory solution (see Appendix A.6.2.1) of the switched system (4.1) is unique and is given by

$$x(t) = e^{A_{\sigma_k}(t-t_k)} \ldots e^{A_{\sigma_0}(t_1-t_0)}x_0, \tag{4.2}$$

where $t_k > \cdots > t_2 > t_1$ are the switching time instances, $t > t_k$, $t_1 > t_0$, and $\sigma_j := \sigma(t_j)$ for all $j \in \{0, \ldots, k\}$; see (Shorten et al., 2007). The uniqueness of the solution results from the fact that at each time instance only one subsystem is active, and for any active mode $i \in \mathcal{L}$ the vector field $f_i := A_i x$ is locally Lipschitz. It is known that differential equations with locally Lipschitz vector fields possess unique solutions (Coddington and Levinson, 1955; Cortes, 2008).

Recall from Problem 1.3.1 that the switched linear system (4.1) is called quadratically stable if and only if there exists a function $V(x) = x^\top Px$ with P symmetric positive definite, such that

$$A_i^\top P + PA_i < 0 \qquad \forall A_i \in \mathcal{A}. \tag{4.3}$$

Now, we attempt to prove that quadratic stability of switched linear systems implies their exponential stability, a fact which was already explored and existed in the literature (Shorten et al., 2007). To see this, assume that all matrices A_i for $i \in \mathcal{L}$ share a quadratic Lyapunov function $V(x) = x^\top Px$. As $A_i^\top P + PA_i < 0$ for all $i \in \mathcal{L}$, then there exists a number $\gamma > 0$, such that

$$\begin{aligned} A_i^\top P + PA_i &< -\gamma I \\ \Rightarrow \dot{V}(x) &= x^\top(A_i^\top P + PA_i)x \leq -\gamma\|x\|^2. \end{aligned}$$

On the other hand, as P is symmetric positive definite, the following inequalities are valid

$$\lambda_{\min}(P)\|x\|^2 \leq x^\top Px = V(x) \leq \lambda_{\max}(P)\|x\|^2. \tag{4.4}$$

Therefore,

$$\begin{aligned}\dot{V}(x) =& x^\top(A_i^\top P + PA_i)x \leq -\gamma\|x\|^2 \leq \frac{-\gamma}{\lambda_{\max}(P)}V(x)\\ &\Rightarrow V(x) \leq V(x_0)e^{-\beta(t-t_0)},\end{aligned}$$

where $\beta := \gamma/\lambda_{\max}(P)$. Now, paying attention to (4.4), we can deduce

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq V(x_0)e^{-\beta(t-t_0)} \Rightarrow \|x\|^2 \leq \frac{V(x_0)}{\lambda_{\min}(P)}e^{-\beta(t-t_0)}.$$

This implies exponential stability of the switched linear system (4.1). Therefore, numerous contributions offering sufficient conditions for the existence of a common Lyapunov solution for a set of Hurwitz matrices have appeared in the literature, including Shorten and Narendra (1998); Narendra and Balakrishnan (1994); Shorten and Cairbre (2001). One of the established results in this context concerns this problem for a set of similar matrices. The following lemma points out this result in detail.

Lemma 4.2.1. *(See also Wulff (2005); Shorten et al. (2007)) Suppose in the switched linear system* (4.1)*, each $A_i \in \mathbb{R}^{n\times n}$ is block similar to $\bar{A}_i \in \mathbb{C}^{n\times n}$ for $i \in \mathcal{L}$, obtained by a similarity transformation matrix $S \in \mathbb{C}^{n\times n}$. Then, there exists a real common Lyapunov solution $P = P^\top > 0$ for all A_i, if and only if there exists a Hermitian common Lyapunov solution $\bar{P} = \bar{P}^* > 0$ for all $\bar{A}_i$, that is, $\bar{A}_i^*\bar{P} + \bar{P}\bar{A}_i < 0$ for all $i \in \mathcal{L}$.*

Proof of sufficiency: As A_i is block similar to $\bar{A}_i$ for all $i \in \mathcal{L}$ by using a similarity transformation $S \in \mathbb{C}^{n\times n}$, one can write

$$\bar{A}_i = S^{-1}A_iS; \tag{4.5}$$

see Appendix A.2.12. Now, suppose there exists a positive definite matrix $P = P^\top$ such that $A_i^\top P + PA_i < 0$. By pre-multiplying this inequality by S^* and post-multiplying it by S, the sign of the inequality does not change. Therefore, we have

$$S^*A_i^\top PS + S^*PA_iS < 0 \Rightarrow \bar{A}_i^*\bar{P} + \bar{P}\bar{A}_i < 0 \qquad \forall i \in \mathcal{L},$$

where $\bar{P} = S^*PS > 0$ is Hermitian.

Proof of necessity: Suppose there exists a Hermitian $\bar{P} > 0$ such that $\bar{A}_i^*\bar{P} + \bar{P}\bar{A}_i < 0$. Then, by pre-multiplying this inequality by S^{-*} and post-multiplying it by S^{-1}, the sign of this inequality does not change. Hence, we can write

$$S^{-*}\bar{A}_i^*\bar{P}S^{-1} + S^{-*}\bar{P}\bar{A}_iS^{-1} < 0 \Rightarrow A_i^\top(S^{-*}\bar{P}S^{-1}) + (S^{-*}\bar{P}S^{-1})A_i < 0 \qquad \forall i \in \mathcal{L}.$$

Now, define $P = \text{Re}\,(S^{-*}\bar{P}S^{-1})$, and consider the function $V(x) = x^\top Px$ for $x \in \mathbb{R}^n$. We show $V(x)$ is indeed a CQLF for the switched linear system (4.1). As $S^{-*}\bar{P}S^{-1}$ is Hermitian, P is symmetric. Moreover, for $x \in \mathbb{R}^n \backslash \{0\}$ we have

$$\begin{aligned} V(x) &= x^\top Px = x^\top S^{-*}\bar{P}S^{-1}x > 0, \\ \dot{V}(x) &= x^\top (A_i^\top P + PA_i)x = x^\top (S^{-*}\bar{A}_i^*\bar{P}S^{-1} + S^{-*}\bar{P}\bar{A}_iS^{-1})x < 0. \end{aligned}$$

The first inequality implies that $P > 0$, and the second inequality implies that $V(x)$ is decreasing along the solutions of the switched system. Hence, $V(x)$ satisfies the criteria of a CQLF. □

In the sequel, we aim at establishing conditions on the A_i's, such that these matrices share a Lyapunov solution. Our approaches rely on the concept of common eigenvectors and invariant subspaces. To gain more intuition about our approach, we outline the following principle. Consider the special case that all Hurwitz matrices in the switched linear system (4.1) share a set of m linearly independent left eigenvectors given by the columns of $W = [w_1 \ \ldots \ w_m] \in \mathbb{C}^{n\times m}$ corresponding to a set of eigenvalues given by the diagonal elements of the matrix $\Lambda_i = \text{diag}([\lambda_{i1}, \ldots, \lambda_{im}])$ for each $i \in \mathcal{L}$. Notice when w_j for $j \in \{1, \ldots, m\}$ is a left eigenvector of A_i corresponding to the eigenvalue λ_{ij}, then w_j is also a left eigenvector of the matrix e^{A_it} corresponding to the eigenvalue $e^{\lambda_{ij}t}$. This fact can be proved via exploiting the Taylor expansion of e^{A_it}

$$e^{A_it} = I + A_it + \frac{(A_it)^2}{2!} + \cdots . \tag{4.6}$$

Then, by pre-multiplication of (4.6) by W^* and considering the identity $W^*A_i = \Lambda_iW^*$, we will have

$$W^*e^{A_it} = \left(I + \Lambda_it + \frac{(\Lambda_it)^2}{2!} + \cdots\right)W^* = e^{\Lambda_it}W^*.$$

Further, pre-multiplying (4.2) by W^*, the following property holds

$$W^*x(t) = e^{\Lambda_{\sigma_k}(t-t_k)}e^{\Lambda_{\sigma_{k-1}}(t_k-t_{k-1})}\ldots e^{\Lambda_{\sigma_0}(t_1-t_0)}W^*x_0.$$

Suppose all eigenvalues λ_{ij} for $i \in \mathcal{L}$ and $j \in \{1, \ldots, m\}$ have negative real parts, that is, $\text{Re}\,(\lambda_{ij}) < 0$. Assuming $\lambda_0 = \max(\text{Re}(\lambda_{ij}))$ for $i \in \mathcal{L}$ and $j \in \{1, \ldots, m\}$, we can write

$$\begin{aligned} \|W^*x(t)\| =& \|e^{\Lambda_{\sigma_k}(t-t_k)}e^{\Lambda_{\sigma_{k-1}}(t_k-t_{k-1})}\ldots e^{\Lambda_{\sigma_0}(t_1-t_0)}W^*x_0\| \\ &\le \|e^{\Lambda_{\sigma_k}(t-t_k)}\|.\|e^{\Lambda_{\sigma_{k-1}}(t_k-t_{k-1})}\|\ldots\|e^{\Lambda_{\sigma_0}(t_1-t_0)}\|.\|W^*x_0\| \\ &\le e^{\lambda_0(t-t_k)}.e^{\lambda_0(t_k-t_{k-1})}\ldots e^{\lambda_0(t_1-t_0)}\|W^*x_0\| = e^{\lambda_0(t-t_0)}\|W^*x_0\|. \end{aligned}$$

This means that all solutions (4.2) converge exponentially to a set defined by

$$\mathcal{X}_{n-m} = \{x \in \mathbb{C}^n : \ \ W^*x = 0\}, \tag{4.7}$$

in the worst case with time constant $-1/\lambda_0$. On the other hand, the following property

$$x \in \mathcal{X}_{n-m} \Rightarrow W^*\dot{x} = W^*A_ix = \Lambda_iW^*x = 0 \qquad \forall i \in \mathcal{L}$$

indicates that the set $\mathcal{X}_{n-m}$ is invariant for the switched system (4.1).

We consider this principle as the initial step for developing our results in this chapter; see also Kouhi and Bajcinca (2011b).

4.2.1 Stability of switched systems with a common invariant subspace

Motivated by the previous discussion, we now assume that all matrices in the switched system (4.1) are Hurwitz and share m linearly independent left eigenvectors corresponding to m eigenvalues with negative real parts. We explored that the solution starting from an initial point x_0 outside of $\mathcal{X}_{n-m}$, defined by (4.7), exponentially converges to this set. Now, as our switched system consists of linear flows, we hope quadratic stability of the system inside the invariant subspace can imply quadratic stability of the switched system in the entire state space. The correctness of this conjecture is established by accomplishing the proof of the next theorem.

Theorem 4.2.1. *Consider the switched linear system defined by* (4.1)*. Let all $A_i \in \mathcal{A}$ be Hurwitz, and share m linearly independent left eigenvectors given by the columns of $W = [w_1 \dots w_m]$ corresponding to the eigenvalues given by the diagonal entries of the matrices $\Lambda_i = diag([\lambda_{i1}, \dots, \lambda_{im}])$ for all $i \in \mathcal{L}$. Furthermore, suppose the columns of $V \in \mathbb{C}^{n\times(n-m)}$ are an orthonormal basis for the common invariant subspace of all $A_i \in \mathcal{A}$, namely $\mathcal{X}_{n-m} \subset \mathbb{C}^n$ defined by* (4.7)*, corresponding to the eigenvalues $\lambda_{i,m+1}, \dots, \lambda_{i,n}$ for $i \in \mathcal{L}$, that is,*

$$A_iV = VL_i, \quad \text{and} \quad W^*V = 0, \tag{4.8}$$

where $L_i \in \mathbb{C}^{(n-m)\times(n-m)}$ for all $i \in \mathcal{L}$. Moreover, assume that the systems $\dot{\bar{x}} = L_i\bar{x}$ share a CQLF, that is, there exist a function $\mathcal{V}(\bar{x}) = \bar{x}^*\mathcal{P}\bar{x}$ with $\mathcal{P} \in \mathbb{C}^{(n-m)\times(n-m)}$ and $\mathcal{P} = \mathcal{P}^* > 0$, such that the following Lyapunov inequalities hold

$$L_i^*\mathcal{P} + \mathcal{P}L_i < 0 \qquad \forall\, i \in \mathcal{L}. \tag{4.9}$$

Then, the switched linear system (4.1) *is quadratically stable.*

Proof: Let us define the reduced QR-factorization of W as

$$W = Q_1R_1, \tag{4.10}$$

where R_1 is an $m \times m$ upper triangular non-singular matrix and Q_1 is an $n \times m$ matrix which has orthonormal columns, that is, $Q_1^*Q = I_m$; see Appendix A.2.8. As the columns of W contain the set of left eigenvectors of A_i, we can write

$$W^*A_i = \Lambda_iW^* \Rightarrow R_1^*Q_1^*A_i = \Lambda_iR_1^*Q_1^* \Rightarrow Q_1^*A_iQ_1 = R_1^{-*}\Lambda_iR_1^*.$$

Since Λ_i is diagonal and its main diagonal elements have negative real parts, I_m is a common Lyapunov solution for the Λ_i's for all $i \in \mathcal{L}$. Then, referring to Lemma 4.2.1, $R_1^{-*}\Lambda_iR_1^*$ also share a Lyapunov solution in the form of $\mathcal{P}_1 = R_1R_1^*$ for all $i \in \mathcal{L}$. On the other hand, it follows from

$$W^*V = 0 \Rightarrow R_1^*Q_1^*V = 0 \Rightarrow Q_1^*V = 0, \tag{4.11}$$

that the matrix $T = [Q_1 \; V]$ is an orthonormal matrix, *i.e.*, $T^*T = I_n$. Considering the transformed matrices $\bar{A}_i := T^* A_i T$, each $A_i \in \mathcal{A}$ is similar to

$$\bar{A}_i = \begin{bmatrix} R_1^{-*} \Lambda_i R_1^* & 0 \\ X_i & L_i \end{bmatrix},$$

where $X_i := V^* A_i Q_1 \in \mathbb{C}^{(n-m)\times m}$. It is inferred from Lemma 4.2.1 that A_i share a Lyapunov solution if and only if $\bar{A}_i$ share a Lyapunov solution for all $i \in \mathcal{L}$. Next, we show that $\bar{A}_i$ share a Lyapunov solution in the form of

$$\bar{P} = \tfrac{1}{2} \begin{bmatrix} R_1 R_1^* & 0 \\ 0 & \epsilon \mathcal{P} \end{bmatrix}, \tag{4.12}$$

where $\mathcal{P}$ satisfies the inequality (4.9) and $\epsilon > 0$ is a scalar which must be properly selected. In fact, we have

$$\bar{A}_i^* \bar{P} + \bar{P} \bar{A}_i = \begin{bmatrix} R_1 \mathrm{Re}(\Lambda_i) R_1^* & \frac{\epsilon}{2} X_i^* \mathcal{P} \\ \frac{\epsilon}{2} \mathcal{P} X_i & \frac{\epsilon}{2}(\mathcal{P} L_i + L_i^* \mathcal{P}) \end{bmatrix}. \tag{4.13}$$

Note that $R_1 \mathrm{Re}(\Lambda_i) R_1^* < 0$ because all $A_i \in \mathcal{A}$ are individually Hurwitz. Thus, the matrix in (4.13) is negative definite if and only if its Schur complement with respect to the upper left block, denoted by $\mathcal{S}_i$, is negative definite; see Appendix A.2.2. The Schur complement,

$$\mathcal{S}_i = \tfrac{\epsilon}{2}(\mathcal{P} L_i + L_i^* \mathcal{P}) - \tfrac{\epsilon^2}{4} \mathcal{P} X_i \, [R_1 \mathrm{Re}(\Lambda_i) R_1^*]^{-1} X_i^* \mathcal{P} \qquad \forall i \in \mathcal{L}, \tag{4.14}$$

is negative definite for small ϵ, for instance with

$$\epsilon = \min_{i \in \mathcal{L}} \left(\frac{\lambda_{\max}(\mathcal{P} L_i + L_i^* \mathcal{P})}{\lambda_{\min}(\mathcal{P} X_i \, [R_1 \mathrm{Re}(\Lambda_i) R_1^*]^{-1} X_i^* \mathcal{P})} \right). \tag{4.15}$$

Thus, referring to Lemma 4.2.1 the existence of a common Lyapunov solution for $A_i \in \mathcal{A}$ is guaranteed if we pick $P = P^\top = \mathrm{Re}(T\bar{P}T^*)$. Now, substituting $\bar{P}$ from (4.12), we get

$$T\bar{P}T^* = \tfrac{1}{2} \, [Q_1 \; V] \begin{bmatrix} R_1 R_1^* & 0 \\ 0 & \epsilon \mathcal{P} \end{bmatrix} \begin{bmatrix} Q_1^* \\ V^* \end{bmatrix} = \tfrac{1}{2} \, (WW^* + \epsilon V \mathcal{P} V^*).$$

This means that the Lyapunov function has the form $V(x) = x^\top P x$, where

$$P = \tfrac{1}{2} \, \mathrm{Re}\,(WW^* + \epsilon V \mathcal{P} V^*). \tag{4.16}$$

A different statement and proof of Theorem 4.2.1 has been presented in Bajcinca et al. (2013). □

4.2.1.1 Stability with $(n-1)$ common right eigenvectors

Now, consider that all matrices $A_i \in \mathcal{A}$ in the switched system (4.1) are Hurwitz and share $(n-1)$ linearly independent right eigenvectors given by the columns of $V =$

$[v_1 \dots v_{n-1}] \in \mathbb{C}^{n\times(n-1)}$ corresponding to $(n-1)$ eigenvalues given by the diagonal entries of $\Lambda_i = \mathrm{diag}([\lambda_{i2}, \dots, \lambda_{in}])$, satisfying

$$A_i V = V\Lambda_i \qquad \forall i \in \mathcal{L}. \tag{4.17}$$

Then, they share a left eigenvector w corresponding to the eigenvalues λ_{i1} with $\mathrm{Re}(\lambda_{i1}) < 0$ for all $i \in \mathcal{L}$, satisfying the property

$$w^* V = 0.$$

Let us define the reduced QR-factorization of V as

$$V = Q_2 R_2,$$

where $R_2 \in \mathbb{C}^{(n-1)\times(n-1)}$ is an upper triangular non-singular matrix and $Q_2 \in \mathbb{C}^{n\times(n-1)}$ has orthonormal columns, that is, $Q_2^* Q_2 = I_{n-1}$. As the columns of V contain a set of eigenvectors of $A_i \in \mathcal{A}$, we have

$$A_i V = V\Lambda_i \Rightarrow A_i Q_2 R_2 = Q_2 R_2 \Lambda_i \Rightarrow A_i Q_2 = Q_2(R_2 \Lambda_i R_2^{-1}).$$

Note again that Λ_i to be diagonal with the entries located in the open left half plane implies that I_{n-1} is a common Lyapunov solution for Λ_i for each $i \in \mathcal{L}$. Then, referring to Lemma 4.2.1, $R_2\Lambda_i R_2^{-1}$ also share a Lyapunov solution in the form of $\mathcal{P} = R_2^{-*} R_2^{-1}$ for all $i \in \mathcal{L}$. Therefore, referring to Theorem 4.2.1, the switched system (4.1) has a CQLF in the form of $V(x) = x^\top P x$ with

$$\begin{aligned} P = \tfrac{1}{2}\ \mathrm{Re}\left(ww^* + \epsilon Q_2\ R_2^{-*} R_2^{-1} Q_2^*\right) = \\ = \tfrac{1}{2}\ \mathrm{Re}\left(ww^* + \epsilon V \left[R_2^{-1} R_2^{-*} R_2^{-1} R_2^{-*}\right] V^*\right), \end{aligned} \tag{4.18}$$

where ϵ is a scalar which can be computed with regard to (4.15).

Quadratic stability of a switched linear system when all its constituent matrices are Hurwitz and share $(n-1)$ right eigenvectors has been previously shown by Shorten and Cairbre (2001).

4.2.1.2 Stability with $(n-1)$ common real left eigenvectors

Now, consider that all A_i's in the switched system (4.1) share $(n-1)$ linearly independent left eigenvectors $w_j \in \mathbb{R}^n$ for all $j \in \{1, \dots, n-1\}$ corresponding to the $(n-1)$ eigenvalues $\lambda_{i1}, \dots, \lambda_{i(n-1)}$ for $i \in \mathcal{L}$. Then, they share a right eigenvector v_n corresponding to the n-th eigenvalues λ_{in} for all $i \in \mathcal{L}$, satisfying the property

$$w_j^\top v_n = 0 \qquad \forall j \in \{1, \dots, n-1\}.$$

Now we plan to compute the analytical expression of the solutions of switched system (4.1) given in (4.2) with respect to its eigenvalues and eigenvectors; see also Kouhi and Bajcinca (2011c). To this end, consider the eigenvalue decomposition

$$A_i = V_i \hat{\Lambda}_i W_i^\top \quad \forall i \in \mathcal{L}, \tag{4.19}$$

where $\hat{\Lambda}_i = \mathrm{diag}([\lambda_{i1}, \dots, \lambda_{in}])$, $V_i = [v_{i1}\ \dots\ v_{i(n-1)}\ v_n]$, and $W_i = [w_1\ \dots\ w_{n-1}\ w_{in}]$. We also assume that V_i and W_i have normal columns. Referring to our discussion in

Section 4.2, the diagonal entries of $e^{\hat{\Lambda}_{\sigma_j}}$ are eigenvalues of $e^{A_{\sigma_j}}$ for any $j \in \{0, \dots, k\}$. Thus, we can write

$$e^{A_{\sigma_j}(t_{j+1}-t_j)} = V_{\sigma_j} e^{\hat{\Lambda}_{\sigma_j}(t_{j+1}-t_j)} W_{\sigma_j}^\top,$$

where by definition of eigenvectors, the following relationships must hold

$$w_r^\top v_{\sigma_j s} = 0,\ w_r^\top v_{\sigma_j r} = 1,\ w_r^\top v_n = 0,\ w_{\sigma_j n}^\top v_n = 1 \qquad \forall r, s \in \{1, \dots, n-1\},\ r \neq s.$$

Then, we have

$$W_{\sigma_j}^\top V_{\sigma_{j-1}} = \begin{bmatrix} w_1^\top \\ \vdots \\ w_{n-1}^\top \\ w_{\sigma_j n}^\top \end{bmatrix} \begin{bmatrix} v_{\sigma_{j-1}1} & \dots & v_{\sigma_{j-1}(n-1)} & v_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \\ w_{\sigma_j n}^\top v_{\sigma_{j-1}1} & w_{\sigma_j n}^\top v_{\sigma_{j-1}2} & \dots & w_{\sigma_j n}^\top v_{\sigma_{j-1}(n-1)} & 1 \end{bmatrix}.$$

Defining $\theta_{jq} := \lambda_{\sigma_j q}(t_{j+1} - t_j)$ for each $q \in \{1, \dots, n\}$ to simplify the notation, a part of the expression (4.2) is computed as follows:

$$e^{A_{\sigma_j}(t_{j+1}-t_j)} e^{A_{\sigma_{j-1}}(t_j-t_{j-1})} = \left(V_{\sigma_j} e^{\hat{\Lambda}_{\sigma_j}(t_{j+1}-t_j)} W_{\sigma_j}^\top\right)\left(V_{\sigma_{j-1}} e^{\hat{\Lambda}_{\sigma_{j-1}}(t_j-t_{j-1})} W_{\sigma_{j-1}}^\top\right) = \tag{4.20}$$

$$= V_{\sigma_j} \begin{bmatrix} e^{\theta_{j1}} & 0 & \dots & 0 & 0 \\ 0 & e^{\theta_{j2}} & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & e^{\theta_{j(n-1)}} & 0 \\ 0 & 0 & \dots & 0 & e^{\theta_{jn}} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \\ w_{\sigma_j n}^\top v_{\sigma_{j-1}1} & w_{\sigma_j n}^\top v_{\sigma_{j-1}2} & \dots & w_{\sigma_j n}^\top v_{\sigma_{j-1}(n-1)} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} e^{\theta_{(j-1)1}} & 0 & \dots & 0 & 0 \\ 0 & e^{\theta_{(j-1)2}} & \dots & 0 & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & \dots & e^{\theta_{(j-1)(n-1)}} & 0 \\ 0 & 0 & \dots & 0 & e^{\theta_{(j-1)n}} \end{bmatrix} W_{\sigma_{j-1}}^\top$$

$$= V_{\sigma_j} \begin{bmatrix} e^{\theta_{j1}+\theta_{(j-1)1}} & \dots & 0 & 0 \\ \vdots & & & \\ 0 & \dots & e^{\theta_{j(n-1)}+\theta_{(j-1)(n-1)}} & 0 \\ e^{\theta_{jn}+\theta_{(j-1)1}} w_{\sigma_j n}^\top v_{\sigma_{j-1}1} & \dots & e^{\theta_{jn}+\theta_{(j-1)(n-1)}} w_{\sigma_j n}^\top v_{\sigma_{j-1}(n-1)} & e^{\theta_{jn}+\theta_{(j-1)n}} \end{bmatrix} W_{\sigma_{j-1}}^\top.$$

Note that for consistency of notation, we assume $t_{k+1} := t$. Now, referring to (4.2) and (4.19) the analytical solution of the switched linear system (4.1) equals

$$\begin{aligned} x(t) &= \left(V_{\sigma_k} e^{\hat{\Lambda}_{\sigma_k}(t-t_k)} W_{\sigma_k}^\top\right)\left(V_{\sigma_{k-1}} e^{\hat{\Lambda}_{\sigma_{k-1}}(t_k-t_{k-1})} W_{\sigma_{k-1}}^\top\right)\cdots\left(V_{\sigma_0} e^{\hat{\Lambda}_{\sigma_0}(t_1-t_0)} W_{\sigma_0}^\top\right)x_0 \\ &= V_{\sigma_k}\left(e^{\hat{\Lambda}_{\sigma_k}(t-t_k)} W_{\sigma_k}^\top V_{\sigma_{k-1}} e^{\hat{\Lambda}_{\sigma_{k-1}}(t_k-t_{k-1})} W_{\sigma_{k-1}}^\top \dots V_{\sigma_0} e^{\hat{\Lambda}_{\sigma_0}(t_1-t_0)}\right)\left(W_{\sigma_0}^\top x_0\right). \end{aligned} \tag{4.21}$$

Now, recalling (4.20), and considering the following computation

$$U := e^{\hat{\Lambda}_{\sigma_k}(t-t_k)} W_{\sigma_k}^\top V_{\sigma_{k-1}} e^{\hat{\Lambda}_{\sigma_{k-1}}(t_k-t_{k-1})} W_{\sigma_{k-1}}^\top \cdots V_{\sigma_0} e^{\hat{\Lambda}_{\sigma_0}(t_1-t_0)}$$

$$= \begin{bmatrix} e^{\theta_{k1}+\dots+\theta_{01}} & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & e^{\theta_{k(n-1)}+\dots+\theta_{0(n-1)}} & 0 \\ \kappa_1(t) & \dots & \kappa_{n-1}(t) & e^{\theta_{kn}+\dots+\theta_{0n}} \end{bmatrix},$$

where

$$\kappa_r(t) = \sum_{p=1}^{k} e^{\theta_{kn}+\dots+\theta_{(k-p+1)n}+\theta_{(k-p)r}+\dots+\theta_{0r}} (w_{\sigma_{k-p+1}n}^\top v_{\sigma_{k-p}r}) \qquad \forall r \in \{1,\dots,n-1\},$$

the analytical solution (4.21) reads

$$x(t) = \begin{bmatrix} v_{\sigma_k 1} & \dots & v_{\sigma_k(n-1)} & v_n \end{bmatrix} U \begin{bmatrix} w_1^\top x_0 \\ \vdots \\ w_{n-1}^\top x_0 \\ w_{\sigma_0 n}^\top x_0 \end{bmatrix}.$$

This leads to the more explicit form of the solution, given by

$$\begin{aligned} x(t) = &\sum_{r=1}^{n-1} (w_r^\top x_0)\, e^{\theta_{kr}+\dots+\theta_{0r}} \quad v_{\sigma_k r} + \\ &+ \left(\sum_{r=1}^{n-1} (w_r^\top x_0)\kappa_r(t) + e^{\theta_{kn}+\dots+\theta_{0n}} \left(w_{\sigma_0 n}^\top x_0\right)\right) v_n. \end{aligned} \tag{4.22}$$

The expression (4.22) reveals that the shape of the solution is determined by the right eigenvectors of the k-th switching mode, while the speed of the solution is governed by the eigenvalues of all switching modes. Further, we show that when all A_i are Hurwitz, the solution $x(t)$ eventually converges exponentially to zero. Defining

$$\lambda_0 = \max_{i \in \mathcal{L}} \lambda(A_i), \quad \Delta t = \min_{j \in \{0,\dots,k-1\}} (t_{j+1} - t_j),$$

where $k \in \mathbb{N}$, and considering that all eigenvalues of $A_i \in \mathcal{A}$ are real negative, and the fact that $\Delta t > 0$, we can derive the following upper bound for $\kappa_r(t)$

$$\kappa_r(t) \le e^{\lambda_0(k\Delta t + (t-t_k))} \sum_{j=1}^{k} w_{\sigma_{k-j+1}n}^\top v_{\sigma_{k-j}r} \le e^{\lambda_0(k\Delta t+(t-t_k))} k \qquad \forall r \in \{1,\dots,n-1\}.$$

Define $\bar{t} = k\Delta t + (t - t_k)$, and notice $t \to \infty$ if and only if $\bar{t} \to \infty$. As $\lim_{\bar{t}\to\infty} e^{\lambda_0 \bar{t}} k = 0$, we must have

$$\lim_{t\to\infty} k_r(t) = 0 \qquad \forall r \in \{1, \dots, n-1\}.$$

This implies that the solution (4.22) converges exponentially to zero if all matrices $A_i \in \mathcal{A}$ are Hurwitz, that is,

$$\lim_{t\to\infty} x(t) = 0.$$

The proof of exponential stability for this switched linear system can alternatively be achieved, by the result of Theorem 4.2.1. Note that a switched linear system with $(n-1)$ real common left eigenvectors share basically a real right eigenvector which defines a common invariant subspace with dimension 1. Then, it immediately follows from Theorem 4.2.1 that the switched system (4.1) is quadratically stable.

Now, we compute a CQLF for this class of switched linear systems. We follow the proof of Theorem 4.2.1. Let us denote the normal real common eigenvector of $A_i \in \mathcal{A}$ corresponding to λ_{in} by v_n, and the reduced QR-factorization of $W := [w_1 \ \dots \ w_{n-1}]$ as $W = Q_1 R_1$, where $R_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ is invertible and $Q_1 \in \mathbb{R}^{n\times(n-1)}$ has orthonormal columns. Now, notice in this case, for using Theorem 4.2.1 we should assume $L_i = \lambda_{in}$, $V = v_n$, and $\Lambda_i = \operatorname{diag}([\lambda_{i1}, \dots, \lambda_{i(n-1)}])$. Let us define the orthonormal transformation matrix $T = [Q_1 \ v_n]$ and the transformed matrices $\bar{A}_i := T^\top A_i T$ for $i \in \mathcal{L}$. By definition, each A_i is similar to

$$\bar{A}_i = \begin{bmatrix} R_1^{-\top} \Lambda_i R_1^\top & 0 \\ v_n^\top A_i Q_1 & \lambda_{in} \end{bmatrix}.$$

Next, we argue that $\bar{A}_i$ share a Lyapunov solution in the form of

$$\bar{P} = \tfrac{1}{2} \begin{bmatrix} R_1 R_1^\top & 0 \\ 0 & \epsilon \end{bmatrix}, \tag{4.23}$$

where $\epsilon > 0$ is a scalar. To this end, the Lyapunov equation for each $\bar{A}_i$ equals:

$$\bar{A}_i^\top \bar{P} + \bar{P}\bar{A}_i = \begin{bmatrix} R_1 \Lambda_i R_1^\top & \frac{\epsilon}{2}\left(v_n^\top A_i Q_1\right)^\top \\ \frac{\epsilon}{2} v_n^\top A_i Q_1 & \epsilon\lambda_{in} \end{bmatrix}. \tag{4.24}$$

The Schur complement of (4.24) with respect to the block $R_1 \Lambda_i R_1^\top$ reads:

$$\mathcal{S}_i = \epsilon\lambda_{in} - \tfrac{\epsilon^2}{4}(v_n^\top A_i Q_1)(R_1 \Lambda_i R_1^\top)^{-1}(v_n^\top A_i Q_1)^\top \qquad \forall i \in \mathcal{L}.$$

Obviously, $\mathcal{S}_i$ is negative definite for small ϵ. For instance, with regard to (4.15) we can select

$$\epsilon = \min_{i\in\mathcal{L}} \left(\frac{2\lambda_{in}}{(v_n^\top A_i Q_1)(R_1 \Lambda_i R_1^\top)^{-1}(v_n^\top A_i Q_1)^\top} \right).$$

Thus, referring to Lemma 4.2.1, we can choose a CQLF as $V(x) = x^\top P x$ with $P = P^\top = T\bar{P}T^\top$. Now, substituting $\bar{P}$ from (4.23), we get

$$P = T\bar{P}T^\top = \tfrac{1}{2}\,[Q_1 \ v_n] \begin{bmatrix} R_1 R_1^\top & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} Q_1^\top \\ v_n^\top \end{bmatrix} = \tfrac{1}{2}\,(WW^\top + \epsilon v_n v_n^\top).$$

This means that the Lyapunov function has the form

$$V(x) = \tfrac{1}{2}\, x^\top \left(WW^\top + \epsilon v_n v_n^\top\right) x = \tfrac{1}{2}\, \sum_{i=1}^{n-1}(w_i^\top x)^2 + \tfrac{1}{2}\, \epsilon(v_n^\top x)^2. \tag{4.25}$$

This Lyapunov function is consistent with the one introduced in (Kouhi and Bajcinca, 2011c) which was derived by using a different approach. The particular case when the dimensions of A_i are two, that is, $n = 2$, has been already studied in Example 3.2.3.

4.3 Robust Stability of switched linear systems

Consider again the switched linear system (4.1). This system features a discontinuous vector field $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ in the form of

$$\dot{x} = A_{\sigma(t)} x =: f(t, x).$$

Such differential equations can be approximated by a convexified differential inclusion

$$\Sigma_F :\ \dot{x} \in F(x) := \overline{\text{co}}\,\{Ax : A \in \mathcal{A}\} = \sum_{i=1}^{\ell} \gamma_i\ A_i x, \tag{4.26}$$

where

$$\sum_{i=1}^{\ell} \gamma_i = 1, \quad \gamma_i \geq 0 \quad \forall i \in \mathcal{L},$$

and $\overline{\text{co}}$ stands for the convex hull (Kouhi and Bajcinca, 2011c; Shorten et al., 2007). Then, the solution (4.2) is contained in the set of Caratheodory solutions of the linear differential inclusions (4.26). The set-valued map $F(x)$ in (4.26) satisfies the so called "basic conditions" as it is outer semi-continuous on $\mathbb{R}^n$, and at any $x \in \mathbb{R}^n$, $F(x)$ is compact and convex; see Teel and Praly (2000) and Appendix A.7. Now, we introduce the notion of robust stability for switched linear system (4.1) based on the definition of robust stability for differential inclusion (4.26).

Definition 4.3.1. (Teel and Praly, 2000; Cai et al., 2007; Clarke et al., 1997), A smooth function $V(x)$ is a robust Lyapunov function for the differential inclusion (4.26) if there exist two increasing positive definite functions $\alpha_1, \alpha_2 : [0, \infty) \to [0, \infty)$ satisfying

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{4.27}$$

$$\max_{f \in F(x)} \langle \nabla V(x), f \rangle \leq -V(x). \tag{4.28}$$

Definition 4.3.2. (Teel and Praly, 2000) The differential inclusion $\dot{x} \in F(x)$ is said to be robustly asymptotically stable if a continuous perturbation function $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $\delta(x) > 0$ for $x \in \mathbb{R}^n \backslash \{0\}$ exists, such that the perturbed differential inclusion

$$\dot{x} \in F_{\delta(x)}(x) := \overline{\text{co}}\ F\left(x + \delta(x)\mathbb{B}\right) + \delta(x)\mathbb{B} \tag{4.29}$$

is asymptotically stable, where $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$.

In Teel and Praly (2000) it has been shown that, in general, robust asymptotic stability of a differential inclusion with F satisfying the basic conditions is equivalent to the existence of a robust Lyapunov function. Specifically, it turns out that the existence of a robust Lyapunov function is a sufficient and necessary condition for robust exponential stability of (4.26); see also Appendix A.7.1.

Now, we aim at establishing conditions under which quadratic stability of a switched linear system implies its robust stability in the sense of Definition 4.3.1 and Definition 4.3.2. Let us assume a CQLF, $V(x) = x^\top Px$ with $P = P^\top > 0$, for the switched system (4.1) exists such that the condition (4.28) also holds. This means

$$A_i^\top P + PA_i + P \le 0,$$

or, equivalently

$$\left(A_i + \tfrac{1}{2}I\right)^\top P + P\left(A_i + \tfrac{1}{2}I\right) \le 0 \qquad \forall i \in \mathcal{L}. \tag{4.30}$$

This implies that all matrices $A_i + \frac{1}{2}I$ share a weak Lyapunov solution. Furthermore, we claim that (4.30) is sufficient for robust stability of the differential inclusion (4.26), and the condition (4.27) automatically holds when the Lyapunov function is quadratic. Indeed, for the CQLF $V(x) = x^\top Px$, convenient functions α_1 and α_2 are

$$\alpha_1(\|x\|) = \lambda_{\min}(P)\|x\|^2, \;\; \alpha_2(\|x\|) = \lambda_{\max}(P)\|x\|^2. \tag{4.31}$$

Then, from the results by Teel and Praly (2000), the differential inclusion (4.26) is robustly exponentially stable, provided that (4.30) holds.

Although the result by Teel and Praly (2000) provides simple criteria for robust stability of differential inclusion (4.26), the perturbation bound δ, introduced by Definition 4.3.2, will be characterized by the eigenvalues of P. Although many articles concerning the computations of upper and lower bounds for eigenvalues of a Lyapunov matrix P exists in the literature, this problem still undergoes more research (Lee, 1997). That means we are not able to describe δ explicitly by the system parameters. Thus, we do not involve ourselves for computing such a perturbation function in the general case. Alternatively, we consider a special case of switched systems, where all constituents $A_i \in \mathcal{A}$ share $(n-1)$ real left eigenvectors. For such a system, the structure of the Lyapunov function allows us to find a perturbation bound $\delta(x)$ which is only represented by the system parameters.

4.3.1 Robust stability with $(n-1)$ common real left eigenvectors

We have shown so far, that the switched system (4.1) is quadratically stable if all matrices $A_i \in \mathcal{A}$ are Hurwitz and share $(n-1)$ real left eigenvectors. In this part, we wish to establish conditions under which this class of switched linear systems is robustly exponentially stable. The next theorem illustrates this result.

Theorem 4.3.1. *(Kouhi and Bajcinca, 2011c) Consider the switched system* (4.1)*. Let all $A_i \in \mathcal{A}$ be Hurwitz, and share $(n-1)$ real linearly independent left eigenvectors given* *by the columns of $W = [w_1 \; \dots \; w_{n-1}]$, such that $w_j^\top w_j = 1$ for all $j \in \{1, \dots, n-1\}$. Then, for a continuous perturbation function $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the convexified linear differential inclusion represented by (4.26) is robustly exponentially stable if the eigenvalues of A_i satisfy: $\lambda_{ij} < -\frac{1}{2}$ for all $i \in \mathcal{L}$ and $j \in \{1, \dots, n\}$.*

Proof: In Section 4.2.1.2 we showed that for a sufficiently small ϵ, $V(x) = x^\top P x$ with $P = \frac{1}{2}WW^\top + \frac{1}{2}\epsilon v_n v_n^\top$ is a CQLF for the switched linear system (4.1). In other words, the inequalities $A_i^\top P + PA_i < 0$ hold for all $i \in \mathcal{L}$. Employing the same structure of P, we need again to show that an ϵ exists, such that the differential inclusion (4.26) is exponentially stable. To this end, we can basically follow the same lines of the previous arguments, by taking into account that

$$\lambda\left(A_i + \tfrac{1}{2}I\right) = \lambda(A_i) + \tfrac{1}{2}, \tag{4.32}$$

and the eigenvectors of A_i and $A_i + \frac{1}{2}I$ are equal. Consequently, it turns out that by referring to (4.15) and choosing

$$\epsilon < \min_{i \in \mathcal{L}} \left(\frac{2(\lambda_{in} + \frac{1}{2})}{(v_n^\top A_i Q_1)(R_1(\Lambda_i + \frac{1}{2}I_{n-1})R_1^\top)^{-1}(v_n^\top A_i Q_1)^\top} \right), \tag{4.33}$$

$V(x)$ is a robust Lyapunov function for the differential inclusion (4.26).

Note that $P = \frac{1}{2}WW^\top + \frac{1}{2}\epsilon v_n v_n^\top$ has one eigenvalue equal $\frac{1}{2}\epsilon$ and its other eigenvalues are eigenvalues of $\frac{1}{2}W^\top W$. This can be verified by observing that

$$\begin{aligned} P\; v_n &= \tfrac{1}{2}\epsilon v_n, \\ P\; W &= W(\tfrac{1}{2}W^\top W). \end{aligned}$$

The first relationship implies that v_n is the eigenvector of P corresponding to the eigenvalue $\frac{1}{2}\epsilon$, and the second one implies that P has an $(n-1)$ dimensional right invariant subspace with columns of W as its basis. Thus, it follows from this relation that $(n-1)$ eigenvalues of P are embedded in $\frac{1}{2}W^\top W$. As W has normalized columns and ϵ can be choose sufficiently small, we can write

$$\lambda_{\min}(P) = \tfrac{1}{2}\epsilon, \text{ and } \lambda_{\max}(P) \leq \text{tr}\,(\tfrac{1}{2}W^\top W) = \tfrac{1}{2}(n-1); \tag{4.34}$$

see also Appendix A.2.11. Now, we propose an upper bound for the uncertainty function $\delta(x)$ such that despite its presence exponential stability of the perturbed differential inclusion defined by (4.29) is ensured. Let's denote the maximum singular value of A_i by $\sigma_{\max}(A_i)$ and introduce Lipschitz functions $\delta_i(x)$ for all $i \in \mathcal{L}$ as

$$\delta_i(x) = \psi_i.\|x\|, \text{ where } \psi_i := \frac{\epsilon}{4(n-1)(\sigma_{\max}(A_i)+1)}. \tag{4.35}$$

We then choose the function $\delta(x)$ in (4.29) as

$$\delta(x) = \psi.\|x\|, \quad \text{where } \; \psi = \min\{\psi_1, \dots, \psi_\ell\}. \tag{4.36}$$

The perturbed set valued map defined in (4.29) equals

$$F_{\delta(x)}(x) = \left\{ \sum_{i=1}^{\ell} \gamma_i A_i x + \psi.\|x\| \left(\left(\sum_{i=1}^{\ell} .\gamma_i A_i \right) + I \right) \mathbb{B} \right\}, \quad \sum_{i=1}^{\ell} \gamma_i = 1,\ \gamma_i \geq 0 \ \ \forall i \in \mathcal{L}.$$

Considering (4.34), (4.30), and the definition of ψ_i in (4.35), for $x \neq 0$ we have

$$\begin{aligned}
\max_{f \in F_\delta(x)} \langle \nabla V(x), f \rangle &= \max_{f \in F_\delta(x)} \langle 2x^\top P, f \rangle \\
&= \max_{v \in \mathbb{B}} \left(\sum_{i=1}^{\ell} \left(\gamma_i x^\top (A_i^\top P + PA_i) x \right) + 2 \sum_{i=1}^{\ell} \left(\gamma_i.\psi.\|x\|.x^\top P.(A_i + I)v \right) \right) \\
&\leq -x^\top P x + 2 \max_{v \in \mathbb{B}} \left(\sum_{i=1}^{\ell} \left(\gamma_i \psi_i.\|x\|.\|x^\top P\|.\|(A_i + I)v\| \right) \right) \\
&\leq -x^\top P x + 2 \sum_{i=1}^{\ell} \left(\gamma_i \psi_i.\|x\|.\lambda_{\max}(P).\|x\|.\left(\sigma_{\max}(A_i) + 1\right) \right) \\
&\leq -x^\top P x + \frac{1}{4} \epsilon.\|x\|.\|x\| \\
&\leq -\frac{1}{2} x^\top P x < 0. \qquad (4.37)
\end{aligned}$$

Furthermore, it follows from (4.37) that

$$\dot{V}(x) \leq -\tfrac{1}{2} x^\top P x = -\tfrac{1}{2} V(x) \Rightarrow V(x) \leq V(x_0) e^{-\frac{1}{2}t}.$$

Now, using the inequality $\lambda_{\min}(P)\|x\|^2 \leq x^\top P x$ and taking into account that $\frac{1}{2}\epsilon = \lambda_{\min}(P)$, we have

$$\|x\|^2 \leq \tfrac{2}{\epsilon} V(x_0) e^{-\frac{1}{2}t}. \qquad (4.38)$$

This is a verification that exponential stability of the perturbed differential inclusion (4.29) is attained. □

Note that for this class of switched linear systems the robust Lyapunov function and the upper bound for the parameter ϵ are explicitly represented by the system parameters.

4.4 Quadratic stabilization of switched linear systems

In the second part of the current chapter, we investigate stabilization of controlled switched linear systems by exploiting the developed concepts in the first part. We consider the controlled switched linear system

$$\dot{x} = A_{\sigma(t)} x + B_{\sigma(t)} u, \qquad (4.39)$$

where $\sigma : t \to \mathcal{L} := \{1, \ldots, \ell\}$ is a piecewise constant switching signal, $A_i \in \mathbb{R}^{n\times n}$, and we assume $B_i \in \mathbb{R}^{n\times m}$ has full column rank for $m \leq n$ and for each $i \in \mathcal{L}$. Our goal is to design local state feedbacks $u = K_{\sigma(t)}x$ such that the switched system

$$\dot{x} = (A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)})x, \tag{4.40}$$

is quadratically stable under arbitrary switching signal, *i.e.*, a common quadratic Lyapunov function (CQLF) $V(x) = x^\top Px$ with $P = P^\top > 0$ exists such that

$$(A_i + B_iK_i)^\top P + P(A_i + B_iK_i) < 0 \qquad \forall i \in \mathcal{L}. \tag{4.41}$$

To develop our results concerning this problem, in the next step, we introduce the concept of controlled block similar matrices for controlled switched linear systems.

4.4.1 Block similar controlled switched linear systems

In this part, we introduce the concept of similarity for controlled switched linear systems. For the definition of block similar controlled systems, see Appendix A.3.2 and the reference therein.

Lemma 4.4.1. *Suppose in the controlled switched linear system* (4.39)*, each controlled pair* $[A_i\ B_i]$ *is block similar to* $[\bar{A}_i\ \bar{B}_i]$ *for* $i \in \mathcal{L}$*, related by a real similarity transformation matrix. Then, there exist real state feedback gains* $K_i \in \mathbb{R}^{m\times n}$ *and a real matrix* $P = P^\top > 0$ *such that the closed loop matrices* $A_i + B_iK_i$ *satisfy the Lyapunov inequalities* (4.41)*, if and only if there exist feedback gains* $\bar{K}_i \in \mathbb{R}^{m\times n}$ *and a real positive definite matrix* $\bar{P} = \bar{P}^\top > 0$ *such that* $(\bar{A}_i + \bar{B}_i\bar{K}_i)^\top \bar{P} + \bar{P}(\bar{A}_i + \bar{B}_i\bar{K}_i) < 0$ *for all* $i \in \mathcal{L}$.

Proof of necessity: Since the controlled pairs $[A_i\ B_i]$ are block similar to $[\bar{A}_i\ \bar{B}_i]$ for each $i \in \mathcal{L}$ obtained by a real similarity transformation matrix, there exist a matrix L, and invertible matrices N and M with appropriate dimensions such that

$$[\bar{A}_i\ \bar{B}_i] = N^{-1}[A_i\ B_i]\begin{bmatrix} N & 0 \\ L & M \end{bmatrix}$$

$$\Rightarrow \bar{A}_i = N^{-1}(A_iN + B_iL),\ \bar{B}_i = N^{-1}B_iM, \tag{4.42}$$

see Appendix A.3.2. Now, suppose there exists a feedback gain K_i for each $i \in \mathcal{L}$ and a symmetric positive definite matrix P such that $(A_i + B_iK_i)^\top P + P(A_i + B_iK_i) < 0$. Then, by pre-multiplying of this inequality by $N^\top$ and post-multiplying it by N and utilizing (4.42), we can write

$$\begin{aligned}
&N^\top(A_i + B_iK_i)^\top PN + N^\top P(A_i + B_iK_i)N < 0\\
&\Rightarrow \left[N^\top(A_i + B_iK_i)^\top N^{-\top}\right] N^\top PN + N^\top PN\left[N^{-1}(A_i + B_iK_i)N\right] < 0\\
&\Rightarrow \left[\bar{A}_i + \bar{B}_iM^{-1}(K_i - LN^{-1})N\right]^\top \bar{P} + \bar{P}\left[\bar{A}_i + \bar{B}_iM^{-1}(K_i - LN^{-1})N\right] < 0,
\end{aligned}$$

where $\bar{P} = N^\top PN$. By choosing $\bar{K}_i = M^{-1}(K_i - LN^{-1})N$, we deduce

$$(\bar{A}_i + \bar{B}_i\bar{K}_i)^\top \bar{P} + \bar{P}(\bar{A}_i + \bar{B}_i\bar{K}_i) < 0 \qquad \forall i \in \mathcal{L}. \tag{4.43}$$

Proof of sufficiency: Suppose there exists a positive $\bar{P} > 0$ such that $(\bar{A}_i + \bar{B}_i\bar{K}_i)^\top \bar{P} + \bar{P}(\bar{A}_i + \bar{B}_i\bar{K}_i) < 0$ for each $i \in \mathcal{L}$. Then, by pre-multiplying of this inequality by $N^{-\top}$ and post multiplying it by N^{-1}, the sign of the inequality does not change. Thus, we have

$$\begin{aligned} &N^{-\top}(\bar{A}_i + \bar{B}_i\bar{K}_i)^\top \bar{P}N^{-1} + N^{-\top}\bar{P}(\bar{A}_i + \bar{B}_i\bar{K}_i)N^{-1} < 0 \\ &\qquad \Rightarrow (A_i + B_iM(\bar{K}_i + M^{-1}L)N^{-1})^\top(N^{-\top}\bar{P}N^{-1}) \\ &\qquad\qquad + (N^{-\top}\bar{P}N^{-1})(A_i + B_iM(\bar{K}_i + M^{-1}L)N^{-1}) < 0. \end{aligned}$$

Selecting $K_i = M(\bar{K}_i + M^{-1}L)N^{-1}$, we have

$$(A_i + B_iK_i)^\top(N^{-\top}\bar{P}N^{-1}) + (N^{-\top}\bar{P}N^{-1})(A_i + B_iK_i) < 0. \tag{4.44}$$

Now, take $P = N^{-\top}\bar{P}N^{-1}$. □

Corollary 4.4.1. *If $[A_i\ B_i]$ for all $i \in \mathcal{L}$ are block similar to a real controllable pair $[A\ B]$, with the same invertible transformation matrix, then the switched system* (4.39) *can be quadratically stabilized.*

Proof: It follows from Lemma 4.4.1 that the quadratic stabilization problem for the switched system (4.39) is equivalent to the stabilization problem for the LTI system $\dot{x} = Ax + Bu$. It is well known that any controllable LTI system can be stabilized by designing an appropriate state feedback $u = Kx$. □

4.4.2 Stabilization and common invariant subspaces

In this part, we aim to use the left eigenvectors assignment technique for stabilization of a class of controlled switched linear systems in the form of (4.39). As stated in Chapter 4.4, we use local state feedbacks $u = K_ix$ for this purpose. Then, we choose a matrix $W = [w_1\ \dots\ w_m]$ consisting of m linearly independent columns with complex conjugate pairs to be the set of desired left eigenvectors for $A_{\text{cl}i} = A_i + B_iK_i$. These eigenvectors correspond to m desired eigenvalues with complex conjugate pairs and negative real parts given by the diagonal entries of the matrix $\Lambda = \text{diag}([\lambda_1, \dots, \lambda_m])$. Recalling Chapter 2.2.2 we can adopt the formula (2.25) for this purpose. This formula then reads

$$K_i = -(W^*B_i)^{-1}(W^*A_i - \Lambda W^*) \qquad \forall i \in \mathcal{L}. \tag{4.45}$$

Such feedbacks are feasible if and only if $\det(W^*B_i)$ for each $i \in \mathcal{L}$ is nonzero. Now, suppose we impose an additional restriction on W. We would like W to meet the condition $W^*V = 0$, where $V \in \mathbb{C}^{n\times(n-m)}$ has $(n-m)$ linearly independent columns. This gives rise to the question under which conditions the feedback gain of the form (4.45) is feasible. Lemma 4.4.2 provides an answer to this question.

Lemma 4.4.2. *(Bajcinca et al., 2013) Consider the matrices B_i for all $i \in \mathcal{L}$ in the controlled switched linear system* (4.39)*. Given $V \in \mathbb{C}^{n\times(n-m)}$ with rank $(V) = n-m$, a matrix $W \in \mathbb{C}^{n\times m}$ satisfying det$(W^*B_i) \neq 0$ and $W^*V = 0$ for all $i \in \mathcal{L}$ exists, if and only if there exists a $j \in \mathcal{L}$ such that for all $i \in \mathcal{L}$*

$$rank(B_j^\top \mathcal{Q}_V B_i) = m, \tag{4.46}$$

where $\mathcal{Q}_V := I - V(V^*V)^{-1}V^*$. *Moreover, if* (4.46) *holds then one can choose*

$$W = \mathcal{Q}_V B_j \quad \text{for a fixed } j \in \mathcal{L}. \tag{4.47}$$

Proof of sufficiency: Choose an index $j \in \mathcal{L}$ such that (4.46) holds for all $i \in \mathcal{L}$, and select $W = \mathcal{Q}_V B_j$. Then, $W^*V = 0$ follows immediately from the definition of W. On the other hand, we have $W^*B_i = B_j^\top \mathcal{Q}_V B_i$ for all $i \in \mathcal{L}$. Thus, (4.46) implies $\text{rank}\,(W^*B_i) = m$.

Proof of necessity: Suppose there exists a W such that $W^*V = 0$ and $\text{rank}\,(W^*B_i) = m$ for all $i \in \mathcal{L}$. It follows from $W^*\mathcal{Q}_V B_i = W^*B_i$ that $\text{rank}\,(\mathcal{Q}_V B_i) = m$ for all $i \in \mathcal{L}$. Now, fix an index $j \in \mathcal{L}$. Due to the relationships $W^*V = 0$ and $B_j^\top \mathcal{Q}_V V = 0$, there must exist a non-singular $M \in \mathbb{C}^{m\times m}$ such that $W = \mathcal{Q}_V B_j M$. Consequently, the equality $\text{rank}\,(W^*B_i) = \text{rank}\,(M^*B_j^\top \mathcal{Q}_V B_i) = m$ implies that $\text{rank}\,(B_j^\top \mathcal{Q}_V B_i) = m$ for all $i \in \mathcal{L}$. □

Taking advantage of Lemma 4.4.2, by choosing $W = \mathcal{Q}_V B_j$ the proposed control gains (4.45) will take the form

$$K_i = -(B_j^\top \mathcal{Q}_V B_i)^{-1}(B_j^\top \mathcal{Q}_V A_i - \Lambda B_j^\top \mathcal{Q}_V) \qquad \forall i \in \mathcal{L},\ \exists j \in \mathcal{L}. \tag{4.48}$$

Now, in the following theorem, we intend to express how using the left eigenvectors assignment technique can be beneficial for stabilization of controlled switched linear systems.

Theorem 4.4.1. *Consider the controlled switched linear system defined by* (4.39). *Suppose all* A_i*'s for* $i \in \mathcal{L}$ *share an* $(n-m)$ *dimensional invariant subspace* $\mathcal{X}_{n-m} \subset \mathbb{C}^n$. *Let the columns of* $V \in \mathbb{C}^{n\times(n-m)}$ *with conjugate pairs of complex vectors be an orthonormal basis for* $\mathcal{X}_{n-m}$, *which by definition satisfy the property*

$$A_i V = V L_i \qquad \forall i \in \mathcal{L}, \tag{4.49}$$

where $L_i \in \mathbb{C}^{(n-m)\times(n-m)}$. *Let all systems* $\dot{\bar{x}} = L_i \bar{x}$ *share a CQLF, that is, there exists a* $\mathcal{V}(\bar{x}) = \bar{x}^*\mathcal{P}\bar{x}$ *with* $\mathcal{P} = \mathcal{P}^* > 0$ *and* $\bar{x} \in \mathbb{C}^{n-m}$ *such that the following Lyapunov inequalities hold*

$$L_i^*\mathcal{P} + \mathcal{P}L_i < 0 \qquad \forall\ i \in \mathcal{L}. \tag{4.50}$$

Moreover, assume the rank properties

$$\text{rank}\ (B_j^\top \mathcal{Q}_V B_i) = m \qquad \forall i \in \mathcal{L},\ \exists j \in \mathcal{L}, \tag{4.51}$$

where $\mathcal{Q}_V = I - VV^*$, *hold. Then the switched system* (4.39) *is quadratically stabilizable.*

Proof: Our stabilization design method is based on the left eigenvectors assignment algorithm. We choose a matrix $W = [w_1\ \ldots\ w_m]$ consisting of m linearly independent columns with conjugate pairs of complex vectors and the property $W^*V = 0$, to be the set of desired left eigenvectors. These left eigenvectors correspond to m numbers including complex conjugate pairs and negative real parts given by the diagonal elements of the matrix $\Lambda = \text{diag}([\lambda_1, \ldots, \lambda_m])$ as the desired eigenvalues for the closed loop matrices $A_{\text{cl}i} = A_i + B_i K_i$. The desired feedback gains then have the forms (4.45). Such feedbacks are feasible if and only if $\det(W^* B_i)$ is non-zero for each $i \in \mathcal{L}$. Lemma 4.4.2 emphasizes that feasibility of the feedbacks is equivalent to conditions (4.51) to hold. Moreover, a desired W has the form $W = \mathcal{Q}_V B_j$ for some $j \in \mathcal{L}$, and the desired feedback gains will be computed by (4.48). Making use of these feedback gains, the following properties are established

$$W^* A_{\text{cl}i} = \Lambda W^*, \;\; A_{\text{cl}i} V = V L_i, \;\text{and}\; W^* V = 0. \tag{4.52}$$

Now, referring to Theorem 4.2.1 we can state that the resulting switched system (4.40) is quadratically stable. A similar idea for stabilization of controlled switched system has been presented by Bajcinca et al. (2013). □

Remark 4.4.1. One can combine the statements of Theorem 4.4.1 and Lemma 4.4.1 for weakening the requirement that the open loop matrices A_i must share an invariant subspace. The weaker statement can be said: "Suppose each pair $[A_i \; B_i]$ is block similar to $[\bar{A}_i \; \bar{B}_i]$ related by a same real similarity transformation matrix, for $i \in \mathcal{L}$. If the pairs $(\bar{A}_i, \bar{B}_i)$ for all $i \in \mathcal{L}$ satisfy the criteria of Theorem 4.4.1, then the switched system (4.39) can be stabilized by local state feedbacks design".

Example 4.4.1. (See also Bajcinca et al. (2013)) Consider the controlled switched linear system (4.39) with the data

$$A_1 = \begin{bmatrix} 3 & -3 & -7 \\ 0 & -4 & 0 \\ 1 & 3 & -5 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 3 & -5 \\ 2 & -6 & -2 \\ 7 & 1 & -11 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

A_2 has all eigenvalues with negative real parts, namely at $-8, -4, -4$, while A_1 has two eigenvalues with negative real parts and an eigenvalue in the right half plane, namely at $2, -4, -4$. We intend to stabilize this controlled switched system. Using the algorithm by Tsatsomeros (2001), we can realize that both A_1 and A_2 share a two dimensional invariant subspace with an orthonormal basis given by the columns of $V = [v_1 \; v_2]$, where

$$v_1 = [-0.7071 \;\; 0 \;\; -0.7071]^\top, \; v_2 = [-0.4082 \;\; -0.8165 \;\; 0.4082]^\top,$$

and (4.49) holds. The corresponding matrices L_1 and L_2 defined in accordance with (4.49) equal

$$L_1 = \begin{bmatrix} -4 & 4.6188 \\ 0 & -4 \end{bmatrix}, \; L_2 = \begin{bmatrix} -4 & 9.2376 \\ 0 & -4 \end{bmatrix}.$$

Notice L_1 and L_2 share a Lyapunov solution, for instance, $\mathcal{P} = \text{diag}([1, 3])$. Now, we follow an algorithm represented in the proof of Theorem 4.4.1. We design local feedback laws which assign a common left eigenvector to the closed loop matrices $A_{\text{cl}1}$ and $A_{\text{cl}2}$. This desired left eigenvector resulting from (4.47) is computed to be $W = [-0.5774 \;\; 0.5774 \;\; 0.5774]^\top$. As $W^\top B_1$ and $W^\top B_2$ are both non-zero, assigning W as the desired left eigenvector is possible. Choosing the desired eigenvalue $\Lambda = -2$, the state feedback gains $K_1 = [2 \;\; -2 \;\; -2]$ and $K_2 = [-6 \;\; 6 \;\; 6]$ are computed, leading to the closed loop matrices

$$A_{\text{cl}1} = \begin{bmatrix} 5 & -5 & -9 \\ 2 & -6 & -2 \\ 5 & -1 & -9 \end{bmatrix}, \; A_{\text{cl}2} = \begin{bmatrix} -5 & 9 & 1 \\ 2 & -6 & -2 \\ -5 & 13 & 1 \end{bmatrix}.$$

4.4.2.1 Stabilization for the case of $(n-1)$ dimensional common invariant subspace

Consider the special case of the controlled switched linear system (4.39) where the dimension of the largest common invariant subspace introduced in Theorem 4.4.1 is $(n-1)$ and the system has a single control input. Then, the A_i's for all $i \in \mathcal{L}$ share a real left eigenvector w characterized by

$$w^\top V = 0. \tag{4.53}$$

Note that the open loop matrices are not necessarily Hurwitz, because some eigenvalues of A_i corresponding to the eigenvector w, denoted by μ_{i1} for $i \in \mathcal{L}$, might lie in the right half plane. Therefore, for stabilization of this controlled switched system we prefer to shift the eigenvalues μ_{i1} to a point on the negative real axis, while maintaining the other eigenvalues of the matrices A_i unchanged. For this purpose, we use the single shift eigenvalue method elaborated in Chapter 2.2.1.3. Let us denote the desired eigenvalue corresponding to the left eigenvector w by λ_1, and define the vectors $b_i := B_i$ with $i \in \mathcal{L}$ for keeping the notation consistent with the one previously introduced. Further, assume $w^\top b_i \neq 0$ for all $i \in \mathcal{L}$. The convenient controller gains then turn out to have the forms

$$K_i = -(\mu_{i1} - \lambda_1)\frac{w^\top}{w^\top b_i} \qquad \forall i \in \mathcal{L}. \tag{4.54}$$

Now with this control design, Theorem 4.4.1 ensures that the controlled switched system (4.39) is quadratically stable.

Example 4.4.2. Consider again the controlled switched linear system defined in Example 4.4.1. As explained in that example, both A_1 and A_2 share a 2 dimensional invariant subspace. Consequently, they share also a left eigenvector w orthogonal to v_1 and v_2. This eigenvector is computed as follows:

$$w = [-0.5774 \;\; 0.5774 \;\; 0.5774]^\top.$$

Now, we design local state feedback gains each of which only shifts a single eigenvalue corresponding to the common left eigenvector w of the open loop matrices A_1 and A_2. As $w^\top B_1$ and $w^\top B_2$ are both non-zero, the single shift eigenvalue procedure is possible. Choosing the desired eigenvalue to be $\lambda_1 = -2$, the same controller gains as the ones derived in Example 4.4.1 result.

4.4.2.2 Stabilization based on $(n-1)$ common real left eigenvectors

In this part, we plan to incorporate the algorithm developed in Chapters 4.2.1.2 and 2.2.2.1 into our stabilization approach for stabilization of controlled switched linear systems with $(n-1)$ control inputs. Our algorithm suggests to assign $(n-1)$ real common left eigenvectors given by the columns of $W = [w_1 \dots w_{n-1}] \in \mathbb{R}^{n\times(n-1)}$ corresponding to the desired real stable eigenvalues given by the diagonal elements of $\Lambda = \mathrm{diag}([\lambda_1, \dots, \lambda_{n-1}])$ to the closed loop matrices $A_{\mathrm{cl}i}$ for all $i \in \mathcal{L}$. For the selection of such W, stability of the closed loop matrices must be taken into account. For this purpose, we define new variables $W_1, \dots, W_\ell$, where $W_i = [w_{i1} \dots w_{i(n-1)}]$ for $i \in \mathcal{L}$ are a set of desired left eigenvectors for $A_{\mathrm{cl}i}$, respectively. A common W is obtained whenever $W := W_1 = \dots = W_\ell$.

We now employ feedback gains in the form of (4.45). For stability of each closed loop subsystem, we need that the last eigenvalue of each $A_{\mathrm{cl}i}$ for $i \in \mathcal{L}$, denoted by λ_{in}, should also be real negative. It has been shown in Chapter 2.2.2.1 that this condition is equivalent to

$$\lambda_{in} = -\mathrm{tr}\left(W_i^\top (a_{i1}/(n-1)\ I + A_i) B_i (W_i^\top B_i)^{-1}\right) < 0 \qquad \forall i \in \mathcal{L}. \tag{4.55}$$

We can determine a set of matrices W_i, denoted by Ω_i, that satisfy the inequalities (4.55) by using the parameterization method introduced in Chapter 2.2.2.1

$$W_i^\top = \mathcal{W}_i^\top + \mu_i \theta_i^\top \qquad \forall i \in \mathcal{L},$$

and follow the algorithm proposed therein. Then, the desired set of W, denoted by Ω, that simultaneously stabilizes all $A_{\mathrm{cl}i}$ for all $i \in \mathcal{L}$, is obtained by intersection of all Ω_i, that is, $\Omega = \cap_{i=1}^{\ell} \Omega_i$ (see Kouhi and Bajcinca (2011c)). The next example illustrates how this technique can be numerically executed.

Example 4.4.3. Suppose we intend to quadratically stabilize the controlled switched linear system (4.39), with the data

$$A_1 = \begin{bmatrix} 3 & -3 & -7 \\ 0 & -4 & 0 \\ 1 & 3 & -5 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 3 & -5 \\ 2 & -6 & -2 \\ 7 & 1 & -11 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Let the columns of $W_1 = [w_{11}\ w_{12}]$ and $W_2 = [w_{21}\ w_{22}]$ be desired left eigenvectors for the closed loop matrices $A_{\mathrm{cl}1}$ and $A_{\mathrm{cl}2}$, respectively. Note that two common left eigenvectors given by the columns of $W = [w_1\ w_2]$ are obtained when $W = W_1 = W_2$. Now, we want to compute a matrix W such that the closed loop matrices $A_{\mathrm{cl}1}$ and $A_{\mathrm{cl}2}$ constructed by the state feedback gains in the form of (4.45) are Hurwitz. To this end, we first use the parameterization

$$W_i^\top = \mathcal{W}_i^\top + \mu_i \theta_i^\top \qquad i \in \{1, 2\},$$

where μ_i and θ_i are defined in Chapter 2.2.2.1. The parameters θ_i are computed directly from the equations $\theta_i^\top B_i = 0$ for $i \in \{1, 2\}$

$$\theta_1 = [-0.8944\ \ 0\ \ 0.4472]^\top,\ \theta_2 = [-0.4082\ \ -0.4082\ \ 0.8165]^\top.$$

Note that this problem includes several unknown parameters, namely $\mathcal{W}_1$, $\mathcal{W}_2$, μ_1, and μ_2. Therefore, for convenience we fix $\mathcal{W}_1$ and μ_2 to be

$$\mathcal{W}_1^\top = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}, \quad \mu_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and compute $\mathcal{W}_2$ and μ_1 afterwards. From the identity $W_1^\top = W_2^\top$, we conclude

$$\begin{aligned} \mathcal{W}_1^\top + \mu_1 \theta_1^\top &= \mathcal{W}_2^\top + \mu_2 \theta_2^\top \\ &\Rightarrow \mathcal{W}_2^\top = \mathcal{W}_1^\top + \mu_1 \theta_1^\top - \mu_2 \theta_2^\top. \end{aligned}$$

This allows us to represent $\mathcal{W}_2^\top$ solely by the unknown parameter $\mu_1 = [\mu_{11} \;\; \mu_{12}]^\top$ as

$$\mathcal{W}_2^\top = \begin{bmatrix} 1.4082 - 0.8944\mu_{11} & 1.4082 & 1.1835 + 0.4472\mu_{11} \\ 1.4082 - 0.8944\mu_{12} & 0.4082 & 1.1835 + 0.4472\mu_{12} \end{bmatrix}.$$

Now, recall from the algorithm presented in Chapter 2.2.2.1 that for stability of the closed loop matrices the following conditions must hold

$$-Y_1^\top \mu_1 - \operatorname{tr}(\mathcal{W}_1 X_1) < 0, \qquad -Y_2^\top \mu_2 - \operatorname{tr}(\mathcal{W}_2 X_2) < 0,$$

where variables X_i and Y_i are defined as follows:

$$X_i = \Big(a_{i1}/(n-1)\; I + A_i\Big) B_i (\mathcal{W}_i^\top B_i)^{-1},$$
$$Y_i^\top = \theta_i^\top \Big(a_{i1}/(n-1)\; I + A_i\Big) B_i (\mathcal{W}_i^\top B_i)^{-1},$$

for each $i \in \{1, 2\}$. X_1 and Y_1 are known parameters as $\mathcal{W}_1$ is given

$$X_1^\top = \begin{bmatrix} -3 & -1 & 3 \\ 1.4 & 1 & -3.6 \end{bmatrix}, \quad Y_1^\top = \begin{bmatrix} 4.0249 & -2.8622 \end{bmatrix}.$$

X_2 and Y_2 are parameterized by μ_1. Now, we find an appropriate μ_1 via solving the following optimization problem:

$$\text{minimize} \qquad \|\mu_1\|^2$$

$$\text{subject to} \begin{cases} -Y_1^\top \mu_1 - \operatorname{tr}(\mathcal{W}_1 X_1) + \gamma \le 0, \\ -Y_2^\top \mu_2 - \operatorname{tr}(\mathcal{W}_2 X_2) + \gamma \le 0, \end{cases}$$

where the positive number γ enters into the problem for ruling out computation of the zero solution for μ_1. In this example we assume $\gamma = 0.2$. Using the MATLAB command "fmincon" we are able to find a solution for μ_1, namely $\mu_1 = [0.6600 \;\; -0.4694]^\top$. This identifies

$$W^\top = \begin{bmatrix} 0.4096 & 1 & 2.2952 \\ 1.4198 & 0 & 1.7901 \end{bmatrix}.$$

Now, we take the eigenvalues corresponding to the columns of W to be the diagonal entries of $\Lambda = \operatorname{diag}([-2, -3])$. Thus, the state feedback gains K_1 and K_2 are computed to be:

$$K_1 = \begin{bmatrix} 5.9656 & -2.5458 & -3.7658 \\ -8.0274 & 2.3236 & 6.4696 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -4.5586 & 2.9411 & 6.6617 \\ -0.7727 & -3.3458 & 0.0079 \end{bmatrix},$$

and the closed loop matrices are given by

$$A_{\text{cl1}} = \begin{bmatrix} 0.9382 & -3.2222 & -4.2962 \\ 5.9656 & -6.5458 & -3.7658 \\ -3.1236 & 2.5557 & 0.4075 \end{bmatrix}, \quad A_{\text{cl2}} = \begin{bmatrix} -5.1040 & -0.7506 & 1.6774 \\ -2.5586 & -3.0589 & 4.6617 \\ 1.6687 & 0.5953 & -4.3305 \end{bmatrix}.$$

The sets of eigenvalues of $A_{\rm cl1}$ and $A_{\rm cl2}$ are $\{-0.2, -2, -3\}$ and $\{-7.4933, -2, -3\}$, respectively. This indicates that both matrices are Hurwitz and they share two left eigenvectors given by the columns of the computed W.

Example 4.4.4. (Kouhi and Bajcinca, 2011c) For the case of $n = 2$ in the controlled switched linear system (4.39), we can obtain a simple expressions for stability of the closed loop matrices. Let's denote $b_i := B_i$ for each $i \in \mathcal{L}$, and $w_i := W_i$, where w_i is the desired left eigenvector for the closed loop matrix $A_{{\rm cl}i}$ for each $i \in \mathcal{L}$. We can choose λ_{i1} any real negative number. Furthermore, recalling (2.8), we must have

$$\lambda_{2i} = -\frac{w_i^\top (a_{i1} I + A_i) b_i}{w_i^\top b_i} < 0 \qquad \forall i \in \mathcal{L}. \tag{4.56}$$

The criteria (4.56) are met if the inner-products $w_i^\top b_i$ and $w_i^\top (a_{i1} I + A_i) b_i$ have the same sign. Define Ω_i to be the set of all vectors w_i that satisfy this condition. Then, the set of a stabilizing common left eigenvector $w = w_1 = \cdots = w_\ell$, denoted by Ω, can be computed by $\Omega = \cap_{i=1}^{\ell} \Omega_i$. The region Ω_i can be represented geometrically similar to what were illustrated in Figure 2.2.1.

4.4.3 Stabilization and perturbed invariant subspaces

In this part, we are interested in enlarging the class of controlled switched linear systems that can be stabilized by left eigenstructure assignment approach. The core of this section is to show that if all open loop matrices A_i's with $i \in \mathcal{L}$ in (4.39) have $p := n - m$ dimensional invariant subspaces which are sufficiently close to each other, we may still be capable of stabilizing the controlled switched system. This argument is substantiated by accomplishing the proof of the next theorem. Before proceeding, we refer the reader to Appendix A.2.13.1 and Truhar (1996) for the definition of "distance between two invariant subspaces".

Theorem 4.4.2. *Define $p = n - m$. Suppose all A_i's in the controlled switched system* (4.39) *possess p-dimensional invariant subspaces $\mathcal{X}_{i,p}$, such that the distance between $\mathcal{X}_{i,p}$ and $\mathcal{X}_{j,p}$ is sufficiently small for each $i, j \in \mathcal{L}$ and $i \neq j$, that is, there exists a small number $\delta \geq 0$ satisfying*

$$dist(\mathcal{X}_{i,p}, \mathcal{X}_{j,p}) \leq \delta. \tag{4.57}$$

Moreover, assume that an index $j \in \mathcal{L}$ and an orthonormal basis with conjugate pairs of complex vectors, given by the columns of a matrix $V_j \in \mathbb{C}^{n\times p}$, for the invariant set $\mathcal{X}_{j,p}$ exist such that

$$rank\left(B_j^\top \mathcal{Q}_{V_j} B_i\right) = m \qquad \forall i \in \mathcal{L}, \tag{4.58}$$

where $\mathcal{Q}_{V_j} = I - V_j V_j^*$. For each $i \in \mathcal{L}$ and $i \neq j$, let the columns of $V_i \in \mathbb{C}^{n\times p}$ be a normal basis (not necessarily orthogonal) for $\mathcal{X}_{i,p}$, satisfying

$$A_i V_i = V_i L_i, \quad \|V_i - V_j\|_2 \leq \sqrt{2p \; dist(\mathcal{X}_{i,p}, \mathcal{X}_{j,p})}, \tag{4.59}$$

where $\|.\|_2$ *indicates the induced 2-norm (see Appendix A.2.11). Then, if there exist scalars* $\gamma_1, \gamma_2 > 0$ *and a Hermitian positive definite matrix* $\mathcal{P} \in \mathbb{C}^{p\times p}$*, i.e.,* $\mathcal{P} = \mathcal{P}^* > 0$*, such that the following Riccati inequalities hold*

$$L_i^* \mathcal{P} + \mathcal{P} L_i + \gamma_1^2 \mathcal{P}^2 + \gamma_2^2 I_p < 0 \qquad \forall i \in \mathcal{L}, \tag{4.60}$$

and the number $\gamma_1\gamma_2$ *is sufficiently large, the controlled switched system* (4.39) *can be stabilized by local state feedback controller design.*

Proof: The proof consists of two parts. In part A, we show that a $V_i \in \mathbb{C}^{n\times p}$ exists, such that its columns form a basis for the invariant subspace $\mathcal{X}_{i,p}$ and simultaneously satisfy (4.59). In Part B we demonstrate that if (4.60) holds and $\gamma_1\gamma_2$ is sufficiently large, then the controlled switched linear system (4.39) can be quadratically stabilized.

Part A: Fix a $j \in \mathcal{L}$ and an orthonormal matrix $V := V_j \in \mathbb{C}^{n\times p}$ such that (4.58) holds. Let's define an orthonormal matrix $\Phi_i \in \mathbb{C}^{n\times m}$ such that its columns are perpendicular to the space $\mathcal{X}_{i,p}$. It is obvious that

$$\mathcal{X}_{i,p} = \{x \in \mathbb{C}^n : \quad {\Phi_i}^* x = 0\} \qquad \forall i \in \mathcal{L}. \tag{4.61}$$

The projection of a vector v_k, the k-th column of V for $k \in \{1, \dots, p\}$, onto the set $\mathcal{X}_{i,p}$ equals

$$\mathrm{Proj}_{\mathcal{X}_{i,p}}(v_k) = (I_n - \Phi_i {\Phi_i}^*) v_k \qquad \forall k \in \{1, \dots, p\};$$

see Appendix A.8.1. Now, we define the following normal vectors

$$v_{ik} = \frac{1}{\|(I_n - \Phi_i {\Phi_i}^*) v_k\|} (I_n - \Phi_i {\Phi_i}^*) v_k \qquad \forall k \in \{1, \dots, p\}, \tag{4.62}$$

and the collection of them as $V_i = [v_{i1} \dots v_{ip}]$ for $i \in \mathcal{L}$. We claim that V_i is defined, its column vectors are linearly independent and form a basis for the set $\mathcal{X}_{i,p}$ for each $i \in \mathcal{L}$. To see this, first notice that the following identity is valid

$$(I_n - \Phi_i {\Phi_i}^*)[V \;\; 0] = (I_n - \Phi_i {\Phi_i}^*)[V \;\; \Phi_i].$$

On the other hand, we should have $\mathrm{rank}([V \;\Phi_i]) = n$; otherwise, there must exist a vector $\phi_i \in \mathbb{C}^n$ such that $\phi_i^*[V \;\; \Phi_i] = 0$, implying that both $\phi_i^* V = 0$ and $\phi_i^* \Phi_i = 0$ must hold. Next, referring to (4.61) the expression $\phi_i^* \Phi_i = 0$ implies that $\phi_i \in \mathcal{X}_{i,p}$, and the expression $\phi_i^* V = 0$ implies that ϕ_i is perpendicular to all vectors in the space $\mathcal{X}_{j,p}$. But this is impossible since the canonical angle between $\mathcal{X}_{i,p}$ and $\mathcal{X}_{j,p}$ is assumed to be small. Now, we have

$$\begin{aligned} \mathrm{rank}\,(V_i) &= \mathrm{rank}\,((I_n - \Phi_i {\Phi_i}^*)V) = \mathrm{rank}\,((I_n - \Phi_i {\Phi_i}^*)[V \; 0]) \\ &= \mathrm{rank}\,((I_n - \Phi_i {\Phi_i}^*)[V \; \Phi_i]) = \mathrm{rank}(I_n - \Phi_i {\Phi_i}^*) = p. \end{aligned} \tag{4.63}$$

Next, note that by the definition of v_{ik} in (4.62), we can deduce

$$\min_{k\in\{1,\dots,p\}} v_k^* v_{ik} = \min_{v_k\in\{v_1,\dots,v_p\}} \max_{\nu_i\in\mathcal{X}_{i,p}} v_k^* \nu_i \geq \min_{v\in\mathcal{X}_{j,p}} \max_{\nu_i\in\mathcal{X}_{i,p}} v^* \nu_i = \cos(\theta_{ij}), \tag{4.64}$$

where θ_{ij} is the maximum canonical angle between two spaces $\mathcal{X}_{i,p}$ and $\mathcal{X}_{j,p}$; see Appendix A.2.13.1. Let's define $\Delta V_i := V - V_i$ for each $i \in \mathcal{L}$ with $i \neq j$. Then,

$$\|\Delta V_i\|_2^2 = \|V - V_i\|_2^2 = \sigma_{\max}(\Delta V_i)^2 \leq \mathrm{tr}\,(\Delta V_i^* \Delta V_i) \leq p \max_{k\in\{1,\dots,p\}} (\|v_k - v_{ik}\|^2);$$

see Appendix A.2.11. Then, we have

$$\begin{aligned} \|\Delta V_i\|_2^2 &\leq p \max_{k\in\{1,\dots,p\}} (\|v_k - v_{ik}\|^2) = p \max_{k\in\{1,\dots,p\}} (2 - 2v_k^* v_{ik}) = \\ &= 2p \left(1 - \min_{k\in\{1,\dots,p\}} (v_k^* v_{ik})\right). \end{aligned} \tag{4.65}$$

Now, referring to (4.64) we can write

$$\|\Delta V_i\|^2 \leq 2p\ (1 - \min_{v\in\mathcal{X}_{j,p}} \max_{\nu_i\in\mathcal{X}_{i,p}} (v^* \nu_i)) = 2p(1 - \cos(\theta_{ij})), \tag{4.66}$$

where θ_{ij} is the greatest canonical angle between $\mathcal{X}_{i,p}$ and $\mathcal{X}_{j,p}$. On the other hand, using $2\sin^2(\theta_{ij}/2) = 1 - \cos(\theta_{ij})$, we deduce that

$$\|\Delta V_i\|^2 \leq 4p\ \sin^2(\theta_{ij}/2).$$

As $\theta_{ij} < \pi/2$, the inequality $\sin(\theta_{ij}/2) \leq \cos(\theta_{ij}/2)$ is valid. Hence,

$$\|\Delta V_i\|^2 \leq 4p\ \sin(\theta_{ij}/2)\ \cos(\theta_{ij}/2) = 2p\ \sin(\theta_{ij}) = 2p\ \mathrm{dist}\,(\mathcal{X}_{i,p}, \mathcal{X}_{j,p})\,. \tag{4.67}$$

Part B: Now, choosing appropriate matrices V_i for all $i \in \mathcal{L}$ with regard to the discussion in Part A, we introduce our stabilization method. We choose the columns of $W = \mathcal{Q}_V B_j$ as the set of desired left eigenvectors corresponding to m desired eigenvalues coming with complex conjugate pairs in the open left half plane given by the diagonal entries of $\Lambda = \mathrm{diag}([\lambda_1, \dots, \lambda_m])$ for the closed-loop matrix $A_{\mathrm{cl}i} = A_i + B_i K_i$. Referring to Lemma 4.4.2, we can argue that W can be assigned to $A_{\mathrm{cl}i}$ if (4.58) holds. Then, the desired controller gain K_i has the form (4.48). Next, define the reduced QR-factorization $W = Q_1 R_1$, where R_1 is an $m \times m$ upper triangular non-singular matrix and Q_1 is an $n \times m$ matrix which has orthonormal columns. Choosing the orthonormal transformation matrix $T = [Q_1\ V]$, we can define a set of $\bar{A}_{\mathrm{cl}i}$'s which are similar to $A_{\mathrm{cl}i}$'s as follows:

$$\bar{A}_{\mathrm{cl}i} = T^* A_{\mathrm{cl}i} T = \begin{bmatrix} R_1^{-*} \Lambda R_1^* & 0 \\ V^* A_{\mathrm{cl}i} Q_1 & V^* A_{\mathrm{cl}i} V \end{bmatrix} \qquad \forall i \in \mathcal{L}.$$

We take a common Lyapunov solution for $\bar{A}_{\mathrm{cl}i}$, $i \in \mathcal{L}$, in the form of

$$\bar{P} = \tfrac{1}{2} \begin{bmatrix} R_1 R_1^* & 0 \\ 0 & \epsilon \mathcal{P} \end{bmatrix},$$

where $\mathcal{P} = \mathcal{P}^* \in \mathbb{C}^{p\times p}$ is defined in accordance with (4.60). The Lyapunov equation for the transformed matrices $\bar{A}_{\mathrm{cl}i}$ equals

$$\bar{A}_{\mathrm{cl}i}^* \bar{P} + \bar{P} \bar{A}_{\mathrm{cl}i} = \begin{bmatrix} R_1 \mathrm{Re}(\Lambda) R_1^* & \frac{\epsilon}{2} Q_1^* A_{\mathrm{cl}i}^\top V \mathcal{P} \\ \frac{\epsilon}{2} \mathcal{P} V^* A_{\mathrm{cl}i} Q_1 & \frac{\epsilon}{2}(\mathcal{P} V^* A_{\mathrm{cl}i} V + V^* A_{\mathrm{cl}i}^\top V \mathcal{P}) \end{bmatrix} \qquad \forall i \in \mathcal{L}. \tag{4.68}$$

As $R_1\mathrm{Re}(\Lambda)R_1^* < 0$, (4.68) is negative definite if and only if $\mathcal{S}_i < 0$, where $\mathcal{S}_i$ stands for its Schur complement with respect to the upper left block as

$$\mathcal{S}_i = \tfrac{\epsilon}{2}(\mathcal{P}V^*A_{\mathrm{cl}i}V + V^*A_{\mathrm{cl}i}^\top V\mathcal{P}) - \tfrac{\epsilon^2}{4}(\mathcal{P}V^*A_{\mathrm{cl}i}Q_1)\left(R_1\mathrm{Re}(\Lambda)R_1^*\right)^{-1}(Q_1^*A_{\mathrm{cl}i}^\top V\mathcal{P}). \quad (4.69)$$

Following the proof of Theorem 4.4.1, for a sufficiently small $\epsilon > 0$, the inequality $\mathcal{S}_i < 0$ is satisfied if and only if

$$\mathcal{P}V^*A_{\mathrm{cl}i}V + V^*A_{\mathrm{cl}i}^\top V\mathcal{P} < 0 \qquad \forall i \in \mathcal{L}. \quad (4.70)$$

We prove the last inequality holds if $\|\Delta V_i\|_2$ is sufficiently small. First, note that using the feedback gain of the form (4.48), the identity

$$\begin{aligned} A_{\mathrm{cl}i}V &= \left[A_i - B_i(B_j^\top \mathcal{Q}_V B_i)^{-1}(B_j^\top \mathcal{Q}_V A_i - \Lambda B_j^\top \mathcal{Q}_V)\right] V \\ &= \left[I - B_i(B_j^\top \mathcal{Q}_V B_i)^{-1}B_j^\top \mathcal{Q}_V\right] A_iV = A_iV + E_iV \end{aligned} \quad (4.71)$$

holds, where we have defined

$$E_i := -B_i(B_j^\top \mathcal{Q}_V B_i)^{-1}B_j^\top \mathcal{Q}_V A_i.$$

Now, recall that ΔV_i is defined as $\Delta V_i := V - V_i$, where V_i is defined by (4.62). Thus, the property (4.59) also holds. Consequently, the matrices L_i for all $i \in \mathcal{L}$ in (4.59) are defined and available. Now, suppose a suitable $\mathcal{P}$ can be determined such that (4.60) holds. Then, part of the Lyapunov equation for this mode can be written in the form of

$$\begin{aligned} \mathcal{P}V^*A_{\mathrm{cl}i}V &= \mathcal{P}V^*(A_i + E_i)V \\ &= \mathcal{P}(V - V_i(V_i^*V_i)^{-1} + V_i(V_i^*V_i)^{-1})^*A_i(V_i + \Delta V_i) + \mathcal{P}V^*E_iV \\ &= \mathcal{P}(V_i^*V_i)^{-1}V_i^*A_iV_i + \mathcal{P}(V - V_i(V_i^*V_i)^{-1})^*A_iV + \mathcal{P}(V_i^*V_i)^{-1}V_i^*A_i\Delta V_i + \mathcal{P}V^*E_iV \\ &= \mathcal{P}(V_i^*V_i)^{-1}V_i^*V_iL_i + \mathcal{P}(V - V_i(V_i^*V_i)^{-1})^*A_iV + \mathcal{P}(V_i^*V_i)^{-1}V_i^*A_i\Delta V_i + \mathcal{P}V^*E_iV \\ &= \mathcal{P}L_i + \mathcal{P}(V - V_i(V_i^*V_i)^{-1})^*A_iV + \mathcal{P}(V_i^*V_i)^{-1}V_i^*A_i\Delta V_i + \mathcal{P}V^*E_iV \\ &= \mathcal{P}L_i + \mathcal{P}F_i, \end{aligned}$$

where we have defined F_i as

$$F_i := (V - V_i(V_i^*V_i)^{-1})^*A_iV + (V_i^*V_i)^{-1}V_i^*A_i\Delta V_i + V^*E_iV. \quad (4.72)$$

Referring to (4.60), we can write

$$\begin{aligned} &\mathcal{P}V^*A_{\mathrm{cl}i}V + V^*A_{\mathrm{cl}i}^\top V\mathcal{P} = \mathcal{P}L_i + L_i^*\mathcal{P} + \mathcal{P}F_i + F_i^*\mathcal{P} < \\ &-\gamma_1^2\mathcal{P}^2 + \mathcal{P}F_i + F_i^*\mathcal{P} - \gamma_2^2 I_p = \\ &-\left(\gamma_1\mathcal{P} - \frac{1}{\gamma_1}F_i\right)^*\left(\gamma_1\mathcal{P} - \frac{1}{\gamma_1}F_i\right) + \left[\tfrac{1}{\gamma_1^2}F_i^*F_i - \gamma_2^2 I_p\right]. \end{aligned}$$

The first summand of the above expression is clearly negative semi-definite. We will show also when ΔV_i is small, the second summand is negative semi-definite. To this end, for a small value of $\|\Delta V_i\|$, we should show

$$\tfrac{1}{\gamma_1^2}F_i^*F_i - \gamma_2^2 I_p \le 0 \Rightarrow \sigma_{\max}(F_i) \le \gamma_1\gamma_2.$$

On the other hand, referring to the definition of F_i in (4.72), $\sigma_{\max}(F_i)$ is bounded by

$$\begin{aligned}\sigma_{\max}(F_i) &\le \sigma_{\max}\left((V - V_i(V_i^*V_i)^{-1})^*A_iV + V^*E_iV\right) + \sigma_{\max}\left((V_i^*V_i)^{-1}V_i^*A_i\Delta V_i\right)\\ &\le \sigma_{\max}\left((V - V_i(V_i^*V_i)^{-1})^*A_iV + V^*E_iV\right) + \sigma_{\max}\left((V_i^*V_i)^{-1}V_i^*A_i\right)\|\Delta V_i\|_2.\end{aligned}$$

Hence, if the norm of the perturbation ΔV_i is bounded by

$$\|\Delta V_i\|_2 = \sigma_{\max}(\Delta V_i) \le \frac{\gamma_1\gamma_2 - \sigma_{\max}\left(V^*(A_i + E_i)V - (V_i^*V_i)^{-1}V_i^*A_iV\right)}{\sigma_{\max}\left((V_i^*V_i)^{-1}V_i^*A_i\right)}, \tag{4.73}$$

then the switched system (4.39) is stable. Recalling (4.71), if $\gamma_1\gamma_2$ is sufficiently large so that the inequalities $\gamma_1\gamma_2 \ge \sigma_{\max}\left(V^*A_{\text{cl}i}V - (V_i^*V_i)^{-1}V_i^*A_iV\right)$ for all $i \in \mathcal{L}$ hold, and the distances between the invariant subspaces $\mathcal{X}_{i,p}$ and $\mathcal{X}_{j,p}$ are less than or equal to some $\delta \ge 0$, for instance,

$$\operatorname{dist}(\mathcal{X}_{i,p}, \mathcal{X}_{j,p}) \le \delta,\ \delta := \frac{1}{2p}\min_{\substack{i\in\mathcal{L}\\ i\ne j}}\left(\frac{\gamma_1\gamma_2 - \sigma_{\max}\left(V^*A_{\text{cl}i}V - (V_i^*V_i)^{-1}V_i^*A_iV\right)}{\sigma_{\max}\left((V_i^*V_i)^{-1}V_i^*A_i\right)}\right)^2, \tag{4.74}$$

then (4.73) and (4.67) are satisfied. Therefore, quadratic stability of the closed loop switched linear system (4.39) is deduced. □

Example 4.4.5. Consider again the controlled switched linear system (4.39), with the same matrices A_1, B_1, and B_2 presented in Example 4.4.1. However, we assume that the matrix A_2 has been slightly perturbed in the form of

$$A_2 = \begin{bmatrix} 1 & 3 & -5\\ 2 & -6 & -2\\ 7 & 1 & -11\end{bmatrix} + 10^{-4}\times\begin{bmatrix} 1 & 2 & -1\\ 2 & -1 & -1\\ 7 & 1 & -2\end{bmatrix}.$$

After finding the two dimensional invariant subspaces of matrices A_1 and A_2, we realize that the two dimensional invariant subspaces $\mathcal{X}_{1,2}$ and $\mathcal{X}_{2,2}$ defined by

$$\mathcal{X}_{i,2} = \{x \in \mathbb{C}^3 :\ {\Phi_i}^*x = 0\} \qquad i \in \{1, 2\},$$

where

$$\Phi_1 = [-0.5774\ \ 0.5774\ \ 0.5774]^\top,\ \Phi_2 = [-0.5658{+}0.1148i\ \ 0.5660{-}0.1149i\ \ 0.5657{-}0.1148i]^\top,$$

corresponding to A_1 and A_2, respectively, have a fairly small distance $\operatorname{dist}(\mathcal{X}_{1,2} - \mathcal{X}_{2,2}) = 3.2314 \times 10^{-4}$. The orthonormal vectors given by the columns of $V = [v_1\ v_2]$ computed in Example 4.4.1 form a basis for $\mathcal{X}_{1,2}$. Likewise, the vectors given by the columns of $V_2 = [v_{21}\ \ v_{22}]$ with $v_{2k} = (I_3 - \Phi_2{\Phi_2}^*)v_k/\|(I_3 - \Phi_2{\Phi_2}^*)v_k\|$ for $k \in \{1, 2\}$ form a normal basis for $\mathcal{X}_{2,2}$. These vectors are numerically given by

$$\begin{aligned} v_{21} &= [-0.7071 \qquad 0 \quad -0.7071]^\top,\\ v_{22} &= [-0.4084 \quad -0.8164 \quad 0.4084]^\top.\end{aligned}$$

Furthermore, one can realize that

$$\|V - V_2\| = 2.8794 \times 10^{-4} < \sqrt{4\ \operatorname{dist}(\mathcal{X}_{1,2} - \mathcal{X}_{2,2})} = 0.0360.$$

The matrices L_1 and L_2 with regard to (4.59) equal

$$L_1 = \begin{bmatrix} -4 & 4.6188 \\ 0 & -4 \end{bmatrix}, \; L_2 = \begin{bmatrix} -4.0003 + 0.0001i & 9.2399 + 0.0003i \\ -0.0002 & -3.9997 + 0.0001i \end{bmatrix}.$$

Moreover, with $\gamma_1 = 0.9487$, $\gamma_2 = 1.0954$, and the positive definite $\mathcal{P} = \mathrm{diag}([1 \;\; 3])$, we have that the inequalities (4.60) are valid. Furthermore, the parameter $\delta = 0.0198$, defined by (4.74), is greater than the distance between two subspaces $\mathcal{X}_{1,2}$ and $\mathcal{X}_{2,2}$. Therefore, designing local state feedbacks that quadratically stabilize this controlled switched linear system is possible. We achieve the design by assigning a common left eigenvector to the closed loop matrices A_{cl1} and A_{cl2}. The desired left eigenvector is selected equal to the one given in Example 4.4.1. As $W^\top B_1$ and $W^\top B_2$ are both non-zero, assigning such left eigenvector is feasible. Choosing again the desired left eigenvalue $\Lambda = -2$, the state feedback gains $K_1 = [2 \;\; -2 \;\; -2]$ and $K_2 = [-6.0006 \;\; 5.9997 \;\; 6]$ are attained, leading to the closed loop matrices

$$A_{\mathrm{cl1}} = \begin{bmatrix} 5 & -5 & -9 \\ 2 & -6 & -2 \\ 5 & -1 & -9 \end{bmatrix}, \; A_{\mathrm{cl2}} = \begin{bmatrix} -5.0005 & 8.9999 & 0.9999 \\ 2.0002 & -6.0001 & -2.0001 \\ -5.0004 & 12.9995 & 0.9997 \end{bmatrix}.$$

4.5 Robust control design with $(n-1)$ control inputs

Now, we consider a perturbed controlled switched linear system

$$\dot{x} = A_{\sigma(t)}\left(x + \Delta_1(t)\|x\|\right) + B_{\sigma(t)}u + \Delta_2(t)\|x\|, \tag{4.75}$$

where $u \in \mathbb{R}^{n-1}$, $\Delta_1(t)$ and $\Delta_2(t)$ are the perturbation vector functions, and $\sigma : t \to \mathcal{L} := \{1, \ldots, \ell\}$ is the switching signal. Our intention is to quadratically stabilize (4.75) by using state feedback $u = K_{\sigma(t)}x$, in the case that, the 2-norms of the perturbation vectors are sufficiently small. For this design, we can employ the feedback gains of the form (4.45) with $W \in \mathbb{R}^{n\times(n-1)}$. Referring to Chapter 2.2.2.1, Section 4.3.1, and Section 4.4.2.2, the $(n-1)$ desired eigenvalues of $A_i + B_iK_i$ corresponding to the left eigenvectors given by the columns of W can be selected as real numbers satisfying $\lambda_j < -\frac{1}{2}$ for $j \in \{1, \ldots, n-1\}$. Then, one must additionally take care of the eigenvalue λ_{in}, which corresponds to the right eigenvector v_n of $A_i + B_iK_i$, for each $i \in \mathcal{L}$. The condition $\lambda_{in} < -\frac{1}{2}$ modifies to

$$-\mathrm{tr}\left(W^\top \left((a_{i1} - \tfrac{1}{2})/(n-1)I_n + A_i\right) B_i(W^\top B_i)^{-1}\right) < 0 \qquad \forall i \in \mathcal{L}, \tag{4.76}$$

where a_{i1} has been defined in Chapter 2.2.2.1. Moreover, referring to our discussion in Section 4.4.2.2 and Chapter 2.2.2.1, having $\|\Delta_1(t)\|_2 \le \psi$ and $\|\Delta_2(t)\|_2 \le \psi$, where ψ is the upper bound for the perturbation defined in (4.35), are the required conditions for guaranteeing stability of the perturbed switched system with this design. Then, referring to (4.35) these conditions are equivalent to have: $\epsilon.I_{n-1} \le W^\top W$, (4.33) holds, and

$$\|\Delta_j(t)\|_2 \le \min_{i\in\mathcal{L}} \; \frac{\epsilon}{4(n-1)\left(\sigma_{\max}(A_i + B_iK_i) + 1\right)} \qquad \forall j \in \{1, 2\}. \tag{4.77}$$

Note that in this control design the upper bounds for $\|\Delta_1(t)\|_2$ and $\|\Delta_2(t)\|_2$ are a priori known. An appropriate W can be found by employing the similar approaches introduced in Section 4.4.2.2 and Example 4.4.3 to satisfy the conditions $\epsilon.I_{n-1} \le W^\top W$, (4.33), (4.76), and (4.77); see also Kouhi and Bajcinca (2011c).

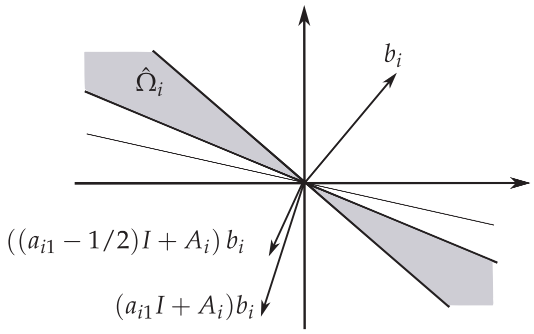

Figure 4.5.1: The interior of colored region, denoted by $\hat{\Omega}_i$, is the proper region for selection of a desired left eigenvector w_i satisfying (4.79).

Example 4.5.1. In the case $n = 2$, for stabilization of the perturbed controlled switched linear system (4.75), we require that the eigenvalues of $A_i + B_iK_i$ satisfy

$$\lambda_{i1} < -\frac{1}{2},\ \lambda_{i2} < -\frac{1}{2} \qquad \forall i \in \mathcal{L}. \tag{4.78}$$

Let us denote $b_1 := B_1$, $b_2 := B_2$, and $w_i = W_i \in \mathbb{R}^2$ for consistency of notation with the one used in Example 4.4.4. We can choose λ_{i1} any real negative number less than $-1/2$. Then, the set of a desired left eigenvector w_i, denoted by $\hat{\Omega}_i$, which satisfy

$$\lambda_{i2} + \frac{1}{2} = -\frac{w_i^\top((a_{i1}-\frac{1}{2})I + A_i)b_i}{w_i^\top b_i} < 0 \qquad \forall i \in \mathcal{L}, \tag{4.79}$$

can be specified geometrically. As a result, in this example the set $\hat{\Omega}_i$ is a subset of the set Ω_i introduced in Example 4.4.4, that is, $\hat{\Omega}_i \subset \Omega_i$. This has been illustrated in Figure 4.5.1. The set of a common left eigenvector $w = w_1 = \cdots = w_\ell$ which satisfy (4.79), denoted by $\hat{\Omega}$, is obtained by $\hat{\Omega} = \cap_{i=1}^{\ell}\hat{\Omega}_i$; see also Kouhi and Bajcinca (2011c).

4.6 Conclusions

In this chapter we have studied the stability problem for the class of switched linear systems whose subsystem matrices share a number of left eigenvectors and an invariant subspace such that a common quadratic Lyapunov function can be associated to this space. Particular cases include sets of Hurwitz matrices which share $(n-1)$ right eigenvectors or $(n-1)$ real left eigenvectors.

Furthermore, several approaches for stabilization of the class of controlled switched linear systems whose open loop matrices share such an invariant subspace have been proposed. The stabilization techniques are based on the concept of left eigenstructure assignment developed in Chapter 2.

We have also discussed the robust stability problem for convexified differential inclusions associated with switched linear systems. In particular, we have derived an interesting result for robust stability of switched linear systems whose matrices share $(n-1)$ real left eigenvectors.

Chapter 5

Rank-m difference switched systems

5.1 Introduction

In this chapter, we introduce a tractable condition on pairs of Hurwitz matrices A_1 and A_2, which guarantees existence of a common Lyapunov solution for them. While it turns out that A_1A_2 having no real negative eigenvalue (spectral condition) is a necessary condition for the existence of a common Lyapunov solution for A_1 and A_2 (Shorten and Narendra, 2003), this condition is not sufficient. An example that this necessary condition fails to be sufficient is as follows:

$$A_1 = \begin{bmatrix} -1.8 & 0.5 & 0.1 & 0.4 \\ 0.1 & -1.8 & 0.1 & 0.9 \\ 0.1 & 0.7 & 0.6 & -2.1 \\ 0.1 & 0.8 & 0.9 & -1.8 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1.8 & 0.1 & 0.9 & 0.3 \\ 0.2 & -0.4 & -1.7 & 0.3 \\ -0.4 & 0.7 & 1.1 & -2 \\ 0.4 & 0.6 & 0.8 & -2.8 \end{bmatrix}. \tag{5.1}$$

A_1 and A_2 are both Hurwitz, and the eigenvalues of the product A_1A_2 are $4.696, 0.6303 \pm 1.5505i, 0.2333$. Thus, no real negative eigenvalue for A_1A_2 exists. However, by using a suitable software, it can be checked that no common solution $P = P^\top > 0$ for the Lyapunov inequalities $A_i^\top P + PA_i < 0$, $i \in \{1, 2\}$, exists.

Therefore, we are interested to characterize a class of matrices that given above spectral condition is also sufficient for the existence of such a solution. An initial result in this direction has been given by Shorten and Narendra (2003), stating that any two matrices which are Hurwitz, differ by a rank-1 matrix, and satisfy the spectral condition possess a common Lyapunov solution. In the proof of this result, a transfer function is associated with A_1 and A_2. This transfer function is demonstrated to be Strictly Positive Real (SPR). Existence of a common Lyapunov solution for the pair of matrices then results from the Kalman-Yakubovic-Popov (KYP) lemma (see Zhou et al. (1996)) which is tight for the class of SPR systems. Despite much effort, it has not been possible to extend this result to more general pairs of matrices until recently (Kouhi et al., 2013a).

In this chapter, we introduce a new pair of matrices for which the above spectral condition is both sufficient and necessary for having a common Lyapunov solution and thus guaranteeing stability of the switched system $\dot{x} = A_{\sigma(t)}x$, $\sigma(t) \in \{1, 2\}$. These are pairs of stable matrices which are related by a symmetric transfer function matrix. For developing the results, we follow a similar approach to the existing one for the class of rank-1 difference matrices.

5.2 Symmetric transfer function matrices

Symmetric transfer function matrices can be found in many applications. For example, symmetric transfer functions are ubiquitous in the study of electrical systems (Helmke et al., 2006; Semlyen and Gustavsen, 2009), in systems with collocated sensors and actuators (Yang and Qiu, 2002), and in chemical process plants (Shinskey, 1984; Hovd and Skogestad, 1994; Kouhi et al., 2014). Due to this wide range of applications, numerous results concerning control design for this class of systems exist in the literature; see, *e.g.*, Shinskey (1984); Hovd and Skogestad (1994); Xie et al. (2004); Fuhrmann (1983). In this part, however, we study this class of systems solely from a mathematical point of view. In fact, the mathematical properties of symmetric transfer function matrices play an integral role for the stability theory of switched linear systems in this chapter.

We begin by exploring conditions on the matrix components of a state space realization (A, B, C, D) characterizing a symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$. Lemma 5.2.1 represents a necessary and sufficient condition for symmetric transfer function matrices with respect to Markov parameters, while Lemma 5.2.2 establishes a link between the symmetry condition and similar state space realizations. In Lemma 5.2.4 we introduce one of the important properties that symmetric transfer function matrices exhibit.

Throughout this chapter we always assume that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ has full column rank, $C \in \mathbb{R}^{m\times n}$ has full row rank, and $D \in \mathbb{R}^{m\times m}$ for some $m \leq n$.

Lemma 5.2.1. *(See,* e.g.*, Kouhi et al. (2013a)) The transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is symmetric, that is, $G(s) = G^\top(s)$ if and only if*

$$\begin{aligned} D &= D^\top, \\ CA^iB &= (CA^iB)^\top \qquad \forall i \in \{0, 1, \ldots, n-1\}. \end{aligned} \tag{5.2}$$

Proof: Let's define the characteristic polynomial of A as

$$\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n.$$

Then, referring to (Kailath (1980), pp.67), the following identity holds

$$C(sI - A)^{-1}B = \frac{1}{\det(sI - A)}[s^{n-1}(CB) + s^{n-2}(CAB + a_1CB) + \\ \cdots + (CA^{n-1}B + \cdots + a_{n-1}CB)].$$

Hence, it is immediately evident that the condition (5.2) and the assumption $D = D^\top$ are necessary and sufficient for the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$

to be symmetric. □

We say that the dynamic systems $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$, where matrices A, B, C, and D satisfy the property of Lemma 5.2.1, are generators of the symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$; see Kouhi et al. (2013a).

Lemma 5.2.2. *See Kailath (1980). Let (A, B, C, D) be a minimal realization of the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$. Then $G(s)$ is symmetric, if and only if there exists a unique invertible matrix $S^\top = S$ (not necessarily positive definite), such that*

$$A^\top S = SA, \quad C = B^\top S, \quad \textit{and } D = D^\top. \tag{5.3}$$

Proof of necessity: This proof can be found in Kailath (1980); Willems (1972). As $G(s)$ and

$$G^\top(s) = B^\top(sI - A^\top)^{-1}C^\top + D^\top$$

are equal, the minimal state space realizations (A, B, C, D) and $(A^\top, C^\top, B^\top, D^\top)$ are similar. This implies that a non-singular unique matrix S exists such that

$$A^\top = SAS^{-1},\ C^\top = SB,\ B^\top = CS^{-1},\ \text{and } D^\top = D.$$

Therefore, $A^\top S = SA$ and $C = B^\top S$. In addition, we have $A^\top S^\top = S^\top A$ and $C = B^\top S^\top$. This means that S is not unique, unless we have $S = S^\top$.

Proof of sufficiency: Suppose there exists a unique invertible symmetric matrix S, such that (5.3) holds. Substituting $C = B^\top S$ into $G(s)$, and paying attention that in

$$G(s) = C(sI - A)^{-1}B + D = B^\top(s\ S^{-1} - AS^{-1})^{-1}B + D \tag{5.4}$$

S^{-1}, AS^{-1}, and D are symmetric, symmetry of $G(s)$ results. Note that (5.4) in fact represents the descriptor form of $G(s)$; see Knockaert et al. (2013). □

As discussed in the introduction of this chapter, we are interested in developing results for switched systems associated with symmetric transfer function matrices. The basic question now is how to specify the class of matrix pairs A_1 and A_2, such that a symmetric transfer function can be associated with them? The following lemma provides a tool for specifying this class. Before proceeding, note that $\otimes$ denotes the Kronecker product of two matrices; see Appendix A.2.4.

Lemma 5.2.3. *(See also Kouhi et al. (2014)) Given two matrices $A_1 \in \mathbb{R}^{n\times n}$ and $A_2 \in \mathbb{R}^{n\times n}$ with the controllable pair $(A_1, A_1 - A_2)$, a symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ with the minimal realization (A, B, C, D) can be associated with A_1 and A_2 such that*

$$A_1 := A \textit{ and } A_2 = A - BD^{-1}C, \tag{5.5}$$

if and only if the matrices

$$\begin{aligned} E_1 := I_n \otimes A_1 - A_1 \otimes I_n \ \textit{and} \\ E_2 := I_n \otimes A_2 - A_2 \otimes I_n \end{aligned} \tag{5.6}$$

share a right eigenvector corresponding to a zero eigenvalue, say

$$vec(Y) = [y_{11} \dots y_{n1} \ y_{12} \dots y_{n2} \dots y_{1n} \dots y_{nn}]^\top,$$

such that

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & \vdots & \dots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \tag{5.7}$$

is symmetric and invertible, and $(A_1 - A_2)Y$ is positive semi-definite.

Proof of necessity: Define $A := A_1$. Suppose there exist matrices B, C, and D such that $G(s) = C(sI - A)^{-1}B + D$ is symmetric, (A, B, C, D) is a minimal realization, and $A_2 = A - BD^{-1}C$. Then, referring to Lemma 5.2.2 there exists a matrix $S = S^\top$, such that

$$A^\top S - SA = 0, \quad B^\top S = C. \tag{5.8}$$

Replacing C from (5.8) into (5.5), we get

$$A_2 = A - BD^{-1}B^\top S \Rightarrow A_2 S^{-1} = AS^{-1} - BD^{-1}B^\top. \tag{5.9}$$

Notice (5.8) reveals that $AS^{-1} = S^{-1}A^\top$. This implies that $AS^{-1} - BD^{-1}B^\top$ is symmetric, and therefore from (5.9) we conclude that A_2S^{-1} is symmetric as well, that is,

$$A_2 S^{-1} - S^{-1}A_2^\top = 0. \tag{5.10}$$

Defining $Y = S^{-1}$, both A and A_2 satisfy the following Sylvester equations

$$AY - YA^\top = 0, \quad A_2Y - YA_2^\top = 0. \tag{5.11}$$

For finding a solution of these equations, utilizing the Kronecker product notation and the vectorization operator $\text{vec}(Y) = [y_{11} \ \dots y_{n1} \ y_{12} \ \dots \ y_{n2} \ \dots \ y_{1n} \ \dots \ y_{nn}]^\top$, we can reformulate (5.11) in the form of

$$\begin{aligned} (I_n \otimes A_1 - A_1 \otimes I_n) \ \text{vec}(Y) = 0, \\ (I_n \otimes A_2 - A_2 \otimes I_n) \ \text{vec}(Y) = 0; \end{aligned} \tag{5.12}$$

see Horn and Johnson (1990). It is obvious that (5.12) implies $E_1 = (I_n \otimes A_1 - A_1 \otimes I_n)$ and $E_2 = (I_n \otimes A_2 - A_2 \otimes I_n)$ share a right eigenvector corresponding to a zero eigenvalue, where its associated matrix Y is symmetric and invertible, and $(A_1 - A_2)Y = BD^{-1}B^\top$

is positive semi-definite. Note that as both E_1 and E_2 are singular, they both have zero eigenvalues.

Proof of sufficiency: Suppose E_1 and E_2 share an eigenvector corresponding to a zero eigenvalue, say $\mathrm{vec}(Y)$, such that the matrix $Y = [y_{ij}] \in \mathbb{R}^{n\times n}$ defined by (5.7) is symmetric and invertible (Note that the assumption for Y to be symmetric is not restrictive. In fact, if X is a solution to $A_1X - XA_1^\top = 0$, then $(XA_1^\top)^\top = (A_1X)^\top$ and $A_1(X + X^\top) = XA_1^\top + (A_1X)^\top = XA_1^\top + X^\top A_1^\top = (X + X^\top)A_1^\top$. It follows that the symmetric matrix $(X + X^\top)$ is also a solution to the equation $A_1X - XA_1^\top = 0$). Then, taking $A := A_1$, Y satisfies (5.11). This implies that the matrix $(A - A_2)Y$ is symmetric. Suppose $\mathrm{rank}\,((A - A_2)Y) = m$. As $(A - A_2)Y$ is symmetric and by the assumption of this lemma is positive semi-definite, its eigenvalue decomposition can be written as

$$(A - A_2)Y = T^\top \Lambda T, \tag{5.13}$$

where $T^\top = T^{-1} \in \mathbb{R}^{n\times n}$ and $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix with exactly $n - m$ zero elements; see Appendix A.2.7. Therefore, there exists a diagonal matrix $\Lambda_1 \in \mathbb{R}^{m\times n}$ such that $\Lambda = \Lambda_1^\top \Lambda_1$. Consequently, we can write

$$(A - A_2)Y = T^\top \Lambda T = (\Lambda_1 T)^\top (\Lambda_1 T).$$

Defining $S = Y^{-1}$, and

$$B = (\Lambda_1 T)^\top,\ C = B^\top S,\ \text{and } D = I, \tag{5.14}$$

we have

$$\begin{aligned} A - A_2 = (\Lambda_1 T)^\top (\Lambda_1 T) Y^{-1} = BD^{-1}C& \\ \Rightarrow A_2 = A - BD^{-1}C,& \end{aligned} \tag{5.15}$$

and $G(s) = C(sI - A)^{-1}B + D$ is symmetric. For proving that (A, B, C, D) is a minimal realization of $G(s)$, first note that

$$\begin{aligned} \mathrm{rank}\ \Phi_c(A, B) &= \mathrm{rank}\ \big([B\ AB\ \ldots\ A^{n-1}B]\big) \\ &\geq \mathrm{rank}\ \big([B\ \ AB\ \ \ldots\ \ A^{n-1}B](I_n \otimes B^\top S)\big) \\ &= \mathrm{rank}\ \big([BB^\top S\ \ ABB^\top S\ \ \ldots\ \ A^{n-1}BB^\top S]\big) \\ &= \mathrm{rank}\ \big([(A - A_2)\ \ A(A - A_2)\ \ \ldots\ \ A^{n-1}(A - A_2)]\big) \\ &= \mathrm{rank}\ \Phi_c(A, A - A_2) = n, \end{aligned}$$

where Φ_c is the controllability matrix; see Appendix A.3. On the other hand, rank of the observability matrix $\Phi_o(C, A)$ equals

$$\begin{aligned} \mathrm{rank}\ \Phi_o(C, A) &= \mathrm{rank}\ \Phi_c(A^\top, C^\top) = \mathrm{rank}\ \Phi_c(A^\top, SB) \\ &= \mathrm{rank}\ \big([SB\ \ A^\top SB\ \ \ldots\ \ (A^\top)^{n-1}SB]\big) \\ &= \mathrm{rank}\ \big([SB\ \ SAB\ \ \ldots\ \ SA^{n-1}B]\big) \\ &= \mathrm{rank}\ (S\ \Phi_c(A, B)) = n; \end{aligned}$$

see Appendix A.4. This implies that (A, B) is controllable and (C, A) is observable, and thus (A, B, C, D) is a minimal realization of $G(s)$. □

Remark 5.2.1. If the eigenvalues of A_1 are $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of E_1 defined in (5.6) are $\lambda_i - \lambda_j$ for all $i, j \in \{1, \ldots, n\}$; see Horn and Johnson (1990). Therefore, E_1 has at least n zero eigenvalues. Likewise, such a property holds for E_2. However, this does not imply that E_1 and E_2 share an eigenvector corresponding to a zero eigenvalue.

As a result, if two real matrices A_1 and A_2 are given and one wishes to check whether a symmetric transfer function matrix can be associated with them, the following algorithm can be utilized:

i) compute a common eigenvector $\text{vec}(Y)$, corresponding to a zero eigenvalue (if one exists) of the matrices $E_1 := (I_n \otimes A_1 - A_1 \otimes I_n)$ and $E_2 = (I_n \otimes A_2 - A_2 \otimes I_n)$.

ii) compute Y by re-arranging $\text{vec}(Y)$. If Y is symmetric and invertible, and $(A - A_2)Y$ is positive semi-definite, then associating a symmetric transfer function matrix with A_1 and A_2 is possible.

iii) define $A := A_1$, and compute the eigenvalue decomposition of $(A - A_2)Y$. This defines matrices T and Λ_1 which satisfy $(A - A_2)Y = T^\top \Lambda_1^\top \Lambda_1 T$.

iv) compute B, C, and D in accordance with (5.14).

The following example illustrates the steps of the algorithm.

Example 5.2.1. Consider the two matrices

$$A_1 = \begin{bmatrix} -1 & 0 \\ 5 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1.5 & -1.3 \\ 7.5 & -5 \end{bmatrix}. \tag{5.16}$$

The matrices $E_1 := (I_2 \otimes A_1 - A_1 \otimes I_2)$ and $E_2 := (I_2 \otimes A_2 - A_2 \otimes I_2)$ are computed as follows:

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 & -1.3 & 1.3 & 0 \\ 7.5 & -6.5 & 0 & 1.3 \\ -7.5 & 0 & 6.5 & -1.3 \\ 0 & -7.5 & 7.5 & 0 \end{bmatrix}. \tag{5.17}$$

A common eigenvector $\text{vec}(Y)$ corresponding to a zero eigenvalue such that Y is symmetric exists and equals

$$\text{vec}(Y) = [0\ 1\ 1\ 5]^\top.$$

The matrix Y resulting from re-arrangement of $\text{vec}(Y)$

$$Y = \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix},$$

is invertible and symmetric. Defining $A := A_1$, we have that

$$(A - A_2)Y = \begin{bmatrix} 1.3 & 4 \\ 4 & 17.5 \end{bmatrix}$$

is positive definite. Next, from the eigenvalue decomposition of $(A - A_2)Y$ and using (5.14), we compute

$$B = \begin{bmatrix} -0.5892783 & -0.9760897 \\ 0.1375708 & -4.1810375 \end{bmatrix}, \quad C = \begin{bmatrix} 3.0839625 & -0.5892783 \\ 0.6994109 & -0.9760897 \end{bmatrix}, \quad D = I.$$

Therefore, we can associate a symmetric transfer function matrix with A_1 and A_2 as

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{(s+1)^2}\begin{bmatrix} s^2 + 0.0102s + 0.838 & 2.33 - 0.546s \\ 2.33 - 0.546s & s^2 + 5.4s + 9.162 \end{bmatrix}.$$

Note that such a symmetric transfer function matrix associated with A_1 and A_2 is not unique. For instance, in this example we can choose

$$B = \begin{bmatrix} 1 & 2 \\ 10 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.2 \\ -1 & 0.4 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0.667 \end{bmatrix}.$$

This leads to the symmetric transfer function matrix

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{(s+1)^2}\begin{bmatrix} 2s^2 + 7s + 6 & 5 + 3s \\ 5 + 3s & 0.667s^2 + 1.333s + 4.667 \end{bmatrix}.$$

Symmetric transfer function matrices and SISO transfer functions share many properties. In particular, we note the following result which is shared by both system classes.

Lemma 5.2.4. *(Kouhi et al., 2014) For any symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ with $D = D^\top$, the following equality holds*

$$\frac{1}{2}\{G(j\omega) + G^\top(-j\omega)\} = D - C(\omega^2 I + A^2)^{-1}AB.$$

Proof: First, by considering the identity

$$(j\omega I - A)^{-1}A = A(j\omega I - A)^{-1},$$

the following relationship holds

$$\begin{aligned} &\tfrac{1}{2}\left\{(j\omega I - A)^{-1} + (-j\omega I - A)^{-1}\right\} = \\ &\quad = \tfrac{1}{2}(-j\omega I - A)^{-1}\left\{(-j\omega I - A) + (j\omega I - A)\right\}(j\omega I - A)^{-1} \\ &\quad = -(-j\omega I - A)^{-1}A(j\omega I - A)^{-1} \\ &\quad = -(-j\omega I - A)^{-1}(j\omega I - A)^{-1}A \\ &\quad = -(\omega^2 I + A^2)^{-1}A. \end{aligned}$$

Therefore, as $D = D^\top$ we can write

$$\begin{aligned} &\frac{1}{2}\{G(j\omega) + G(-j\omega)\} = \\ &\quad = \frac{1}{2}\{D + C(j\omega I - A)^{-1}B\} + \frac{1}{2}\{D + C(-j\omega I - A)^{-1}B\} = \\ &\quad = D + \frac{1}{2}C\{(j\omega I - A)^{-1} + (-j\omega I - A)^{-1}\}B = D - C(\omega^2 I + A^2)^{-1}AB. \qquad \square \end{aligned}$$

5.3 Strictly Positive Real systems (SPR)

Now, we describe the concept of strictly positive real systems, which plays a crucial role in our stability analysis of switched linear systems. For this reason, in this section we study some properties of such systems in the context of linear time invariant systems.

Definition 5.3.1. (See, *e.g.*, Corless and Shorten (2010)) An $m \times m$ rational transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is said to be strictly positive real (SPR), if there exists an $\alpha > 0$ such that $G(s)$ is analytic in the region of the complex plane which includes all s for which $\text{Re}(s) \geq -\alpha$ and

$$G(j\omega - \alpha) + G^\top(-j\omega - \alpha) \geq 0 \qquad \forall \omega \in \mathbb{R}. \tag{5.18}$$

The following characterization, inspired principally by Narendra and Taylor (1973), provides a more convenient description of an SPR transfer function matrix.

Lemma 5.3.1. *(Corless and Shorten, 2010) Given A Hurwitz, an $m\times m$ rational transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is strictly positive real if and only if*

$$G(j\omega) + G^\top(-j\omega) > 0 \qquad \omega \in \mathbb{R}, \tag{5.19}$$

and

$$\lim_{\omega\to\infty} \omega^{2(m-p)} det\big(G(j\omega) + G^\top(-j\omega)\big) > 0, \tag{5.20}$$

where $p = rank\big(G(\infty) + G^\top(\infty)\big)$. □

Now, we present some results concerning strictly positive real systems which have symmetric transfer function matrices.

5.3.1 Symmetric SPR systems with nonsingular D

For symmetric transfer function matrices with invertible D, we have the following simple result.

Theorem 5.3.1. *(Kouhi et al., 2013a; Semlyen and Gustavsen, 2009) Given a Hurwitz matrix A, the symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ with $D = D^\top > 0$ is SPR if and only if $A(A - BD^{-1}C)$ has no real negative eigenvalue.*

Proof of sufficiency: Suppose $A(A - BD^{-1}C)$ has no real negative eigenvalue, we demonstrate that $G(s)$ is SPR. By continuity of $\det\left(\omega^2 I + A(A - BD^{-1}C)\right)$ with respect to ω everywhere, we can write

$$\begin{aligned} &\det(\omega^2 I + A(A - BD^{-1}C)) > 0 \\ &\Rightarrow \det(\omega^2 I + A^2)\det(I - (\omega^2 I + A^2)^{-1}ABD^{-1}C) > 0 \\ &\Rightarrow \det(\omega^2 I + A^2)\det(I - C(\omega^2 I + A^2)^{-1}ABD^{-1}) > 0 \\ &\Rightarrow \det(\omega^2 I + A^2)\det(D^{-1})\det(D - C(\omega^2 I + A^2)^{-1}AB) > 0. \end{aligned} \tag{5.21}$$

In the third line of the above expression we used the general identity

$$\det(I_n - XY) = \det(I_m - YX);$$

see Appendix A.2.3. As A has no eigenvalue on the $j\omega$-axis, the identity

$$\det(\omega^2 I + A^2) = \det(j\omega I + A).\det(-j\omega I + A),$$

implies that $\det(\omega^2 I + A^2) \neq 0$. Thus, by continuity of $\det(\omega^2 I + A^2)$ with respect to ω everywhere, we can deduce

$$\det(\omega^2 I + A^2) > 0 \qquad \forall \omega \in \mathbb{R}.$$

On the other hand, the assumption on D to be positive definite implies that D^{-1} is also positive definite and thus $\det(D^{-1}) > 0$. Consequently, by (5.21) we have that

$$\det(D - C(\omega^2 I + A^2)^{-1} AB) > 0 \qquad \forall \omega \in \mathbb{R}. \tag{5.22}$$

Using the fact that $G(j\omega)$ is symmetric, one can conclude from (5.22) and Lemma 5.2.4 that

$$\det\left(\frac{1}{2}\{G(j\omega) + G^\top(-j\omega)\}\right) > 0 \qquad \forall \omega \in \mathbb{R}. \tag{5.23}$$

Furthermore, $G(j\omega) + G^\top(-j\omega)$ is a Hermitian matrix, implying that its eigenvalues are all real. Therefore, if for some frequency it fails to be positive definite, then there must exist an $\omega = \omega_1$ such that at least one eigenvalue of this matrix equals zero, that is,

$$\det\left(\frac{1}{2}\{G(j\omega_1) + G^\top(-j\omega_1)\}\right) = 0.$$

This is obviously in contradiction with (5.23).

Proof of necessity: Suppose $G(s)$ is SPR. Then, from Lemma 5.3.1 and Lemma 5.2.4

$$\det\left(\frac{1}{2}\{G(j\omega) + G^\top(-j\omega)\}\right) > 0$$
$$\Rightarrow \det\big(D - C(\omega^2 I + A^2)^{-1} AB\big) > 0 \qquad \forall \omega \in \mathbb{R}.$$

Using the fact $\det(\omega^2 I + A^2) > 0$ for $\omega \in \mathbb{R}$, $\det(D^{-1}) > 0$, and following the computation (5.21) in reverse, we have

$$\det(\omega^2 I + A(A - BD^{-1}C)) > 0 \qquad \forall \omega \in \mathbb{R}.$$

This verifies that $A(A - BD^{-1}C)$ has no real negative eigenvalue. □

Theorem 5.3.1 has been originally stated by (Semlyen and Gustavsen, 2009). One advantage of using Theorem 5.3.1 is to replace the SPR condition expressed in terms of conditions on transfer function matrices (over an infinite set of frequencies) with a point condition expressed by the spectral condition (Kouhi et al., 2013a; Semlyen and Gustavsen, 2009).

5.3.2 Symmetric SPR systems with singular D

Now, consider the symmetric transfer function matrix $G(s)$ with a singular matrix $D = D^\top \geq 0$. Our aim is to study under which conditions $G(s)$ is SPR. Obviously, we can not employ Lemma 5.3.1 as D may not be invertible. Therefore, we look for another alternative eigenvalue condition.

Theorem 5.3.2. *(Kouhi et al., 2014) Suppose A is Hurwitz. Then, a symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ with $D = D^\top \geq 0$, and rank$(D) = p \leq m$, is SPR if and only if $D - CA^{-1}B > 0$, and*

$$M = A^{-1}\left(A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}\right)$$

has no real negative eigenvalue and has exactly $m - p$ eigenvalues equal to zero.

Proof: Theorem 5.3.2 can be proved by developing the ideas in Shorten et al. (2009) presented in Kouhi et al. (2014). For an alternative proof see Bajcinca and Voigt (2013).

Proof of necessity: Suppose $G(s)$ is SPR. Then, always equation (5.19) in Lemma 5.3.1 must hold. In particular for $\omega = 0$, the following inequality is correct

$$\frac{1}{2}\{G(0) + G^\top(0)\} > 0 \Rightarrow D - CA^{-1}B > 0. \tag{5.24}$$

Then, using Lemma 5.2.4 we can write

$$\begin{aligned} \det\left(\tfrac{1}{2}\{G(j\omega) + G^\top(-j\omega)\}\right) &= \det(D - C(\omega^2 I + A^2)^{-1}AB) \\ &= \det(D - CA^{-1}B - C[(\omega^2 I + A^2)^{-1}A - A^{-1}]B) \\ &= \det\left(D - CA^{-1}B + \omega^2 CA^{-1}(\omega^2 I + A^2)^{-1}B\right) > 0. \end{aligned} \tag{5.25}$$

As $D - CA^{-1}B$ is positive definite, $\det(D - CA^{-1}B) > 0$. Thus,

$$\begin{aligned} &\det\left(D - CA^{-1}B + \omega^2 CA^{-1}(\omega^2 I + A^2)^{-1}B\right) > 0 \\ &\Rightarrow \det(D - CA^{-1}B).\det\left(I + \omega^2(D - CA^{-1}B)^{-1}CA^{-1}(\omega^2 I + A^2)^{-1}B\right) > 0 \\ &\Rightarrow \det\left(I + \omega^2 B(D - CA^{-1}B)^{-1}CA^{-1}(\omega^2 I + A^2)^{-1}\right) > 0 \\ &\Rightarrow \det\left((\omega^2 I + A^2)^{-1}\right)\det\left(\omega^2 I + A^2 + \omega^2 B(D - CA^{-1}B)^{-1}CA^{-1}\right) > 0 \\ &\Rightarrow \omega^{2n}\det\left(A^2\right)\frac{\det\left(\frac{1}{\omega^2}I + A^{-1}\left(A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}\right)\right)}{\det\left(\omega^2 I + A^2\right)} > 0 \\ &\Rightarrow \omega^{2n}\det\left(A\right)^2\frac{\det(\frac{1}{\omega^2}I + M)}{\det(\omega^2 I + A^2)} > 0 \\ &\Rightarrow \omega^{2n}\frac{\det(\frac{1}{\omega^2}I + M)}{\det(\omega^2 I + A^2)} > 0. \end{aligned} \tag{5.26}$$

Now, referring again to Lemma 5.3.1, the conditions (5.19) and (5.20) must hold. Notice, as A does not have any eigenvalue on the $j\omega$ axis, and with continuity of $\det(\omega^2 I + A^2)$ everywhere in $\omega \in \mathbb{R}$, we have $\det(\omega^2 I + A^2) > 0$. On the other hand, it is obvious that

$$\lim_{\omega\to\infty}\det(\omega^2 I + A^2) = \omega^{2n}. \tag{5.27}$$

Therefore, substituting ω by $\frac{1}{\omega}$, the conditions (5.19) and (5.20) are equivalent to

$$\det\left(\omega^2 I + M\right) > 0 \qquad \forall \omega \in \mathbb{R}\backslash\{0\}, \tag{5.28}$$

$$\lim_{\omega\to 0} \frac{1}{\omega^{2(m-p)}} \det(\omega^2 I + M) > 0. \tag{5.29}$$

Now, choosing the parameter $\lambda = -\omega^2$, equations (5.28) and (5.29) lead to

$$\det(\lambda I - M) \neq 0 \qquad \text{for } \lambda \in \mathbb{R}_{<0}, \tag{5.30}$$

$$\lim_{\lambda\to 0} \frac{1}{\lambda^{m-p}} \det(\lambda I - M) \neq 0. \tag{5.31}$$

This means M does not have any real negative eigenvalue and at most has $m-p$ zero eigenvalues. Furthermore, as $\mathrm{rank}(D) = p$, there exists a matrix $D^{\perp} \in \mathbb{R}^{m\times(m-p)}$ such that $DD^{\perp} = 0$, and consequently

$$-CA^{-1}BD^{\perp} = (D - CA^{-1}B)D^{\perp}.$$

Then, we have

$$\begin{aligned} MBD^{\perp} &= A^{-1}(A^{-1}BD^{\perp} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}BD^{\perp}) \\ &= A^{-1}(A^{-1}BD^{\perp} - A^{-1}B(D - CA^{-1}B)^{-1}(D - CA^{-1}B)D^{\perp}) = 0. \end{aligned}$$

This fact implies that the columns of $BD^{\perp}$ are eigenvectors of M. Therefore, M contains at least $m-p$ zero eigenvalues. As a result, it has exactly $m-p$ zero eigenvalues.

Proof of sufficiency: Assuming that $D - CA^{-1}B > 0$ and M has no real negative eigenvalue and has exactly $m-p$ zero eigenvalues implies that $\det(\lambda I - M) = \lambda^{(m-p)}q(\lambda)$, with $q(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}_{\leq 0}$. Consequently, $\det(\omega^2 I + M) \neq 0$ for all $\omega \neq 0$. Due to continuity of the determinant with respect to ω everywhere, we have $\det(\omega^2 I + M) > 0$ for all $\omega \neq 0$, implying that (5.28) holds. With the similar argument, we can state that (5.29) holds, and also we have $\det(\omega^2 I + A^2) > 0$ as A is Hurwitz. Now, from (5.26), (5.25) and the assumption $\bar{D} = D - CA^{-1}B > 0$, it immediately follows that

$$\det\left(G(j\omega) + G^{\top}(-j\omega)\right) > 0 \qquad \forall\, \omega \in \mathbb{R}, \tag{5.32}$$

$$\lim_{\omega\to\infty} \omega^{2(m-p)} \det\left(G(j\omega) + G^{\top}(-j\omega)\right) > 0. \tag{5.33}$$

Now, we demonstrate that (5.32) implies $G(j\omega) + G^{\top}(-j\omega) > 0$. Indeed, $G(j\omega) + G^{\top}(-j\omega)$ is a Hermitian matrix, implying that its eigenvalues are all real and at $\omega = 0$ is positive definite. Therefore, if for some frequency it fails to be positive definite, then there must exist an $\omega = \omega_1 \in \mathbb{R}$ such that at least one eigenvalue of this matrix equals zero, that is,

$$\det\left(G(j\omega_1) + G^{\top}(-j\omega_1)\right) = 0.$$

This is obviously in contradiction with (5.32). Now, the requirements of Lemma 5.3.1 are fulfilled and $G(s)$ to be SPR is inferred. $\square$

Remark 5.3.1. Note that if a state space realization of $G(s)$ are represented by (A, B, C, D) then a state space realization of $G(\frac{1}{s})$ is represented by

$$(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = (A^{-1}, -A^{-1}B, CA^{-1}, D - CA^{-1}B); \tag{5.34}$$

see Shorten et al. (2008). Thus, the matrix M introduced in Theorem 5.3.2 is indeed equal to

$$M = \bar{A}(\bar{A} - \bar{B}\bar{D}^{-1}\bar{C}).$$

Hence, Theorem 5.3.2 reflects the fact that $G(s)$ is SPR if any only if the matrix $\bar{A}(\bar{A} - \bar{B}\bar{D}^{-1}\bar{C})$ does not have any real negative eigenvalue and has exactly $m - p$ eigenvalues equal to zero.

5.4 Stability of a class of switched linear systems

We now present our results on the existence of a common quadratic Lyapunov function for a certain class of switched linear systems, namely the class of switched systems that a symmetric transfer function matrix can be associated with.

5.4.1 Quadratic stability

Consider the switched linear system

$$\dot{x} = (A - \sigma(t)BD^{-1}C)x \qquad \sigma(t) \in \{0, 1\}, \tag{5.35}$$

where σ is an arbitrary switching signal. We emphasize again that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ has full column rank, $C \in \mathbb{R}^{m\times n}$ has full row rank, and $D \in \mathbb{R}^{m\times m}$ for some $m \leq n$. Both matrices A and $A - BD^{-1}C$ are assumed to be Hurwitz. Exponential stability of the switched system (5.35) is guaranteed if there exists a $P = P^\top > 0$ such that

$$A^\top P + PA < 0, \tag{5.36}$$

$$(A - BD^{-1}C)^\top P + P(A - BD^{-1}C) < 0. \tag{5.37}$$

We are interested to explore under which condition such a CQLF $V(x) = x^\top Px$ exists.

Theorem 5.4.1. *(Kouhi et al., 2013a) Given two Hurwitz matrices A and $A - BD^{-1}C$ with (A, B) controllable and (C, A) observable, satisfying $D = D^\top > 0$ and*

$$CA^iB = (CA^iB)^\top \qquad \forall i \in \{0, 1, \dots, n-1\}, \tag{5.38}$$

namely the LTI systems $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$ are generators of a symmetric transfer function matrix. Then the switched system (5.35) is quadratically stable if and only if $A(A - BD^{-1}C)$ has no real negative eigenvalue.

Proof of necessity: Suppose that $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$ share a quadratic Lyapunov function (CQLF). Then, by pre multiplying the inequality $A^\top P + PA < 0$ by the non-singular matrix $A^{-\top}$ and post multiplying it by A^{-1}, we get

$$A^{-\top}P + PA^{-1} < 0. \tag{5.39}$$

This means that $\dot{x} = A^{-1}x$, $\dot{x} = (A - BD^{-1}C)x$, and consequently the family of systems

$$\dot{x} = (\omega^2 A^{-1} + (A - BD^{-1}C))x \tag{5.40}$$

share the CQLF $V(x) = x^\top Px$ for all $\omega \in \mathbb{R}$, that is,

$$\begin{aligned}&\left[\omega^2 A^{-1} + (A - BD^{-1}C)\right]^\top P + \\ &\qquad P\left[\omega^2 A^{-1} + (A - BD^{-1}C)\right] < 0 \qquad \forall w \in \mathbb{R};\end{aligned}$$

see also Shorten and Narendra (2003). Hence, it follows from Lyapunov's second theorem that the matrix $\omega^2 A^{-1} + (A - BD^{-1}C)$ is Hurwitz for all $\omega \in \mathbb{R}$ and thus is non-singular, that is,

$$\begin{aligned}&\det\left(\omega^2 A^{-1} + (A - BD^{-1}C)\right) \neq 0 \\ \Rightarrow\ &\det(A^{-1})\ \det\left(\omega^2 I + A(A - BD^{-1}C)\right) \neq 0 \qquad \forall w \in \mathbb{R}.\end{aligned}$$

As A^{-1} is Hurwitz the latter implies that

$$\det\left(\omega^2 I + A(A - BD^{-1}C)\right) \neq 0 \qquad \forall w \in \mathbb{R},$$

or equivalently that $A(A - BD^{-1}C)$ has no real negative eigenvalue. Note that for the proof of necessity the symmetry conditions given in (5.38) are not demanded.

Proof of sufficiency: Recall from Theorem 5.3.1 that $A(A - BD^{-1}C)$ having no real negative eigenvalue and the symmetry conditions stated in Theorem 5.4.1 imply that $G(s) = C(sI - A)^{-1}B + D$ is SPR. Then, referring to the Kalman-Yakubovic-Popov lemma (see Appendix A.4.1), there must exist a matrix $P = P^\top > 0$, a scalar $\alpha > 0$, and matrices L and W satisfying

$$A^\top P + PA = -L^\top L - \alpha P, \tag{5.41}$$

$$B^\top P + W^\top L = C, \tag{5.42}$$

$$D + D^\top = W^\top W. \tag{5.43}$$

Now, we show the function $V(x) = x^\top Px$ is a Lyapunov function for both $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$. Notice that $A^\top P + PA < 0$ immediately follows from the equation (5.41). Furthermore, $(A - BD^{-1}C)^\top P + P(A - BD^{-1}C) < 0$ also holds. To see that, we have

$$\begin{aligned}&(A - BD^{-1}C)^\top P + P(A - BD^{-1}C) = \\ &\quad = -\alpha P - L^\top L - C^\top D^{-\top}(C - W^\top L) - (C - W^\top L)^\top D^{-1}C \\ &\quad = -\alpha P - (L - WD^{-1}C)^\top (L - WD^{-1}C) < 0,\end{aligned} \tag{5.44}$$

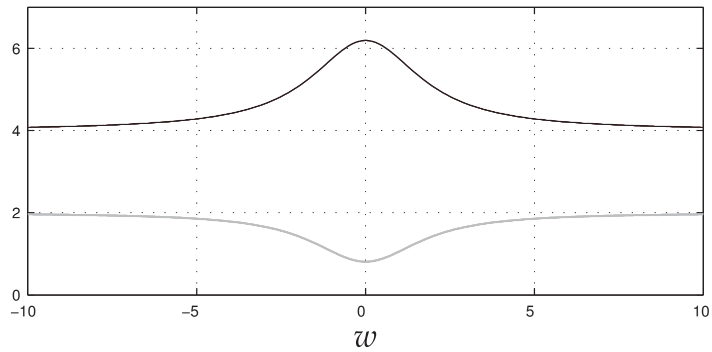

Figure 5.4.1: The eigenvalues of $G(j\omega) + G^\top(-j\omega)$ as a function of ω in Example 5.4.1.

where we used the identities (5.41), (5.42), and (5.43). This implies that $V(x)$ is a CQLF for the switched linear system (5.35). Hence, the proof of Theorem 5.4.1 is completed. □

Note that as A and $A - BD^{-1}C$ in Theorem 5.4.1 are both Hurwitz, we have $\det(A(A - BD^{-1}C)) \neq 0$. This indicates that zero can never be in the spectrum of $A(A - BD^{-1}C)$. Moreover, the next corollary is a direct result of the proof of Theorem 5.4.1.

Corollary 5.4.1. *Given two Hurwitz matrices A and $A - BD^{-1}C$ with (A, B) controllable, (C, A) observable, and the symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$, the switched system* (5.35) *is quadratically stable if and only if $G(s)$ is SPR.*

Example 5.4.1. (Kouhi et al., 2013a) Consider the LTI systems

$$\dot{x} = A_1 x = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x, \quad \dot{x} = A_2 x = \begin{bmatrix} -1 & -1 \\ -0.5 & -3 \end{bmatrix} x,$$

where A_1 and A_2 are Hurwitz, and $A_1 - A_2$ has rank 2. Note that with $A := A_1$,

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \; C = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

we have that $A_2 = A - BD^{-1}C$. Furthermore, the transfer function matrix

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} \frac{s+1}{s+2} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{2(s+3)}{s+2} \end{bmatrix}$$

is symmetric. In addition, it is easily verified that (A, B) and (C, A) are controllable and observable, respectively.

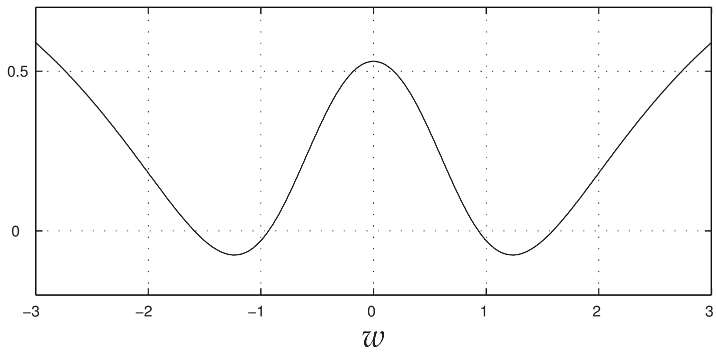

Figure 5.4.2: The minimum eigenvalue of $G(j\omega) + G^{\top}(-j\omega)$ in Example 5.4.2.

The eigenvalues of the matrix $G(j\omega) + G^{\top}(-j\omega)$ are depicted in Figure 5.4.1 as functions of ω. It is evident from this figure that $G(s)$ is SPR. Consequently, using the KYP lemma, quadratic stability of the switched system (5.35) can be deduced.

Alternatively, a much simpler method of establishing the above conclusion is to use the spectral condition presented in Theorem 5.4.1. To this end, note that the eigenvalues of the matrix product $A(A - BD^{-1}C)$ are 1.55 and 6.45, respectively. Consequently, from our discussion, $G(s)$ is SPR and the switched system $\dot{x} = (A - \sigma(t)BD^{-1}C)x$ is quadratically stable for $\sigma(t) \in \{0, 1\}$.

Example 5.4.2. (Kouhi et al., 2013a) Consider now the following two LTI systems:

$$\dot{x} = A_1 x = \begin{bmatrix} -1 & 0 \\ 5 & -1 \end{bmatrix}, \; \dot{x} = A_2 x = \begin{bmatrix} 1.7 & -1.38 \\ 8 & -5.2 \end{bmatrix} x.$$

A_1 and A_2 are Hurwitz, $A_1 - A_2$ has rank 2, and with $A := A_1$,

$$B = \begin{bmatrix} 1 & 2 \\ 10 & 5 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0.2 \\ -1 & 0.04 \end{bmatrix}, \; D = \begin{bmatrix} 2 & 0 \\ 0 & 0.625 \end{bmatrix},$$

it follows that $A_2 = A - BD^{-1}C$. Moreover, the transfer function matrix associated with A_1 and A_2 is symmetric

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{(s+1)^2} \begin{bmatrix} 2s^2 + 7s + 6 & 5 + 3s \\ 5 + 3s & 0.625s^2 + 1.25s + 4.625 \end{bmatrix}.$$

In addition, it is easily verified that (A, B) and (C, A) are controllable and observable, respectively.

Figure 5.4.2 depicts a part of the minimum eigenvalue of $G(j\omega) + G^{\top}(-j\omega)$ as a function of ω. Clearly, the transfer function matrix is not SPR as the eigenvalue is less than zero for some frequencies. In this case, the KYP lemma cannot be used to deduce existence of a CQLF.

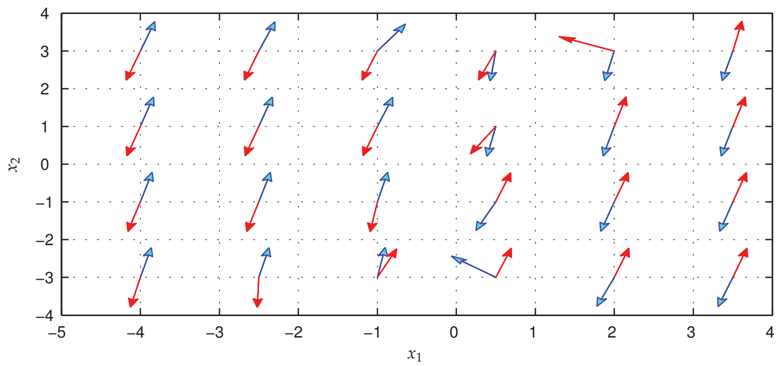

Figure 5.4.3: Directions of the vector fields for the two subsystems in Example 5.4.3.

Alternatively, note that the eigenvalues of $A(A - BD^{-1}C)$ are $-1.5, -1.5$. Since the eigenvalues are negative, it follows that $G(s)$ cannot be SPR, and also that a CQLF cannot exist. Consequently, the switched system $\dot{x} = (A - \sigma(t)BD^{-1}C)x$ is not quadratically stable for all $\sigma(t) \in \{0, 1\}$.

Example 5.4.3. (Kouhi et al., 2013a) Now, consider the pair of matrices defined in Example 5.4.2 and the associated switched system:

$$\dot{x} = A_{\sigma(t)}x(t) \qquad A_{\sigma(t)} \in \{A_1^{-1}, A_2\}.$$

Since the matrix product has a real negative eigenvalue, it follows that the determinant of $\omega^2 A_1^{-1} + A_2$ is zero for some finite ω and the vector fields of the systems $\dot{x} = A^{-1}x$ and $\dot{x} = (A - BD^{-1}C)x$ consists of some points that the angles between the vector fields are 180° degree; see Figure 5.4.3. Roughly speaking, by switching sufficiently fast between the vector fields associated with these matrices in the context of the above switched system, one should arrive at a situation where the state does not converge to the origin for an appropriate initial condition. To verify this, we use Floquet theory (Khalil, 2002) under the assumption of periodic switching. Note that

$$e^{A_1^{-1}t_1}e^{A_2t_2} = (I + A_1^{-1}t_1 + \cdots)(I + A_2t_2 + \cdots) \approx I + A_1^{-1}t_1 + A_2t_2$$

has an eigenvalue whose magnitude is greater than unity for small t_1 and t_2. For example, with $t_1 = 0.0016$ and $t_2 = 0.001$ the eigenvalues of the product of the exponentials are $1.0001, 0.9933$. Since one of the eigenvalues is greater than unity, we have an unstable switching sequence.

5.4.2 Weak quadratic stability

Consider again the switched linear system (5.35), where A, B, C, and D are defined in accordance with the assumptions stated in Section 5.4.1. In this part, we assume A is Hurwitz, but $A - BD^{-1}C$ has all eigenvalues with negative real parts except some eigenvalues to be equal to zero. We want to study under which conditions the switched system (5.35) possesses a weak CQLF in the sense that there exists a function $V(x) = x^\top Px$ with $P = P^\top > 0$, such that

$$A^\top P + PA < 0, \tag{5.45}$$

$$(A - BD^{-1}C)^\top P + P(A - BD^{-1}C) \leq 0. \tag{5.46}$$

If such a CQLF exists, then all solutions of the switched system (5.35) are bounded. For this problem we cannot directly exploit Theorem 5.4.1 due to the existence of the equality sign in (5.46). In the next lemma we demonstrate that if (5.45) and (5.46) hold, then $A - BD^{-1}C$ can have at most m eigenvalues equal to zero without having any generalized eigenvector. In addition, $A - BD^{-1}C$ cannot have any eigenvalue on the $j\omega$ axis for $\omega \neq 0$ provided that $G(j\omega) = C(j\omega I - A)^{-1}B + D$ is symmetric for all $\omega \in \mathbb{R}$.

Lemma 5.4.1. *(Kouhi et al., 2014) If* (5.45) *and* (5.46) *for a symmetric positive definite P hold, $G(s) = C(sI - A)^{-1}B + D$ is symmetric, A is Hurwitz, and D is positive definite, then $A - BD^{-1}C$ cannot have any eigenvalue on the $j\omega$ axis for $\omega \in \mathbb{R}\backslash\{0\}$. Moreover, $A - BD^{-1}C$ can have at most m eigenvalues equal to* 0*, and all zero eigenvalues must have linearly independent eigenvectors.*

Proof: Suppose $A-BD^{-1}C$ has a non-zero eigenvalue on the $j\omega$ axis, that is, there exists an eigenvalue $j\omega_1$ with $\omega_1 \neq 0$. Note that A being Hurwitz implies $\det(j\omega_1 I - A) \neq 0$, and $D > 0$ implies $\det(D^{-1}) \neq 0$. Then, we have

$$\begin{aligned}
&\det\big(j\omega_1 I - (A - BD^{-1}C)\big) = 0\\
&\Rightarrow \det(j\omega_1 I - A)\det\big(I + (j\omega_1 I - A)^{-1}BD^{-1}C\big) = 0\\
&\Rightarrow \det(D^{-1})\det\big(D + C(j\omega_1 I - A)^{-1}B\big) = 0\\
&\Rightarrow \det\big(D + C(j\omega_1 I - A)^{-1}B\big) = 0,
\end{aligned} \tag{5.47}$$

which implies that the matrix $G(j\omega_1)$ is singular. Let's denote the complex eigenvector corresponding to eigenvalue 0 of $G(j\omega_1)$ by $v \in \mathbb{C}^m$. Similarly, we can show that the matrix $G(-j\omega_1) = G^\top(-j\omega_1) = G^*(j\omega_1)$ is also singular and has a left eigenvector v^* corresponding to the eigenvalue 0. Thus, recalling Lemma 5.2.4, the following holds:

$$\begin{aligned}
&\frac{1}{2}v^*[G(j\omega_1) + G^\top(-j\omega_1)]v =\\
&\qquad = v^*[D - C(\omega_1^2 I + A^2)^{-1}AB]v = 0.
\end{aligned} \tag{5.48}$$

On the other hand, similar to the situation discussed in the proof of necessity in Theorem 5.3.1, a necessary condition for (5.45) and (5.46) to hold is that the matrix $\omega^2 A^{-1} + (A - BD^{-1}C)$ for $\omega \neq 0$ is non-singular. Hence, by continuity of $\det(\omega^2 I + A(A - BD^{-1}C))$ with respect to ω and the fact that $\det(\omega^2 I + A^2) > 0$, we can deduce

$$\begin{aligned}
&\det\big(\omega^2 I + A(A - BD^{-1}C)\big) > 0\\
&\quad\Rightarrow \det\big(\omega^2 I + A^2\big)\ \det(D - C(\omega^2 I + A^2)^{-1}AB) > 0\\
&\quad\Rightarrow \det\left(\frac{1}{2}\{G(j\omega) + G^\top(-j\omega)\}\right) > 0.
\end{aligned}$$

Now, continuity of the determinant with respect to ω indicates that $G(j\omega) + G^\top(-j\omega)$ is positive definite for all $\omega \in \mathbb{R}\backslash\{0\}$, in particular for $\omega = \omega_1$. However, this is a contradiction with (5.48).

Now, we prove $A - BD^{-1}C$ can have at most m eigenvalues equal to 0. This proof is similar to the proof given in Shorten et al. (2009). Since the inequality (5.46) holds, the system $\dot{x} = (A - BD^{-1}C)x$ is stable, that is, the algebraic multiplicity and geometric multiplicity of the zero eigenvalues are the same. Now, suppose that we have more than m linearly independent eigenvectors corresponding to zero eigenvalues. Let us define the functions $g_1(x) = x^\top(A^\top P + PA)x$ and $g_2(x) = x^\top\left((A - BD^{-1}C)^\top P + P(A - BD^{-1}C)\right)x$. Obviously, at least for $n - m$ linearly independent vectors $v_i \in \mathbb{R}^n$ with $i \in \{1, \dots, n-m\}$, we have $g_1(v_i) = g_2(v_i)$, for instance, for those v_i's that satisfy $v_i^\top PB = 0$. Suppose there are more than m linearly independent vectors x such that $g_2(x)$ is zero. Then, the two spaces $\{x : g_2(x) = 0\}$ and $\{x : g_2(x) = g_1(x)\}$ intersect at least at one non-zero point, implying that there exists a non-zero vector x such that

$$g_1(x) = g_2(x) = 0.$$

The condition $g_1(x) = 0$ is in contradiction with the choice of P in (5.45). □

Now, the following theorem provides sufficient and necessary conditions for the existence of a weak CQLF for the switched system (5.35).

Theorem 5.4.2. *(Kouhi et al., 2014) Assume A is Hurwitz and all eigenvalues of the matrix $A - BD^{-1}C$ have negative real parts or are zero. Furthermore, assume that the zero eigenvalue has a multiplicity of $m - p$, and associated with the zero eigenvalue is a full set of $m - p$ linearly independent eigenvectors. Suppose also that*

$$CA^iB = (CA^iB)^\top \qquad \forall i \in \{0, 1, \dots, n-1\}, \tag{5.49}$$

$D = D^\top > 0$, (A, B) *is controllable, and* (C, A) *is observable. Then, the switched system* (5.35) *is weakly quadratically stable if and only if* $A(A - BD^{-1}C)$ *has no real negative eigenvalue and has exactly* $m - p$ *eigenvalues equal to zero.*

Proof of sufficiency: Recall that Lemma 5.2.1 implies that the symmetry conditions (5.49) and $D = D^\top$ suffice for the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ to be symmetric. The remainder of the proof of sufficiency consists of two parts. In Part A, we prove that $G(0) \geq 0$ and $\text{rank}(G(0)) = p$. Afterwards, in Part B we use $G(\frac{1}{s})$ in combination with the KYP lemma to complete the proof of sufficiency.

Part A: Following the proof of sufficiency in Theorem 5.3.1, $A(A - BD^{-1}C)$ having no real negative eigenvalue implies that

$$\begin{aligned} &\det(\omega^2 I + A(A - BD^{-1}C)) > 0 \\ &\qquad \Rightarrow \det(D - C(\omega^2 I + A^2)^{-1}AB) > 0 \qquad \forall \omega \in \mathbb{R} \setminus \{0\}. \end{aligned} \tag{5.50}$$

Therefore, analogous to the proof of sufficiency in Theorem 5.3.1, we can argue that

$$\frac{1}{2}\{G(j\omega) + G^\top(-j\omega)\} = D - C(\omega^2 I + A^2)^{-1}AB > 0 \qquad \forall \omega \in \mathbb{R} \setminus \{0\}.$$

By continuity of the eigenvalues with respect to ω around zero, we conclude that the matrix

$$G(0) = \lim_{\omega \to 0} \frac{1}{2}\{G(j\omega) + G^\top(-j\omega)\} = D - CA^{-1}B$$

does not have any eigenvalue in the open left half plane, that is, $G(0) = D - CA^{-1}B \geq 0$. Next, we prove that $G(0)$ has exactly $m - p$ eigenvalues equal to zero. To this end, consider the equality

$$(A - BD^{-1}C)A^{-1}B = BD^{-1}(D - CA^{-1}B),$$

then exploiting the Sylvester rank inequality (see Appendix A.2.5.1 and Horn and Johnson (1990)), we have

$$\begin{aligned}\operatorname{rank}\big(BD^{-1}(D - CA^{-1}B)\big) &= \operatorname{rank}\big((A - BD^{-1}C)A^{-1}B\big) \geq \\ &\operatorname{rank}(A - BD^{-1}C) + \operatorname{rank}(A^{-1}B) - n = \\ &[n - (m - p)] + m - n = p.\end{aligned}$$

As $\operatorname{rank}(BD^{-1}) = m \geq p$, we must have $\operatorname{rank}(D - CA^{-1}B) \geq p$. On the other hand, $A - BD^{-1}C$ has exactly $m - p$ eigenvalues equal to zero with a full set of eigenvectors, so there exists a matrix $\mathcal{W}^\top \in \mathbb{R}^{(m-p)\times n}$ constructed from the left eigenvectors of $A - BD^{-1}C$ corresponding to the zero eigenvalue, such that $\mathcal{W}^\top(A - BD^{-1}C) = 0$. Note that $\operatorname{rank}(\mathcal{W}^\top BD^{-1}) = m - p$ since $\mathcal{W}^\top A$ has full rank and $\mathcal{W}^\top A = \mathcal{W}^\top BD^{-1}C$. It thus follows from

$$\mathcal{W}^\top(A - BD^{-1}C)A^{-1}B = 0 \Rightarrow \mathcal{W}^\top BD^{-1}(D - CA^{-1}B) = 0 \tag{5.51}$$

that the columns of $D^{-1}B^\top\mathcal{W} \in \mathbb{C}^{m\times(m-p)}$ are indeed left eigenvectors of $G(0) = D - CA^{-1}B$ corresponding to the zero eigenvalues. Note also that the symmetry of $G(0)$ excludes the possibility of the existence of any generalized eigenvector for $G(0)$. Consequently, as rank of $G(0) \geq p$ and (5.51) holds, we have $\operatorname{rank}(D - CA^{-1}B) = p$, indicating that this matrix has exactly $m - p$ zero eigenvalues.

Part B: Define the system $\bar{G}(s) := G(\frac{1}{s}) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$, with the state space representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$,

$$\bar{A} = A^{-1},\ \bar{B} = -A^{-1}B,\ \bar{C} = CA^{-1}, \bar{D} = D - CA^{-1}B.$$

Consequently, the symmetry of $\bar{G}(s)$ follows from symmetry of $G(s)$. On the other hand, $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{A})$ by construction are controllable and observable, respectively, and the matrix $\bar{D} \geq 0$ has rank p. Now, Theorem 5.3.2 implies that $A(A - BD^{-1}C)$ having no real negative and exactly $m - p$ zero eigenvalues, is equivalent to $\bar{G}(s)$ being SPR. Then, the Kalman-Yakubovic-Popov lemma (see Zhou et al. (1996) and Appendix A.4.1) implies that there exist a scalar $\alpha > 0$ and matrices $P = P^\top > 0$, L, and W such that

$$\bar{A}^\top P + P\bar{A} = -L^\top L - \alpha P, \tag{5.52}$$

$$\bar{B}^\top P + W^\top L = \bar{C}, \tag{5.53}$$

$$\bar{D} + \bar{D}^\top = W^\top W. \tag{5.54}$$

The first equation in (5.52) ensures that

$$A^{-\top}P + PA^{-1} < 0. \tag{5.55}$$

By pre- and post-multiplying of this equation by $A^\top$ and A, respectively, we get $A^\top P + PA < 0$, which verifies that (5.45) holds. Next, we show that (5.46) also holds. To this end, first we claim that

$$\begin{bmatrix} A^{-\top}P + PA^{-1} & P\bar{B} - \bar{C}^\top \\ \bar{B}^\top P - \bar{C} & -(\bar{D} + \bar{D}^\top) \end{bmatrix} \leq 0. \tag{5.56}$$

Note that $\bar{D} \geq 0$ and we shall demonstrate that the Schur complement of the above matrix is less than or equal to zero (see Appendix A.2.2), in other words,

$$\bar{\mathcal{S}} = -(\bar{D} + \bar{D}^\top) - (\bar{B}^\top P - \bar{C})(A^{-\top}P + PA^{-1})^{-1}(P\bar{B} - \bar{C}^\top) \leq 0.$$

Using equations (5.52-5.54), we get

$$\begin{aligned} \bar{\mathcal{S}} &= -(\bar{D} + \bar{D}^\top) - (\bar{B}^\top P - \bar{C})(A^{-\top}P + PA^{-1})^{-1}(P\bar{B} - \bar{C}^\top) \\ &= -W^\top W + W^\top L\,(\alpha P + L^\top L)^{-1}L^\top W = \\ &= -W^\top \left[I - L(\alpha P + L^\top L)^{-1}L^\top\right] W. \end{aligned} \tag{5.57}$$

In (5.57), the matrix $I - L(\alpha P + L^\top L)^{-1}L^\top$ is the Schur complement of the positive definite matrix

$$\begin{bmatrix} \alpha P + L^\top L & L^\top \\ L & I \end{bmatrix}$$

with respect to the upper left block, which is also positive definite and subsequently $\bar{\mathcal{S}} < 0$. Now, let reformulate (5.56) as

$$\begin{aligned} &\begin{bmatrix} A^{-1} & \bar{B} \\ -\bar{C} & -\bar{D} \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} + \\ &\qquad \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & \bar{B} \\ -\bar{C} & -\bar{D} \end{bmatrix} \leq 0. \end{aligned} \tag{5.58}$$

Since $\bar{D} - \bar{C}A\bar{B} = D$, we have

$$\begin{aligned} &\begin{bmatrix} A^{-1} & \bar{B} \\ -\bar{C} & -\bar{D} \end{bmatrix}^{-1} = \\ &\begin{bmatrix} A - A\bar{B}(-\bar{D} + \bar{C}A\bar{B})^{-1}\bar{C}A & -A\bar{B}(-\bar{D} + \bar{C}A\bar{B})^{-1} \\ (-\bar{D} + \bar{C}A\bar{B})^{-1}\bar{C}A & (-\bar{D} + \bar{C}A\bar{B})^{-1} \end{bmatrix} \\ &\qquad = \begin{bmatrix} A - BD^{-1}C & -BD^{-1} \\ -D^{-1}C & -D^{-1} \end{bmatrix}; \end{aligned} \tag{5.59}$$

see Appendix A.2.1. By pre-multiplying the inequality (5.58) by the transpose of the above matrix and by post-multiplying by the above matrix itself, we end up with

$$\begin{bmatrix} (A - BD^{-1}C)^\top P + P(A - BD^{-1}C) & -(PB + C^\top)D^{-1} \\ -D^{-1}(B^\top P + C) & -2D^{-1} \end{bmatrix} \leq 0,$$

from which it immediately follows that

$$(A - BD^{-1}C)^\top P + P(A - BD^{-1}C) \leq 0. \tag{5.60}$$

Thus, we can say that $V(x) = x^\top Px$ is a weak Lyapunov function for both $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$. This establishes the proof of sufficiency.

Proof of necessity: The proof of the fact that $A(A - BD^{-1}C)$ does not have any non-zero real negative eigenvalue is similar to the proof of necessity given in Theorem 5.4.1.

The last part of the proof is concerned with proving that the product AA_2, with $A_2 = A - BD^{-1}C$, has exactly $m - p$ zero eigenvalues equal to zero. To show this holds, first notice that AA_2 has exactly $m - p$ zero eigenvalues with a full set of eigenvectors. This fact can be derived by considering that

$$\text{rank}(AA_2) = \text{rank}(A_2).$$

Therefore, if AA_2 has more than $m - p$ zero eigenvalues, then it must contain at least one generalized eigenvector, say $v_2 \in \mathbb{R}^n$ satisfying $AA_2v_2 = v_1$, and $AA_2v_1 = 0$; see Appendix A.2.7. This implies that

$$A_2v_2 = A^{-1}v_1, \text{ and } A_2v_1 = 0. \tag{5.61}$$

Moreover, it can be inferred from the inequality $A^{-\top}P + PA^{-1} < 0$ that the function $f(x) = x^\top(A^{-\top}P + PA^{-1})x$ is always negative for all non-zero $x \in \mathbb{R}^n$, in particular for $x = v_1$. Recalling (5.61), we have

$$\begin{aligned} &v_1^\top(A^{-\top}P + PA^{-1})v_1 < 0 \\ &\qquad \Rightarrow v_2^\top A_2^\top Pv_1 + v_1^\top PA_2v_2 < 0. \end{aligned} \tag{5.62}$$

On the other hand, with regard to (5.46) the inequality $g(x) = x^\top(A_2^\top P + PA_2)x \leq 0$ must hold for all $x \in \mathbb{R}^n$. Now, choosing $x = \beta v_1 + v_2$ with $\beta \in \mathbb{R}$ as a parameter and considering $A_2v_1 = 0$, we should have

$$\begin{aligned} &(\beta v_1 + v_2)^\top(A_2^\top P + PA_2)(\beta v_1 + v_2) \leq 0 \\ &\Rightarrow \beta(v_2^\top A_2^\top Pv_1 + v_1^\top PA_2v_2) + v_2^\top(A_2^\top P + PA_2)v_2 \leq 0. \end{aligned} \tag{5.63}$$

As the inequalities $e = v_2^\top A_2^\top Pv_1 + v_1^\top PA_2v_2 < 0$ and $g(v_2) = v_2^\top(A_2^\top P + PA_2)v_2 \leq 0$ both hold, there exists always a $\beta \in \mathbb{R}$ which makes the expression $\beta e + g(v_2)$ positive. For instance, one can choose $\beta < -g(v_2)/e$. However, this is evidently in contradiction with (5.63). □

Example 5.4.4. (Kouhi et al., 2014) Consider two matrices

$$A_1 = \begin{bmatrix} -1 & 0 \\ 5 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & -1 \\ 0 & -5 \end{bmatrix}.$$

A_1 is Hurwitz and A_2 has all eigenvalues in the open left half plane except one eigenvalue at zero. In addition $A_1 - A_2$ has rank two, and with $A = A_1$

$$B = \begin{bmatrix} 1 & 2 \\ 10 & 5 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0.2 \\ -1 & 0.4 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have $A_2 = A - BD^{-1}C$. Furthermore, the transfer function matrix

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{(s+1)^2} \begin{bmatrix} s^2 + 5s + 5 & 3s + 5 \\ 3s + 5 & s^2 + 2s + 5 \end{bmatrix}$$

is symmetric. Note that (A, B) and (C, A) are controllable and observable, respectively. The eigenvalues of the matrix A_1A_2 are $0, 0$ indicating that there is no weak CQLF for the switched system associated with the two matrices A_1 and A_2.

Example 5.4.5. (Kouhi et al., 2014) Consider two matrices

$$A_1 = \begin{bmatrix} -0.9 & -0.1 \\ 4.5 & -1.4 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & -1 \\ 0 & -5 \end{bmatrix}.$$

A_1 is Hurwitz and A_2 has all eigenvalues in the open left half plane except one eigenvalue at zero. In addition $A_1 - A_2$ has rank two, and with $A = A_1$,

$$B = \begin{bmatrix} 0.9 & 1.8 \\ 9 & 4.5 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0.2 \\ -1 & 0.4 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have $A_2 = A - BD^{-1}C$. Furthermore, the transfer function matrix

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{s^2 + 2.3s + 1.71} \begin{bmatrix} s^2 + 5s + 4.5 & 2.7s + 4.5 \\ 2.7s + 4.5 & s^2 + 2.3s + 4.5 \end{bmatrix}$$

is symmetric. Note that (A, B) and (C, A) are controllable and observable, respectively. The eigenvalues of the matrix A_1A_2 are $0, 2.5$, implying that there is a weak CQLF $V(x) = x^\top Px$ for the switched system (5.35). For instance, one can choose

$$P = \begin{bmatrix} 1.48 & -0.296 \\ -0.296 & 0.3089 \end{bmatrix}.$$

Example 5.4.6. (Kouhi et al., 2014) Consider the switched electrical circuit illustrated in Figure 5.4.4. The variables $v_1(t)$ and $v_2(t)$ are voltages of two capacitors with capacities C_1 and C_2, respectively. Let $x(t) := [v_1(t) \; v_2(t)]^\top$ indicate the vector of the system states. Depending on the status of the switch S, the system model can be represented by a switched linear system with two modes $\dot{x} = A_i x$ for $i \in \{1, 2\}$, where

$$A_1 = \begin{bmatrix} \frac{-1}{R_1C_1} & \frac{1}{R_1C_1} \\ \frac{1}{R_1C_2} & \frac{-(R_1+R_2)}{R_1R_2}\frac{1}{C_2} \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{R_2C_2} \end{bmatrix}. \tag{5.64}$$

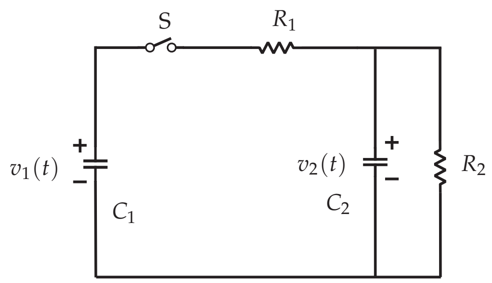

Figure 5.4.4: A switched electrical circuit.

In (5.64) A_1 refers to the situation when S is closed and A_2 refers to the condition when S is open. Note that A_1 is Hurwitz and A_2 is singular including exactly one zero eigenvalue and one real negative eigenvalue. In addition, A_1 and A_2 have rank one difference. Defining $A := A_1$, $B := [-1/C_1 \ 1/C_2]^\top$, $C := [1/R_1 \ -1/R_1]$, and $D := I$, we have $A_2 = A - BD^{-1}C$ and the transfer function $G(s) = C(sI - A)^{-1}B + D$ is trivially symmetric. Furthermore, (A, B) and (C, A) are controllable and observable, respectively. Now, consider that the matrix

$$A_1 A_2 = \begin{bmatrix} 0 & \frac{-1}{R_1 R_2 C_1 C_2} \\ \\ 0 & \frac{(R_1+R_2)}{R_1 R_2^2 C_2^2} \end{bmatrix}$$

has no real negative eigenvalue and has exactly one zero eigenvalue. Hence, according to Theorem 5.4.2 this electrical circuit is weakly quadratically stable, implying that the voltages of both capacitors remain bounded in spite of all possible switching events and each initial condition.

5.5 Stabilization of controlled switched linear systems

In this part, we employ the theory elaborated for stability of switched systems associated with symmetric transfer functions, for stabilization of controlled switched linear systems. We consider the problem of stabilization of a class of switched linear systems defined by

$$\dot{x} = Ax + \sigma(t)Bu \qquad \sigma(t) \in \{0, 1\}, \tag{5.65}$$

where $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix, $B \in \mathbb{R}^{n\times m}$, and the pair (A, B) is controllable. The goal is to find a set of feedback controls in the form of $u = Kx$, such that the switched system

$$\dot{x} = (A + \sigma(t)BK)x \qquad \sigma(t) \in \{0, 1\}, \tag{5.66}$$

is quadratically stable, that is, there exists a matrix $P = P^\top > 0$ such that

$$(A + \sigma(t)BK)^\top P + P(A + \sigma(t)BK) < 0 \qquad \sigma(t) \in \{0, 1\}. \tag{5.67}$$

For this purpose, we suppose K has a structure of the form $K = -D^{-1}C$. This modifies (5.66) into the form

$$\dot{x} = (A - \sigma(t)BD^{-1}C)x \qquad \sigma(t) \in \{0, 1\}. \tag{5.68}$$

Now, our wish is to design the unknown matrices D and C by utilizing Theorem 5.4.1. To this end, we require $D = D^\top > 0$, and the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ to be symmetric and to be strictly positive real. Referring to Lemma 5.2.2 the transfer function $G(s)$ is symmetric and has the minimal state space realization (A, B, C, D) if and only if there exists a non-singular matrix $S = S^\top$ such that

$$A^\top S = SA, \quad B^\top S = C, \text{ and } D = D^\top. \tag{5.69}$$

Therefore, defining $Y = S^{-1}$, the matrix A satisfies the following Sylvester equation

$$AY - YA^\top = 0. \tag{5.70}$$

For finding a solution of this equation, similar to the proof of Lemma 5.2.3 we use the Kronecker product and the vectorization operator $\text{vec}(Y) = [y_{11} \ \dots y_{n1} \ y_{12} \ \dots y_{n2} \ \dots \ y_{1n} \ \dots \ y_{nn}]^\top$. We can then reformulate (5.70) in the form of

$$(I_n \otimes A - A \otimes I_n) \ \text{vec}(Y) = 0. \tag{5.71}$$

Define $S := Y^{-1}$ and $C := B^\top S$. As (A, B) is controllable, (C, A) is also observable. Furthermore, we must have $G(j\omega) + G^\top(-j\omega) > 0$. This is equivalent to the condition

$$\frac{1}{2}\{G(j\omega) + G^\top(-j\omega)\} = D - C(\omega^2 I + A^2)^{-1}AB > 0. \tag{5.72}$$

Therefore, a sufficient condition for the last equation to hold is that

$$D > \left(\sup_\omega \sigma_{\max}\left(C(\omega^2 I + A^2)^{-1}AB\right)\right).I_m. \tag{5.73}$$

Note that $C(\omega^2 I + A^2)^{-1}AB$ is finite as $\omega^2 I + A^2$ is always non-singular, and $\lim_{\omega\to\infty} C(\omega^2 I + A^2)^{-1}AB = 0$. Consequently, the right hand side of the inequality (5.73) is always finite. This verifies that there exists infinitely many matrices $D = D^\top > 0$ such that the inequality (5.73) holds true. This shows that the switched system (5.66) is always quadratically stabilizable.

In a different context using a distinct approach, output feedback design for an LTI system has been introduced for making the closed loop system to be SPR (Huang et al., 1999).

Remark 5.5.1. Assume a Hurwitz matrix A is given. One advantage of using the discussed approach is to specify a set of matrices A_2 which differ by a rank-m matrix and share a common Lyapunov solution with A.

Example 5.5.1. Suppose we want to find a set of feedbacks $u = Kx$, where $K \in \mathbb{R}^{2\times 2}$, which stabilizes the switched linear system (5.66) with the data

$$A = \begin{bmatrix} -0.9 & -0.1 \\ 4.5 & -1.4 \end{bmatrix}, \ B = \begin{bmatrix} 0.9 & 1.8 \\ 9 & 4.5 \end{bmatrix}.$$

Note that A is Hurwitz and (A, B) is controllable. Now, a matrix S satisfying $A^\top S - SA = 0$ equals

$$S = \begin{bmatrix} -0.0025 & 0.7049 \\ 0.7049 & -0.0783 \end{bmatrix}.$$

This allows us to compute C as

$$C = B^\top S = \begin{bmatrix} 6.3422 & -0.07 \\ 3.1677 & 0.9167 \end{bmatrix}.$$

Now, we derive an appropriate matrix D with regard to (5.73). Computing $\sup_\omega \sigma_{\max}\left(C(\omega^2 I + A^2)^{-1}AB\right) = 28.41$, we can pick D as any two dimensional symmetric positive matrix whose minimum eigenvalue is greater than 28.41. For instance, we can choose

$$D = \begin{bmatrix} 30 & 0 \\ 0 & 40 \end{bmatrix}.$$

Then, the resulting state feedback gain and the closed loop matrix are

$$K = -D^{-1}C = \begin{bmatrix} -0.2114 & 0.0023 \\ -0.0792 & -0.0229 \end{bmatrix}, \quad \text{and } A - BD^{-1}C = \begin{bmatrix} -1.2328 & -0.1392 \\ 2.2410 & -1.4821 \end{bmatrix}.$$

The eigenvalues of $A(A - BD^{-1}C)$ are equal to $1.170 \pm 1.515i$, which confirms that a CQLF $V(x) = x^\top Px$ for the switched linear system (5.66) exists. Using the MATLAB LMI Toolbox one such P can be computed as

$$P = \begin{bmatrix} 5.6408 & -1.2073 \\ -1.2073 & 1.4448 \end{bmatrix}.$$

5.6 Conclusions

In this chapter, we explored conditions under which a symmetric transfer function matrix can be associated with a given pair of Hurwitz matrices. Furthermore, we showed that if the symmetric transfer function matrix associated with this pair is strictly positive real, then a common Lyapunov solution for such pair of matrices exists. On this basis, we introduced an approach for stabilization of a class of switched linear systems whose constituent matrices differ by a rank m matrix. In the case that one matrix is Hurwitz, and the other one has all eigenvalues with negative real parts and some eigenvalues equal to zero, we extended our result in terms of a weak common quadratic Lyapunov function.

Chapter 6

Control of hybrid linear systems

6.1 Introduction

In this chapter, we investigate control of hybrid linear systems including linear flow and jump dynamics. We start with stability of such hybrid systems and establish a link between their stability and stability of switched linear systems discussed in the previous chapters. Moreover, we derive convenient conditions for robust stability of such hybrid systems. It turns out that the concept of stability for hybrid linear systems is very similar to the concept of stability for switched linear systems. In addition, we study some problems concerning optimal control of hybrid linear systems.

In the first optimal control problem in Section 6.4, we find analytic expressions for lower and upper bounds of the optimal value of the cost function for a linear quadratic problem in a class of hybrid linear systems. The optimization problem involves state space constraints and switches between the continuous evolution and jumps at fixed time instances on the boundaries of flow and jump sets. The basic idea for our approach uses analytical solutions to continuous- and discrete-time LQR problems with fixed initial and final states over fixed time intervals; see, for instance, Lewis and Syrmos (1995); Bryson and Ho (1975). We show that the optimal cost of each time interval can be represented by its initial and final state variables. Using this fact, we propose a parameterization method with states corresponding to the initial and end points of each generalized time domain as decision variables, and show that we can reformulate the problem as a static optimization problem. The lower bound for the optimal value of the cost function can be computed analytically, whereas the upper bound can be computed numerically via solving a quadratic programming problem. Throughout Section 6.4, we use the restrictive assumption that switching times are fixed.

Therefore, in Section 6.5, we tackle a free switching times optimal control problem for a class of hybrid linear systems. We employ the same approach as the one for the fixed switching times optimal control problem to describe the switching points with respect to switching time instances. Moreover, we show that the problem of finding switching time instances gives rise to solving a differential algebraic equation with given boundary conditions. The resulting problem, of course, can only be solved numerically.

6.2 Stability of hybrid linear systems with single flow and jump dynamics

Consider a simple form of hybrid linear systems consisting of a single flow and a jump dynamics given by

$$\mathcal{H} : \begin{cases} \dot{x} &= Ax \qquad x \in C, \\ x^{+} &= Gx \qquad x \in D, \end{cases} \tag{6.1}$$

where the sets $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ denote the *flow* and *jump* sets, respectively. In general, C and D can be open or closed sets, but in this section we assume both are closed. Furthermore, we assume $C, D \subseteq \mathbb{R}^n$, $C \cup D = \mathbb{R}^n$, A is Hurwitz, and all eigenvalues of G are inside the unit circle.

Given a number $K \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the domain of x for such a system is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}_0$ defined by time instances

$$0 := t_0 \leq t_1 \leq t_2 \ldots \leq t_K, \tag{6.2}$$

as follows:

$$\mathbf{T}_K := \bigcup_{k=0}^{K-1} \mathbb{T}_k,$$

where

$$\mathbb{T}_k := [t_k, t_{k+1}] \times \{k\} \qquad \forall k \in \{0, \ldots, K-1\}. \tag{6.3}$$

This time domain is called "hybrid time domain"; see, *e.g.*, Goebel et al. (2012). Note that in (6.2) K and t_K are allowed to go beyond all bounds.

Given a number $K \in \mathbb{N}_0$ and $x_0 := x(t_0, 0) \in C \cup D$, we say $x : \mathbf{T}_K \to \mathbb{R}^n$ is a solution of the system (6.1) if

a) for each $(t, k) \in \mathbb{T}_k$ where (t_k, t_{k+1}) has nonempty interior and $x(t, k) \in C$, the derivative of $x(t, k)$ with respect to t is defined and the following differential equation holds

$$\frac{d}{dt} x(t, k) = Ax(t, k) \qquad \forall t \in (t_k, t_{k+1}).$$

b) for each $(t, k) \in \mathbf{T}_K$ such that $(t, k+1) \in \mathbf{T}_K$, and $x(t_k, k) \in D$, the following difference equation holds

$$x(t_{k+1}, k+1) = Gx(t_{k+1}, k).$$

Note that from item a) in this definition, one can realize that the mapping $(t, k) \mapsto x(t, k)$ is absolutely continuous within any interval $\mathbb{T}_k$ with nonempty interior.

We say the hybrid system $\mathcal{H}$ is quadratically stable if a quadratic function $V(x) = x^\top P x$ with $P = P^\top > 0$ exists such that

$$A^\top P + PA < 0, \tag{6.4}$$

$$G^\top PG - P < 0. \tag{6.5}$$

Analogous to the concept of stability for switched linear systems, one would expect if such a P exists, exponential stability of the system (6.1) is implied. We now give a short proof for this argument. Without loss of generality assume that $x(t,k) \in C$ for some $(t,k) \in \mathbb{T}_k$. For exponential stability there must exist numbers $\alpha, \beta > 0$ such that

$$\|x(t,k)\| \leq \alpha e^{-\beta(t+k-t_0)} x(t_0, 0). \tag{6.6}$$

We show now that (6.6) holds. If both (6.4) and (6.5) hold, there exists a number $0 < \gamma < \lambda_{\max}(P)$, such that

$$A^\top P + PA < -\gamma I,$$
$$G^\top PG - P < -\gamma I.$$

This implies that for each $(t,k) \in \mathbb{T}_k$, and $x \in C$

$$\dot{V}(x) = x^\top (A^\top P + PA)x \leq -\gamma \|x\|^2. \tag{6.7}$$

As P is symmetric positive definite, we can write

$$\lambda_{\min}(P)\|x\|^2 \leq x^\top P x = V(x) \leq \lambda_{\max}(P)\|x\|^2. \tag{6.8}$$

Therefore, defining $\gamma_c := \gamma/\lambda_{\max}(P)$, (6.7) gives us the following upper bound for V:

$$\dot{V}(x) = x^\top (A^\top P + PA)x \leq -\gamma\|x\|^2 \leq \frac{-\gamma}{\lambda_{\max}(P)} V(x)$$

$$\Rightarrow V(x(t,j)) \leq V(x(t - t_j, j)) e^{-\gamma_c(t-t_j)} \qquad \forall (t,j) \in \mathbb{T}_j,\ j \leq k. \tag{6.9}$$

On the other hand, for each $(t,j) \in \mathbf{T}_K$, such that $x(t,j+1) \in \mathbf{T}_K$ and $x(t,j) \in D$ for some $j \leq k$, we have

$$V\left(x(t,j+1)\right) - V\left(x(t,j)\right) = x^\top (G^\top PG - G)x \leq -\gamma\|x(t,j)\|^2.$$

Again, recalling (6.8) we can write

$$-\gamma\|x(t,j)\|^2 \leq -\frac{\gamma V(x(t,j))}{\lambda_{\max}(P)} \qquad \forall (t,j) \in \mathbf{T}_K.$$

Now, defining $\gamma_d := -\ln\left(1 - \gamma/\lambda_{\max}(P)\right) > 0$, we have

$$V\left(x(t,j+1)\right) - V\left(x(t,j)\right) \leq -\frac{\gamma}{\lambda_{\max}(P)} V\left(x(t,j)\right)$$
$$\Rightarrow V\left(x(t,j+1)\right) \leq \left(1 - \frac{\gamma}{\lambda_{\max}(P)}\right) V\left(x(t,j)\right)$$
$$\Rightarrow V\left(x(t,j+1)\right) \leq e^{-\gamma_d} V(x(t,j)) \qquad \forall (t,j), (t,j+1) \in \mathbf{T}_K,\ j < k. \tag{6.10}$$

As a result, combining (6.9) and (6.10), we get

$$V(x(t,k)) \leq V(x(t_0, 0)) e^{-[\gamma_c(t-t_0) + \gamma_d k]}. \tag{6.11}$$

Now, considering (6.8) and defining $\beta := \min(\gamma_c, \gamma_d)$, we can write

$$\|x(t,k)\|^2 \leq \frac{V(x(t_0,0))}{\lambda_{\min}(P)} e^{-\beta(t-t_0+k)} \qquad \forall (t,k) \in \mathbf{T}_K. \tag{6.12}$$

This immediately implies exponential stability of the hybrid linear system (6.1). For such reason, finding a convenient condition under which a CQLF for this system exists is our point of interest. Note that (6.4) and (6.5) represent continuous and discrete Lyapunov inequalities, respectively. Nevertheless, we can use bilinear transformation as proposed in Mason and Shorten (2004) to unify the two inequalities in the form of continuous Lyapunov inequalities.

Lemma 6.2.1. *Let all eigenvalues of G be inside the unit circle. Then, $P = P^\top > 0$ is a solution of the discrete Lyapunov equation*

$$G^\top P G - P = -Q_d \qquad \text{with } Q_d > 0, \tag{6.13}$$

if and only if P is the solution of the following continuous Lyapunov equation

$$C(G)^\top P + P\ C(G) = -\bar{Q}_d, \tag{6.14}$$

where

$$C(G) = (G - I)(G + I)^{-1}, \quad \text{and } \bar{Q}_d = 2(G + I)^{-\top} Q_d (G + I)^{-1} > 0.$$

The mapping $G \mapsto C(G) = (G - I)(G + I)^{-1}$ is known as the bilinear transformation; see Mason and Shorten (2004).

Proof of sufficiency: Note that (6.14) is equivalent to

$$(G + I)^{-\top}(G - I)^\top P + P(G - I)(G + I)^{-1} = -2(G + I)^{-\top} Q_d (G + I)^{-1}. \tag{6.15}$$

By pre- multiplying (6.15) by $(G + I)^\top$ and post- multiplying it by $(G + I)$, we deduce

$$(G - I)^\top P(G + I) + (G + I)^\top P(G - I) = -2Q_d. \tag{6.16}$$

Thus, after simplification (6.13) results. Due to simplicity, proof of necessity is not given. □

Lemma 6.2.1 indicates that for checking the existence of a CQLF for the hybrid linear system (6.1), it is necessary and sufficient to consider the common Lyapunov solution existence problem for the matrix pair $(A, C(G))$. This problem was extensively discussed in Chapters 4 and 5, on stabilization and stability problems for switched linear systems.

6.3 Robust Stability of hybrid linear systems

Assume A is Hurwitz and all eigenvalues of G are inside the unit circle. We define the following concept of robust stability for the hybrid linear system (6.1).

Definition 6.3.1. (See Appendix A.7, Cai et al. (2007); Goebel et al. (2012)) A smooth function $V(x)$ is a robust Lyapunov function for the hybrid linear system (6.1), if there exist two increasing positive definite functions $\alpha_1, \alpha_2 : [0,\infty) \to [0,\infty)$ satisfying

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|), \tag{6.17}$$

$$\langle \nabla V(x), Ax\rangle \le -V(x) \qquad \forall x \in C, \tag{6.18}$$

$$V(Gx) \le e^{-1}V(x) \qquad \forall x \in D. \tag{6.19}$$

Now, the following definition of robustness of the hybrid linear system (6.1) is adopted from the general definition of robustness for hybrid systems introduced, for example, by Cai et al. (2007); Goebel et al. (2012).

Definition 6.3.2. (See Appendix A.7 and Cai et al. (2007); Goebel et al. (2012)) The hybrid linear system (6.1) is said to be robustly asymptotically stable if a continuous perturbation function $\delta : \mathbb{R}^n \to \mathbb{R}^n_{\ge 0}$ with $\delta(x) > 0$ for $x \in \mathbb{R}^n\backslash\{0\}$ exists, such that the perturbed hybrid system given by the differential and difference inclusions

$$\mathcal{H}_\delta := \begin{cases} \dot{x} \in F_\delta(x) & x \in C_\delta, \\ x^+ \in G_\delta(x) & x \in D_\delta, \end{cases} \tag{6.20}$$

with

$$\begin{aligned} &C_\delta := \{x : (x + \delta(x)\mathbb{B}) \cap C \neq \emptyset\}, \\ &F_\delta(x) := \{A.\left((x + \delta(x)\mathbb{B}) \cap C\right) + \delta(x)\mathbb{B}\} && \forall x \in C_\delta, \\ &D_\delta := \{x : (x + \delta(x)\mathbb{B}) \cap D \neq \emptyset\}, \\ &G_\delta(x) := \ \{v : v \in g + \delta(g)\mathbb{B},\ g \in G.\left((x + \delta(x)\mathbb{B}) \cap D\right)\} && \forall\, x \in D_\delta, \end{aligned} \tag{6.21}$$

is asymptotically stable, where $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$.

From the results by Cai et al. (2007); Goebel et al. (2012), robust asymptotic stability of the hybrid linear system (6.1) is equivalent to the existence of a robust Lyapunov function; see Theorem A.7.2 in Appendix A. Now, we study the robust stability problem for the hybrid linear system (6.1) when the robust Lyapunov function is quadratic, that is, there exist $V(x) = x^\top Px$ with $P = P^\top > 0$ such that (6.18-6.19) hold. Note that condition (6.17) does not limit the selection of such a Lyapunov function. Indeed, for any $P = P^\top > 0$, two appropriate functions α_1 and α_2 exist. For instance, one can take

$$\alpha_1(\|x\|) = \lambda_{\min}(P)\|x\|^2, \quad \alpha_2(\|x\|) = \lambda_{\max}(P)\|x\|^2,$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of P, respectively. Now, (6.18-6.19) can be rewritten as

$$A^\top P + PA + P \leq 0, \tag{6.22}$$
$$G^\top PG - P + (1 - e^{-1})P \leq 0. \tag{6.23}$$

It follows from (6.22-6.23) that the matrix $\left(A + \frac{1}{2}I\right)$ does not have any eigenvalue in the open right half plane, and all eigenvalues of $(e^{\frac{1}{2}}G)$ lie within the closed unit disc. Therefore, intuitively one may expect that the system (6.1) exhibits robustness properties in the presence of some bounded uncertainties. To verify this formally, we introduce a Lipschitz uncertainty function in the form of $\delta(x) = \psi\|x\|$. Then, (6.21) becomes:

$$C_\delta := \{x : (x + \psi\|x\|.\mathbb{B}) \cap C \neq \emptyset\}, \tag{6.24}$$
$$F_\delta(x) := \{A((x + \psi\|x\|.\mathbb{B}) \cap C) + \psi\|x\|.\mathbb{B}\} \qquad \forall\, x \in C_\delta, \tag{6.25}$$
$$D_\delta := \{x : (x + \psi\|x\|.\mathbb{B}) \cap D \neq \emptyset\}, \tag{6.26}$$
$$G_\delta(x) := \ \{v : v \in g + \psi.\|g\|.\mathbb{B},\ g \in G.\left((x + \psi.\|x\|\mathbb{B}) \cap D\right)\} \qquad \forall\, x \in D_\delta. \tag{6.27}$$

In the following lemma we introduce an upper bound for such a function $\delta(x)$ by taking into account that the perturbed system $\mathcal{H}_\delta$ should be asymptotically stable. In addition, we show that the perturbed hybrid system is exponentially stable. Note that our approach for the proof of this lemma is different from the one used in Cai et al. (2007); Goebel et al. (2012) for general nonlinear differential/difference inclusions. We exploit some characteristics of systems with linear flow and jump dynamics.

Lemma 6.3.1. *Suppose there exists a function $V(x) = x^\top Px$ with $P = P^\top > 0$ satisfying* (6.22−6.23)*. Then, the perturbed hybrid system H_δ defined by* (6.20) *and* (6.24−6.27) *is exponentially stable for*

$$\delta(x) = \psi.\|x\| = min\Big(\delta_c(x), \delta_d(x)\Big), \tag{6.28}$$

where

$$\delta_c(x) = \psi_c.\|x\|, with\ \psi_c := \frac{\lambda_{min}(P)}{4\left[\lambda_{max}(P)(2\sigma_{max}(A) + 3)\right]}, \tag{6.29}$$
$$\delta_d(x) = \psi_d.\|x\|, with\ \psi_d := \frac{0.02\ \lambda_{min}(P)}{3\lambda_{max}(P)(\sigma_{max}(G)^2 + 2)}, \tag{6.30}$$

and $\lambda_{max}(.)$ and $\sigma_{max}(.)$ denote the maximum eigenvalue and singular value of a matrix, respectively.

Proof: Our proof consists of three parts. In Part A, we present a proof for continuous dynamics, while in Part B we state a proof for discrete counterpart. Finally, in Part C, we prove exponential stability of the perturbed hybrid system $\mathcal{H}_\delta$.

Part A: Define C_{δ_c} and F_{δ_c} as C_δ and F_δ in (6.24) and (6.25) by setting ψ_c instead of

ψ. We demonstrate that the derivative of $V(x)$ for all perturbations bounded by $\delta_c(x)$ is negative. In other words, we show for each $x \neq 0$

$$\max_{f \in F_{\delta_c}(x)} \langle \nabla V(x), f \rangle < 0 \qquad \forall x \in C_{\delta_c}.$$

First, note that by definition of C_{δ_c}, for any $x \in C_{\delta_c}$ and $y \in C$ such that $(y - x) \in \delta_c(x)\mathbb{B}$, there exists a $v \in \mathbb{B}$ satisfying

$$y = x + \delta_c(x)v = x + \psi_c\|x\|v. \tag{6.31}$$

On the other hand, by definition of ψ_c in (6.29), we have $0 < \psi_c < 1$. This means that the closest point of the set $\{x \in C_{\delta_c} : x + \psi_c\|x\|\mathbb{B}\}$ to the origin is indeed the point $x_1 := x(1 - \psi_c)$. This implies

$$\|y\| \geq (1 - \psi_c)\|x\|, \quad \text{and } (1 - \psi_c)\|x\|.\mathbb{B} \subseteq \|y\|.\mathbb{B}. \tag{6.32}$$

Paying attention to (6.32) and (6.31), one can write

$$x \in y + \psi_{c1}\|y\|.\mathbb{B}, \text{ where } \psi_{c1} := \frac{\psi_c}{1 - \psi_c}. \tag{6.33}$$

In addition, from (6.32), we can write

$$\|x\| \leq \frac{1}{1 - \psi_c}\|y\| \Rightarrow \psi_c\|x\| \leq \frac{\psi_c}{1 - \psi_c}\|y\| = \psi_{c_1}\|y\|, \tag{6.34}$$

and also $y \neq 0$ when $x \neq 0$. Now, note that referring to (6.25) and (6.33) any vector inside the set $F_{\delta_c(x)}$ can be written as

$$f = Ay + \psi_c.\|x\|v_3 = Ay + \psi_{c1}.\|y\|v_2, \tag{6.35}$$

for $x \in C_{\delta_c}$, $y \in C$, and some $v_2, v_3 \in \mathbb{B}$. Utilizing (6.18), (6.35), and (6.33) the derivative of the Lyapunov function is bounded by:

$$\begin{aligned}
\max_{f \in F_{\delta_c}(x)} \langle \nabla V(x), f \rangle &= \max_{v_2 \in \mathbb{B},\ x \in C_{\delta_c}} \langle 2x^\top P, Ay + \psi_{c1}.\|y\|v_2 \rangle \\
&= \max_{v_1, v_2 \in \mathbb{B}} \langle 2(y + \psi_{c1}.\|y\|.v_1)^\top P \ , \ Ay + \psi_{c1}.\|y\|v_2 \rangle \\
&= y^\top (A^\top P + PA)y + 2\psi_{c1}.\|y\|.y^\top P v_2 + 2\psi_{c1}.\|y\|.v_1^\top P.Ay \\
&\qquad + 2\psi_{c1}^2.\|y\|^2.v_1^\top P v_2 \\
&\leq -y^\top P y + 2\psi_{c1}.\|y\|.\|y\|.\|Pv_2\| + 2\psi_{c1}.\|y\|.\|v_1^\top P\|.\|Ay\| \\
&\qquad + 2\psi_{c1}^2.\|y\|^2.\|v_1\|.\|Pv_2\|.
\end{aligned}$$

As P is symmetric and positive definite, its maximum eigenvalue equals its maximum singular value. Thus, $\|Pv_1\| \leq \lambda_{\max}(P)$ and $\|Pv_2\| \leq \lambda_{\max}(P)$. Moreover, the inequality $\|Ay\| \leq \sigma_{\max}(A)\|y\|$, in general, holds true. By the definition of ψ_c in (6.29), we have also $\psi_c \leq \frac{1}{2}$ and therefore, $2\psi_{c1}^2 \leq \psi_{c1}$. Furthermore, by the definition of ψ_{c1} from (6.33), we have $\psi_{c1} \leq 2\psi_c$. Consequently, the following upper bound for the derivative of the Lyapunov function results

$$\max_{f \in F_{\delta_c}(x)} \langle \nabla V(x), f \rangle \leq -y^\top P y + 2\psi_c\Big(3 + 2\sigma_{\max}(A)\Big)\|y\|^2\lambda_{\max}(P).$$

Substituting ψ_c from (6.29) into the last equation and having $y \neq 0$ lead to

$$\max_{f \in F_{\delta_c}(x)} \langle \nabla V(x), f \rangle \leq -y^\top P y + \frac{1}{2}\lambda_{\min}(P).\|y\|^2 \leq -\frac{1}{2} y^\top P y < 0. \tag{6.36}$$

Part B: Let us define D_{δ_d} and G_{δ_d} similar to what defined for D_δ and G_δ in (6.26) and (6.27), respectively, by changing the role of δ to δ_d. We show that during a jump the following Lyapunov inequality holds

$$\max_{g \in G_{\delta_d(x)}} V(g) - V(x) < 0 \qquad \forall x \in D_{\delta_d} \backslash \{0\}. \tag{6.37}$$

With the similar argument as presented in Part A, for each $x \in D_{\delta_d}$ and $y \in D$ such that $(x - y) \in \delta_d(x)\mathbb{B}$, there exists a vector $v_1 \in \mathbb{B}$ satisfying

$$x = y + \psi_{d1}.\|y\|v_1, \text{ where } \psi_{d1} = \frac{\psi_d}{1 - \psi_d}. \tag{6.38}$$

Notice also any vector $g \in G_{\delta_d}$, in accordance with (6.27), can be formulated as

$$g = Gy + \psi_d.\|Gy\|v_2, \qquad y \in D, \ \exists v_2 \in \mathbb{B}. \tag{6.39}$$

Thus, recalling (6.39) we have

$$\begin{aligned} \max_{g \in G_{\delta_d}(x)} V(g) - V(x) &= \max_{v_1, v_2 \in \mathbb{B}} \Big(Gy + \psi_d.\|Gy\|v_2\Big)^\top P \Big(Gy + \psi_d.\|Gy\|v_2\Big) \\ &\qquad - (y + \psi_{d1}.\|y\|v_1)^\top P (y + \psi_{d1}.\|y\|v_1) \\ &= y^\top G^\top P G y - y^\top P y \\ &\qquad + \max_{v_1, v_2 \in \mathbb{B}} \Big(2\psi_d.\|Gy\|v_2^\top P.Gy + \psi_d^2.\|Gy\|^2 v_2^\top P v_2 \\ &\qquad\qquad - 2\psi_{d1}.\|y\|v_1^\top P y - \psi_{d1}^2.\|y\|^2 v_1^\top P v_1\Big). \end{aligned} \tag{6.40}$$

Note that from the definition of ψ_d in (6.30), we have $\psi_d \leq 1/2$. Therefore, from the definition of ψ_{d1} in (6.38), we can deduce $\psi_{d1} \leq 2\psi_d \leq 1$. Moreover, the following inequalities are valid

$$\begin{aligned} &\max_{v_2 \in \mathbb{B}} \psi_d \|Gy\| v_2^\top P.Gy \leq \psi_d \ \sigma_{\max}(G)^2 \lambda_{\max}(P).\|y\|^2, \\ &\qquad\qquad \max_{v_2 \in \mathbb{B}} \psi_d^2 \|Gy\|^2 v_2^\top P v_2 \leq \psi_d \ \sigma_{\max}(G)^2 \lambda_{\max}(P).\|y\|^2, \\ &\max_{v_1 \in \mathbb{B}} \ \big(-\psi_{d1}.\|y\|v_1^\top P y\big) \leq 2\psi_d.\lambda_{\max}(P).\|y\|^2, \\ &\qquad\qquad \max_{v_1 \in \mathbb{B}} \ \big(-\psi_{d1}^2.\|y\|^2 v_1^\top P v_1\big) \leq 2\psi_d.\lambda_{\max}(P).\|y\|^2. \end{aligned}$$

Then by substituting the value of ψ_d from (6.30) into (6.40) and using (6.19) for $x \in D_{\delta_d} \backslash \{0\}$ and consequently $y \in D \backslash \{0\}$, we have

$$\begin{aligned} \max_{g \in G_\delta(x)} V(g) - V(x) &\leq e^{-1} V(y) - V(y) + 3\psi_d.\|y\|^2.(\sigma_{\max}(G)^2 + 2)\lambda_{\max}(P) \\ &\leq (e^{-1} - 1)V(y) + 0.02V(y) \leq -e^{\frac{-1}{2}} V(y) < 0. \end{aligned} \tag{6.41}$$

Now, we observe that if $\delta(x)$ is less than $\delta_c(x)$ and $\delta_d(x)$ defined by (6.29) and (6.30), respectively, then the two inequalities (6.36) and (6.41) are valid. Therefore, the Lyapunov function $V(x)$ and the perturbed hybrid system $\mathcal{H}_\delta$ satisfy the requirements of Theorem A.7.1. Hence, asymptotically stability of the origin for this system is guaranteed. Thus, we confine the perturbation function to the uncertainty δ defined by

$$\delta(x) = \min\Big(\delta_c(x), \delta_d(x)\Big).$$

Part C: Another interesting characteristic of the system (6.1) satisfying Lemma 6.3.1 is that robust asymptotic stability is equivalent to the robust exponential stability for this system, which indeed is a more appreciated notion of stability in control systems theory. To show this fact, we can again define a hybrid time domain for $\mathcal{H}_\delta$, denoted by $\mathbf{T}_K$, for a given number $K \in \mathbb{N}_0$. For exponential stability, for each given $(t, k) \in \mathbf{T}_K$, we must show there exist positive scalars α and β such that (6.6) holds. Let's define $\gamma_c = (1-\psi_c)^2 \frac{\lambda_{\min}(P)}{2\lambda_{\max}(P)}$. Referring to (6.36) and then (6.32), in any time interval $(t, j) \in \mathbf{T}_K$ such that $x(t, j) \in C_\delta$, we can write

$$\begin{aligned} \dot{V}(x(t,j)) \le -\frac{1}{2}V(y) \le -\frac{1}{2}\lambda_{\min}(P)\|y\|^2 \le -(1-\psi_c)^2\frac{\lambda_{\min}(P)}{2\lambda_{\max}(P)}V(x(t,j)) \\ \Rightarrow V(x) \le V(x(t-t_j, j))e^{-\gamma_c(t-t_j)} \qquad \forall (t,j) \in \mathbf{T}_K, \ \ j \le k. \end{aligned} \tag{6.42}$$

Now, recalling (6.41) and (6.38) for the points $(t, j) \in \mathbf{T}_K$ such that $(t, j+1) \in \mathbf{T}_K$ and $x(t, j) \in D_\delta$, and defining

$$\gamma_d = \ -\ln\Big(1-(1-\psi_d)^2\frac{\lambda_{\min}(P)}{2\lambda_{\max}(P)}\Big),$$

we can write

$$\begin{aligned} V(x(t,j+1)) - V(x(t,j)) \le -e^{\frac{-1}{2}}V(y) \le -(1-\psi_d)^2\frac{\lambda_{\min}(P)}{2\lambda_{\max}(P)}V(x(t,j)) \\ \Rightarrow V(x(t,j+1)) \le V(x(t,j))e^{-\gamma_d} \qquad \forall (t,j) \in \mathbf{T}_K, \ j < k. \end{aligned} \tag{6.43}$$

Thus, from (6.42) and (6.43) and defining $\beta = \min(\gamma_c, \gamma_d)$, the norm of the states of $\mathcal{H}_\delta$ is bounded by

$$\|x(t,k)\|^2 \le \frac{1}{\lambda_{\min}(P)}V(x(t_0,0))e^{-\beta(t+k-t_0)} \qquad \forall (t,k) \in \mathbf{T}_k.$$

This implies exponential stability of $\mathcal{H}_\delta$. □

6.4 LQR design for a class of hybrid linear systems (Scenario I)

In this part, we discuss a linear quadratic control problem for a class of hybrid linear systems. The results of this section have been published in Kouhi et al. (2013b).

Consider a controlled hybrid linear system defined by

$$\mathcal{H} : \begin{cases} \dot{x} & = Ax + Bu \qquad x \in C, \\ x^+ & = Gx + Hv \qquad x \in D. \end{cases} \tag{6.44}$$

Suppose the pairs (A, B) and (G, H) are controllable. Let us assume that the sets $C, D \subseteq \mathbb{R}^n$ are given by

$$C = \cup_{i\in\mathbf{I}} C_i, \quad D = \cup_{i\in\mathbf{I}} D_i, \tag{6.45}$$

where $\mathbf{I}$ is a finite index set, and C_i and D_i satisfy the following conditions:

1) for each $i \in \mathbf{I}$, C_i and D_i are polyhedral sets; namely, there exists matrices E_i, F_i and vectors e_i, f_i with appropriate dimensions such that

$$\begin{aligned} C_i &:= \{x \in \mathbb{R}^n : E_i x + e_i \leq 0\}, \\ D_i &:= \{x \in \mathbb{R}^n : F_i x + f_i \leq 0\}. \end{aligned} \tag{6.46}$$

2) $\cup_{i\in\mathbf{I}} (C_i \cup D_i) = \mathbb{R}^n$.

3) for each $i, i' \in \mathbf{I}$, the intersection between the interiors of C_i and $D_{i'}$, between C_i and $C_{i'}$ when $i \neq i'$, and between D_i and $D_{i'}$ when $i \neq i'$ are empty.

This particular form of the polyhedral sets implies that there exist matrices W_q and vectors w_q such that for each point x on the boundary between two polyhedral sets C_i and $D_{i'}$ with nonempty intersection the following relationship holds

$$W_q x + w_q = 0 \quad x \in C_i \cap D_{i'}, \tag{6.47}$$

where $i,\ i' \in \mathbf{I}$ and $q = (i, i')$.

Although the hybrid time domain discussed in Section 6.2 can be used as domain of x, we introduce a different form of generalized time domain for the hybrid linear system (6.44), which is more suitable for our problem in this section. We define the domain of x for the hybrid system (6.44) as:

$$\mathbb{T}_K := \bigcup_{k=0}^{K-1} (\mathbb{T}_{c,k} \ \cup \ \mathbb{T}_{d,k}) \quad \subseteq \mathbb{R}_{\geq 0} \times \mathbb{N}_0,$$

with the time intervals $\mathbb{T}_{c,k}$ and $\mathbb{T}_{d,k}$ defined by

$$\mathbb{T}_{c,k} := [t_k, t_{k+1}] \times \{j_k\}, \tag{6.48}$$

$$\mathbb{T}_{d,k} := \{t_{k+1}\} \times \{j_k, \dots j_{k+1}\}, \tag{6.49}$$

where we assume the time instances

$$0 := t_0 < t_1 < t_2 \dots < t_K := T, \tag{6.50}$$

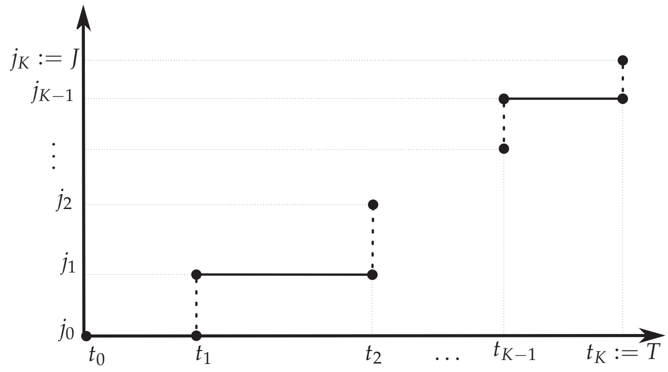

Figure 6.4.1: Pictorial description of a generalized time domain in Section 6.4.

the jump indices

$$0 := j_0 < j_1 < j_2 \ldots < j_K := J, \tag{6.51}$$

and the number $K \in \mathbb{N}$ are given; see Figure 6.4.1. Then, we use the notation (t, j) for expressing any time instance, where t indicates the flow time and j refers to the jump index. Note that the definition of $\mathbb{T}_K$ is different from the notion of hybrid time domain in Goebel et al. (2012).

Although many classes of solutions can be investigated for the hybrid system (6.44), we only study a particular form of trajectories which are characterized by Definition 6.4.1. Figure 6.4.2 depicts such a desired trajectory schematically.

Definition 6.4.1. Given inputs u and v, and the fixed indices $i_k, i'_k \in \mathbf{I}$ such that $C_{i_k} \cap D_{i'_k}$ and $D_{i'_k} \cap C_{i_{k+1}}$ are nonempty for $k \in \{0, \ldots, K-1\}$, we say $x : \mathbb{T}_K \to \mathbb{R}^n$ is a *desired* trajectory of the system (6.44) if

a) x starts at a given point x_0 in C_{i_0}, *i.e.*,

$$x_0 = x(t_0, j_0) \in C_{i_0} = \{x : E_{i_0}x + e_{i_0} \leq 0\}.$$

b) x ends at a given point x_f in $D_{i'_{K-1}}$, *i.e.*,

$$x_f = x(t_K, j_K) \in D_{i'_{K-1}} = \left\{x : F_{i'_{K-1}}x + f_{i'_{K-1}} \leq 0\right\}.$$

c) for each $k \in \{0, 1, \ldots, K-1\}$, we have

c1) $x(t, j_k) \in C_{i_k} = \{x : E_{i_k}x + e_{i_k} \leq 0\} \quad \forall t \in [t_k, t_{k+1}]$.

c2) $(t, j_k) \mapsto x(t, j_k)$ is continuously differentiable for all $t \in (t_k, t_{k+1})$.

c3) $\frac{d}{dt}x(t, j_k) = Ax(t, j_k) + Bu(t, j_k)$ for all $t \in (t_k, t_{k+1})$.

d) for each $k \in \{0, 1, \ldots, K-1\}$, and $(t_{k+1}, j) \in \mathbb{T}_{d,k}$ such that $(t_{k+1}, j+1) \in \mathbb{T}_{d,k}$, we have

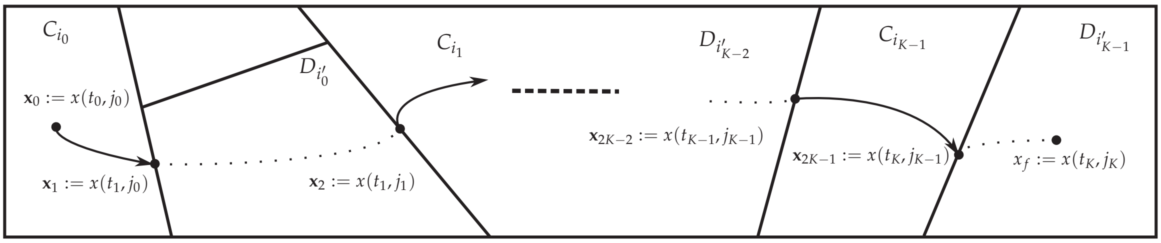

Figure 6.4.2: Pictorial description of a desired hybrid trajectory.

d1) $x(t_{k+1}, j) \in D_{i'_k} = \{x : F_{i'_k} x + f_{i'_k} \leq 0\}$.

d2) $x(t_{k+1}, j+1) = Gx(t_{k+1}, j) + Hv(t_{k+1}, j)$.

e) for each (t_{k+1}, j_k) with $k \in \{0, 1, \dots, K-1\}$, and (t_{k+1}, j_{k+1}) with $k \in \{0, 1, \dots, K-2\}$, we have

e1) $x(t_{k+1}, j_k) \in C_{i_k} \cap D_{i'_k}$.

e2) $x(t_{k+1}, j_{k+1}) \in D_{i'_k} \cap C_{i_{k+1}}$.

Then, recalling item e) in Definition 6.4.1 and Equation (6.47), with some abuse of notation, we define $q_1, q_2, \dots, q_{2K-1}$ to index the matrices $W_{q_1}, \dots, W_{q_{2K-1}}$ and vectors $w_{q_1}, \dots, w_{q_{2K-1}}$ such that the following relations hold

$$W_{q_{2k+1}} x + w_{q'_{2k+1}} = 0 \qquad x \in C_{i_k} \cap D_{i'_k}, \quad k \in \{0, 1, \dots, K-1\}, \tag{6.52}$$

$$W_{q_{2k+2}} x + w_{q'_{2k+2}} = 0 \qquad x \in D_{i'_k} \cap C_{i_{k+1}}, \quad k \in \{0, 1, \dots, K-2\}. \tag{6.53}$$

Now, interpreting x as $x(t, j_k)$, u as $u(t, j_k)$, x_j as $x(t_{k+1}, j)$, and v_j as $v(t_{k+1}, j)$ in the sequel, we define the LQ problem for the hybrid system (6.44) as follows:

Problem 6.4.1. Given $Q_c \geq 0$, $R_c > 0$, $Q_d \geq 0$, $R_d > 0$, $K \in \mathbb{N}$, time instances as in (6.50), jump indices as in (6.51), and the symmetric matrices $S_c(t_{k+1}, j_k) \geq 0$, $S_d(t_{k+1}, j_{k+1}) \geq 0$, $k \in \{0, 1, \dots, K-1\}$, find controls u and v such that $x(t, j)$ with $(t, j) \in \mathbb{T}_K$ is a desired trajectory for the hybrid system (6.44), and the following optimization problem is solved:

$$\text{minimize} \quad \mathbf{J} = \sum_{k=0}^{K-1} (\mathbf{J}_{c,k} + \mathbf{J}_{d,k}) \tag{6.54}$$

$$\text{subject to} \begin{cases} \mathcal{H} \text{ defined by (6.44)}, \\ x(t_0, j_0) = x_0, \\ x(t_K, j_K) = x_f, \end{cases} \tag{6.55}$$

where

$$\mathbf{J}_{c,k} = \frac{1}{2}\, x(t_{k+1}, j_k)^\top\, S_c(t_{k+1}, j_k)\, x(t_{k+1}, j_k) + \\ + \frac{1}{2} \int_{t_k}^{t_{k+1}} [x^\top\, Q_c\, x + u^\top\, R_c u]\, dt, \tag{6.56}$$

$$\mathbf{J}_{d,k} = \frac{1}{2}\, x\,(t_{k+1}, j_{k+1})^\top\; S_d(t_{k+1}, j_{k+1})\, x(t_{k+1}, j_{k+1}) + \\ + \frac{1}{2} \sum_{j=j_k}^{j_{k+1}-1} [x_j^\top\, Q_d\, x_j + v_j^\top\, R_d\, v_j]. \quad \square \tag{6.57}$$

In this problem setting, $\mathbf{J}_{c,k}$ and $\mathbf{J}_{d,k}$ are the cost variables associated to the time intervals $\mathbb{T}_{c,k}$ and $\mathbb{T}_{d,k}$ for $k \in \{0, \ldots, K-1\}$, respectively. The symmetric matrices $S_c(t_{k+1}, j_k)$ with $k \in \{0, \ldots, K-1\}$, and $S_d(t_{k+1}, j_{k+1})$ with $k \in \{0, \ldots, K-2\}$ are used to specify the costs on the boundaries of the flow and jump sets, and $S_d(t_K, j_K)$ is used to specify the cost at the terminal point $x(T, J) = x_f$. Note that Problem 6.4.1 incorporates two control objectives: it seeks controls u and v which move the trajectory from the initial condition x_0 to the final destination x_f with $2K-1$ switching between flow and jump dynamics, and additionally minimizes the cost $\mathbf{J}$ defined by (6.54).

Due to the state space constraints and hybrid nature of the system (6.44) solving Problem 6.4.1 is non-trivial. Consequently, we instead determine controls u and v which provide suboptimal solutions for the cost (6.54). The approach we follow for finding these controls, is first to consider the LQR problems associated to each hybrid time interval parameterized by their initial and end states. For these problems, we introduce analytical suboptimal controls by neglecting the inequality constraints arising from the description of the polyhedral sets in (6.46). We further show that the closed-loop system can be written in affine form with respect to unknown parameters. Later, introducing static optimization problems, we compute the parameters and hence derive suboptimal solutions to Problem 6.4.1.

6.4.1 Suboptimal solutions to flow equations

Consider a piece of a desired trajectory that evolves on the flow set C_{i_k} within the interval $\mathbb{T}_{c,k}$ for some $k \in \{0, \ldots, K-1\}$. Recalling the principle of optimality, given the initial and end conditions in $\mathbb{T}_{c,k}$ the control u which solves Problem 6.4.1 must also minimize the cost $\mathbf{J}_{c,k}$ associated to this time interval. On the other hand, if we assume the initial state vector $\mathbf{x}_{2k} := x(t_k, j_k)$ and final state vector $\mathbf{x}_{2k+1} := x(t_{k+1}, j_k)$ as parameters, then finding the control u which addresses item c) of Definition 6.4.1 as a constraint and solves the following optimization problem, is motivated by Problem 6.4.1:

$$\text{minimize} \qquad \mathbf{J}_{c,k} \tag{6.58}$$

$$\text{subject to} \begin{cases} \dot{x} &= Ax + Bu, \\ x(t_k, j_k) &= \mathbf{x}_{2k}, \\ x(t_{k+1}, j_k) &= \mathbf{x}_{2k+1}, \\ E_{i_k}x + e_{i_k} &\leq 0. \end{cases}$$

Here, we formally need the convention $\mathbf{x}_0 = x_0$ to allow k taking value 0.

Given $S_c(t_{k+1}, j_k) \geq 0$, $Q_c \geq 0$, and $R_c > 0$, a lower bound for this problem can be given in analytical form by neglecting the inequality constraint in (6.58) and only considering the fixed initial and end states as the constraints. This solution can be written as (see, *e.g.*, Lewis and Syrmos (1995), pp.224)

$$u = -(K_c - R_c^{-1}B^\top V_c P_c^{-1} V_c^\top)x - R_c^{-1}B^\top V_c P_c^{-1}\mathbf{x}_{2k+1}, \tag{6.59}$$

where

$$\begin{aligned} -\dot{S}_c &= A^\top S_c + S_c A - S_c B R_c^{-1} B^\top S_c + Q_c, \\ K_c &= R_c^{-1} B^\top S_c, \\ -\dot{V}_c &= (A - BK_c)^\top V_c, \\ \dot{P}_c &= V_c^\top B R_c^{-1} B^\top V_c, \end{aligned} \tag{6.60}$$

for $(t, j_k) \in \mathbb{T}_{c,k}$, and with the boundary conditions

$$V_c(t_{k+1}, j_k) = I, \;\; P_c(t_{k+1}, j_k) = 0,$$

and the given $S_c(t_{k+1}, j_k)$; see Appendix A.8.3.

In (6.60), the auxiliary variable $V_c \in \mathbb{R}^{n\times n}$ is a "modified state transition matrix" for the adjoint of the linear time varying closed loop system, and $-P_c(t, j_k) \in \mathbb{R}^{n\times n}$ is a sort of weighted reachability gramian. If $\det(P_c(t, j_k)) = 0$ for all $(t, j_k) \in \mathbb{T}_{c,k}$, the problem is abnormal and no solution exists. For this reason, we assume P_c is non-singular within these time intervals. Note that if $Q_c = 0$, non-singularity of P_c is implied by controllability of the pair (A, B); see Lewis and Syrmos (1995). Moreover, the variables

$$\theta_{c,k} := \theta_c(t, j_k) = -P_c(t, j_k)^{-1} \left[V_c(t, j_k)^\top \, x - \mathbf{x}_{2k+1}\right]$$

are constant in the interval $\mathbb{T}_{c,k}$, and the costate parameters $\lambda_c(t, j_k)$ are given by

$$\lambda_c(t, j_k) = S_c(t, j_k)\, x + V_c(t, j_k)\, \theta_{c,k}. \tag{6.61}$$

Furthermore, referring to (6.59) and (6.61), the relationship between the costate and the control is given by

$$u = -R_c^{-1} B^\top \lambda_c(t, j_k).$$

Note that in (6.60), the first Riccati equation for S_c, as well as the differential equations for V_c and P_c, are solved backwards in time up to (t_k, j_k), within any time interval $\mathbb{T}_{c,k}$; see Lewis and Syrmos (1995).

Now, a suboptimal value of the cost $\mathbf{J}_{c,k}$ with the state feedback control given by (6.59) can be analytically computed in the following. Note that throughout this chapter the symbol $*$ denotes an optimal value of a variable and must be distinguished by the conjugate transpose symbol used in the previous chapters.

Lemma 6.4.1. *(Kouhi et al., 2013b; Bryson and Ho, 1975)) A lower bound for the optimal value of the cost* $\mathbf{J}_{c,k}$ *in Problem* (6.58) *with the control* (6.59) *is given by*

$$\begin{aligned}\mathbf{J}^{l*}_{c,k} = \tfrac{1}{2}\, \mathbf{x}_{2k}^\top S_c(t_k, j_k)\, \mathbf{x}_{2k} - \tfrac{1}{2}\left[V_c(t_k, j_k)^\top \mathbf{x}_{2k} - \mathbf{x}_{2k+1}\right]^\top \\ \times P_c(t_k, j_k)^{-1}\, [V_c(t_k, j_k)^\top\, \mathbf{x}_{2k} - \mathbf{x}_{2k+1}].\end{aligned} \tag{6.62}$$

Proof: Rewrite the control (6.59) in the more convenient form

$$u = -R_c^{-1} B^\top \Big(S_c x - V_c P_c^{-1} z\Big), \tag{6.63}$$

where for reducing the computation we defined the variable

$$z(t, j_k) = V_c(t, j_k)^\top\, x - \mathbf{x}_{2k+1}.$$

From the definitions of K_c, V_c, and P_c in (6.60), and the definition of u in (6.59), we have

$$\begin{aligned}\dot{z} &= \dot{V}_c^\top x + V_c^\top \dot{x} \\ &= -V_c^\top (A - BK_c) + V_c^\top (Ax + Bu) \\ &= V_c^\top B R_c^{-1} B^\top V_c P_c^{-1}\, [V_c(t_k, j_k)^\top\, x - \mathbf{x}_{2k+1}] \\ &= \dot{P}_c P_c^{-1} z.\end{aligned}$$

Consequently, we can obtain the following equality

$$\begin{aligned}\tfrac{d}{dt}\Big[z^\top P_c^{-1} z\Big] &= 2z^\top P_c \dot{z} - z^\top P_c^{-1} \dot{P}_c P_c^{-1} z \\ &= z^\top P_c^{-1} \dot{P}_c P_c^{-1} z.\end{aligned} \tag{6.64}$$

Then, by utilizing (6.60), (6.63) and (6.64), we have

$$\begin{aligned}& x^\top Q_c x + u^\top R_c u = \\ &= x^\top Q_c x + \Big[S_c\, x - V_c P_c^{-1} z\Big]^\top B R_c^{-1} B^\top \Big[S_c\, x - V_c P_c^{-1} z\Big] \\ &= x^\top (-\dot{S}_c - A^\top S_c - S_c A + S_c B R_c^{-1} B^\top S_c)\, x \\ &\qquad + x^\top\, S_c B R_c^{-1} B^\top S_c\, x - 2x^\top\, S_c B R_c^{-1} B^\top V_c P_c^{-1}\, z + z^\top\, P_c^{-1} \dot{P}_c P_c^{-1}\, z \\ &= \Big[-x^\top \dot{S}_c\, x - (Ax + Bu)^\top S_c\, x - x^\top\, S_c (Ax + Bu)\Big] + \tfrac{d}{dt}[z^\top\, P_c^{-1}\, z] \\ &= \tfrac{d}{dt}[-x^\top\, S_c\, x + z^\top\, P_c^{-1}\, z].\end{aligned} \tag{6.65}$$

Hence, from the definition of $\mathbf{J}_{c,k}$ in (6.56), the suboptimal cost equals

$$\begin{aligned}\mathbf{J}^{l*}_{c,k} &= \tfrac{1}{2}x(t_{k+1},j_k)^\top\ S_c(t_{k+1},j_k)\ x(t_{k+1},j_k)\\ &\quad -\tfrac{1}{2}[x(t,j_k)^\top\ S_c(t,j_k)\ x(t,j_k)]_{t_k}^{t_{k+1}}\\ &\quad +\tfrac{1}{2}\Big[z(t,j_k)^\top\ P_c(t,j_k)^{-1}\ z(t,j_k)\Big]_{t_k}^{t_{k+1}}\\ &= \tfrac{1}{2}\mathbf{x}_{2k}^\top S_c(t_k,j_k)\,\mathbf{x}_{2k} - \tfrac{1}{2}\,z(t_k,j_k)^\top\ P_c(t_k,j_k)^{-1}\,z(t_k,j_k).\end{aligned}$$

Note that in the last equation we used the fact

$$z(t_{k+1},j_k)^\top\ P_c(t_{k+1},j_k)^{-1}\ z(t_{k+1},j_k) = 0. \qquad \square$$

Now, with the control (6.59), the resulting closed-loop system turns out to be a linear time varying system

$$\dot{x} = M_c(t,j_k)\ x + N_c(t,j_k)\ \mathbf{x}_{2k+1}, \tag{6.66}$$

with

$$M_c(t,j_k) = A - BR_c^{-1}B^\top\Big(S_c(t,j_k) - V_c(t,j_k)\ P_c(t,j_k)^{-1}\ V_c(t,j_k)^\top\Big),$$

and

$$N_c(t,j_k) = -BR_c^{-1}B^\top V_c(t,j_k)\ P_c(t,j_k)^{-1}.$$

As a consequence, the solution of (6.66) is affine with respect to $\mathbf{x}_{2k}$ and $\mathbf{x}_{2k+1}$; namely

$$x(t,j_k) = \mathbf{M}_c(t,t_k,j_k)\ \mathbf{x}_{2k} + \mathbf{N}_c(t,j_k)\ \mathbf{x}_{2k+1}, \tag{6.67}$$

where $\mathbf{M}_c(t,t_k,j_k)\in\mathbb{R}^{n\times n}$ and $\mathbf{N}_c(t,j_k)\in\mathbb{R}^{n\times n}$ are given by

$$\dot{\mathbf{M}}_c(t,t_k,j_k) = M_c(t,j_k)\ \mathbf{M}_c(t,t_k,j_k) \qquad \mathbf{M}_c(\tau,\tau,j_k) = I_n,$$

$$\mathbf{N}_c(t,j_k) = \int_{t_k}^{t}\mathbf{M}_c(t,\tau,j_k)\ N_c(\tau,j_k)\ \ d\tau; \tag{6.68}$$

see Appendix A.5. The function $\mathbf{M}_c$ represents the state-transition matrix. The computation of $\mathbf{M}_c(t,t_k,j_k)$ can be accomplished by solving the corresponding differential equation forward in time.

6.4.2 Suboptimal solutions for jumps

Consider a piece of a desired trajectory which evolves on the jump set $D_{i'_k}$ and satisfies item d) in Definition 6.4.1 corresponding to the interval $\mathbb{T}_{d,k}$. Then, the optimal control v which solves Problem 6.4.1 must minimize the cost $\mathbf{J}_{d,k}$ in (6.56) associated to this time interval. On the other hand, if we consider the initial state vector $\mathbf{x}_{2k+1} := x(t_{k+1},j_k)$

and final state vector $\mathbf{x}_{2k+2} := x(t_{k+1}, j_{k+1})$ as parameters, then finding a control v which minimizes the cost $\mathbf{J}_{d,k}$ in the following problem is required in Problem 6.4.1:

$$\text{minimize} \qquad \mathbf{J}_{d,k}$$

$$\text{subject to} \begin{cases} x^+ &= Gx + Hv, \\ x(t_{k+1}, j_k) &= \mathbf{x}_{2k+1}, \\ x(t_{k+1}, j_{k+1}) &= \mathbf{x}_{2k+2}, \\ F_{i'_k} x + f_{i'_k} &\leq 0. \end{cases} \tag{6.69}$$

Here, we need the formal convention $\mathbf{x}_{2K} = x_f$ to allow k taking value $K-1$. Neglecting the inequality constraint in (6.69), Problem (6.69) will be again a standard discrete LQR problem with initial and final states as parameters. Its solution is known and available analytically (see Lewis and Syrmos (1995), p.p 250). For simplicity of notation, we denote

$$S_{d,j} = S_d(t_{k+1}, j),\ P_{d,j} = P_d(t_{k+1}, j),\ V_{d,j} = V_d(t_{k+1}, j).$$

Then, given $S_d(t_{k+1}, j_{k+1}) \geq 0$, $Q_d \geq 0$, and $R_d > 0$ the suboptimal control reads

$$v_j = -K_j\ x + K_j^v\ V_{d,j+1}\ P_{d,j}^{-1}\ [V_{d,j}^\top\ x_j - \mathbf{x}_{2k+2}], \tag{6.70}$$

where

$$\begin{aligned} K_j &= (H^\top\ S_{d,j+1}\ H + R_d)^{-1}\ H^\top\ S_{d,j+1}\ G, \\ S_{d,j} &= G^\top\ S_{d,j+1}\ (G - HK_j) + Q_d, \\ V_{d,j} &= (G - H\ K_j)^\top V_{d,j+1}, \\ P_{d,j} &= P_{d,j+1} - V_{d,j+1}^\top H\,(H^\top\ S_{d,j+1}\,H + R_d)^{-1}\,H^\top\,V_{d,j+1}, \\ K_j^v &= (H^\top\ S_{d,j+1}\ H + R_d)^{-1}\ H^\top, \end{aligned} \tag{6.71}$$

and the boundary conditions

$$P_d(t_{k+1}, j_{k+1}) = P_{d,j_{k+1}} = 0,\ V_d(t_{k+1}, j_{k+1}) = V_{d,j_{k+1}} = I,\ S_{d,j_{k+1}} = S_d(t_{k+1}, j_{k+1}),$$

hold; see Appendix A.8.4. Note that $S_d(t_{k+1}, j_{k+1})$ according to the assumption of Problem 6.4.1 is given.

In (6.70), the auxiliary variables $V_{d,j} \in \mathbb{R}^{n\times n}$ are the "modified state transition matrices" for the adjoint of the time varying closed-loop system, and $-P_{d,j} \in \mathbb{R}^{n\times n}$ is a sort of weighted reachability gramian; see Lewis and Syrmos (1995). The problem has a solution if and only if $\det(P_d(t_{k+1}, j_k)) \neq 0$. Thus, it is natural to assume that non-singularity of P_d holds within these time intervals. Note that if $Q_d = 0$, controllability of (G, H) suffices for non-singularity of P_d; see Lewis and Syrmos (1995). If for some $j_k < j \leq j_{k+1}$, $\det(P_{d,j}) = 0$, then the control (6.70) needs to be modified to

$$v_j = -K_j\ x_j + K_j^v\ V_{d,j+1}\ P_{d,j_k}^{-1}\ [V_{d,j_k}^\top\ \mathbf{x}_{2k+1} - \mathbf{x}_{2k+2}].$$

Moreover, in this form of solutions the co-state parameters $\lambda_d(t_k, j)$ are given in the form of

$$\lambda_{d,j} = S_{d,j}\ x_j + V_{d,j}\ \theta_{d,j}, \tag{6.72}$$

where the variables

$$\theta_{d,k} = \theta_{d,j} := -P_{d,j}^{-1}\ [V_{d,j}^\top\ x_j - \mathbf{x}_{2k+2}], \tag{6.73}$$

are constant in each discrete interval $\mathbb{T}_{d,k}$. Furthermore, referring to (6.70) and (6.72), the relationship between the co-state and the control is given by

$$v_j = -R_d^{-1}\ G^\top\ \lambda_{d,j}.$$

Similarly to the continuous evolution, the first Riccati equation for S_d, as well as the difference equations for V_d and P_d in (6.71) are solved backwards in time up to (t_{k+1}, j_k) within any interval $\mathbb{T}_{d,k}$. Now, the suboptimal value of the cost $\mathbf{J}_{d,k}$ with the state feedback control given by (6.69) can be given in analytical form as follows:

Lemma 6.4.2. *(Kouhi et al., 2013b)) A lower bound for the optimal value of the cost* $\mathbf{J}_{d,k}$ *in Problem* (6.69) *with the control* (6.70) *is given by*

$$\begin{aligned}\mathbf{J}_{d,k}^{l*} = \tfrac{1}{2}\ \mathbf{x}_{2k+1}^\top\ S_d(t_{k+1}, j_k)\ \mathbf{x}_{2k+1} - \tfrac{1}{2}\ [V_d(t_{k+1}, j_k)^\top\ \mathbf{x}_{2k+1} - \mathbf{x}_{2k+2}]^\top \times \\ \times\ P_d(t_{k+1}, j_k)^{-1}\ [V_d(t_{k+1}, j_k)^\top\ \mathbf{x}_{2k+1} - \mathbf{x}_{2k+2}].\end{aligned} \tag{6.74}$$

Proof: Let us define a new variable

$$z_j = V_{d,j}^\top\ x_j - \mathbf{x}_{2k+2}.$$

Then, recalling the equations for $V_{d,j}$ from (6.71) and v_j in (6.70), we can deduce

$$\begin{aligned} z_j - z_{j+1} &= V_{d,j}^\top\ x_j - V_{d,j+1}^\top\ x_{j+1} \\ &= V_{d,j+1}^\top\ (G - HK_j)\ x_j - V_{d,j+1}^\top (Gx_j + Hv_j) \\ &= -V_{d,j+1}^\top\ H\ K_j^v\ V_{d,j+1}\ P_{d,j}^{-1}\ z_j. \end{aligned}$$

Therefore, using the difference equation for $P_{d,j}$ and the equation of K_j^v from (6.71), we can compute z_{j+1} as

$$\begin{aligned} z_{j+1} &= (P_{d,j} + V_{d,j+1}^\top\ HK_j^v\ V_{d,j+1})\ P_{d,j}^{-1}\ z_j \\ &= P_{d,j+1}\ P_{d,j}^{-1}\ z_j. \end{aligned}$$

It follows that

$$\begin{aligned} z_j^\top\ P_{d,j}^{-1}\ (P_{d,j} - P_{d,j+1})\ P_{d,j}^{-1}\ z_j = \\ = z_j^\top\ P_{d,j}^{-1}\ z_j - z_{j+1}^\top\ P_{d,j+1}^{-1}\ z_{j+1}. \end{aligned} \tag{6.75}$$

Furthermore, from (6.71) we have

$$\begin{aligned} x_{j+1}^\top\ S_{d,j+1}\ x_{j+1} =& \\ &= (Gx_j + Hv_j)^\top\ S_{d,j+1}\ (Gx_j + Hv_j) \\ &= x_j^\top\ G^\top\ S_{d,j+1}\ G\ x_j + 2x_j^\top\ G^\top\ S_{d,j+1}\ H\ v_j \\ &- v_j^\top\ R_d\ v_j + v_j^\top\ (H^\top\ S_{d,j+1}\ H + R_d)\ v_j. \end{aligned} \tag{6.76}$$

Then, from the last equation we can compute

$$\begin{aligned} v_j^\top\ R_d\ v_j &= -x_{j+1}^\top\ S_{d,j+1}\ x_{j+1} + x_j^\top\ G^\top\ S_{d,j+1}\ G\ x_j \\ &+ 2x_j^\top\ G^\top\ S_{d,j+1} H\ v_j + v_j^\top (H^\top S_{d,j+1} H + R_d) v_j. \end{aligned} \tag{6.77}$$

To prove (6.74), we now exploit (6.70), (6.71), (6.75), and (6.77) for the evaluation of the summing terms in the cost variable. To this end, introduce a new variable

$$y_j = x_j^\top\ S_{d,j}\ x_j - x_{j+1}^\top\ S_{d,j+1}\ x_{j+1}$$

to simplify the computations. Then, we have

$$\begin{aligned} &x_j^\top\ Q_d\ x_j + v_j^\top\ R_d\ v_j = \\ &= x_j^\top \Big[S_{d,j} - G^\top S_{d,j+1} (G - H\ K_j)\Big] x_j - x_{j+1}^\top S_{d,j+1} x_{j+1} \\ &\quad + x_j^\top\ G^\top\ S_{d,j+1}\ G\ x_j + 2x_j^\top\ G^\top\ S_{d,j+1}\ H\ v_j + v_j^\top\ (H\ S_{d,j+1}^\top\ H + R_d)\ v_j \\ &= y_j + x_j^\top\ G^\top\ S_{d,j+1}\ H\ \underbrace{(K_j\ x_j + v_j)}_{K_j^v\ V_{d,j+1}\ P_{d,j}^{-1}\ z_j} \\ &\quad + [v_j^\top + x_j^\top\ \underbrace{G^\top S_{d,j+1}\ H\ (H^\top\ S_{d,j+1}\ H + R_d)^{-1}}_{K_j^\top}\ ]\ (H^\top\ S_{d,j+1}\ H + R_d)\ v_j \\ &= y_j + [\underbrace{x_j^\top\ G^\top\ S_{d,j+1}\ H + v_j^\top\ (H^\top\ S_{d,j+1}\ H + R_d)}_{z_j^\top\ P_{d,j}^{-1}\ V_{d,j+1}^\top\ K_j^{v\top}\ (H^\top\ S_{d,j+1}\ H + R_d)}\ ]\ K_j^v\ V_{d,j+1}\ P_{d,j}^{-1}\ z_j \\ &= y_j + z_j^\top P_{d,j}^{-1} \underbrace{V_{d,j+1}^\top H (H^\top S_{d,j+1} H + R_d)^{-1} H^\top V_{d,j+1}}_{P_{d,j+1} - P_{d,j}} P_{d,j}^{-1} z_j \\ &= y_j - z_j^\top\ P_{d,j}^{-1}\ z_j + z_{j+1}^\top\ P_{d,j+1}^{-1}\ z_{j+1}. \end{aligned}$$

Now summing up the terms resulted from the previous equation with the terminal cost of this time interval, $\mathbf{J}_{d,k}$ defined by (6.57) equals

$$\begin{aligned} \mathbf{J}_{d,k} &= \tfrac{1}{2}\ x\,(t_{k+1}, j_{k+1})^\top\ \ S_d(t_{k+1}, j_{k+1})\ x(t_{k+1}, j_{k+1}) + \\ &\quad + \tfrac{1}{2} \sum_{j=j_k}^{j_{k+1}-1} [x_j^\top\ Q_d\ x_j + v_j^\top\ R_d\ v_j] \\ &= \tfrac{1}{2}\ x\,(t_{k+1}, j_{k+1})^\top\ \ S_d(t_{k+1}, j_{k+1})\ x(t_{k+1}, j_{k+1}) \\ &\quad + \tfrac{1}{2} \sum_{j=j_k}^{j_{k+1}-1} [y_j - z_j^\top\ P_{d,j}^{-1}\ z_j + z_{j+1}^\top\ P_{d,j+1}^{-1}\ z_{j+1}] \\ &= \tfrac{1}{2}\ \mathbf{x}_{2k+1}^\top\ S_d(t_{k+1}, j_k)\ \mathbf{x}_{2k+1} \\ &\quad - \tfrac{1}{2}\ z(t_{k+1}, j_k)^\top\ P_d(t_{k+1}, j_k)^{-1}\ z(t_{k+1}, j_k), \end{aligned}$$

where we used the identity

$$z(t_{k+1}, j_{k+1})^\top \; P_d(t_{k+1}, j_{k+1})^{-1} \; z(t_{k+1}, j_{k+1}) = 0. \quad \square$$

Note that the closed-loop system with the affine control defined in (6.70) is again described by a linear time varying difference equation:

$$x_{j+1} = M_d(t_{k+1}, j) \; x_j + N_d(t_{k+1}, j) \; \mathbf{x}_{2k+2}, \tag{6.78}$$

with the coefficients

$$M_d(t_{k+1}, j) = G - H \; K_j^v \; (S_{d,j+1} \; G - V_{d,j+1} \; P_{d,j}^{-1} \; V_{d,j}^\top),$$

and

$$N_d(t_{k+1}, j) = -H \; K_j^v \; V_{d,j+1} \; P_{d,j}^{-1}.$$

The solution to (6.78) is an affine function in $\mathbf{x}_{2k+1}$ and $\mathbf{x}_{2k+2}$, that is,

$$x_j = \mathbf{M}_d(t_{k+1}, j) \; \mathbf{x}_{2k+1} + \mathbf{N}_d(t_{k+1}, j) \; \mathbf{x}_{2k+2}, \tag{6.79}$$

where the coefficients are given by

$$\mathbf{M}_d(t_{k+1}, j) = \prod_{r=0}^{j-j_k} M_d(t_{k+1}, j-r), \quad \mathbf{M}_d(t_{k+1}, j_k) = I_n,$$

$$\mathbf{N}_d(t_{k+1}, j) = \sum_{r=j_k}^{j-1} \prod_{p=1}^{j-r-1} M_d(t_{k+1}, j-p) \; N_d(t_{k+1}, r),$$

and $\mathbf{N}_d(t_{k+1}, j_k) = 0$. This fact will be utilized in the next section for imposing the inequality constraints for deriving a desired trajectory.

6.4.3 Constrained QP problems for the hybrid linear system

Having discussed the analytical suboptimal solutions to the optimal control problems separately for the flow and jump dynamics, in this section we consider them jointly for establishing a link to Problem 6.4.1. From the elaborations in the previous two subsections, we know that neglecting the inequality constraints of the polyhedral sets, the suboptimal cost in each hybrid time interval can be parametrized quadratically by parameters $\mathbf{x}_{2k}$ and $\mathbf{x}_{2k+1}$ given by (6.62), or $\mathbf{x}_{2k+1}$ and $\mathbf{x}_{2k+2}$ given by (6.74) for $k \in \{0, 1, \ldots, K-1\}$. Hence, the overall suboptimal cost equals

$$\bar{\mathbf{J}} = \sum_{k=0}^{K-1} \left(\mathbf{J}_{c,k}^{l*} + \mathbf{J}_{d,k}^{l*} \right) = \frac{1}{2} X^\top P X + Q X + R, \tag{6.80}$$

where

$$X := [\mathbf{x}_1^\top \;\; \mathbf{x}_2^\top \;\; \dots \;\; \mathbf{x}_{2K-1}^\top]^\top,$$

includes all unknown parameters, and $P = [P_{ij}] \in \mathbb{R}^{(2K-1)n\times(2K-1)n}$ is a symmetric matrix of the form

$$P = \begin{bmatrix} P_{11} & P_{12} & 0 & \dots & 0 \\ P_{21} & P_{22} & P_{23} & \dots & 0 \\ 0 & P_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & P_{2K-2,2K-2} & P_{2K-2,2K-1} \\ 0 & 0 & \dots & P_{2K-1,2K-2} & P_{2K-1,2K-1} \end{bmatrix}$$

with the symmetric matrix elements

$$P_{kk} = \begin{cases} -P_c(t_{k-1}, j_{k-1})^{-1} + S_d(t_k, j_{k-1}) - V_d(t_k, j_{k-1})\, P_d(t_k, j_{k-1})^{-1}\, V_d(t_k, j_{k-1})^\top, & k \text{ odd}, \\ -P_d(t_k, j_{k-1})^{-1} + S_c(t_k, j_k) - V_c(t_k, j_k)\, P_c(t_k, j_k)^{-1}\, V_c(t_k, j_k)^\top, & k \text{ even}, \end{cases}$$

and

$$P_{k,k+1} = P_{k+1,k}^\top = \begin{cases} V_d(t_k, j_{k-1})\;\, P_d(t_k, j_{k-1})^{-1}, & k \text{ odd} \\ V_c(t_k, j_k)\;\, P_c(t_k, j_k)^{-1}, & k \text{ even}. \end{cases}$$

The element $QX + R$ in the cost (6.80) appears when $k = 0$ and $k = K$ are considered. The row vector $Q \in \mathbb{R}^{1\times[(2K-1)n]}$ is given by

$$Q = [x_0^\top V_c(t_0, j_0) P_c(t_0, j_0)^{-1} \quad 0 \quad \cdots \quad 0 \quad x_f^\top V_d(t_K, j_{K-1}) P_d(t_K, j_{K-1})^{-1}],$$

and R is a scalar which is given by

$$\begin{aligned} R = \tfrac{1}{2}\, x_0^\top\; S_c(t_0, j_0)\; x_0 - \tfrac{1}{2}\, x_0^\top\; V_c(t_0, j_0)\; P_c(t_0, j_0)^{-1} \\ \times V_c(t_0, j_0)^\top\; x_0 - \tfrac{1}{2}\, x_f^\top\; P_d(t_K, j_{K-1})^{-1}\; x_f. \end{aligned}$$

Now, the cost defined in (6.80) is parameterized only by the decision variable X. Note that since $\mathbf{J}_{c,k}^{l*} + \mathbf{J}_{d,k}^{l*} > 0$ for each $k \in \{1, \dots, K-1\}$, we have that $\frac{1}{2}X^\top PX + QX + R > 0$. In particular, when $x_0 = 0$ and $x_f = 0$, we have that $\frac{1}{2}X^\top PX > 0$ for all $X \neq 0$. This implies that $P > 0$.

6.4.3.1 Lower bound for optimal control problem

Now, we intend to determine the unknown parameters $\mathbf{x}_1, \dots, \mathbf{x}_{2K-1}$ in a way that we obtain a lower bound for the optimal cost of Problem 6.4.1. To this end, we consider the Equations (6.52) and (6.53) as constraints for the minimization of the cost (6.80). Thus, we define a static optimization problem as follows:

$$\text{minimize } \bar{\mathbf{J}} = \tfrac{1}{2}X^\top PX + QX + R$$

$$\text{subject to } \; \mathbf{C}X + \mathbf{R} = 0, \tag{6.81}$$

where matrix $\mathbf{C}$ and vector $\mathbf{R}$ are given by

$$\begin{aligned}\mathbf{C} &= \operatorname{diag}\left([W_{q_1}, W_{q_2}, \ldots, W_{q_{2K-1}}]\right), \\ \mathbf{R} &= \begin{bmatrix} w_{q_1}^\top & w_{q_2}^\top & \ldots & w_{q_{2K-1}}^\top \end{bmatrix}^\top. \end{aligned} \tag{6.82}$$

The Lagrangian for Problem (6.81) then reads

$$L(X, \lambda) = \tfrac{1}{2} X^\top P X + QX + R + \lambda^\top (\mathbf{C}X + \mathbf{R}),$$

where λ is the Lagrange multiplier. As $P > 0$, Problem (6.81) has always a minimum. For finding this optimal value, we assign $\partial L / \partial X = 0$ and use equality constraint in (6.81) to compute an analytical solution for X, denoted by X_l^*, as follows:

$$X_l^* = -P^{-1}\left[Q^\top - \mathbf{C}^\top (\mathbf{C}P^{-1}\mathbf{C}^\top)^{-1}(\mathbf{C}P^{-1}Q^\top - \mathbf{R})\right]. \tag{6.83}$$

Hence, the optimal value of $\bar{\mathbf{J}}$ in Problem (6.81), denoted by $\bar{\mathbf{J}}_l^*$, is given by

$$\begin{aligned}\bar{\mathbf{J}}_l^* = \tfrac{1}{2} &\left[Q + (QP^{-1}\mathbf{C}^\top - \mathbf{R}^\top)(\mathbf{C}P^{-1}\mathbf{C}^\top)^{-1}\mathbf{C}\right] P^{-1} \\ \times &\left[-Q^\top + \mathbf{C}^\top (\mathbf{C}P^{-1}\mathbf{C}^\top)^{-1}(\mathbf{C}P^{-1}Q^\top - \mathbf{R})\right] + R. \end{aligned} \tag{6.84}$$

Note that $\bar{\mathbf{J}}_l^*$ is indeed the optimal cost for Problem 6.4.1 when the inequality constraints given in Problems (6.58) and (6.69) for $k = 0, \ldots, K-1$, which capture the flow and jump sets, are neglected. Therefore, we have $\bar{\mathbf{J}}_l^* \leq \mathbf{J}^*$.

6.4.3.2 Algorithm to compute solution and an upper bound

Now, we aim at determining the controls u and v, such that the solution of the hybrid system (6.44) becomes a desired trajectory, and an upper bound for the cost $\mathbf{J}^*$ defined in Problem 6.4.1 is derived. To this end, we consider the inequality constraints given in Problems (6.58) and (6.69) for minimization of the cost (6.80) in order to ensure that all criteria (a-f) in Definition 6.4.1 hold. Notice (6.67) and (6.79) indicate that each point of the suboptimal trajectory has the affine representation with respect to the decision variables $\mathbf{x}_1, \ldots, \mathbf{x}_{2K-1}$. Now, we insert $x(t, j_k)$ from (6.67) into the inequality constraints (6.58),

$$E_{i_k}\ x(t, j_k) + e_{i_k} \leq 0 \tag{6.85}$$

for all time $(t, j_k) \in \mathbb{T}_{c,k}$, and insert $x(t_{k+1}, j)$ from (6.79) into the inequality constraint given in (6.69), namely

$$F_{i'_k}\ x(t_{k+1}, j) + f_{i'_k} \leq 0, \tag{6.86}$$

for all $(t_{k+1}, j) \in \mathbb{T}_{d,k}$. Thus, for each $k \in \{0, \ldots, K-1\}$, (6.85) and (6.86) can be expressed in the matrix form as

$$\mathbf{X}_c(t, j) X + \mathbf{Y}_c(t, j) \leq 0, \tag{6.87}$$

and

$$\mathbf{X}_d(t,j)X + \mathbf{Y}_d(t,j) \leq 0, \tag{6.88}$$

where

$$\mathbf{X}_c(t,j) = \begin{cases} [E_{i_0}\mathbf{N}_c(t,j_0) \;\; 0 \; \dots \; 0] & \forall (t,j) \in \mathbb{T}_{c,0}, \\ [0 \; \dots \; E_{i_k}\mathbf{M}_c(t,t_k,j_k) \quad E_{i_k}\mathbf{N}_c(t,j_k) \; \dots \; 0] & \forall (t,j) \in \mathbb{T}_{c,k}, \; k \geq 1, \end{cases}$$

$$\mathbf{Y}_c(t,j) = \begin{cases} [x_0^\top \mathbf{M}_c(t,t_0,j_0)^\top E_{i_0}^\top + e_{i_0}^\top \quad 0 \; \dots \; 0]^\top & \forall (t,j) \in \mathbb{T}_{c,0}, \\ [0 \; \dots \; e_{i_k}^\top \; \dots \; 0]^\top & \forall (t,j) \in \mathbb{T}_{c,k}, \; k \geq 1, \end{cases}$$

$$\mathbf{X}_d(t,j) = \begin{cases} [0 \; \dots \; F_{i'_k}\mathbf{M}_d(t_k,j) \;\; F_{i'_k}\mathbf{N}_d(t_k,j) \;\; 0 \; \dots \; 0] & \forall (t,j) \in \mathbb{T}_{d,k}, \; k \leq K-2, \\ [0 \; \dots \; 0 \;\; F_{i'_{K-1}}\mathbf{M}_d(t_K,j)] & \forall (t,j) \in \mathbb{T}_{d,K-1}, \end{cases}$$

and

$$\mathbf{Y}_d(t,j) = \begin{cases} [0 \; \dots \; f_{i'_k}^\top \; \dots \; 0]^\top & \forall (t,j) \in \mathbb{T}_{d,k}, \; k \leq K-2, \\ [0 \; \dots \; 0 \;\; x_f^\top \mathbf{N}_d(t_K,j)^\top F_{i'_{K-1}}^\top + f_{i'_{K-1}}^\top]^\top & \forall (t,j) \in \mathbb{T}_{d,K-1}. \end{cases}$$

Now, considering the inequality constraints (6.87) and (6.88) for minimization of the cost (6.80), and for $k \in \{0,\dots,K-1\}$ we define the following static optimization problem:

$$\text{minimize } \bar{\mathbf{J}} = \tfrac{1}{2}X^\top PX + QX + R \tag{6.89}$$

$$\text{subject to} \begin{cases} \mathbf{X}_c(t,j)X + \mathbf{Y}_c(t,j) \leq 0 & \forall (t,j) \in \mathbb{T}_{c,k}, \\ \mathbf{X}_d(t,j)X + \mathbf{Y}_d(t,j) \leq 0 & \forall (t,j) \in \mathbb{T}_{d,k}. \end{cases}$$

Problem (6.89) consists of a quadratic cost and a set of infinitely many affine inequalities due to the continuity of time. However, note that the matrices $\mathbf{X}_c(t,j)$ and the vectors $\mathbf{Y}_c(t,j)$ implicitly depend on the variables S_c, P_c, and V_c which are computed numerically via solving the ODE in (6.60) with certain sample times. Therefore, the number of inequalities in (6.89) is finite from a numerical point of view. Suppose the number of the time samples are $N \in \mathbb{N}$. Therefore, (6.89) can be rewritten as

$$\text{minimize } \bar{\mathbf{J}} = \tfrac{1}{2}X^\top PX + QX + R \tag{6.90}$$

$$\text{subject to} \begin{cases} \mathbb{X}_c \, X + \mathbb{Y}_c & \leq 0, \\ \mathbb{X}_d \, X + \mathbb{Y}_d & \leq 0, \end{cases}$$

where the matrix $\mathbb{X}_c \in \mathbb{R}^{N\times(2K-1)n}$ and the vector $\mathbb{Y}_c \in \mathbb{R}^N$, the matrix $\mathbb{X}_d \in \mathbb{R}^{J\times(2K-1)n}$ and the vector $\mathbb{Y}_d \in \mathbb{R}^J$ are defined by

$$\mathbb{X}_c = \begin{bmatrix} \mathbf{X}_c(0,0) \\ \vdots \\ \mathbf{X}_c(T, j_{K-1}) \end{bmatrix}, \ \mathbb{Y}_c = \begin{bmatrix} \mathbf{Y}_c(0,0) \\ \vdots \\ \mathbf{Y}_c(T, j_{K-1}) \end{bmatrix},$$

$$\mathbb{X}_d = \begin{bmatrix} \mathbf{X}_d(0,0) \\ \vdots \\ \mathbf{X}_d(T, J) \end{bmatrix}, \qquad \mathbb{Y}_d = \begin{bmatrix} \mathbf{Y}_d(0,0) \\ \vdots \\ \mathbf{Y}_d(T, J) \end{bmatrix}.$$

Problem (6.90) consists of a quadratic cost and a finite set of inequalities defining a standard quadratic program (QP). It is well known that if P is positive definite, then the entire problem is convex and can be solved in polynomial time; otherwise, the problem is NP hard. Quadratic problems are well understood and plenty of established numerical and analytical algorithms are available, including interior point methods, active set, etc; see Nocedal and Wright (2006); Bayon et al. (2010).

Next we want to provide a numerical algorithm for solving Problem (6.90) by employing the dual optimal control theory and gradient projection method (see, *e.g.*, Boyd and Vandenberghe (2004)), provided that the problem is feasible. To this end, the Lagrangian of the cost reads

$$\begin{aligned} L(X, \lambda_c, \lambda_d) = \tfrac{1}{2} X^\top P X + QX + R + \\ + \lambda_c^\top (\mathbb{X}_c X + \mathbb{Y}_c) + \lambda_d^\top (\mathbb{X}_d X + \mathbb{Y}_d), \end{aligned}$$

where

$$\lambda_c \geq 0, \ \lambda_d \geq 0,$$

are Lagrange multipliers. Then, the dual problem of (6.90) is given by

$$\begin{aligned} &\text{maximize } g(\lambda_c, \lambda_d), \\ &\text{subject to } \lambda_c, \lambda_d \geq 0, \end{aligned} \tag{6.91}$$

where the function g is defined by

$$g(\lambda_c, \lambda_d) = \inf_X L(X, \lambda_c, \lambda_d).$$

Since all inequalities in (6.90) are affine with respect to X and $P > 0$, strong duality is given by the Slater condition (Boyd and Vandenberghe, 2004), implying that optimal solutions to the dual and primal problems are identical. Starting from an arbitrary initial condition $\lambda_c^{(0)}$ and $\lambda_d^{(0)}$, the solution to the dual problem is determined by an iterative algorithm for computation of the Lagrange multipliers

$$\begin{aligned} \lambda_c^{(r+1)} &= \lambda_c^{(r)} - \alpha(\mathbb{X}_c \, X_u^{(r)} + \mathbb{Y}_c), \\ \lambda_d^{(r+1)} &= \lambda_d^{(r)} - \alpha(\mathbb{X}_d \, X_u^{(r)} + \mathbb{Y}_d). \end{aligned} \tag{6.92}$$

where $0 < \alpha < 1$ represents a fixed step size, r represents the r^{th} update of the variables, and the index u in X_u refers to the upper bound control algorithm. After each iteration, the Lagrange multipliers $\lambda_c^{(r)}$ and $\lambda_d^{(r)}$ are projected in accordance with

$$[\lambda_c^{(r+1)},\ \lambda_d^{(r+1)}] = [\max(0, \lambda_c^{(r+1)}),\ \max(0, \lambda_d^{(r+1)})]. \tag{6.93}$$

The variable $X_u^{(r)}$ in the last equation can be analytically computed in terms of the Lagrange multipliers by differentiating the Lagrangian $L(X_u^{(r)}, \lambda_c^{(r)}, \lambda_d^{(r)})$ with respect to $X_u^{(r)}$ and setting it to zero, that is,

$$\begin{aligned} &\mathbb{X}_c^\top\ \lambda_c^{(r)} + \mathbb{X}_d^\top\ \lambda_d^{(r)} + Q^\top + P\ X_u^{(r)} = 0 \\ &\quad \Rightarrow X_u^{(r)} = -P^{-1}(\mathbb{X}_c^\top \lambda_c^{(r)} + \mathbb{X}_d^\top \lambda_d^{(r)} + Q^\top). \end{aligned} \tag{6.94}$$

Now, let us define

$$\Lambda^{(r)} := [\ \lambda_c^{(r)\top}\quad \lambda_d^{(r)\top}\ ]^\top.$$

Then, replacing (6.94) into (6.92), the update law for the gradient projection method reads (see Boyd and Vandenberghe (2004)):

$$\Lambda^{(r+1)} = \begin{bmatrix} I + \alpha \mathbb{X}_c P^{-1} \mathbb{X}_c^\top & \alpha \mathbb{X}_c P^{-1} \mathbb{X}_d^\top \\ \alpha \mathbb{X}_d P^{-1} \mathbb{X}_c^\top & I + \alpha \mathbb{X}_d P^{-1} \mathbb{X}_d^\top \end{bmatrix} \Lambda^{(r)} - \alpha \begin{bmatrix} \mathbb{Y}_c - \mathbb{X}_c P^{-1} Q^\top \\ \mathbb{Y}_d - \mathbb{X}_d P^{-1} Q^\top \end{bmatrix}. \tag{6.95}$$

Equation (6.93) contains a nonlinear operation, thus we are not able to compute $\Lambda^{(r)}$ analytically. However, we can describe the equation (6.94) in the matrix form as

$$X_u^{(r)} = -P^{-1}\left(Q^\top + \mathbb{X}\ \Lambda^{(r)}\right), \tag{6.96}$$

where $\mathbb{X} := [\mathbb{X}_c^\top\ \ \mathbb{X}_d^\top]$. It follows from the gradient projection method that

$$\lim_{r\to\infty} X_u^{(r)} \to X_u^*, \text{ and } \lim_{r\to\infty} \Lambda^{(r)} \to \Lambda^*.$$

Therefore, the optimal cost results to be:

$$\bar{\mathbf{J}}_u^* = \frac{1}{2}(-Q^\top + \mathbb{X}\Lambda^*)^\top P^{-1}(Q^\top + \mathbb{X}\Lambda^*) + R. \tag{6.97}$$

The cost $\bar{\mathbf{J}}_u^*$ indeed provides an upper bound for the optimal cost $\mathbf{J}^*$ in Problem 6.4.1. The reason is that the trajectory which uses the solutions of Problem (6.90) with controls u, defined by (6.59), and v, defined by (6.70), satisfies all conditions of Definition 6.4.1. Nevertheless, the equality $\bar{\mathbf{J}}_u^* = \mathbf{J}^*$ does not necessarily hold since the controls u and v in (6.59) and (6.70) are computed via neglecting the inequalities given on (6.58) and (6.69). Thus, this trajectory is not necessarily optimal in the sense of Problem 6.4.1.

The following theorem summarizes this result.

Theorem 6.4.1. *Consider the linear quadratic problem for the hybrid linear system* (6.44) *defined by Problem 6.4.1. Suppose that the optimization problem* (6.90) *is feasible. Then, the optimal cost* $\mathbf{J}^*$ *defined in Problem 6.4.1 is bounded by*

$$\bar{\mathbf{J}}_l^* \leq \mathbf{J}^* \leq \bar{\mathbf{J}}_u^*, \tag{6.98}$$

with $\mathbf{J}_l^*$ *and* $\mathbf{J}_u^*$ *given by expressions* (6.84) *and* (6.97)*, respectively.* □

Example 6.4.1. (See also (Kouhi et al., 2013b)) Consider the hybrid linear system

$$\mathcal{H}: \begin{cases} \dot{x} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} u & x \in C, \\ \\ x^+ = \begin{bmatrix} 0.7 & 0.5 \\ 0 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0.125 & 0.5 \\ 0.5 & 0.2 \end{bmatrix} v & x \in D, \end{cases}$$

where denoting $x = [x_1 \; x_2]^\top$, the sets C and D are specified by

$$C = \{x \in \mathbb{R}^2 : 2x_1 - x_2 \leq 0\}, \; D = \overline{\mathbb{R}^2 \backslash C}.$$

The corresponding cost variables are defined by

$$\mathbf{J}_{c,k} = \tfrac{1}{2} \int_{t_k}^{t_{k+1}} \left(x^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + u^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \right) dt,$$

$$\mathbf{J}_{d,k} = \frac{1}{2} \sum_{j=j_k}^{j_{k+1}-1} x_j^\top \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x_j + v_j^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v_j,$$

where $K = 2$ is a fixed parameter, indicating that only three switching between the flow and jump sets must occur. Then, the overall cost $\mathbf{J}$ is defined by (6.54). The switching time instances are fixed and given by

$$(t_1, j_0) = (0.2, 0), \; (t_1, j_1) = (0.2, 4), \; \text{and} \; (t_2, j_1) = (1, 4),$$

and the fixed final time equals $(t_2, j_2) := (T, J) = (1, 8)$. Thus, the time domain equals

$$\begin{aligned} \mathbb{T}_K = & ([0, 0.2] \times \{0\}) \cup (\{0.2\} \times \{0, \ldots, 4\}) \\ & \cup ([0.2, 1] \times \{4\}) \cup (\{1\} \times \{4, \ldots, 8\}). \end{aligned}$$

The fixed initial and final states are given by $x_0 = [3 \;\; 7]^\top \in C$ and $x_f = [3 \;\; -1]^\top \in D$, respectively. The cost associated to the switching points and to the terminal point are zero. Two trajectories computed via our algorithm are depicted in Figure 6.4.3. The flow evolution is represented by the curves with solid lines, and the jump evolution by small circles. The top picture refers to the solution obtained by lower bound control for the hybrid system with neglected inequality constraints arising from the definition of the flow and jump sets. One can observe that inequality constraints in definition of the flow

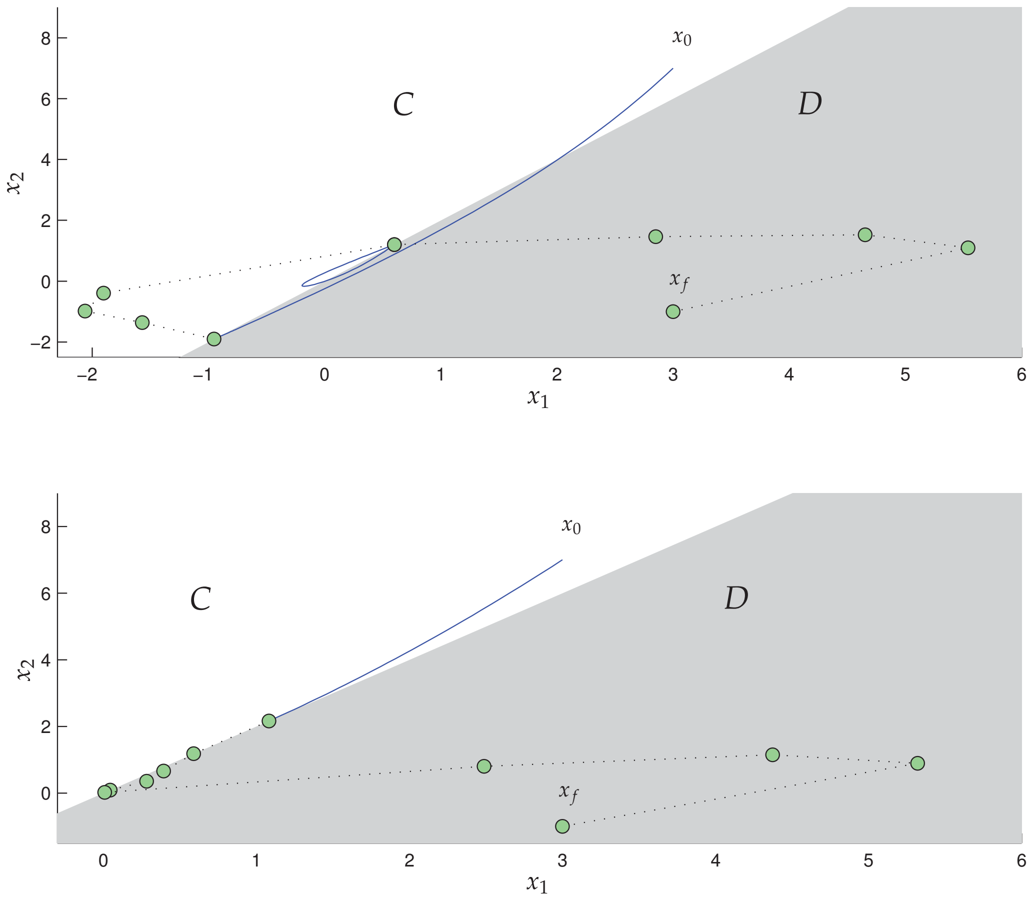

Figure 6.4.3: (Top:) The trajectory resulting from the control policy achieves the provided lower bound of the cost by ignoring the inequality constraints. (Bottom:) The system trajectory resulting from the control policy achieves the provided upper bound of the cost.

and jump sets are violated. Note that this trajectory principally does not belong to the solution set of this system since a trajectory of the hybrid system cannot jump within the set C. In contrary, in the bottom picture the solution resulting from the upper bound control is a desired trajectory of the hybrid system. The parameters $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ in the lower bound optimal control problem defined in Section 6.4.3.1 are computed to be

$$\mathbf{x}_1 = [-0.9512 \;\; -1.9024]^\top,\; \mathbf{x}_2 = [0.60\; 1.20]^\top,\; \mathbf{x}_3 = [0.60\; 1.20]^\top,$$

incidentally, with two latter variables being identical. The parameters for the upper bound control problem defined in Section 6.4.3.2 are computed to be

$$\mathbf{x}_1 = [1.0820\; 2.1640]^\top,\, \mathbf{x}_2 = [0.0446\; 0.0893]^\top,\, \mathbf{x}_3 = [0.0095\; 0.0189]^\top.$$

For this problem the following upper and lower bounds for the optimal cost $\mathbf{J}^*$ are computed as

$$585.3010 \leq \mathbf{J}^* \leq 658.3335.$$

6.5 LQR design for a class of hybrid linear systems (Scenario II)

In this section, we address a problem that relates maximum principle (Sussmann, 1999; Caines et al., 2006; Johansen et al., 2002; Liberzon, 2011) to a class of hybrid linear systems.

Let us consider a hybrid linear system $\mathcal{H}$ consisting of different linear flow dynamics given by

$$\mathcal{H} : \begin{cases} \dot{x} & = A_{\sigma(t,x)}x + B_{\sigma(t,x)}u \qquad x \in C, \\ x^+ & = x \qquad\qquad\qquad\qquad\quad x \in D, \end{cases} \tag{6.99}$$

where $\sigma(t, x) \in \mathcal{L} := \{1, \ldots, \ell\}$ is the switching signal between different dynamics. We assume that switching to a new dynamics can only occur when the states of the system belong to the jump set. Suppose also

$$A_{\sigma(t,x)} \in \mathcal{A} \in \{A_1, \ldots, A_\ell\}, \qquad B_{\sigma(t,x)} \in \mathcal{B} := \{B_1, \ldots, B_\ell\}, \tag{6.100}$$

where $u(t) \in \mathbb{R}^m$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, and the pairs (A_i, B_i) are controllable for all $i \in \mathcal{L}$. Further, we assume $C = \mathbb{R}^n$ and

$$D = \cup_{i\in\mathbf{I}}\, D_i, \tag{6.101}$$

where the jump set D is the union of switching manifolds given by the affine equations in the state space as

$$D_i = \{x \in \mathbb{R}^n : c_i x + r_i = 0\} \qquad \forall i \in \mathbf{I}. \tag{6.102}$$

In (6.102), $\mathbf{I}$ is a finite index set, c_i are matrices, and r_i are vectors with appropriate dimensions, for all $i \in \mathbf{I}$. Given $K \in \mathbb{N}$ and $T \in \mathbb{R}_{>0}$, assume the switching time instances $t_1, \ldots, t_{K-1}$ between different dynamics exist such that

$$0 := t_0 < t_1 \ldots < t_{K-1} < t_K =: T. \tag{6.103}$$

For this system, we define the hybrid time domain as follows:

$$\mathbf{T}_K := \bigcup_{k=0}^{K-1} \mathbb{T}_k,$$

where

$$\mathbb{T}_k = [t_k, t_{k+1}] \times \{k\} \qquad \forall k \in \{0, \ldots, K-1\}. \tag{6.104}$$

Note that with these assumptions and the definition of the hybrid system in (6.99), the mapping $(t, k) \mapsto x(t, k)$ is absolutely continuous over the hybrid time domain. For a definition of absolute continuity see Appendix A.6.1. Interpreting x as $x(t, k)$, we define an LQ problem for the hybrid system (6.99) as follows:

Problem 6.5.1. Given a number $K \in \mathbb{N}$, matrices $Q_k \geq 0$ and $R_k > 0$ for each $k \in \{0, \dots, K-1\}$, symmetric matrices $S_c(t_{k+1}, k) \geq 0$ for each $k \in \{0, 1, \dots, K-1\}$, and a sequence of switching, that is, $A_{\sigma(t,x)} = A_{j_k} \in \mathcal{A}$ and $B_{\sigma(t,x)} = B_{j_k} \in \mathcal{B}$ for all $(t, k) \in \mathbf{T}_K$ are fixed. Moreover, assume that the final time $t_K = T$ is given and either:

a) the time instances $t_1, \dots, t_{K-1}$ satisfying (6.103) are given.

b) the time instances $t_1, \dots, t_{K-1}$ satisfying (6.103) are unknown.

Find the control u (and the switching time instances for case b)), such that the following optimization problem is solved:

$$\text{minimize} \qquad \mathbf{J} = \sum_{k=0}^{K-1} \mathbf{J}_k \tag{6.105}$$

$$\text{subject to} \begin{cases} \mathcal{H} \text{ defined by (6.99)}, & \\ x(t_0, 0) \in D_{i_0} & i_0 \in \mathbf{I}, \\ x(t_{k+1}, k) \in D_{i_{k+1}} & \forall k \in \{0, \dots, K-1\},\ i_k \in \mathbf{I}, \end{cases} \tag{6.106}$$

where

$$\mathbf{J}_k = \frac{1}{2} x(t_{k+1}, k)^\top S(t_{k+1}, k) x(t_{k+1}, k) + \frac{1}{2} \int_{t_k}^{t_{k+1}} [x^\top\ Q_k\ x + u^\top\ R_k\ u]\ dt, \tag{6.107}$$

for all $k \in \{0, \dots, K-1\}$. □

In (6.107) the given symmetric matrices $S(t_{k+1}, k)$ for each $k \in \{0, 1, \dots, K-2\}$ are used to specify the cost variables corresponding to the switching time instances, and $S(t_K, K-1)$ is used to specify the cost variable at an end point $x(T, K-1) = x_f \in D_{i_K}$. Problem 6.5.1 is fairly general and we can solve the following scenarios by finding a solution to this problem.

i) The switching times with regard to case a) in Problem 6.5.1 are fixed. We call this problem "fixed switching times problem".

ii) The switching times are free as represented by case b) of Problem 6.5.1. We refer to this problem as "free switching times problem".

iii) The initial state vector x_0 and the terminal state vector x_f are fixed. This condition is equivalent to have

$$c_{i_0} = I, \quad r_{i_0} = x_0, \qquad c_{i_K} = I,\ r_{i_K} = x_f.$$

iv) Item iii), and the conditions $D_{i_k} = \mathbb{R}^n$ for all $k \in \{1, \dots, K-1\}$ and $i_k \in \mathbf{I}$ hold. Then, $\sigma(t, x) = \sigma(t)$ and Problem 6.5.1 is reduced to computation of an optimal control u and switching time instances $t_1, \dots, t_{K-1}$, which minimize the cost (6.105). In this case, a switching can occur at each point of the state space.

Our solution approach is based on parameterization of the optimal control problem with respect to initial and end points of each switching time interval, similar to the approach for solving Problem 6.4.1. To this end, first we represent the LQR problem for a piece of trajectory and parameterize the problem by the initial and end points of each switching time interval. We derive an analytical expression for the optimal cost using the sweeping method presented by Lewis and Syrmos (1995). This allows us to find an analytical expression for the entire cost $\mathbf{J}$. Afterwards, we determine the switching times and states in the next step.

6.5.1 Optimal solution for a piece of a trajectory

Consider a piece of hybrid trajectory that evolves with the dynamics as described by (6.99) within a time interval $(t, k) \in \mathbb{T}_k$ for some $k \in \{0, \ldots, K-1\}$. Referring to the principle of optimality, if Problem 6.5.1 has an optimal solution, then each piece of the optimal trajectory between two consecutive switching points $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ must be optimal. In other words, if we assume the initial state vector $\mathbf{x}_k := x(t_k, k)$ and the final state vector $\mathbf{x}_{k+1} := x(t_{k+1}, k)$ are parameters, then solving the following optimization problem for a trajectory $x(t, k)$ that satisfies the differential equation (6.99) for $\sigma(t, x) = j_k \in \mathcal{L}$, is motivated by Problem 6.5.1:

$$\text{minimize} \qquad \mathbf{J}_k$$

$$\text{subject to} \begin{cases} \dot{x} & = A_{j_k}\, x + B_{j_k}\, u, \\ x(t_k, k) & = \mathbf{x}_k, \\ x(t_{k+1}, k) & = \mathbf{x}_{k+1}. \end{cases} \tag{6.108}$$

Now, Problem (6.108) represents an optimal control problem with initial and end points as parameters. The solution is given by (see Lewis and Syrmos (1995), pp.224)

$$u = -R_k^{-1}\, B_{j_k}^\top\, \left(Sx - VP^{-1}\, [V^\top\, x - \mathbf{x}_{k+1}] \right), \tag{6.109}$$

where

$$\begin{aligned} -\dot{S} &= A_{j_k}^\top\, S + S\, A_{j_k} - S\, B_{j_k}\, R_k^{-1}\, B_{j_k}^\top\, S + Q_k, \\ K_k &= R_k^{-1}\, B_{j_k}^\top\, S, \\ -\dot{V} &= (A_{j_k} - B_{j_k}\, K_k)^\top\, V, \\ \dot{P} &= V^\top\, B_{j_k}\, R_k^{-1}\, B_{j_k}^\top V, \end{aligned} \tag{6.110}$$

for each $(t, k) \in \mathbb{T}_k$ and with the boundary conditions

$$V(t_{k+1}, k) = I, \;\; P(t_{k+1}, k) = 0, \;\; \text{and given } S(t_{k+1}, k) \geq 0.$$

In the solution (6.110), the auxiliary variable $V(t, k) \in \mathbb{R}^{n \times n}$ is a "modified state transition matrix" for the adjoint of the linear time varying closed loop system, and $-P(t, j) \in \mathbb{R}^{n \times n}$ is a sort of weighted reachability gramian. If $\det(P(t, k)) = 0$ for all

$(t,k) \in \mathbb{T}_k$, the problem is abnormal and no solution exists; see Lewis and Syrmos (1995). However, if $Q_k = 0$ for all $k \in \{0,\ldots,K-1\}$, controllability of the pairs (A_{j_k}, B_{j_k}) is sufficient for the existence of solutions. Moreover, in this form of the solutions the co-state parameters $\lambda(t,k)$ are given in the form of

$$\lambda(t,k) = S(t,k)\; x - V(t,k)\; P(t,k)^{-1}\; \left[V(t,k)^\top\; x(t,k) - \mathbf{x}_{k+1}\right], \tag{6.111}$$

where the elements

$$\theta_k := \theta(t,k) = -P(t,k)^{-1}\; [V(t,k)^\top\; x(t,k) - \mathbf{x}_{k+1}], \tag{6.112}$$

are constant in the interval $(t,k) \in \mathbb{T}_k$. Furthermore, referring to (6.109) and (6.111), the relationship between the co-state and the control is given by

$$u(t,k) = -R_k^{-1}\; B_{j_k}^\top\; \lambda(t,k) \qquad \forall (t,k) \in \mathbb{T}_k;$$

see Lewis and Syrmos (1995). Note that in (6.110), the first Riccati equation for S, as well as the differential equations for V and P, are solved backwards in time within any time interval.

Now, for simplification of notation, in the sequel, we signify the values of the functions S, V, and P at the specific time instance (t_k, k) by the index k. For example, S_k should be interpreted as $S(t_k, k)$. As stated by Lemma 6.4.1 the optimal value of the cost $\mathbf{J}_k$ with the state-feedback given by (6.109) can be represented in analytical form as follows:

$$\mathbf{J}_k^* = \tfrac{1}{2}\mathbf{x}_k^\top\; S_k\; \mathbf{x}_k - \tfrac{1}{2}\; \left[V_k^\top\; \mathbf{x}_k - \mathbf{x}_{k+1}\right]^\top P_k^{-1}\; [V_k^\top\; \mathbf{x}_k - \mathbf{x}_{k+1}]. \tag{6.113}$$

6.5.2 A QP problem for finding the switching points

Having discussed the analytical solutions to optimal control for each piece of the trajectory, in this section we consider all of them together. Hence, by (6.113) the overall cost equals

$$\mathbf{J} = \sum_{k=0}^{K-1} \mathbf{J}_k^* = \frac{1}{2} X^\top M X, \tag{6.114}$$

where $X := [\mathbf{x}_0^\top \;\; \mathbf{x}_1^\top \;\; \ldots \;\; \mathbf{x}_K^\top]^\top$ includes an initial point $\mathbf{x}_0 = x(t_0, 0) \in D_{i_0}$, switching points $\mathbf{x}_1, \ldots, \mathbf{x}_{K-1}$, and a terminal point $\mathbf{x}_K = x(t_K, K-1) \in D_{i_K}$. $M = [M_{ij}] \in \mathbb{R}^{[n(K+1)]\times[n(K+1)]}$ is a symmetric matrix of the form:

$$M = \begin{bmatrix} M_{0,0} & M_{0,1} & 0 & \ldots & 0 \\ M_{1,0} & M_{11} & M_{12} & \ldots & 0 \\ 0 & M_{21} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & M_{K-1,K-1} & M_{K-1,K} \\ 0 & 0 & \ldots & M_{K,K-1} & M_{K,K} \end{bmatrix},$$

with the matrix elements

$$\begin{cases} M_{0,0} = M_{0,0}^\top = S_0 - V_0\ P_0^{-1}\ V_0^\top \\ M_{jj} = M_{jj}^\top = S_j - V_j P_j^{-1} V_j^\top - P_{j-1}^{-1} & \forall j \in \{1, \dots, K-1\}, \\ M_{KK} = M_{KK}^\top = -P_{K-1}^{-1}, \\ M_{j,j+1} = M_{j+1,j}^\top = V_j P_j^{-1} & \forall j \in \{0, \dots, K-1\}. \end{cases}$$

Note that the cost defined in (6.114) is parameterized by the decision variable X. Now, we add the required constraints to the optimal cost $\mathbf{J}$ to ensure that $\mathbf{x}_k \in D_{i_k}$ for each $k \in \{0, \dots, K\}$. This defines a static optimization problem as

$$\begin{aligned} &\text{minimize } \mathbf{J} = \frac{1}{2} X^\top M X \\ &\text{subject to } \ \mathbf{C}X + \mathbf{R} = 0, \end{aligned} \tag{6.115}$$

where the matrix $\mathbf{C}$ and the vector $\mathbf{R}$ include the parameters characterizing the equations of the jump sets D_{i_k} for $i_k \in \{0, \dots, K\}$, namely

$$\mathbf{C} = \text{blockdiag}([c_{i_0}, \dots, c_{i_K}]), \quad \mathbf{R} = [r_{i_0}^\top \ \dots \ r_{i_K}^\top]^\top. \tag{6.116}$$

Now, the cost defined in (6.115) is parameterized only by the decision variable X. Note that since $\mathbf{J}_k > 0$ for each $k \in \{0, \dots, K-1\}$, we have that $\frac{1}{2} X^\top M X > 0$. This implies that $M > 0$ and Problem (6.115) is convex.

Now, introducing the Lagrange multipliers in the form of Λ = blockdiag $([\Lambda_0, \dots, \Lambda_K])$, the Lagrangian of the cost (6.115) reads

$$L(X, \Lambda) = \tfrac{1}{2} X^\top M X + \Lambda^\top (\mathbf{C}X + \mathbf{R}). \tag{6.117}$$

6.5.3 Computing optimal switching points

Now, if we set the derivative of the Lagrangian (6.117) with respect to X to zero, then the optimal switching points can be computed. Consequently, as $M > 0$, Problem (6.115) has a globally optimal value X given by

$$X^* = -M^{-1}\mathbf{C}^\top (\mathbf{C}M^{-1}\mathbf{C}^\top)^{-1}\mathbf{R} := \mathbf{G}(\mathbf{S}, \mathbf{V}, \mathbf{P}), \tag{6.118}$$

where $\mathbf{G}$ is a function of $\mathbf{S} := (S_0, \dots, S_{K-1})$, $\mathbf{V} := (V_0, \dots, V_{K-1})$, and $\mathbf{P} := (P_0, \dots, P_{K-1})$. Hence, the optimal cost equals

$$\mathbf{J}^* = \frac{1}{2} \mathbf{R}^\top (\mathbf{C}M^{-1}\mathbf{C}^\top)^{-1}\mathbf{R}. \tag{6.119}$$

Note that the matrix M is a function of switching times $t_1, \dots, t_{K-1}$. Thus, for the problem of fixed switching times which is specified by case a) in Problem 6.5.1, M is given. However, for the problem of free switching times M is not known so far, and has to be determined in the following.

6.5.3.1 Transversality condition and solutions of fixed switching times problem

Now, we show that our optimal solution u, given by (6.109), satisfies the so called "transversality conditions"(see Sussmann (1999); Caines et al. (2006); Liberzon (2011)) for the points belonging to the jump sets. To this end, recalling (6.113) the Lagrangian of Problem (6.115) can be reformulated as

$$L(X,\Lambda) = \frac{1}{2}X^\top MX + \Lambda^\top(\mathbf{C}X + \mathbf{R}) =$$
$$= \tfrac{1}{2}\left\{\sum_{j=0}^{K-1} \mathbf{x}_k^\top S_k \mathbf{x}_k - \left[V_k^\top \mathbf{x}_k - \mathbf{x}_{k+1}\right]^\top P_k^{-1}[V_k^\top \mathbf{x}_k - \mathbf{x}_{k+1}] + \Lambda_k^\top(c_{i_k}\mathbf{x}_k + r_{i_k})\right\} + \Lambda_K^\top(c_{i_K}\mathbf{x}_K + r_{i_K}).$$

Recall that the optimal value of X in Section 6.5.3 was computed by differentiating the Lagrangian with respect to $\mathbf{x}_k$ and setting it to zero. For an initial condition, this is equivalent to have

$$\frac{\partial L(X,\Lambda)}{\partial \mathbf{x}_0} = 0 \Rightarrow \left(S_0\ \mathbf{x}_0 - V_0\ P_0^{-1}\ [V_0^\top\ \mathbf{x}_0 - \mathbf{x}_1]\right) + c_{i_0}^\top\ \Lambda_0 = 0. \tag{6.120}$$

Likewise, for a point belonging to the jump set D_{i_k}, we should have

$$\begin{aligned}\frac{\partial L(X,\Lambda)}{\partial \mathbf{x}_k} = 0 \Rightarrow &\left(S_k\ \mathbf{x}_k - V_k\ P_k^{-1}\ [V_k^\top\ \mathbf{x}_k - \mathbf{x}_{k+1}]\right) + \\ &+ \left(P_{k-1}^{-1}\ [V_{k-1}^\top\ \mathbf{x}_{k-1} - \mathbf{x}_k]\right) + c_k^\top\ \Lambda_k = 0 \qquad \forall k \in \{1,\dots,K-1\},\end{aligned} \tag{6.121}$$

and finally for a terminal point

$$\frac{\partial L(X,\Lambda)}{\partial \mathbf{x}_K} = 0 \Rightarrow \left(P_{K-1}^{-1}\ [V_{K-1}^\top\ \mathbf{x}_{K-1} - \mathbf{x}_K]\right) + c_{i_K}^\top\ \Lambda_K = 0. \tag{6.122}$$

As discussed previously, the co-states are defined by (6.111) and the variables $\theta(t,k)$ in (6.112) are constant in the time interval $\mathbb{T}_k$ for all $k \leq K-1$. Therefore, the co-states can be rewritten in the form of

$$\begin{aligned}\lambda(t_k,k) &= S_k\ \mathbf{x}_k - V_k\ P_k^{-1}\ [V_k^\top \mathbf{x}_k - \mathbf{x}_{k+1}], && \forall k \in \{0,\dots,K-1\},\\ \lambda(t_k,k-1) &= S(t_k,k-1)\ \mathbf{x}_k - P_{k-1}^{-1}\ [V_{k-1}^\top\ \mathbf{x}_{k-1} - \mathbf{x}_k], && \forall k \in \{1,\dots,K\}.\end{aligned} \tag{6.123}$$

Thus, (6.121) and (6.123) reveal that

$$\lambda(t_0,0) + c_{i_0}^\top\ \Lambda_0 = 0, \tag{6.124}$$
$$\lambda(t_k,k) - \lambda(t_k,k-1) + S(t_k,k-1)\ \mathbf{x}_k + c_{i_k}^\top\ \Lambda_k = 0, \tag{6.125}$$
$$-\lambda(t_K,K-1) + S(t_K,K-1) + c_{i_K}^\top\ \Lambda_K = 0. \tag{6.126}$$

In the literature, conditions (6.124-6.126) are referred to as transversality conditions on the initial point, switching points (jump points), and terminal point, respectively (Sussmann, 1999; Liberzon, 2011). Now, we can present a theorem concerning the fixed switching times problem.

Theorem 6.5.1. *The control* (6.109)*, computed by the ordinary differential equations* (6.110)*, and switching parameters, computed by* (6.118)*, solve the case of the fixed switching times in Problem 6.5.1. Moreover, the optimal cost* (6.105) *is equal to* (6.119)*.* □

Now, solving the free switching times case of Problem 6.5.1 entails deriving the appropriate switching time instances $t_1, \ldots, t_{K-1}$. This will be presented in the sequel.

6.5.4 Computation of optimal switching time instances

For computing the optimal switching times, we set the partial derivative of the cost function in (6.115) with respect to the switching time t_k to zero. In other words,

$$\frac{\partial \mathbf{J}^*}{\partial t_k} = 0 \Rightarrow \frac{\partial \mathbf{J}^*_{k-1}}{\partial t_k} + \frac{\partial \mathbf{J}^*_k}{\partial t_k} = 0 \qquad \forall k \in \{1, \ldots, K-1\}. \tag{6.127}$$

Now, we aim at showing that the following relations hold

$$\begin{aligned}\frac{\partial \mathbf{J}^*_k}{\partial t_k} &= -H\Big(\lambda, \mathbf{x}_k, (t_k, k)\Big) = -\Big[\frac{1}{2}\left(\mathbf{x}_k^\top Q_k \mathbf{x}_k + u(t_k,k)^\top\ R_k\ u(t_k,k)\right) + \\ &\qquad \lambda(t_k,k)^\top\ (A_{j_k}\ \mathbf{x}_k + B_{j_k}\ u(t_k,k))\Big], \\ \frac{\partial \mathbf{J}^*_{k-1}}{\partial t_k} &= H\Big(\lambda, \mathbf{x}_k, (t_k, k-1)\Big) = \frac{1}{2}\left(\mathbf{x}_k^\top\ Q_{k-1}\ \mathbf{x}_k + u(t_k,k-1)^\top\ R_{k-1}\ u(t_k,k-1)\right) \\ &\qquad + \lambda(t_k,k-1)^\top\ \left(A_{j_{k-1}}\ \mathbf{x}_k + B_{j_{k-1}}\ u(t_k,k-1)\right).\end{aligned} \tag{6.128}$$

Note that (6.128) in fact represents the Hamilton-Jacobi-Bellman equation at the time instances (t_k, k) and $(t_k, k-1)$ (see Appendix A.8.2 and Lewis and Syrmos (1995)). For this purpose, similar to the computation (6.65), at time instance (t_k, k), we can write

$$\begin{aligned}\tfrac{1}{2}\ \mathbf{x}_k^\top\ Q_k\ \mathbf{x}_k + \tfrac{1}{2}\ u(t_k,k)^\top R_k u(t_k,k) &= -\tfrac{1}{2}\mathbf{x}_k^\top\ \dot{S}_k\ \mathbf{x}_k - \mathbf{x}_k^\top\ S_k(A_{j_k}\ \mathbf{x}_k + B_{j_k} u(t_k,k)) \\ &\quad + \tfrac{1}{2}\ [V_k^\top\ \mathbf{x}_k - \mathbf{x}_{k+1}]^\top\ P_k^{-1}\ V_k^\top\ B_{j_k}\ R_k^{-1}\ B_{j_k}^\top\ V_k\ P_k^{-1}\ [V_k^\top\ \mathbf{x}_k - \mathbf{x}_{k+1}] \\ &= -\tfrac{1}{2}\ \mathbf{x}_k^\top\ \dot{S}_k \mathbf{x}_k - \lambda(t_k,k)^\top (A_{j_k}\ \mathbf{x}_k + B_{j_k} u(t_k,k)) + \\ &\quad \theta_k^\top V_k^\top (A_{j_k}\ \mathbf{x}_k + B_{j_k} u(t_k,k)) + \tfrac{1}{2}\theta_k V_k^\top B_{j_k}\ R_k^{-1}\ B_{j_k}^\top V_k\ \theta_k,\end{aligned}$$

where θ_k is given by (6.112). Using (6.109) for u and (6.113) for the cost $\mathbf{J}^*_k$, we deduce

$$\begin{aligned}&\tfrac{1}{2}\ \mathbf{x}_k^\top Q_k \mathbf{x}_k + \tfrac{1}{2} u(t_k,k)^\top R_k u(t_k,k) + \lambda(t_k,k)^\top (A_{j_k}\mathbf{x}_k + B_{j_k} u(t_k,k)) = \\ &\quad = -\tfrac{1}{2}\mathbf{x}_k^\top\ \dot{S}_k\ \mathbf{x}_k + \theta_k^\top\ V_k^\top\ (A_{j_k}\ \mathbf{x}_k - B_{j_k}\ K_k\ \mathbf{x}_k) - \tfrac{1}{2}\ \theta_k^\top\ V_k^\top\ B_{j_k}\ R_k^{-1}\ B_{j_k}^\top\ V_k\ \theta_k \\ &\quad = -\tfrac{1}{2}\ \mathbf{x}_k^\top\ \dot{S}_k\ \mathbf{x}_k - \theta_k^\top\ \dot{V}_k^\top\ \mathbf{x}_k - \tfrac{1}{2}\ \theta_k^\top\ \dot{P}_k\ \theta_k \\ &\quad = -\frac{\partial \mathbf{J}^*_k}{\partial t_k}.\end{aligned}$$

Furthermore, at time instance $(t_k, k-1)$, we observe that

$$\frac{\partial \mathbf{J}^*_{k-1}}{\partial t_k} = \tfrac{1}{2}\ \mathbf{x}_k^\top\ \dot{S}(t_k,k-1)\ \mathbf{x}_k + \theta_{k-1}^\top\ \dot{V}(t_k,k-1)^\top\ \mathbf{x}_k + \tfrac{1}{2}\ \theta_{k-1}^\top\ \dot{P}(t_k,k-1)\ \theta_{k-1}.$$

Then, similar to the above computation we can show $\partial \mathbf{J}^*_{k-1}/\partial t_k = H\Big(\lambda, \mathbf{x}_k, (t_k, k-1)\Big)$. Consequently, (6.127) and (6.128) indicate that the Hamiltonian is continuous with respect to time. This fact complies with the maximum principle for hybrid systems; see Sussmann (1999); Liberzon (2011). Now, employing (6.123) for $\lambda(t_k, k)$ and $\lambda(t_k, k-1)$, and substituting into the following equation

$$H\Big(\lambda, \mathbf{x}_k, (t_k, k)\Big) = H\Big(\lambda, \mathbf{x}_k, (t_k, k-1)\Big),$$

condition (6.127) can be expressed by the following boundary condition

$$X^\top \mathbf{M}^{(k)}(\mathbf{S}, \mathbf{V}, \mathbf{P}) X = 0, \tag{6.129}$$

where the matrix $\mathbf{M}^{(k)}$ for each $k \in \{1, \dots, K-1\}$ has the form

$$\mathbf{M}^{(k)} = \begin{bmatrix} M^{(k)}_{0,0} & M^{(k)}_{0,1} & \dots & M^{(k)}_{0,K} \\ M^{(k)}_{1,0} & M^{(k)}_{1,1} & \dots & M^{(k)}_{1,K} \\ \vdots & \ddots & \ddots & \vdots \\ M^{(k)}_{K,1} & M^{(k)}_{K,2} & \dots & M_{K,K} \end{bmatrix} \qquad \forall k \in \{1, \dots, K-1\}.$$

$$\begin{cases} \mathbf{M}^{(k)}_{k,k} & = Q_k - [S_k - V_k\, P_k^{-1}\, V_k^\top]^\top B_{j_k}\, R_k^{-1} B_{j_k}^\top\, [S_k - V_k\, P_k^{-1}\, V_k^\top] \\ & \quad +[S_k - V_k\, P_k^{-1}\, V_k^\top]^\top A_{j_k} + A_{j_k}^\top\, [S_k - V_k\, P_k^{-1} V_k^\top] - Q_{k-1} \\ & \quad +[S(t_k, k-1) + P_{k-1}^{-1}\,]^\top\, B_{j_{k-1}}\, R_{k-1}^{-1}\, B_{j_{k-1}}^\top\, [S(t_k, k-1) + P_{k-1}^{-1}] \\ & \quad -[S(t_k, k-1) + P_{k-1}^{-1}\,]^\top A_{j_{k-1}} - A_{j_{k-1}}^\top\, [S(t_k, k-1) + P_{k-1}^{-1}], \\ \mathbf{M}^{(k)}_{k,k+1} & = \mathbf{M}^{(k)\top}_{k+1,k} = [S_k - V_k\, P_k^{-1}\, V_k^\top]^\top\, B_{j_k}\, R_k^{-1}\, B_{j_k}^\top\, V_k\, P_k^{-1} - P_k^{-1}\, V_k^\top\, A_{j_k}, \\ \mathbf{M}^{(k)}_{k+1,k+1} & = -P_k^{-1}\, V_k^\top\, B_{j_k} R_k^{-1} B_{j_k}^\top\, V_k\, P_k^{-1}, \\ \mathbf{M}^{(k)}_{k-1,k-1} & = V_{k-1}\, P_{k-1}^{-1}\, B_{j_{k-1}}\, R_{k-1}^{-1}\, B_{j_{k-1}}^\top\, P_{k-1}^{-1}\, V_{k-1}^\top, \\ \mathbf{M}^{(k)}_{k-1,k} & = \mathbf{M}^{(k)\top}_{k,k-1} = -P_{k-1}^{-1}\, V_{k-1}^\top\, B_{j_{k-1}}\, R_{k-1}^{-1}\, B_{j_{k-1}}^\top [S(t_k, k-1) - P_{k-1}^{-1}] \\ & \qquad\qquad\qquad + A_{j_{k-1}}^\top\, V_{k-1}\, P_{k-1}^{-1}, \\ \mathbf{M}^{(k)}_{j,l} & = 0, \quad j, l \in \{0, \dots, K\} \text{ otherwise.} \end{cases}$$

Note that in the above formulation all matrices with index "-1", and "$K+1$" should be set to zero. Now we can substitute the optimal value $X^* = \mathbf{G}(\mathbf{S}, \mathbf{V}, \mathbf{P})$ from (6.118) into (6.129), and obtain an expression which is independent of the variable X and only depends on the time instances $t_1, \dots, t_{K-1}$. The difficulty still remains that in (6.110) several ODE's exist which are defined on different time scales and different initial and end times. To overcome this problem, for each time interval $\mathbb{T}_k$ we define a positive variable α_k as

$$\alpha_k^2 = (t_{k+1} - t_k)/T, \quad \sum_{k=0}^{K-1} \alpha_k^2 = 1.$$

and K set of variables $(\bar{S}^{(k)}, \bar{V}^{(k)}, \bar{P}^{(k)})$ for $k \in \{0, \dots, K-1\}$ as

$$\bar{S}^{(k)}(t) = S(\alpha_k^2\, t + t_k, k),\;\; \bar{V}^{(k)}(t) = V(\alpha_k^2\, t + t_k, k),\;\; \bar{P}^{(k)}(t) = P(\alpha_k^2\, t + t_k, k), \qquad \forall t \in [0, T].$$

This definition establishes the following boundary conditions for the variables $(\bar{S}^{(k)}, \bar{V}^{(k)}, \bar{P}^{(k)})$:

$$\begin{aligned}
\bar{S}^{(k)}(T) &= S(t_{k+1}, k) \text{ given}, & \bar{S}^{(k)}(0) &= S(t_k, k),\\
\bar{V}^{(k)}(T) &= V(t_{k+1}, k) = I, & \bar{V}^{(k)}(0) &= V(t_k, k),\\
\bar{P}^{(k)}(T) &= P(t_{k+1}, k) = 0, & \bar{P}^{(k)}(0) &= P(t_k, k),
\end{aligned} \tag{6.130}$$

for each $k \in \{0, \dots, K-1\}$. Now, for finding the switching times $t_1, \dots, t_{K-1}$, we need only to solve one differential algebraic equation, with given boundary conditions and unknown parameters α_k, as follows:

$$\begin{aligned}
-\dot{\bar{S}}^{(k)} &= \alpha_k^2\; (A_{j_k}^\top\; \bar{S}^{(k)} + \bar{S}^{(k)}\; A_{j_k} - \bar{S}^{(k)}\; B_{j_k}\; R_k^{-1}\; B_{j_k}^\top\; \bar{S}^{(k)} + Q_k) \qquad \forall\, t \in [0, T],\\
\bar{K}^{(k)} &= R_k^{-1}\; B_{j_k}^\top\; \bar{S}^{(k)},\\
-\dot{\bar{V}}^{(k)} &= \alpha_k^2\; (A_{j_k} - B_{j_k}\bar{K}^{(k)})^\top\; \bar{V}^{(k)},\\
\dot{\bar{P}}^{(k)} &= \alpha_k^2\; \bar{V}^{(k)\top}\; B_{j_k}\; R_k^{-1}\; B_{j_k}^\top\; \bar{V}^{(k)} \qquad \forall k \in \{0, \dots, K-1\}.
\end{aligned} \tag{6.131}$$

$$\text{subject to:} \begin{cases}
\bar{S}^{(k)}(T) = S(t_{k+1}, k) \text{ given}, \quad \bar{V}^{(k)}(T) = I_n, \quad \bar{P}^{(k)}(T) = 0,\\
\mathbf{G}(\bar{\mathbf{S}}, \bar{\mathbf{V}}, \bar{\mathbf{P}})^\top\; \mathbf{M}^{(k)}(\bar{\mathbf{S}}, \bar{\mathbf{V}}, \bar{\mathbf{P}})\; \mathbf{G}(\bar{\mathbf{S}}, \bar{\mathbf{V}}, \bar{\mathbf{P}}) = 0,\\
\sum_{k=0}^{K-1} \alpha_k^2\; = 1,
\end{cases}$$

where $\bar{\mathbf{S}} = \big(\bar{S}^{(0)}, \dots, \bar{S}^{(K-1)}\big)$, $\bar{\mathbf{V}} = \big(\bar{V}^{(0)}, \dots, \bar{V}^{(K-1)}\big)$, and $\bar{\mathbf{P}} = \big(\bar{P}^{(0)}, \dots, \bar{P}^{(K-1)}\big)$.

The algorithm (6.131) is solved backwards until its boundary conditions and the criteria for the numbers α_i are satisfied. Solving this differential algebraic equation can be done by the Matlab command "bvp5c". Now, we present the next theorem concerning the free switching times problem.

Theorem 6.5.2. *The solution of the differential algebraic equation* (6.131) *determines the switching time instances for Problem 6.5.1. Then, the control* (6.109)*, computed by the ordinary differential equations* (6.110)*, and the switching points, specified by* (6.118)*, solve the free switching times case of Problem 6.5.1. Moreover, the optimal cost* (6.105) *is equal to* (6.119). □

Remark 6.5.1. For the free switching times case of Problem 6.5.1, when the switching signal does not depend on the states, and the initial state vector $x(t_0, 0) = x_0$ and the end state vector $x(t_K, K-1) = x_f$ are fixed, that is,

$$\mathbf{C} = \text{blockdiag}([I, 0, \ldots, 0\ I]), \quad \mathbf{R} = [x_0^\top \ 0 \ \ldots \ 0 \ x_f^\top]^\top, \tag{6.132}$$

the optimal switching times problem can be computed similar to (6.118), and the whole algorithm can be followed similar to what discussed in this section. In this case, the transversality condition (6.125) has a simpler form as:

$$\lambda(t_k, k-1) - \lambda(t_k, k) = S(t_k, k-1)\ \mathbf{x}_k. \tag{6.133}$$

Example 6.5.1. Consider the hybrid linear system (6.99) with $\ell = 3$, and the system parameters

$$A_1 = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

The corresponding cost variables are given by

$$\mathbf{J}_0 = \frac{1}{2}\int_{t_0}^{t_1} \left(x^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + u^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \right) dt,$$

$$\mathbf{J}_1 = \frac{1}{2} x^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}\int_{t_1}^{t_2} \left(x^\top \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x + u^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \right) dt,$$

$$\mathbf{J}_2 = \frac{1}{2} x^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}\int_{t_2}^{t_3} \left(x^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \right) dt,$$

where $K = 3$ has been chosen, indicating that two switches between the different flow dynamics occur. The aim is to minimize the overall cost $\mathbf{J}$ defined by (6.105). Define $x = [x_1\ x_2]^\top$. Let's assume that $t_0 = 0$, $t_K = 3$, $x_0 = [2 \ \ 7]^\top$, $x_f = [4 \ \ -1]^\top$, and the equations of the manifolds are given by

$$D_0 = \{x_0\}, \ D_1 = \{x \in \mathbb{R}^2 : 2x_1 - x_2 + 2 = 0\},$$
$$D_2 = \{x \in \mathbb{R}^2 : x_1 - x_2 = 0\}, \ D_3 = \{x_f\}.$$

The optimal trajectory computed via our algorithm is depicted in Figure 6.5.1. The

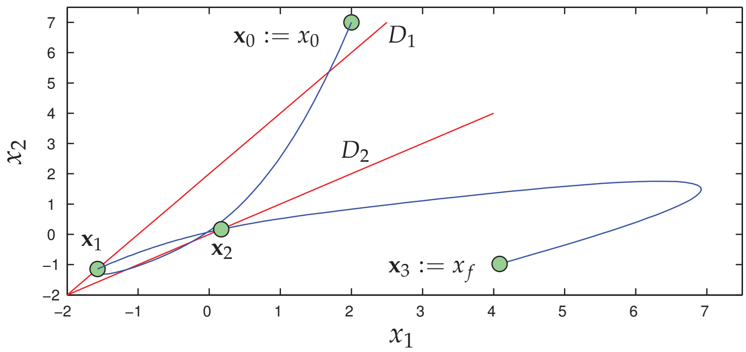

Figure 6.5.1: The optimal solution to the control policy in Example 6.5.1.

switching times computed via a differential algebraic equation are given as $t_1 = 1.0942$ and $t_2 = 1.5123$. Hence, this defines the hybrid time domain as

$$\mathbf{T}_K = ([0, 1.0942] \times \{0\}) \cup ([1.0942, 1.5123] \times \{1\}) \cup ([1.5123, 3] \times \{2\}).$$

The switching points for this problem are computed to be

$$\mathbf{x}_1 = [-1.5716 \ \ -1.1432]^\top, \ \mathbf{x}_2 = [0.0169 \ 0.0169]^\top.$$

The optimal costs equal $\mathbf{J}_0^* = 13.2904$, $\mathbf{J}_1^* = 1.9566$, and $\mathbf{J}_2^* = 67.2008$. Consequently, $\mathbf{J}^* = 82.4478$.

6.6 Conclusions

This chapter considers stability, robust stability, and optimal control of several classes of hybrid linear systems. Concerning the stability problem, we established a link between stability of hybrid and switched linear systems. Furthermore, we found an upper bound for the maximum uncertainty under which stability of a perturbed hybrid linear system having a quadratic robust Lyapunov function is guaranteed.

We discussed two problems regarding optimal control of hybrid linear systems. In the first problem, an algorithm for analytical computation of lower and upper bounds for the fixed initial and end states LQR problem in a class of hybrid linear systems including a single linear flow and linear jump dynamics was proposed. In the next scenario, we considered the LQ problem for a different class of hybrid linear systems with free switching time instances. We showed that the switching times can be determined by solving a differential algebraic equation with given boundary conditions.

Acknowledgement: The results of Sections 6.2, 6.3, and 6.5 have been developed through discussions with Prof. Sanfelice at the University of Arizona, and have not yet been published anywhere.

Chapter 7

Conclusions

In Chapter 2 we have dealt with distinct problems concerning left eigenstructure assignment for multi input systems, and partial and complete pole placement for single input systems. Special attention has been devoted to systems with $(n-1)$ control inputs due to the nice mathematical properties. The results of this chapter are particularly suitable for stabilization of certain classes of switched linear systems.

In Chapter 3 we have studied stability and stabilization of switched linear systems with state dependent switching signals. In our problem setting, we have assumed certain restrictions on switching manifolds. The stability analysis and stabilization of switched systems have been based on the concept of common left eigenvectors and left eigenstructure assignment introduced in Chapter 2. The main challenge here is to appropriately select a set of desired common left eigenvectors for guaranteeing the simultaneous stabilization of all linear subsystems, while avoiding the intersection of the common invariant subspace of the closed loop subsystems with the given switching manifold in the state space. These results are restrictive as an adequate number of control inputs and special assumptions on the geometry of the switching manifolds are required.

In Chapter 4 we have employed the concept of a common invariant subspace and left eigenvectors assignment for stability and stabilization of switched linear systems with arbitrary time switching signals. Our main result in this chapter discusses stabilization of controlled switched linear systems whose open loop constituent matrices share an invariant subspace to which a common quadratic Lyapunov function can be associated. Then, we have broadened this class by assuming that open loop matrices have invariant subspaces with sufficiently small distances from each other, such that a positive definite matrix which satisfies special forms of Riccati inequalities can be associated to those subspaces. For this approach, in addition to the required assumptions on the open loop matrices, an adequate number of control inputs is needed. Moreover, we have discussed robust stability of switched linear systems when their Hurwitz matrices share $(n-1)$ real left eigenvectors.

In Chapter 5 we have investigated the problem of quadratic stability and weak quadratic stability of a class of switched linear systems with two modes. We have shown that the results by Shorten and Narendra (2003) and Shorten et al. (2009) on stability of rank-1 difference switched systems can be extended to the more general form of switched linear systems whose constituent matrices have rank $m \geq 1$ difference, provided that a symmetric transfer function matrix can be associated with the pair of matrices. Moreover, we have defined sufficient and necessary conditions for a pair of matrices for which a suitable symmetric transfer function matrix exists. Finally, we have introduced an approach for computing a set of stabilizing control inputs for a class of switched systems whose constituent matrices have rank m difference.

In Chapter 6 we have studied stability and stabilization of hybrid linear systems. We have shown that the stability problem for a hybrid linear system is equivalent to the stability problem for a switched linear system, by using bilinear transformation for converting the discrete evolution of jumps to continuous dynamics. The main results of this chapter, however, have been related to the optimal control of hybrid linear systems. We have investigated two scenarios. In the first scenario, we have considered a problem in which a fixed sequence of switching between flow and jump sets occurs at fixed time instances. In this problem, the hybrid linear system is specified by a single flow and jump dynamics, and state space constraints are represented by polyhedral sets. We have found upper and lower bounds for the optimal value of the cost function via solving a constrained quadratic programming (QP) problem in which the state variables at switching time instances are the unknown parameters. The lower bound value has been obtained by relaxing the inequality constraints, whereas the upper bound value has been computed by solving the dual problem by means of gradient projection method. In the next scenario, using a similar approach we have considered an optimal control problem for a class of hybrid linear systems including multiple flows and a fixed sequence of switching with free switching time instances. The optimal switching times have been computed via solving a differential algebraic equation. High dimensions and large number of constraints in these optimal control problems cause difficulties in computing the numerical solutions.

Appendix A

Preliminaries

A.1 Vectors

Throughout the dissertation, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of natural, integer, real, and complex numbers, respectively. We denote n-dimensional real (complex) Euclidean space by $\mathbb{R}^n$ ($\mathbb{C}^n$) and the space of $n \times n$ matrices with real (complex) entries by $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$). Consider a complex number $a = \alpha + j\beta$, then we denote the real part of a by $\mathrm{Re}(a) = \alpha$, and its imaginary part by $\mathrm{Im}(a) = \beta$. The complex conjugate of a, denoted by a^*, is defined by $a^* = \alpha - j\beta$.

Consider a vector $v = [\nu_1 \ \dots \ \nu_n]^\top \in \mathbb{C}^n$. Then, $\mathrm{Re}(v) = [\mathrm{Re}(\nu_1) \ \dots \ \mathrm{Re}(\nu_n)]^\top$ and $\mathrm{Im}(v) = [\mathrm{Im}(\nu_1) \ \dots \ \mathrm{Im}(\nu_n)]^\top$ denote the real and imaginary parts of v, respectively. The conjugate transpose of v is defined by $v^* = [\nu_1^* \ \dots \ \nu_n^*]$. The vector 2-norm is defined by

$$\|v\| = \sqrt{\sum_{i=1}^{n} |\nu_i|^2}, \tag{A.1}$$

where $|\nu_i| = \sqrt{\mathrm{Re}(\nu_i)^2 + \mathrm{Im}(\nu_i)^2}$. Then, the infinity norm of v is defined by

$$\|v\|_\infty = \max_i |\nu_i| \qquad i \in \{1, \dots, n\}. \tag{A.2}$$

The following inequality between 2-norm and infinity norm holds

$$\|v\|_\infty \leq \|v\| \leq n\|v\|_\infty. \tag{A.3}$$

A.2 Matrix properties

In this part we give some elementary concepts concerning matrix properties.

A.2.1 Inverse of a matrix

Inverse properties of a matrix can be found, for example, in Horn and Johnson (1990). Consider a non-singular matrix $A \in \mathbb{C}^{n\times n}$. The inverse of A, denoted by A^{-1}, equals

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}, \tag{A.4}$$

where $\mathrm{adj}(A)$ is the adjoint of A. Consider a partitioned matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{A.5}$$

the inverse of A is computed by

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}\mathcal{S}^{-1}A_{21}A_{11}^{-1} & -A_{11}A_{12}\mathcal{S}^{-1} \\ -\mathcal{S}^{-1}A_{21}A_{11}^{-1} & \mathcal{S}^{-1} \end{bmatrix}, \tag{A.6}$$

where $\mathcal{S}$ is the Schur complement of A with respect to the block A_{11}, that is, $\mathcal{S} = A_{22} - A_{21}A_{11}^{-1}A_{12}$; see Horn and Johnson (1990).

A.2.2 Positive definite matrices

See Horn and Johnson (1990). Consider a Hermitian partitioned matrix

$$A = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}, \tag{A.7}$$

Then A is positive definite ($A > 0$), if and only if $X > 0$ and its Schur complement with respect to the block X is positive definite, that is, $\mathcal{S} = Z - Y^*X^{-1}Y > 0$. A Slightly different statement can be presented for a matrix A to be negative definite ($A < 0$), by considering that $-A > 0$.

A.2.3 Some determinant properties

See Horn and Johnson (1990). Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and $A_2 \in \mathbb{C}^{n\times n}$. Then, the following relationships hold

$$\det(A^{-1}) = 1/\det(A), \tag{A.8}$$

$$\det(A^{\top}) = \det(A), \tag{A.9}$$

$$\det(AA_2) = \det(A)\,\det(A_2). \tag{A.10}$$

If $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, then

$$\det(I_n - AB) = \det(I_m - BA). \tag{A.11}$$

A.2.4 Kronecker product

Suppose A is an $m \times n$ matrix and B is a $p \times q$ matrix. Then, the Kronecker product of A and B, denoted by $A \otimes B$, is an $mp \times nq$ block matrix defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}. \tag{A.12}$$

A.2.5 Matrix rank

The rank of a matrix A, denoted by $\text{rank}(A)$, is the number of the largest collection of linearly independent columns of A.

A.2.5.1 Sylvester rank inequality

See Horn and Johnson (1990). Suppose $A \in \mathbb{C}^{n \times m}$, $B \in \mathbb{C}^{m \times q}$, and $m \geq n$. Then, the following property for the rank of matrices holds

$$\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - m. \tag{A.13}$$

A.2.6 Eigenvalues and eigenvectors

For the concept of eigenvalues and eigenvectors see, for example, Horn and Johnson (1990). Consider a matrix $A \in \mathbb{C}^{n \times n}$. A number λ_1 is an eigenvalue of A if

$$\det(\lambda_1 I - A) = 0. \tag{A.14}$$

For each eigenvalue λ_1 there exists a vector $v \in \mathbb{C}^n$ (called a right eigenvector of A corresponding to the eigenvalue λ_1) satisfying the property

$$Av = \lambda_1 v,$$

and there exists a vector $w \in \mathbb{C}^n$ (called a left eigenvector of A corresponding to λ_1) which satisfies the equation

$$w^* A = w^* \lambda_1.$$

The polynomial

$$\det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n, \tag{A.15}$$

is called the characteristic polynomial of A. The roots of the characteristic polynomial specify the eigenvalues of A. Therefore, if $\lambda_1, \dots, \lambda_n$ are eigenvalues of A, the following relationships hold

$$\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(A) \quad = -a_1, \tag{A.16}$$

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \lambda_i \lambda_j \quad = a_2, \tag{A.17}$$

$$\prod_{i=1}^{n} \lambda_i = \det(A) \quad = (-1)^n a_n. \tag{A.18}$$

$A \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if its eigenvalues have negative real parts, *i.e.*, $\mathrm{Re}(\lambda_i) < 0$ for all $i \in \{1, \dots, n\}$.

A.2.7 Eigenvalue decomposition

See Horn and Johnson (1990). Any diagonalizable matrix $A \in \mathbb{R}^{n\times n}$ has a decomposition of the form

$$A = T^{-1} \Lambda T = V \Lambda W^*, \tag{A.19}$$

where Λ is a diagonal matrix whose diagonal entries are eigenvalues of A. $T^{-1} = V \in \mathbb{C}^{n\times n}$ is a matrix whose columns are eigenvectors of A. $W \in \mathbb{C}^{n\times n}$ is a matrix whose columns are the set of left eigenvectors of A.

When A is not diagonalizable, that is, the algebraic multiplicity of at least one eigenvalue, say λ_1, is greater than its geometric multiplicity, A has a generalized eigenvector. Then, there exist nonzero vectors v_1 and v_2 such that

$$(A - \lambda_1 I)\ v_1 = 0, \qquad (A - \lambda_1 I)\ v_2 = v_1, \tag{A.20}$$

where in this formulation v_1 is the eigenvector of A corresponding to λ_1, and v_2 is the generalized eigenvector of A corresponding to λ_1.

A.2.8 QR- decomposition

The concept of QR-decomposition can be found, for instance, in Horn and Johnson (1990). Any matrix $A \in \mathbb{C}^{n\times m}$ with $n \geq m$ can be factorized as the product of an $n \times n$ unitary matrix Q, that is, $Q^*Q = I$, and an $n \times m$ upper triangular matrix R, in the form of

$$A = QR = [Q_1\ Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1, \tag{A.21}$$

where R_1 is an $m \times m$ upper triangular matrix, Q_1 is an $n \times m$ matrix, and Q_2 is $n \times (n - m)$ matrix. In this description, Q_1 and Q_2 both have orthonormal columns. The Q_1R_1-factorization is called reduced QR- factorization (decomposition).

A.2.9 Real Schur decomposition

For the concept of real Schur decomposition, one is referred to, *e.g.*, Arbenz and Kressner (2010). For each $A \in \mathbb{R}^{n\times n}$ there exists a decomposition

$$A = QRQ^{-1}, \tag{A.22}$$

with an orthonormal matrix $Q \in \mathbb{R}^{n\times n}$ and a quasi triangular matrix $R \in \mathbb{R}^{n\times n}$ having the form

$$R = \begin{bmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & R_{m,m} \end{bmatrix}, \tag{A.23}$$

where any R_{ii} for each $i \in \{1, \dots, m\}$ is either a 1×1 matrix equal to a real eigenvalue of A, or is a 2×2 matrix having a pair of complex conjugate eigenvalues corresponding to two eigenvalues of A.

A.2.10 Singular value decomposition

The concept of singular value decomposition can be found, *e.g.*, in Horn and Johnson (1990). Any matrix $A \in \mathbb{C}^{n\times m}$ with $m \le n$ can be factorized in the form of

$$A = U\Sigma V^*, \tag{A.24}$$

where $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ are unitary matrices and $\Sigma \in \mathbb{C}^{n\times m}$ is a matrix with real non-negative entries in the form of

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \sigma_m \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \tag{A.25}$$

where $\sigma_1 \ge \sigma_2 \dots \sigma_m \ge 0$ are referred to as singular values of A.

A.2.11 Matrix norm

See Horn and Johnson (1990). Various definitions for matrix norms exist in the literature. However, we only introduce the norm of a matrix $A \in \mathbb{C}^{n\times m}$ induced by Euclidean vectors (2 norm), that is,

$$\|A\|_2 := \max_{w\neq 0} \frac{\|Aw\|}{\|w\|} = \max_{\|w\|=1} \|Aw\| = \sigma_{\max}(A), \tag{A.26}$$

where $\sigma_{\max}(A)$ is the maximum singular value of A, and $\|w\|$ is the vector 2-norm of w. Note that the following inequality for the matrix norm holds

$$\|A\|_2 \le \sqrt{\mathrm{tr}(A^* A)}. \tag{A.27}$$

A.2.12 Similarity

The definition of similarity can be found, for example, in Horn and Johnson (1990). Two matrices $A \in \mathbb{C}^{n\times n}$ and $\bar{A} \in \mathbb{C}^{n\times n}$ are similar if there exists an invertible matrix T such that

$$\bar{A} = T^{-1}AT. \tag{A.28}$$

A.2.13 Invariant subspace of a matrix

This information can be found, for example, in Gohberg et al. (2006). An m dimensional subspace $\mathcal{X}_m$ of $\mathbb{C}^n$ is said to be A invariant if for each $v \in \mathcal{X}_m$, we have $Av \in \mathcal{X}_m$. Let the columns of $V = [v_1 \ldots, v_m]$ be a basis for $\mathcal{X}_m$, then there exists a matrix $L \in \mathbb{C}^{m\times m}$, such that

$$AV = VL. \tag{A.29}$$

Moreover, the set of eigenvalues of L is a subset of the set of eigenvalues of A.

A.2.13.1 Distance between subspaces

For this part we refer the readers to Truhar (1996). Let V_i and V_j for some $i, j \in \mathbb{N}$ be bases for the two m dimensional subspaces $\mathcal{X}_{i,m}$ and $\mathcal{X}_{j,m}$, respectively. Define the orthogonal projection matrices $P_{\mathcal{X}_{i,m}} = V_i(V_i^*V_i)^{-1}V_i^*$ and $P_{\mathcal{X}_{j,m}} = V_j(V_j^*V_j)^{-1}V_j^*$. Then, the distance between the two m dimensional subspaces $\mathcal{X}_{i,m}$ and $\mathcal{X}_{j,m}$ is defined by the norm of the difference between the corresponding orthogonal projection matrices

$$\operatorname{dist}(\mathcal{X}_{i,m}, \mathcal{X}_{j,m}) := \|P_{\mathcal{X}_{i,m}} - P_{\mathcal{X}_{j,m}}\|.$$

Moreover, we have

$$\|P_{\mathcal{X}_{i,m}} - P_{\mathcal{X}_{j,m}}\| = \sin\theta_{ij},$$

where θ_{ij} is the greatest canonical angle between $\mathcal{X}_{i,m}$ and $\mathcal{X}_{j,m}$ which is defined by

$$\cos\theta_{ij} = \min_{\substack{x\in\mathcal{X}_{i,m}\\ x\neq 0}} \max_{\substack{y\in\mathcal{X}_{j,m}\\ y\neq 0}} \frac{y^*x}{\|x\|\|y\|}. \tag{A.30}$$

A.3 Controlled linear systems

Consider an LTI system

$$\dot{x} = Ax + Bu, \tag{A.31}$$

where $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. The controllability matrix $\Phi_c(A, B)$ for this system is defined by

$$\Phi_c(A, B) = [B \ AB \ \ldots \ A^{n-1}B]. \tag{A.32}$$

The pair (A, B) is said to be controllable if $\operatorname{rank}(\Phi_c(A, B)) = n$.

A.3.1 $[A\ B]$ invariant subspace

This part is taken from Gohberg et al. (2006). An m dimensional subspace $\mathcal{X}_m$ of $\mathbb{C}^n$ is said to be $[A\ B]$ invariant, if there exists a matrix $F \in \mathbb{R}^{n\times m}$ such that for each $\nu \in \mathcal{X}_m$, $(A+BF)\nu \in \mathcal{X}_m$. When $F=0$, this is interpreted as the familiar relation $A\mathcal{X}_m \subseteq \mathcal{X}_m$, for A invariant subspace $\mathcal{X}_m$.

A.3.2 Controlled block similarity

This definition has been taken from Gohberg et al. (2006). We say two controlled pairs $[A\ B] \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ and $[\bar{A}\ \bar{B}] \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ are block similar, if there exists an invertible transformation matrix $S \in \mathbb{C}^{(m+n)\times(m+n)}$ in the form of

$$S = \begin{bmatrix} N & 0 \\ L & M \end{bmatrix}, \tag{A.33}$$

such that

$$[\bar{A}\ \bar{B}] = N^{-1}[A\ B] \begin{bmatrix} N & 0 \\ L & M \end{bmatrix}. \tag{A.34}$$

A.4 Input/Output linear systems

Consider an LTI system

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{cases} \tag{A.35}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$, and $D \in \mathbb{C}^{m\times m}$. The observability matrix is defined by

$$\Phi_o(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{A.36}$$

Then, (C, A) is called observable if $\operatorname{rank}(\Phi_o(C, A)) = n$; see Kailath (1980). The transfer function matrix of the system (A.35) equals

$$G(s) = C(sI - A)^{-1}B + D. \tag{A.37}$$

We say (A, B, C, D) is a realization of the transfer function matrix $G(s)$. This realization is minimal if and only if (A, B) is controllable and (C, A) is observable.

A.4.1 Kalman-Yakubovic-Popov (KYP) lemma

The KYP lemma gives algebraic conditions for the existence of a certain type of Lyapunov functions for strictly positive real systems; see Boyd et al. (1994), and Zhou et al. (1996).

Lemma A.4.1. *Consider the LTI system* (A.35). *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, (A, B) be controllable, and (C, A) be observable. Then, the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is strictly positive real if and only if there exist matrices $P = P^\top > 0$, matrices L and W, and a number $\alpha > 0$, satisfying*

$$A^\top P + PA = -L^\top L - \alpha P, \tag{A.38}$$

$$B^\top P + W^\top L = C, \tag{A.39}$$

$$D + D^\top = W^\top W. \tag{A.40}$$

A.5 Linear time varying systems

The concept of solutions for linear time varying systems can be found, for instance, in (Kailath, 1980). Consider the linear time varying system

$$\dot{x} = A(t)x(t) + B(t)u(t) \qquad x(t_0) = x_0. \tag{A.41}$$

The solution to this system is given by

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)\ d\tau. \tag{A.42}$$

In this representation, ϕ is called state transition matrix and satisfies the conditions

$$\frac{\partial \phi(t, t_0)}{\partial t} = A(t)\phi(t, t_0), \tag{A.43}$$

$$\phi(\tau, \tau) = I. \tag{A.44}$$

A.6 Differential equations and inclusions

Here, we summarize some important concepts concerning the theory of differential equations and inclusions. They can be found, for example, in Cortes (2008); Smirnov (2001).

A.6.1 Absolute continuity

The function $\gamma : [a, b] \to \mathbb{R}$ is absolutely continuous in the interval $[a, b]$, if there exists a Lebesgue integral function $k : [a, b] \to \mathbb{R}$ such that

$$\gamma(t) = \gamma(a) + \int_{a}^{t} k(\tau)\ d\tau \qquad t \in [a, b]. \tag{A.45}$$

Absolute continuity is a stronger notion than continuity and weaker than differentiability. Every continuous differentiable function is absolutely continuous, but the inverse is not true. However, every absolutely continuous function is differentiable almost everywhere.

A.6.2 Solutions of differential equations

For a definition of the different classes of solutions for differential equations see, *e.g.*, Cortes (2008). Consider the differential equation

$$\dot{x} = f(x(t)) \qquad x(t_0) = x_0, \tag{A.46}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ represents the direction of the vector field at each point x. If f is continuous, then the solutions to (A.46) are continuously differentiable. These solutions are called classical solutions. If f is not continuous, then different notions of solutions for differential equation (A.46) are defined. In the following, we point out couple of them.

A.6.2.1 Caratheodory solutions

See Cortes (2008). Caratheodory solutions are absolutely continuous solutions $x(t)$, which result from integrating (A.46)

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau))\, d\tau. \tag{A.47}$$

Caratheodory solutions may not always follow the direction of vector fields, for example, at points of measure zero.

A.6.2.2 Filippov solutions

See Cortes (2008) and Filippov (1988). The Filippov set valued map considers the directions of vector fields around any point in the state space. For defining the Filippov solutions, the vector field in the right hand side of (A.46) is replaced by the so-called Filippov set-valued map defined by

$$F(x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{\text{co}}\, f(\mathbb{B}(x,\delta)\backslash S), \tag{A.48}$$

where $\overline{\text{co}}$ denotes the convex closure and μ denotes the Lebesgue measure. Then, the Filippov solutions are solutions of the following differential equations

$$\dot{x} \in F(x). \tag{A.49}$$

For instance, the differential equation

$$\dot{x} = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0, \end{cases}$$

is modified to the differential inclusion defined by

$$\dot{x} \in \begin{cases} 1 & x > 0, \\ [-1, 1] & x = 0, \\ -1 & x < 0. \end{cases}$$

Consider a differential inclusion $\dot{x} \in F(x)$. In the theory of differential inclusions, often some restrictions on set valued map F are assumed. We explain some related concepts; See Cai et al. (2008) and Goebel et al. (2012).

A.6.3 Outer semi-continuous set valued map

A set valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is said to be outer semi-continuous at the point $\bar{x}$, if

$$\limsup_{x \to \bar{x}} F(x) \subset F(\bar{x}). \tag{A.50}$$

A.6.4 Convex set valued map

A set valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is said to be convex, if the set is convex for all x. In other words, for each $x_0, x_1 \in \mathbb{R}^n$

$$F((1-\alpha)x_0 + \alpha x_1) \supset (1-\alpha)F(x_0) + \alpha F(x_1) \qquad \forall\ \alpha \in [0, 1]. \tag{A.51}$$

A.6.5 Locally bounded set valued map

A set valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is said to be locally bounded at the point $\bar{x}$, if for some neighborhood $\mathcal{N}(\bar{x})$, for each $V \in \mathcal{N}(\bar{x})$ the set $F(V) \subset \mathbb{R}^n$ is bounded. It is called locally bounded on $\mathbb{R}^n$ if it is locally bounded at every point $\bar{x} \in \mathbb{R}^n$.

A.7 Hybrid systems

Consider a hybrid system described by the four tuples (C, F, D, G), *i.e.*, a differential and a difference inclusion in the form of

$$\mathcal{H} : \begin{cases} \dot{x} & \in F(x) \qquad x \in C, \\ x^+ & \in G(x) \qquad x \in D. \end{cases} \tag{A.52}$$

The sets C and D are referred to as flow and jump sets, respectively. Typically, it is assumed that the set valued map $F : C \to 2^{\mathbb{R}^n}$, which contains the set of all possible flow directions within the closed set C, is nonempty, convex, and outer semi-continuous. The set valued map $G : D \to 2^{\mathbb{R}^n}$, which represents the set-valued jump function, is assumed to be nonempty and outer semi-continuous. These assumptions on $\mathcal{H}$ are called "basic assumptions"; see Goebel et al. (2012).

A.7.1 Perturbed hybrid systems

Given a hybrid system (A.52) and a continuous function $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the hybrid system H_δ described by the four tuple $(C_\delta, F_\delta, D_\delta, G_\delta)$ and

$$\mathcal{H}_\delta : \begin{cases} \dot{x} \in F_\delta(x) & x \in C_\delta, \\ x^+ \in G_\delta(x) & x \in D_\delta, \end{cases} \tag{A.53}$$

with the data

$$\begin{aligned} &C_\delta := \{x : (x + \delta(x)\mathbb{B}) \cap C \neq \emptyset\}, \\ &F_\delta(x) := \overline{\text{co}}\, F\left((x + \delta(x)\mathbb{B}) \cap C\right) + \delta(x)\mathbb{B} \qquad \forall x \in C_\delta, \\ &D_\delta := \{x : (x + \delta(x)\mathbb{B}) \cap D \neq \emptyset\}, \\ &G_\delta(x) := \ \{v : v \in g + \delta(g)\mathbb{B},\ g \in G((x + \delta(x)\mathbb{B}) \cap D)\} \qquad \forall\ x \in D_\delta, \end{aligned} \tag{A.54}$$

is called the perturbed form of $\mathcal{H}$ with respect to the uncertainty function δ. In this form, $\mathbb{B}$ denotes the closed unit ball and the uncertainty δ is typically a continuous function.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be increasing positive definite or belong to the class $\mathcal{K}_\infty$, if it is continuous, zero at zero, and strictly increasing. We make use of the following theorems, concerning robust stability of the hybrid system $\mathcal{H}$ or stability of the perturbed hybrid system $\mathcal{H}_\delta$.

Theorem A.7.1. *See Cai et al. (2008) and Goebel et al. (2012). Consider the hybrid system $\mathcal{H}$ defined by* (A.52) *satisfying the basic assumptions and $C \cup D = \mathbb{R}^n$. Suppose a compact set $\mathcal{A} \subset \mathbb{R}^n$ exists such that $G(D \cap \mathcal{A}) \subset \mathcal{A}$. If there exists a smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for the pair $(\mathcal{H}, \mathcal{A})$, that is, V is positive on $(C \cup D)\backslash\mathcal{A}$,* $\lim_{x \to \mathcal{A}} V(x) = 0$*, and*

$$\begin{aligned} &\langle \nabla V(x), f \rangle < 0 && x \in C\backslash\mathcal{A},\ f \in F(x) \\ &V(g) - V(x) < 0 && x \in D\backslash\mathcal{A},\ g \in G(x)\backslash\mathcal{A}, \end{aligned} \tag{A.55}$$

then the set $\mathcal{A}$ is asymptotically stable, and the basin of attraction contains every forward invariant compact set.

Theorem A.7.2. *See Cai et al. (2008) and Goebel et al. (2012). Consider the hybrid system $\mathcal{H}$ defined by* (A.52)*. If the compact set $\mathcal{A}$ is globally asymptotically stable, then there exist a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and α_1, $\alpha_2 \in \mathcal{K}_\infty$, such that*

$$\begin{aligned} &\alpha_1(\|x\|_\mathcal{A})) \leq V(x) \leq \alpha_2(\|x\|_\mathcal{A}) && \forall x \in \mathbb{R}^n, \\ &\underset{f \in F(x)}{max} \langle \nabla V(x), f \rangle \leq -V(x), && \forall x \in C, \\ &\underset{g \in G(x)}{max} V(g) \leq e^{-1} V(x) && \forall x \in D, \end{aligned} \tag{A.56}$$

where $\|x\|_\mathcal{A}$ denotes $inf_{y \in \mathcal{A}}(\|x - y\|)$. Moreover, the set $\mathcal{A}$ is asymptotically stable for the perturbed hybrid system $\mathcal{H}_\delta = (C_\delta, F_\delta, D_\delta, G_\delta)$ with some continuous function $\delta : \mathbb{R}^n \to \mathbb{R}^n_{\geq 0}$, where $\delta(x)$ is positive for $x \in (C \cup D) \backslash \mathcal{A}$.

A particular case of Theorem A.7.2 is obtained by assigning $D = \emptyset$ and $C = \mathbb{R}^n$. In this case,

$$F_\delta(x) := \overline{\text{co}}\ F\left(x + \delta(x)\mathbb{B}\right) + \delta(x)\mathbb{B} \qquad \forall x \in \mathbb{R}^n, \tag{A.57}$$

and the equations (A.56) concerning the Lyapunov function will be modified to

$$\begin{aligned} &\alpha_1(\|x\|_{\mathcal{A}})) \leq V(x) \leq \alpha_2(\|x\|_{\mathcal{A}}), \\ &\max_{f \in F(x)} \langle \nabla V(x), f \rangle \leq -V(x); \end{aligned} \tag{A.58}$$

see Teel and Praly (2000). We use the former description for robust stability of hybrid systems, and the latter illustration for robust stability of switched linear systems.

A.8 Optimization

Now, we present some basic concepts of optimization.

A.8.1 Projection onto a linear subspace

See Boyd and Vandenberghe (2004). The projection of a point x_0 onto the set

$$\mathcal{X} = \{x \in \mathbb{R}^n : Ax - b = 0\},$$

denoted by $\text{Proj}_{\mathcal{X}}(x_0)$, equals

$$\text{Proj}_{\mathcal{X}}(x_0) = x_0 - A^\top (AA^\top)^{-1}(Ax_0 - b). \tag{A.59}$$

A.8.2 Hamilton-Jacobi-Bellman equation

See, *e.g.*, Lewis and Syrmos (1995). Consider the following problem

$$\mathbf{J}(x(0), 0) = \min_u \left\{ \int_0^T C(x(t), u(t))\ dt + D\left(x(T)\right) \right\} \tag{A.60}$$

$$\text{subject to} \qquad \dot{x}(t) = f(x(t), u(t)), \tag{A.61}$$

where $C[.]$ is the scalar cost function, $D[.]$ is a function that specifies the cost at the final state $x(T)$, $x(t)$ is the system state vector, $x(0)$ is assumed to be given, and $u(t)$ for $0 \leq t \leq T$ is the control input vector that must be computed. For this problem, the Hamilton Jacobi Bellman partial differential equation is defined by

$$\dot{\mathbf{J}}(x, t) + \min_u \{\nabla \mathbf{J}(x, t) \cdot f(x, u) + C(x, u)\} = 0, \tag{A.62}$$

subject to the terminal condition

$$\mathbf{J}(x, T) = D(x), \tag{A.63}$$

where ∇ is the gradient operator.

A.8.3 Optimal control with fixed time and fixed final state in continuous time

See Lewis and Syrmos (1995) and Bryson and Ho (1975). Consider the optimal control problem

$$\mathbf{J}(t_0) = \frac{1}{2}x(T)^\top\ S(T)\ x(T) + \frac{1}{2}\int_{t_0}^{T}(x^\top\ Q\ x + u^\top\ R\ u)\ dt, \tag{A.64}$$

$$\text{subject to} \begin{cases} \dot{x} & = Ax + Bu \qquad t \geq t_0, \\ x(t_0) & = x_0, \\ C\ x(T) & = r, \end{cases} \tag{A.65}$$

with the given initial condition x_0, final time T, $S(T) \geq 0$, $Q \geq 0$, $R > 0$, the matrix C, and the vector r. Then, the optimal control law which minimizes the cost (A.64) is given as follows:

$$u(t) = -\left(K(t) - R^{-1}B^\top V(t)P(t)^{-1}V(t)^\top\right)x - R^{-1}B^\top V(t)P(t)^{-1}\ r(T), \tag{A.66}$$

where

$$\begin{aligned} -\dot{S} &= A^\top S + SA - SBR^{-1}B^\top S + Q, \\ K &= R^{-1}B^\top S, \\ -\dot{V} &= (A - BK)^\top V \qquad t \leq T, \\ \dot{P} &= V^\top BR^{-1}B^\top V \qquad t \leq T, \end{aligned} \tag{A.67}$$

and the following boundary conditions hold

$$V(T) = C^\top,\ P(T) = 0,\ S(T) \geq 0 \text{ is given.}$$

If $\det(P(t)) = 0$ for all $t \in (t_0, T)$, the problem is said to be abnormal and there is no solution. Note that the parameter

$$\theta := P(t)^{-1}\left(r(T) - V(t)^\top x(t)\right), \tag{A.68}$$

is fixed for any $t \in [0, T]$, where $\det(P(t)) \neq 0$.

A.8.4 Optimal control with fixed time and fixed final state in discrete time

See Lewis and Syrmos (1995). Consider the optimization problem

$$\mathbf{J} = \frac{1}{2}\ x_K^\top\ S_K\ x_K + \frac{1}{2}\sum_{k=i}^{K-1}[x_k^\top\ Q\ x_k + u_k^\top\ R\ u_k], \tag{A.69}$$

$$\text{subject to} \begin{cases} x^+ & = Ax_k + Bu_k \quad k > i, \\ x_i & \text{given,} \\ C\ x_K & = r_K, \end{cases}$$

with the given final time K, $S_K \geq 0$, $Q \geq 0$, $R > 0$, the matrix C, and the vector r_K. Then, the optimal control is given by

$$u_k = -K_k\ x_k + K_k^u\ V_{k+1}\ P_k^{-1}\ [V_k^\top\ x_k - r_K], \tag{A.70}$$

where

$$\begin{aligned}
K_k &= (B^\top\ S_{k+1}\ B + R)^{-1}\ B^\top\ S_{k+1}\ A, \\
S_k &= A^\top\ S_{k+1}\ (A - BK_k) + Q, \\
V_k &= (A - B\ K_k)^\top V_{k+1}, \\
P_k &= P_{k+1} - V_{k+1}^\top\ B\ (B^\top\ S_{k+1}\ B + R)^{-1}\ B^\top\ V_{k+1}, \\
K_k^u &= (B^\top\ S_{k+1}\ B + R)^{-1}\ B^\top,
\end{aligned} \tag{A.71}$$

with the boundary conditions

$$V_K = C^\top,\ P_K = 0,\ S_K \geq 0 \text{ is given.}$$

If $\det(P_i) = 0$ for all $i \leq k \leq K$, the problem has no solution in the interval $\{i, i+1, \ldots, K\}$. In this case the problem is said to be abnormal. Note that the parameter

$$\theta = P_k^{-1}(r_K - V_k^\top x_k), \tag{A.72}$$

is fixed for any $k \in \{i, i+1, \ldots, K\}$, where $\det(P_k) \neq 0$.

Bibliography

A. A. Agrachev and D. Liberzon. Lie-algebraic stability criteria for switched systems. *SIAM Journal on Control and Optimization*, 40(1):253–269, 2001.

A. Ahmadi, R. Jungers, P. Parrilo, and M. Roozbehani. Joint spectral radius and path-complete graph Lyapunov functions. *SIAM Journal on Control and Optimization*, 52 (1):687–717, 2014.

P. Arbenz and D. Kressner. *Lecture Notes on Solving Large Scale Eigenvalue Problems.* 2010.

V. Azhmyakov, S. A. Attia, D. Gromov, and J. Raisch. Necessary optimality conditions for a class of hybrid optimal control problems. In *Hybrid Systems: Computation and Control*, volume 4416 of *Lecture Notes in Computer Science*, pages 637–640. Springer, 2007.

V. Azhmyakov, R. Galvan-Guerra, and M. Egerstedt. Hybrid LQ-optimization using dynamic programming. In *Proc. of the American Control Conference*, pages 3617–3623, 2009.

N. Bajcinca and M. Voigt. Spectral conditions for symmetric positive real and negative imaginary systems. *In Proc. of the European Control Conference*, pages 809–814, 2013.

N. Bajcinca, D. Flockerzi, and Y. Kouhi. On a geometrical approach to quadratic Lyapunov stability and robustness. In *Proc. of the 52th Conference on Decision and Control*, pages 1–6, 2013.

G. Balas, J. Bokor, and Z. Szabo. Invariant subspaces for LPV systems and their applications. *IEEE Transactions on Automatic Control*, 48(11):2065–2069, 2003.

N. Barabanov. Lyapunov exponent and joint spectral radius: some known and new results. In *Proc. of the 44th IEEE Conference on Decision and Control and European Control Conference*, pages 2332–2337, 2005.

L. Bayon, J. M. Grau, M. M. Ruiz, and P. M. Suarez. An analytic solution for some separable convex quadratic programming problems with equality and inequality constraints. *Journal of Mathematical inequalities*, 4(2):453–465, 2010.

F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.

S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge University Press, 2004.

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15. PA: SIAM, 1994.

M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, 1998.

B. Brogliato. *Nonsmooth Mechanics: Models, Dynamics and Control*. Springer-Verlag, New York, 1999.

A. E. Bryson and Y. C. Ho. *Applied optimal control*. Hemisphere, New York, 1975.

C. Cai, A. R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems-Part I: Existence is equivalent to robustness. *IEEE Transactions on Automatic Control*, 52 (7):1264–1277, 2007.

C. Cai, A. R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems Part II: (pre)asymptotically stable compact sets. *IEEE Transactions on Automatic Control*, 53(3):734–748, 2008.

P. E. Caines, F. H. Clarke, X. Lie, and R. B. Vinter. A maximum principle for hybrid optimal control problems with pathwise state constraints. In *Proc. of the 45th IEEE Conference on Decision and Control*, pages 4821–4825, 2006.

D. Cheng. Stabilization of planar switched systems. *Systems and Control Letters*, 51(2): 79–88, 2004.

G. Chesi, P. Colaneri, J. C. Geromel, R. Middleton, and R. Shorten. A nonconservative LMI condition for stability of switched systems with guaranteed dwell time. *IEEE Transactions on Automatic Control*, 57(5):1297–1302, 2012.

J. W. Choi. Left eigenstructure assignment via sylvester equation. *Journal of Mechanical Science and Technology*, 12(6):1034–1040, 1998a.

J. W. Choi. A simultaneous assignment methodology of right/left eigenstructures. *IEEE Transactions on Aerospace and Electronic Systems*, 34(2):625–634, 1998b.

F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin. Asymptotic controllability implies feedback stabilization. *IEEE Transactions on Automatic Control*, 42(10):1394–1407, 1997.

E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. New York: McGraw-Hill, 1955.

M. Corless and R. Shorten. On the characterization of strict positive realness for general matrix transfer functions. *IEEE Transactions on Automatic Control*, 55(8):1899–1904, 2010.

J. Cortes. Discontinuous dynamical systems: A tutorial on solutions, nonsmooth analysis and stability. *IEEE Control Systems Magazine*, 28(3):36–73, 2008.

T. Craven and G. Csordas. A sufficient condition for strict total positivity of a matrix. *Linear Multilinear Algebra*, 45(2):19–34, 1998.

J. Daafouz, P. Riedinger, and C. Iung. Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. *IEEE Transactions on Automatic Control*, 47(11):1883–1887, 2002.

B. De Schutter and W. P. M. H. Heemels. *Modeling and control of hybrid systems.* Lecture notes of the DISC course, 2004.

D. K. Dimitrov and J. M. Peña. Almost strict total positivity and a class of Hurwitz polynomials. *Journal of Approximation Theory*, 132(2):212–223, 2005.

F. El Hachemi, M. Sigalotti, and J. Daafouz. Stability of planar singularly perturbed switched systems. In *Proc. of the American Control Conference (ACC), 2011*, pages 1464–1469, 2011.

E. Feron. Quadratic stabilizability of switched systems via state and output feedback. Technical report, Center for Intelligent Control Systems, MIT, Cambridge, MA, 1996.

A. F. Filippov. *Differential equations with discontinuous righthand sides: Control Systems (Mathematics and its Applications).* Springer, 1st edition, 1988.

E. Fornasini and M. E. Valcher. On the stability of continuous-time positive switched systems. In *Proc. of the American Control Conference (ACC), 2010*, pages 6225–6230, 2010.

P. A. Fuhrmann. On symmetric rational transfer functions. *Linear Algebra and its Applications*, 50(0):167–250, 1983.

J. Geromel and P. Colaneri. Stability and stabilization of continuous time switched linear systems. *SIAM Journal on Control and Optimization*, 45(5):1915–1930, 2006.

R. Goebel, R. G. Sanfelice, and A. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, 2009.

R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness.* Princeton University Press, New Jersey, 2012.

I. Gohberg, P. Lancaster, and L. Rodman. *Invariant Subspaces of Matrices with Applications.* Society for Industrial and Applied Mathematics, 2006.

W. M. Griggs, C. K. King, R. Shorten, O. Mason, and K. Wulff. Quadratic Lyapunov functions for systems with state-dependent switching. *Linear Algebra and its Applications*, 433(1):52– 63, 2010.

H. Haimovich and J. H. Braslavsky. Feedback stabilisation of switched systems via iterative approximate eigenvector assignment. In *Proc. of the 49th Conference on Decision and Control*, pages 1269–1274, 2010.

W. P. M. H. Heemels, D. Lehmann, J. Lunze, and B. De Schutter. *Handbook of Hybrid Systems Control.* Cambridge University Press, 2009.

U. Helmke, J. Rosenthal, and X. A. Wang. Output feedback pole assignment for transfer functions with symmetries. *SIAM J. of Control and Optimization*, 45(5):1898–1914, 2006.

J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. In *Proc. of the 38th IEEE Conference on Decision and Control*, volume 3, pages 2655–2660, 1999.

R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990.

M. Hovd and S. Skogestad. Control of symmetrically interconnected plants. *Automatica*, 30(6):957–973, 1994.

C. H. Huang, P. A. Ioannou, J. Maroulas, and M. G. Safonov. Design of strictly positive real systems using constant output feedback. *IEEE Transactions on Automatic Control*, 44(3):569–573, 1999.

T. A. Johansen, I. Petersen, and O. Slupphaug. Explicit suboptimal linear quadratic regulation with state and input constraints. *Automatica*, 38(7):1099–1111, 2002.

A. A. Julius and A. J. Van der Schaft. The maximal controlled invariant set of switched linear systems. In *Proc. of the 41st IEEE Conference on Decision and Control*, volume 3, pages 3174–3179, 2002.

T. Kailath. *Linear Systems*. Prentice Hall, Englewood Cliffs, NJ, 1980.

O. M. Katkova and A. M. Vishnyakova. A sufficient condition for a polynomial to be stable. *Journal of Mathematical Analysis and Applications*, 347(1), 2008.

H. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.

C. King and M. Nathanson. On the existence of a common quadratic Lyapunov function for a rank one difference. *Linear Algebra and its Applications*, 419(2–3):400–416, 2006.

L. Knockaert, F. Ferranti, and T. Dhaene. Generalized eigenvalue passivity assessment of descriptor systems with applications to symmetric and singular systems. *International Journal of Numerical Modeling: Electronic Networks, Devices and Fields*, 26(1):1–14, 2013.

Y. Kouhi and N. Bajcinca. On the left eigenstructure assignment and state feedback design. In *Proc. of the American Control Conference*, pages 4326–4327, 2011a.

Y. Kouhi and N. Bajcinca. Nonsmooth control design for stabilizing switched linear systems by left eigenstructure assignment. In *Proc. of the IFAC World Congress*, pages 380–385, 2011b.

Y. Kouhi and N. Bajcinca. Robust control of switched linear systems. In *Proc. of the 50th IEEE Conference on Decision and Control and European Control Conference*, pages 4735–4740, 2011c.

Y. Kouhi, N. Bajcinca, J. Raisch, and R. Shorten. A new stability result for switched linear systems. In *Proc. of the European Control Conference*, pages 2152–2156, 2013a.

Y. Kouhi, N. Bajcinca, and R. G. Sanfelice. Suboptimality bounds for linear quadratic problems in hybrid linear systems. In *Proc. of the European Control Conference*, pages 2663–2668, 2013b.

Y. Kouhi, N. Bajcinca, J. Raisch, and R. Shorten. On the quadratic stability of switched linear systems associated with symmetric transfer function matrices. *Automatica*, 50 (11):2872–2879, 2014.

T. J. Laffey. Common Lyapunov solutions for two matrices whose difference has rank one. *Linear Algebra and its Applications*, 431(1–2):228–240, 2009.

C. H. Lee. New results for the bounds of the solution for the continuous Riccati and Lyapunov equations. *IEEE Transactions on Automatic Control*, 42(1):118–123, 1997.

F. L. Lewis and V. L. Syrmos. *Optimal Control*. John Wiley and Sons, 2nd edition, 1995.

D. Liberzon. *Calculus of variations and optimal control theory: A concise introduction*. Princeton University Press, 2011.

D. Liberzon, J. P. Hespanha, and A. S. Morse. Stability of switched systems: a Lie-algebraic condition. *Systems and Control Letters*, 37(3):117–122, 1999.

H. Lin and P. J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2):308–322, 2009.

G. P. Liu and R. J. Patton. *Eigenstructure Assignment for Control System Design*. John Wiley and Sons, 1998.

J. Lygeros. *Lecture notes on hybrid systems*. 2004.

O. Mason and R. Shorten. On common quadratic Lyapunov functions for stable discrete time LTI systems. *IMA Journal of Applied Mathematics*, 69(3):271–283, 2004.

A. P. Molchanov and Ye. S. Pyatnitskiy. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Systems and Control Letters*, 13(1):59–64, 1989.

K. S. Narendra and J. Balakrishnan. A common Lyapunov function for stable LTI systems with commuting A-matrices. *IEEE Transactions on Automatic Control*, 39 (12):2469–2471, 1994.

K. S. Narendra and J. H. Taylor. *Frequency domain criteria for absolute stability*. Academic Press Inc., 1973.

J. Nocedal and S. Wright. *Numerical Optimization*. Springer, 2nd edition, 2006.

R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9): 1520–1533, 2004.

B. Passenberg, M. Leibold, O. Stursberg, and M. Buss. The minimum principle for time-varying hybrid systems with state switching and jumps. In *Proc. of the 50th IEEE Conference on Decision and Control and European Control Conference*, pages 6723–6729, 2011.

J. Raisch, E. Klein, C. Meder, A. Itigin, and S. O Young. Approximating automata and discrete control for continuous systems - two examples from process control. In *Hybrid Systems V*, volume 1567 of *Lecture Notes in Computer Science*, pages 279–303. Springer Berlin Heidelberg, 1999.

A. Rantzer and M. Johansson. Piecewise linear quadratic optimal control. *IEEE Transactions on Automatic Control*, 45(2):629–637, 2000.

W. Ren and R. W. Beard. *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*. Springer Publishing Company, 2007.

P. Riedinger, F. Kratz, C. Iung, and C. Zanne. Linear quadratic optimization for hybrid systems. In *Proc. of the 38th IEEE Conference on Decision and Control*, volume 3, pages 3059–3064, 1999.

F. Rossi, P. Colaneri, and R. Shorten. Padé discretization for linear systems with polyhedral Lyapunov functions. *IEEE Transactions on Automatic Control*, 56(11):2717–2722, 2011.

Y. Saad. A projection method for partial pole assignment in linear state feedback. Technical Report 449, YALEU/DCS, 1986.

J. Schiffer, A. Anta, T. D. Trung, J. Raisch, and T. Sezi. On power sharing and stability in autonomous inverter-based microgrids. In *Proc. of the 51st IEEE Conference on Decision and Control*, pages 1105–1110, 2012.

A. Semlyen and B. Gustavsen. A half-size singularity test matrix for fast and reliable passivity assessment of rational models. *IEEE Transactions on Power Delivery*, 24(1): 345–351, 2009.

F. G. Shinskey. *Distillation Control*. McGraw-Hill, 1984.

R. Shorten and F. O. Cairbre. On the stability of pairwise triangularisable and related switching systems. In *Proc. of the American Control Conference*, volume 3, pages 1882–1883, 2001.

R. Shorten and K. S. Narendra. On the stability and existence of common Lyapunov functions for stable linear switching systems. In *Proc. of the 37th IEEE Conference on Decision and Control*, volume 4, pages 3723–3724, 1998.

R. Shorten and K. S. Narendra. Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for a finite number of stable second order linear time-invariant systems. *International Journal of Adaptive Control and Signal Processing*, 16(10):709–728, 2002.

R. Shorten and K. S. Narendra. On common quadratic Lyapunov functions for pairs of stable LTI systems whose system matrices are in companion form. *IEEE Transactions on Automatic Control*, 48(4):618–621, 2003.

R. Shorten, F. Wirth, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM*, 49(4):545–592, 2007.

R. Shorten, P. Curran, K. Wulff, and E. Zeheb. A note on spectral conditions for positive realness of transfer function matrices. *IEEE Transactions on Automatic Control*, 53 (5):1258–1261, 2008.

R. Shorten, M. Corless, K. Wulff, S. Klinge, and R. Middleton. Quadratic stability and singular SISO switching systems. *IEEE Transactions on Automatic Control*, 54(11): 2714–2718, 2009.

I. Simeonova. *On-line periodic scheduling of hybrid chemical plants with parallel production lines and shared resources.* PhD thesis at Universite catholique de Louvain, 2008.

I. Simeonova, F. Warichet, G. Bastin, D. Dochain, and Y. Pochet. Feedback stabilization of the operation of an hybrid chemical plant. In *Proc. of the 2nd IFAC Conference on Analysis and Design of Hybrid Systems*, pages 191–198, 2006.

J. D. Simon and S. K. Mitter. A theory of modal control. *Information and Control*, 13 (4):316–353, 1968.

H. Sira-Ramirez. A geometric approach to pulse-width modulated control in nonlinear dynamical systems. *IEEE Transactions on Automatic Control*, 34(2):184–187, 1989.

G. Smirnov. *Introduction to the Theory of Differential Inclusions (Graduate Studies in Mathematics).* American Mathematical Society, 1st edition, 2001.

V. Solo. On the stability of slowly time-varying linear systems. *Mathematics of Control, Signals and Systems*, 7(4):331–350, 1994.

G. W. Stewart. *Matrix Algorithms, Volume II: Eigensystems.* SIAM, 2001.

Z. Sun and D. Zheng. On reachability and stabilization of switched linear systems. *IEEE Transactions on Automatic Control*, 46(2):291–295, 2001.

H. J. Sussmann. A maximum principle for hybrid optimal control problems. In *Proc. of the 38th IEEE Conference on Decision and Control*, pages 425–430, 1999.

A. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESAIM: Control, Optimisation and Calculus of Variations*, pages 313–367, 2000.

N. Truhar. Perturbation of invariant subspaces. *Lecture presented at the Mathematical COLLOQUIUM*, 1996.

M. Tsatsomeros. A criterion for the existence of common invariant subspace of matrices. *Linear Algebra and its Applications*, pages 51–59, 2001.

M. Wicks, P. Peleties, and R. DeCarlo. Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems. *European Journal of Control*, 4(2): 140–147, 1998.

J. C. Willems. Dissipative dynamical systems Part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45(5):352–393, 1972.

F. Wirth. A converse Lyapunov theorem for linear parameter varying and linear switching systems. *SIAM Journal on Control and Optimization*, 44(1):210–239, 2005a.

F. Wirth. The generalized spectral radius is strictly increasing. *Linear Algebra and its Applications*, 395(0):141–153, 2005b.

W. M. Wonham. On pole assignment in multi-input controllable linear systems. *IEEE Transactions on Automatic Control*, 12(6):660–665, 1967.

K. Wulff. *Quadratic and non-quadratic stability criteria for switched linear systems.* PhD thesis, Hamilton Institute, NUI Maynooth, 2005.

G. Xie, Q. Fu, and L. Wang. Stabilization of switched symmetric systems. In *Proc. of the American Control Conference*, volume 5, pages 4535–4536, 2004.

X. Xu and P. J. Antsaklis. Optimal control of switched systems based on parameterization of the switching instants. *IEEE Transactions on Automatic Control*, 49(1): 637–640, 2004.

G. H. Yang and L. Qiu. Optimal symmetric H2 controllers for systems with collocated sensors and actuators. *IEEE Transactions on Automatic Control*, 47(12):2121–2125, 2002.

E. Yurtseven, W. P. M. H. Heemels, and M. K. Camlibel. Disturbance decoupling of switched linear systems. In *Proc. of the 49th IEEE Conference on Decision and Control*, pages 6475–6480, 2010.

K. Zhou, J. C. Doyle, and K. Glover. *Robust and optimal control.* Prentice Hall, Upper Saddle River, 1996.