

# Kinetic modelling of temperature equilibration rates in the plasma

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## Abstract

Using the new Fokker-Planck code KIPP (KInetic code for Plasma Periphery) we examined the accuracy of the common expressions for temperature equilibration rates. Our simulations give new insights into the role of slow electrons in the equilibration process and show that deviations from the common theory are significant especially for the temperature equilibration between two ion species. The second part of the present work deals with the equalization rate of parallel and perpendicular temperatures of a charged species in a magnetic field. Again the simulations show that significant deviations from analytical results arise. Finally we suggest a correction to the analytical formula which better describes our results.

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## 1. Introduction

The important but challenging topic of tokamak edge plasma modelling is usually, e.g. in the SOLPS package, tackled by using Braginskii fluid equations [1]. However their applicability depends on a high collisionality of the plasma and recent results from SOLPS modelling and comparison to experiment suggest that kinetic effects may be important [2, 3]. To further investigate the influence of supra-thermal electrons a new kinetic module for SOLPS, KIPP (KInetic code for Plasma Periphery), is under development.

KIPP is based on the nonlinear Fokker-Planck equation in two velocity dimensions ( $v_{\parallel}$  and  $v_{\perp}$ ) and in dimensionless variables (also used in the figures below) as described in [4]. The equation, for currently up to two colliding species (electrons and ions), is solved by a finite volume discretization in velocity space and an implicit time discretization. The implicit scheme permits large time steps. The presented results were calculated on grids with a typical cell widths of  $0.015 v_{th,e}$ . For small values of  $\epsilon = \frac{m_e T_i}{m_i T_e}$  (see description of formula 1) an inhomogeneous grid with smaller cells towards  $v = 0$  was used.

Initial tests succeeded and directed our attention towards the problem of temperature relaxation rates [4]. These rates are determined in KIPP from the change in temperature. As initial condition Maxwellian distribution functions with different temperatures are used.

## 2. Two particle species equipartition

A solution to the problem of temperature relaxation was given by Spitzer a long time ago [5]. This solution is obtained by calculating the second moment of the collision

operator for Maxwellian electron and ion distribution functions:

$$\begin{aligned} \frac{3}{2}n_e \frac{dT_e}{dt} = Q_e = \int \frac{m_e}{2} v^2 C^{ei} 4\pi v^2 dv = \\ - n_e n_i \frac{4\sqrt{2\pi} \Lambda_c Z_e^2 Z_i^2 e^4}{(4\pi\epsilon_0)^2 \sqrt{m_e} T_e^{3/2}} \frac{m_e}{m_i} \frac{T_e - T_i}{\left(1 + \frac{m_e T_i}{m_i T_e}\right)^{3/2}} \end{aligned} \quad (1)$$

where  $n_{e/i}$  is the electron/ion density,  $T_{e/i}$  are the temperatures,  $m_{e/i}$  the masses,  $Z_{e/i}$  the charge numbers,  $\Lambda_c$  is the Coulomb logarithm and  $C^{ei}$  the (Fokker-Planck) electron-ion collision term. The choice of Maxwellian distribution functions is a simplification justified by the argument that like particle collisions create such distributions on a time scale which is much shorter than the time for the temperature equilibration.

However, the slow electrons at  $v \leq v_{th,i} = \sqrt{\frac{m_e T_i}{m_i T_e}} v_{th,e}$  (with  $v_{th,e/i} = \sqrt{\frac{T_{e/i}}{m_{e/i}}}$  being the thermal velocities) strongly interact with the ions. The high collision frequency of these electrons with the ions is due to the strong negative dependence of the Coulomb scattering cross section on the relative velocity. The fast energy exchange of the slow electrons with the ions takes place within  $\Delta\tau \sim \frac{m_i}{m_e} \cdot \left(\frac{v_{th,i}}{v_{th,e}}\right)^3 \approx \frac{m_i}{m_e} \cdot \left(\frac{m_e}{m_i}\right)^{3/2} = \left(\frac{m_e}{m_i}\right)^{0.5}$  (in units of e-e collisions time). This interaction gives rise to a distortion of the electron distribution function which is moderated by e-e collisions and established within one e-e collision time.

15 years ago Bobylev et al. calculated this distortion of the distribution function [6]. They derived an expression for the distribution function and a new formula for the energy exchange including the deviations from a Maxwellian:

$$Q_e = n_e n_i \frac{4\sqrt{2\pi} \Lambda_c Z_e^2 Z_i^2 e^4}{(4\pi\epsilon_0)^2 \sqrt{m_e} T_e^{3/2}} \frac{m_e}{m_i} \frac{T_e - T_i}{1 + 2.9 \left(\frac{m_e T_i}{m_i T_e}\right)^{2/3}} \quad (2)$$

The deviation from Spitzer's formula  $\frac{\dot{T}_{Bobylev}}{\dot{T}_{Spitzer}} - 1 = \frac{(1+\epsilon)^{3/2}}{1+2.9\epsilon^{2/3}} - 1$  depends only on  $\epsilon := \frac{v_{th,i}^2}{v_{th,e}^2} = \frac{m_e T_i}{m_i T_e}$ . Figure 1 shows that the deviation from a Maxwellian distribution reduces the rate of energy equipartition. The figure exhibits the dependence on the temperature

ratio, however it should be remembered that  $\epsilon$  depends on the (inverse) mass ratio in the same way.

With our code we could retrieve Bobylev's results. This means that after exact replication of Spitzer's value, for the initial condition of a Maxwellian distribution function, the distribution function gets distorted in less than one collision time and the relaxation rate approaches Bobylev's result (see also figure 3). Figure 2 shows a comparison of Bobylev's analytic result with results from KIPP for the steady state equipartition rate. We find that  $\sqrt{\epsilon}$  gives a very good approximation (practically as good as Bobylev's formula) for the deviation from Spitzer's expression.

The physical mechanism behind the distortion of the electron distribution function, causing the reduction of the relaxation rate, is explained as follows: The slow electrons strongly interact with the ions and are quickly pushed to higher (being heated by hot ions) or pulled to lower velocities (loosing energy to cold ions). Our simulations show (in agreement with the estimate given at the beginning of this section) that this process is faster than the e-e collision time by a factor  $\sqrt{\frac{m_i}{m_e}}$  and distorts the electron distribution function.

For the impact of the distortion on the equipartition rate another peculiarity of the e-i energy exchange is important: only very slow electrons (with  $v_e < 4.3 v_{th,i}$  for deuterium ions<sup>1</sup>, corresponding to 0.01% of the electrons for  $T_e \approx T_i$ ) receive energy from the ions while all the rest of the electrons loose energy to the ions. Firstly, this shows again the strong interaction of the slow electrons with the ions. Secondly, the distortion of the electron distribution function decreases the number of slow electrons for  $T_e < T_i$  and increases this number for  $T_e > T_i$  explaining the reduction of the equipartition rate in both cases.

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<sup>1</sup>From Trubnikov [7]:  $\epsilon_e = \frac{T_i}{390}$  with  $\epsilon_e = \frac{m_e v_e^2}{2}$  and  $T_i = v_{i,th}^2 m_i$ .

For the e-i equipartition rate a significant deviation<sup>2</sup> emerges only in cases of extreme  $T_i/T_e$  ratios. The particle species are denoted as *electrons* and *ions* up to now but it is important to note that the theory applies as well to the interaction of two ion species. In that case the mass ratio is much closer to one and the correction may become significant even for moderate temperature ratios.

Ion charge also influences the equipartition rate. Figure 3 shows the relative deviation from Spitzer's expression versus time for a proton (m) - Calcium (M) system ( $Z_M = 20$  and  $M/m = 40$ ), starting with Maxwellian distribution functions at  $t = 0$ . The deviation is about 20% for  $Z = 1$  with  $n_M = n_m$  (as in all previous results), roughly 60% for  $Z = 20$  with  $n_M = n_m$  and finally 40% for  $Z = 20$  with  $n_M = n_m/Z$ . To enable a comparison despite the different speed of the equipartition the time for  $Z = 1$  is rescaled by the factor  $Z^2 = 400$ . In reality the  $Z = 1$  process is a factor  $Z^2$  slower than the  $Z = 20$ ,  $n_m = n_M$  case and a factor  $Z$  slower than the  $n_M = n_m/Z$  case. Reducing the density of the heavier species ( $n_M = n_m/Z$ ) reduces the deviation in comparison to the case with equal densities but cannot compensate for the effect of a larger  $Z$ .

### 3. Equilibration of the temperatures parallel and perpendicular to the magnetic field

The above results also question the validity of the relaxation rate for parallel and perpendicular temperatures of a charged species in a moderate magnetic field (with  $r_c \gg b$  where  $r_c \equiv \bar{v}/\Omega_c$  is the cyclotron radius and  $b \equiv e^2/T$  is the distance of closest approach). The concept of different parallel and perpendicular temperatures is used e.g. in the description of SOL flows [8, p. 387] or in the presence of certain heating methods. An initial

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<sup>2</sup>i.e. larger than the error of  $1/\Lambda_c$  inherent in the Fokker-Planck equation due to the neglect of close collisions

distribution function of a charged species with different parallel and perpendicular temperatures relaxes to a usual Maxwellian with temperature  $T_{tot} = \frac{T_{\parallel} + 2T_{\perp}}{3}$ . The (equivalent) results given by Kogan [9] and Ichimaru [10] for the relaxation rate are calculated for (bi-)Maxwellian distribution functions  $f(v_{\parallel}, v_{\perp}) = \left(\frac{m}{2\pi T_{\perp}}\right) \sqrt{\frac{m}{2\pi T_{\parallel}}} \exp\left(-\frac{v_{\perp}^2 m}{2T_{\perp}} - \frac{v_{\parallel}^2 m}{2T_{\parallel}}\right)$ . As there is no small factor in this interaction, like the mass ratio, a deviation from this expression is to be expected. To our knowledge this has not yet been investigated.

As KIPP is based on parallel and perpendicular velocity coordinates it can also be used to calculate this equalization rate  $\frac{dT_{\parallel/\perp}}{dt}$ . For this calculation only one species with a bi-Maxwellian initial distribution function is used, in one case with  $T_{\parallel} > T_{\perp}$  and in the second case with  $T_{\parallel} < T_{\perp}$ . The lower of both temperatures was set to  $T_{small} = 0.01$  and the larger was adjusted so that  $T_{total} = \frac{T_{\parallel} + 2T_{\perp}}{3} = 1$ . Figure 4 shows the time evolution of the warmer degree of freedom for both cases. The figure shows the expected approach of  $T = 1$  and serves mainly as reference for comparison to figure 5.

Figure 5 shows, again for the two cases, the time evolution of the deviation from Kogan's temperature relaxation rate. Furthermore it contains a fit which is discussed later in this text. For  $t = 0$  the simulation uses a bi-Maxwellian and has to reproduce Kogan's result. Collisions change the distribution function and cause a deviation from Kogan's formula. The final deviation from the analytical (bi-Maxwellian) result is rather large. The period covered by this figure is restricted by the simulation. When the two temperatures become equal (cf. figure 5) the change in temperature becomes very small and inevitably round-off errors deteriorate the numerical result.

The reason for the different behavior for  $T_{\parallel} > T_{\perp}$  and  $T_{\parallel} < T_{\perp}$  can be explained as follows: The parallel component contains less energy and thus should be easier to distort. At least for  $T_{\parallel} > T_{\perp}$  the deviation in  $f_{\parallel}$  is larger than in  $f_{\perp}$  and it is not smaller for  $T_{\parallel} < T_{\perp}$ . From the result on the two species temperature equipartition we know that

the broader (electron) distribution function has more influence on the relaxation rate. Combining these arguments the deviation from the analytical relaxation rate is expected to be larger for  $T_{\parallel} > T_{\perp}$ , where  $f_{\parallel}$  is broader, than for  $T_{\parallel} < T_{\perp}$ , where  $f_{\perp}$  is broader.

The choice of initial temperatures is rather general because  $T_{total} = 1$  can always be obtained by changing the units and the temperature ratio  $\frac{T_{\parallel}}{T_{\perp}}$  adopts all intermediate values during the relaxation process. Nevertheless, an analysis of the dependence on  $\frac{T_{\parallel}}{T_{\perp}}$  is not possible. This is due to the slow development of the deviation from the analytical result. The time scale for the temperature equalization is somewhat shorter than that on which the deviation develops (see figures 4 and 5). This is in sharp contrast to the case of two particle equipartition.

The slow approach to the disturbed state requires a time dependent relaxation rate for an accurate description. As one can see from the fit in figure 5 it is possible to find a rather simple expression for the time dependent relaxation rate:

$$\frac{dT}{dt} = (1 - a(1 - e^{-bt})) \cdot \frac{dT}{dt} \Big|_{Kogan} \quad (3)$$

with  $a \approx 0.7$ ,  $b \approx 0.08$  for  $T_{\perp,0} = \frac{1}{300}T_{\parallel,0}$  and  $a \approx 0.4$ ,  $b \approx 0.1$  for  $T_{\perp,0} = 150T_{\parallel,0}$  (the second fit is not shown in the plot). The value of  $a$  corresponds to the saturated size of the deviation, the time constant  $b$  is similar for both cases. A comparison of different initial temperatures (e.g.  $a \approx 0.6$  for  $T_{\perp,0} = \frac{1}{2}T_{\parallel,0}$ ) showed only a small dependence on this quantity. Hence equation 3 is a rather solid upgrade of Kogan's result.

Up to now the situation for an initial temperature difference and subsequent equilibration was treated. Alternatively  $\frac{T_{\parallel}}{T_{\perp}}$  might be kept constant by cooling one degree of freedom while heating the other one. Such a case would be suited to study the  $\frac{T_{\parallel}}{T_{\perp}}$  dependence of the relaxation rate. Nevertheless a detailed analysis is not straightforward because it would require the implementation of an artificial cooling and heating in the code. This is possible and indeed is used to correct numerical energy loss for heat conduc-

tion calculations ([4]; in the present paper only data is shown where the numerical error is assumed to be irrelevant). However, experience shows that details of this implementation may change results, especially as the slow approach (lasting several collision times) to the disturbed state indicates that the responsible force is rather weak. This sensitivity to a detailed mechanism probably carries over to reality. Therefore reducing the relaxation rate to about 50% can be a rough approximation but an accurate simulation of situations with a steady source or sink of energy in one degree of freedom always requires special analysis.

An important example for such a situation is the scrape-off-layer of tokamaks, where ions can have different parallel and perpendicular temperatures [8, p. 387], [11]. The consequence of using the analytic formula is clearly an underestimation of the difference between parallel and perpendicular temperature for a given energy influx, or the overestimation of an energy influx for a given temperature difference. The relaxation rate enters the Braginskii equations [1] via viscosity. Furthermore, the deviations from a Maxwellian observed in the simulations may even require corrections to other coefficients in the Braginskii equations. This once more demonstrates the need for Fokker-Planck models, which inherently avoids such problems.

In an experiment of Hyatt et al. [12], a deviation of less than 10% from the equation given by Ichimaru [10] was found. However Hyatt compared to a simplified version of the equation which yields somewhat too low values for the equipartition rate. Furthermore his result is dominated by the phase in which the deviation is still growing. Therefore Hyatt's experimental finding is not in a contradiction to our new results predicting a larger deviation.



## 4. Summary

The new Fokker-Planck code KIPP could confirm Bobylev's result on temperature equilibration rates. The reduction by a factor  $\sqrt{\epsilon}$  for singly charged particles in comparison to Spitzer's expression is caused by the strong interaction of slow electrons with the ions. This effect increases with the charge number of the heavier species. A second study, concerning the equilibration of the temperatures parallel and perpendicular to a magnetic field, found an even larger deviation of about 50% from Kogan's theory. The deviation develops on a somewhat slower time scale than the equilibration. Therefore an accurate description requires a time dependent relaxation rate, like the fit to our numerical results given in section 3.

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Figure 1: Temperature equipartition rate for  $m_e/m_i = 1/2000$ ; temperature is in units of the initial electron temperature, time in units of the e-e collision time

Figure 2: Deviation of Bobylev's and numerical results from Spitzer's formula; the results from KIPP were obtained using various combinations of ion temperature and mass ratio

Figure 3: The deviation from Spitzer's theory versus time (in units of m-m collision time) for initial condition of Maxwellian distribution functions. All results are for mass ratio 1/40.

Figure 4: Temperature of the hotter degree of freedom versus time ( $t$  is in units of the m-m collision time) for two different initial conditions. The temperature of the second degree of freedom is given by  $\frac{T_{\parallel} + 2T_{\perp}}{3} = 1$ .

Figure 5: Deviation of numerical results from Kogan's analytical expression versus time (in units of the m-e collision time).

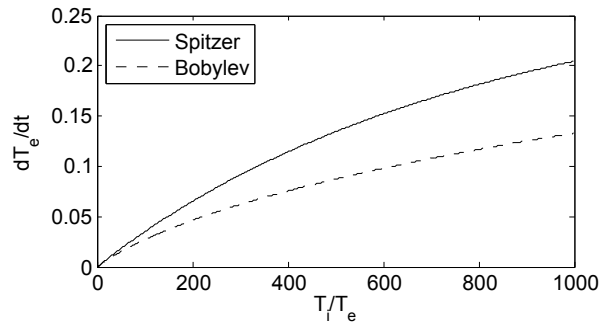


Fig. 1

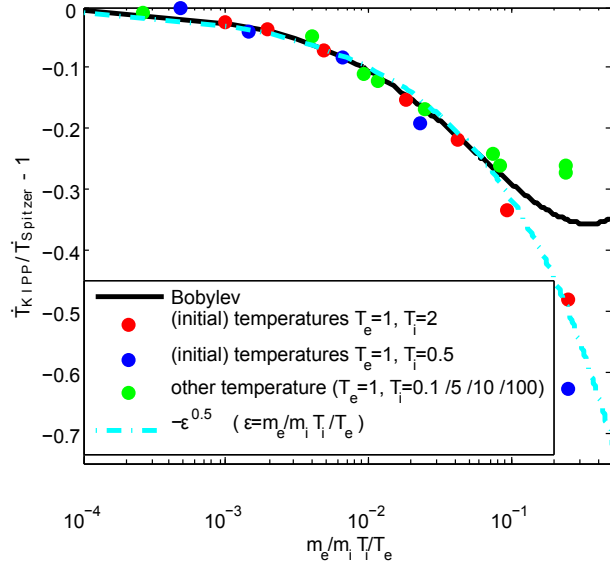


Fig. 2

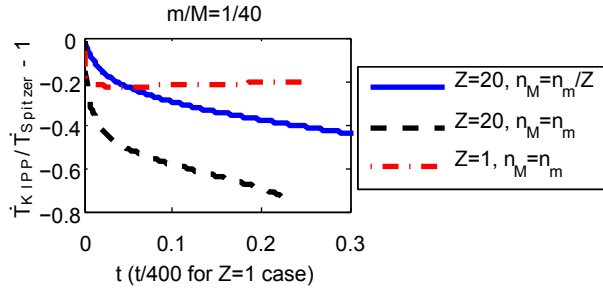


Fig. 3

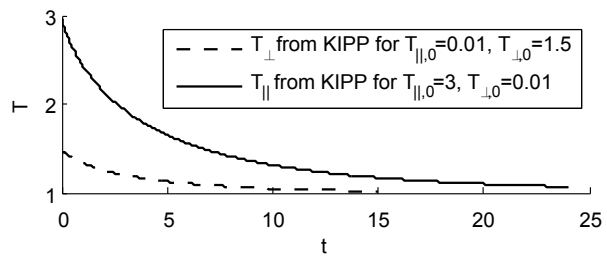


Fig. 4

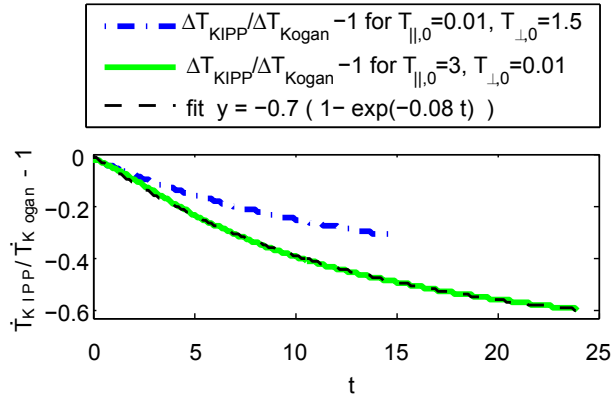


Fig. 5