Role of helicities for the dynamics of turbulent magnetic fields

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Abstract

Investigations of the inverse cascade of magnetic helicity are conducted with pseudospectral, three-dimensional direct numerical simulations of forced and decaying incompressible magnetohydrodynamic turbulence. The high-resolution simulations which allow for the necessary scale-separation show that the observed self-similar scaling behavior of magnetic helicity and related quantities can only be understood by taking the full nonlinear interplay of velocity and magnetic fluctuations into account. With the help of the eddy-damped quasi-normal Markovian approximation a probably universal relation between kinetic and magnetic helicities is derived that closely resembles the extended definition of the prominent dynamo pseudoscalar α. This unexpected similarity suggests an additional nonlinear quenching mechanism of the current-helicity contribution to α.

1 Introduction

Understanding large-scale magnetic structure formation in the Universe is one of the challenging problems in modern astrophysics. In this context, mean-field dynamo theory is a prominent approach [Moffatt1978, Biskamp2003, Brandenburg and Subramanian 2005]. Based on a homogenization formalism, it describes the generation of large-scale magnetic fields by smaller-scale turbulent fluctuations of a magnetofluid. As a result, this classical two-scale closure [Krause and Rädler1980] yields, next to a turbulent diffusivity, a scalar, $\alpha \sim \tau H^K$, that expresses the nonlinear interaction of large-scale field and smaller-scale turbulence. Here, τ stands for a correlation time of the turbulent fluctuations and $H^K = \frac{1}{2V}\int_V dV \mathbf{v} \cdot \boldsymbol{\omega}$ is

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the kinetic helicity of the associated velocity field $\mathbf{v}$ with V being the volume under consideration and $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ defining the vorticity. Statistical closure theory [Pouquet *et al.*1976] more specifically the eddy-damped quasi-normal Markovian (EDQNM) approximation, suggests a more complex expression, $\alpha \sim \tau(H^K - H^J)$, that introduces the current helicity $H^J = \frac{1}{2V}\int_V dV \mathbf{b}\cdot\mathbf{j}$ with $\mathbf{j}$ denoting the electric current density, see also [Blackman and Field 2002, Field and Blackman2002, Subramanian and Brandenburg2004, Brandenburg and Subramanian 2005]. Its name is actually misleading as H^J expresses the helicity of the magnetic field and is in this respect a close relative of the kinetic helicity and, furthermore, also proportional to the total resistive dissipation rate of magnetic helicity (see below).

While H^K is ideally conserved and is spectrally cascading towards smaller scales in the inertial range of three-dimensional Navier-Stokes turbulence, the current helicity has apparently no comparable significance for turbulent dynamics apart from its meaning for the turbulent dynamo. However, as $\mathbf{j} = \nabla \times \mathbf{b} = -\Delta\mathbf{a}$, $\mathbf{b} = \nabla \times \mathbf{a}$ standing for the magnetic field and $\mathbf{a}$ denoting the magnetic vector potential (both dimensionless), a link to an ideal invariant of three-dimensional incompressible magnetohydrodynamics (MHD), the magnetic helicity, $H^M = \frac{1}{2V}\int_V dV \mathbf{a}\cdot\mathbf{b}$, emerges. This quantity characterizing the topology of the magnetic field [Moffatt1969] is prone to an inverse cascade. The cascade is a robust nonlinear mechanism that creates large scale order out of the chaotic randomness of small-scale magnetic turbulence presupposing a sufficient separation of large and turbulent small scales in the system in combination with a small-scale supply of magnetic helicity.

The present work is motivated by the potential importance of magnetic helicity for the dynamics of large-scale dynamo configurations. This is not to be confused with the related issue of the effect of boundary conditions on the magnetic helicity evolution and the consequences for the dynamo process, a topic that has been subject of a number of investigations, see, e.g. [Brandenburg 2009] and references therein. In this work, an idealized system, homogeneous incompressible MHD turbulence with triply periodic boundary conditions, is investigated by three-dimensional direct numerical simulations in combination with statistical closure theory.

2 Model equations and numerical Setup

The dimensionless incompressible MHD equations giving a concise single-fluid description of a plasma read:

$$\partial_t \boldsymbol{\omega} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega} - \mathbf{b} \times \mathbf{j}) + \mu_n (-1)^{n/2-1} \nabla^n \boldsymbol{\omega} + \mathbf{F_v} + \lambda \Delta^{-1} \boldsymbol{\omega} \quad (1)$$

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{v} \times \mathbf{b}) + \eta_n (-1)^{n/2-1} \nabla^n \mathbf{b} + \mathbf{F_b} + \lambda \Delta^{-1} \mathbf{b} \quad (2)$$

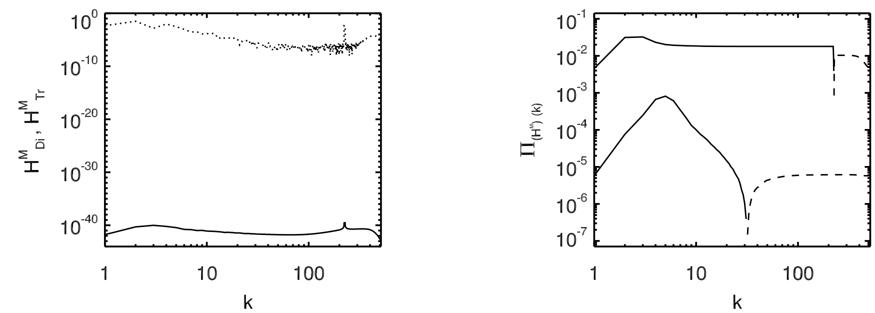
$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{b} = 0. \quad (3)$$

Relativistic effects are neglected and the mass density is assumed to be uniformly unity throughout the system. Other effects such as convection, radiation and rotation are also neglected. Direct numerical simulations are performed by solving the set of model equations by a standard pseudospectral method [Canuto *et al.*1988] in combination with leap-frog integration on a cubic box of linear size 2π that is discretized with 1024 collocation points in each spatial dimension. Spherical mode truncation is used for alleviating aliasing errors. By solving the equations in Fourier space, the solenoidality of $\mathbf{v}$ and $\mathbf{b}$ is maintained algebraically.

To observe clear signatures of an inverse cascade of magnetic helicity the system has to contain a source of this quantity at small scales. This is achieved in two different ways resulting in two main configurations: a driven system and a decaying one. In the driven case, the forcing terms $\mathbf{F_v}$ and $\mathbf{F_b}$ are delta-correlated random processes acting in a band of wavenumbers $203 \leq k_0 \leq 209$. They create a small-scale background of fluctuations with adjustable amount of magnetic and kinetic helicity. The results reported in this paper do not change if kinetic helicity injection is finite. The theoretical results presented in the following do not depend on the setup of the forcing as they presuppose an existing self-similar distribution of energies and helicities. For obtaining such spectra in numerical experiments the magnetic source term $\mathbf{F_b}$ is necessary while a finite momentum source $\mathbf{F_v}$ speeds up the spectral development significantly. In the decaying case the forcing terms are set to zero and the initial condition represents an ensemble of smooth and random fluctuations of maximum magnetic helicity with respect to the energy content (see below) and a characteristic wavenumber $k_0 = 70$.

To reduce finite-size effects, the simulations are run for 6.7 (forced) and 9.2 (decaying) large-eddy turnover times of the system, respectively. The time unit is defined using the system size and its total energy. Additionally, a large scale energy sink $\lambda\Delta^{-1}$ with $\lambda = 0.5$ is present for both fields. In the decaying case $\lambda = 0$. The hyperdiffusivities μ_n and η_n are dimensionless dissipation coefficients of order n (always even in these simulations), with $n = 8$ in both runs. They act like higher-order realizations of viscosity and magnetic diffusivity, respectively. The magnetic hyperdiffusive Prandtl number $Pr_{mn} = \frac{\mu_n}{\eta_n}$ is set to unity.

The initial conditions to these simulations are smooth fluctuations with random phases having a Gaussian energy distribution peaked around k_0 in the decaying and the forced cases. Magnetic and kinetic helicity of the initial state can be controlled in the same way as for the forcing terms, cf. [Biskamp and Müller2000]. The initial/force-supplied ratio of kinetic to magnetic energy is unity with an amplitude of 0.05 in the forced case and an amplitude of unity in the decaying case. Hyperviscosity of order $n = 8$ is chosen in the simulations to obtain sufficient scale-separation. It is difficult to define an unambiguous Reynolds number owing to the use of hyperviscosity ([Malapaka2009] and the references there in). With the above mentioned simulation set up, the equations are solved both for decaying and forced cases separately and the results obtained are discussed below.



a) b)

Figure 1: *a*) Transmission $H^M_{Tr} = \tilde{\mathbf{b}}^* \cdot \widetilde{\mathbf{v} \times \mathbf{b}}$ (dotted line) and dissipation H^M_{Di} $= \eta_n k^6 \tilde{\mathbf{b}}^* \cdot \tilde{\mathbf{j}}$ (solid line) of magnetic helicity space-angle-integrated in Fourier space in the forced case (similar for the decaying case, not shown), *b*) spectral flux of magnetic helicity in the forced (top) and decaying cases (bottom), dashed curves: direct flux, solid curves: inverse flux.

3 Simulation results

Using the simulation setup described in the previous section, inverse cascading of magnetic helicity with a clear scale separation between large and small scales is established in both forced and decaying cases for wavenumbers $k < k_0$. This is indicated by the spectral flux, $\Pi^{H^M}_k = \int_0^k dk' \int d\Omega [\tilde{\mathbf{b}}^* \cdot \widetilde{\mathbf{v} \times \mathbf{b}}]_{|\mathbf{k}'|=k'}$, in both cases depicted in Fig.1b and taken at $t = 6.7$ and 9.2, respectively as dissipation of magnetic helicity is negligible (see Fig.1a). The tilde indicates Fourier transformation and * stands for complex conjugate. The inverse flux in the driven case is constant over a significant spectral interval, indicating equilibrium of source and sink, while the temporal decay of the magnetic helicity reservoir in the decaying case is reflected by the associated non constant inverse flux. In both cases the characteristic wavenumber of the H^M-source can be identified as the separation between inverse and direct flux regions. The spectral flux of magnetic helicity has been extensively studied in earlier numerical simulations e.g. [Brandenburg 2001] and [Alexakis *et al.*2006]. These works, however, are lacking the necessary scale separation to observe self-similar scaling laws. The spectrum of magnetic helicity exhibits scaling behavior $\sim k^q$ with $q \approx -3.3$ and $q \approx -3.6$ (forced and decaying case, respectively) which cannot be explained by the straightforward constant-flux reasoning à la Kolmogorov adopted in [Pouquet *et al.*1976] to interprete their EDQNM results.

In fact, the involved dimensional argument (Alfvénic units), $[H^M_k] = L^4/T^2$ (spectrum), $[\varepsilon_M] = L^3/T^3$ (spectral flux), in combination with the assumption of spectral self-similarity, $H^M_k \sim \varepsilon^a_M k^b$, yields $a = 2/3$, $b = -2$, but does not explicitly include the nonlinear interaction of velocity and magnetic fields. As a first step in the necessary refinement of the theoretical modeling additional consideration of the kinetic helicity H_k^K seems appropriate.

As a consequence of the inverse spectral transfer of magnetic helicity, all magnetic quantities should inherit the observed spectral inverse transfer property. This is indeed the case for the magnetic energy, the electric current density, and the current helicity. These quantities also show self-similar scaling that however, differs to some degree between the two investigated configurations. It is particularly interesting, that the residual helicity $H^R = \left| H^V \ - \ k^2 \, H^M \right|$, also shows self-similar scaling with $q \approx -1.4$ and $q \approx -1.8$ in the forced and decaying cases respectively (see [Malapaka2009] for further details). The interaction of the magnetic field with the velocity in a progressing inverse cascade of magnetic helicity appears to be of importance for a better understanding of the observed scaling laws. At high Reynolds numbers, the process of large-scale magnetic structure formation by the inverse cascade is accompanied by a continuous stirring of the velocity field caused by the expanding magnetic field structure. The magnetic stirring of the MHD-fluid leads to a transfer of magnetic to kinetic energy and generates ever larger velocity fluctuations. These also show self-similar scaling, as, for example, reflected by the kinetic helicity spectrum with $q \approx -0.4$ (forced case) and $q \approx 0.4$ (decaying case).

With regard to the finding, e.g. [Alexakis *et al.*2006], of the pronounced spectral non-locality of the nonlinear interactions underlying $\Pi_k^{H^M}$ a few words about the physical picture of the inverse cascade are in order. The cascading process is realized as a merging of positively-aligned and thus mutually attracting current carrying structures, cf. [Biskamp and Bremer1993]. It is not necessary that the structures grow in size as they indeed do in the decaying case, as long as the corresponding current densities increase. This is observed in the simulation with small-scale forcing. As there is no obvious fluid-dynamical constraint on the merging of two current filaments with regard to their size, this picture is consistent with a spectrally non-local inverse cascade of magnetic helicity.

4 Spectral relationship between kinetic and magnetic helicities

A link between kinetic and magnetic helicities can be constructed with the help of dimensional analysis of the magnetic helicity evolution equation in the EDQNM approximation, a statistical closure model discussed, e.g., in [Pouquet *et al.*1976]. Such an approach was successful earlier, in describing the turbulent residual energy spectrum, $E_k^R = |E_k^M - E_k^V|$ yielding $E_k^R \sim kE_k^2$ [Müller and Grappin2005] with $E_k = E_k^M + E_k^K$, which also turns out to be valid in the present simulations, where E_k^M and E_k^V are magnetic and kinetic energies respectively.

Assuming that the most important nonlinearities involve the turbulent velocity and stationarity of the spectral scaling range of H_k^M, a dynamical equilibrium of turbulent advection and the H^M-increasing effect of helical fluctuations is proposed. This can be formulated straightforwardly using the corresponding dimensionally approximated nonlinear terms from the EDQNM model (for a more detailed derivation see [Müller *et al.*2012]), yielding:

$$H_k^K \sim \left(\frac{E_k^K}{E_k^M}\right) k^2 H_k^M \,. \tag{4}$$

This statement about the spectral dynamics of kinetic and magnetic helicities (or, equivalently, kinetic and current helicities since $H_k^J \sim k^2 H_k^M$) is also valid for $E_k^K/E_k^M \neq 1$. The agreement of Relation(4) with the numerical experiments is however significantly improved by a modification (Relation.(5) below) whose justification is beyond the scope of the presented equilibrium ansatz which basically assumes spectral locality of the inverse cascade:

$$H_k^K \sim \left(\frac{E_k^K}{E_k^M}\right)^2 H_k^J \,. \tag{5}$$

Relation.(5) is a significant improvement over the earlier relations of similar kind [Pouquet *et al.*1976, Pouquet *et al.*2010, Müller and Malapaka 2010]. This is shown in Figs. 2 and 3, a, where $\Theta = (E_k^K/E_k^M)^\gamma H_k^J/H_k^K$ is shown with $\gamma = 0, 1$ and 2 (corresponding to Θ, Θ_1 and Θ_2) for the forced and decaying cases respectively. It is remarkable that Relation.(5) is only fulfilled in wavenumber intervals where the flux of magnetic helicity is spectrally constant.

This relation brings back the ratio of energies (kinetic to magnetic) into the picture, which, under the assumption of equipartition of energies was ignored in previous work [Pouquet *et al.*1976], while linking the magnetic/current and kinetic helicities. Another interpretation for this expression is the partial Alfvénization of the turbulent flow [Pouquet *et al.*2010]. Further, it also highlights the influence of kinetic helicity in the inverse cascade of magnetic helicity.

Relation (5) belongs to a class of probably highly universal expressions which are based statistically on the quasi-normal approximation of nonlinear fluxes. It is interesting to note that relation (5) also allows to determine the spectral scaling exponent of magnetic helicity from astronomical current helicity measurements using vector magnetograms (see e.g. [Brandenburg and Subramanian 2005] and references therein) if kinetic and magnetic energy spectra are also measurable or can be estimated with sufficient accuracy.

The modification of the current helicity contribution present in relation (5) suggests a corresponding modification to the residual helicity, $H^R = H^K - H^J$, and accordingly to the mean-field dynamo α. This, however, has to be taken with care as the present simulations are energetically dominated by the magnetic field although the modifying factor $(E^K/E^M)^2$ should compensate for this. Figs. 2 and 3, b, allow to roughly estimate the respective scale-dependent influence of kinetic and magnetic helicity on the modified residual helicity. The spectrum of residual helicity closely follows the spectral kinetic helicity with growing systematic deviations due to the influence of magnetic helicity at large wavenumbers, in both cases. Thus, the modified residual helicity complies with the earlier definitions of α [Krause and Rädler1980, Pouquet *et al.*1976] at large scales.

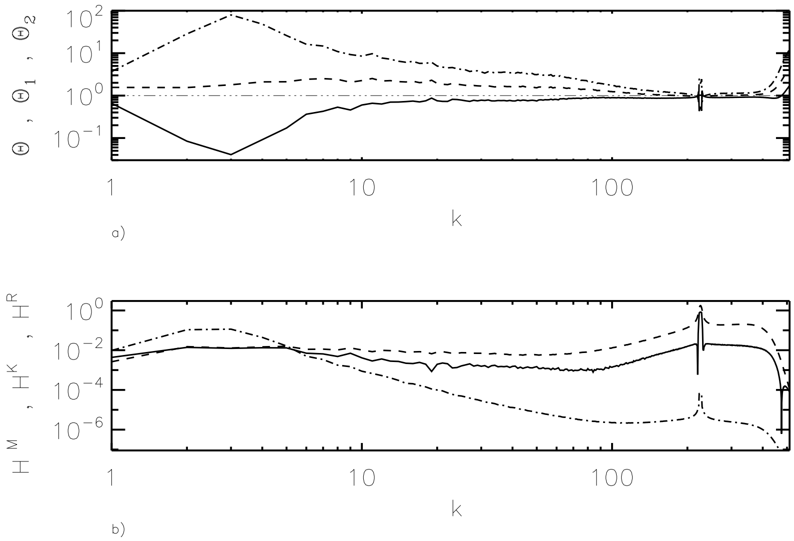


Figure 2: Plots of Relation.5, and kinetic, magnetic and residual helicities for forced turbulence at t=6.7. a) Relation.5 $\Theta = (E_k^K/E_k^M)^\gamma H_k^J/H_k^K$. $\gamma = 0$ (dash-dot curve) $\gamma = 1$ (dashed curve), and $\gamma = 2$ (solid curve). b) Magnetic helicity (dash-dot curve), kinetic helicity (dashed curve) and residual helicity (solid curve).

5 Conclusions

In high-resolution direct numerical simulations of forced and decaying magnetically helical homogeneous MHD turbulence, the nonlinear dynamics of active inverse cascade of magnetic helicity is studied. The simulation results, in particular the observed self-similar spectral scaling of magnetic helicity which contradicts an earlier theoretical explanation [Pouquet *et al.*1976], motivate the consideration of velocity field characteristics for the nonlinear evolution of this purely magnetic quantity. This is done with the help of statistical closure theory yielding a possibly universal relation between kinetic and current helicities. The relation is corroborated by the numerical results. Its form, $H_k^K - (E_k^K/E_k^M)^2 H_k^J \sim$ const., closely resembles the extended definition of the pseudoscalar $\alpha \sim H_k^K - H_k^J$ known from mean-field dynamo theory. The inverse cascade of magnetic helicity is not a dynamo itself as dimensionally $H_k^M \sim kE_k^M$, but a spectral transport process and not even a turbulent cascade in the strict sense [Müller *et al.*2012]. It can as a robust and efficient spectral transporter, nevertheless, play a role in the actual realization of turbulent large-scale dynamos like the α-dynamo. In this respect it is interesting that the newly obtained relation includes the squared ratio of kinetic and magnetic energies. This leads to a purely nonlinear quenching of the current helicity contribution to α that has

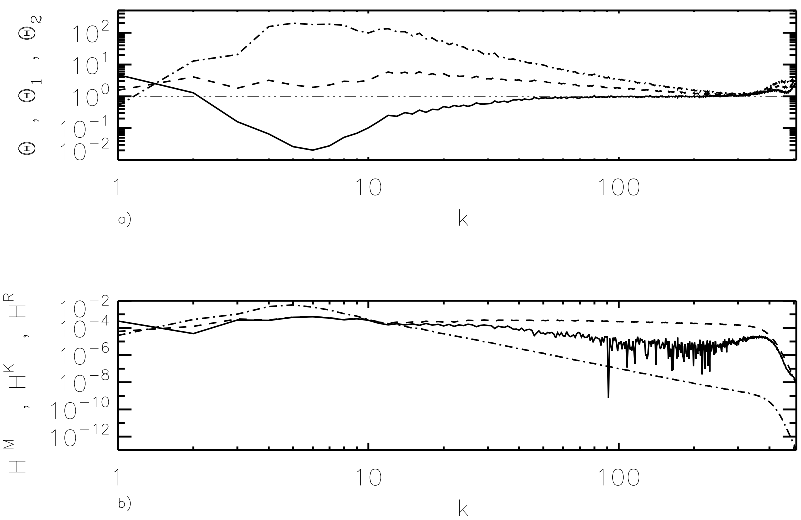


Figure 3: Plots of Relation.5, and kinetic, magnetic and residual helicities for decaying turbulence at t=9.2. a) Relation.5 $\Theta = (E_k^K/E_k^M)^\gamma H_k^J/H_k^K$. $\gamma = 0$ (dash-dot curve) $\gamma = 1$ (dashed curve), and $\gamma = 2$ (solid curve). b) Magnetic helicity (dash-dot curve), kinetic helicity (dashed curve) and residual helicity (solid curve).

no direct connection to the dynamo-quenching mechanisms considered so far (of order $(E^M)^{-1}$) in the literature which are seemingly consequences of a combination of boundary conditions and the approximate conservation of magnetic helicity. In this respect, it is encouraging that [Rheinhardt and Brandenburg2010] for a homogeneous mean flow with Roberts forcing using a test field method observe α-quenching with an $(E^M)^{-2}$ signature. The comparison with this work assumes equivalence of their imposed mean field with the root-mean-square large-scale magnetic fluctuations in the present simulations.

The present relation (5) needs further investigation as it is an additional possible mechanism for dynamo quenching. This new link between kinetic and magnetic helicity in the inverse cascade of magnetic helicity has to be verified in more complex numerical setups such as mean field dynamos, as well as anisotropic 3D-MHD and isotropic 3D-MHD turbulence with different initial conditions and forcing mechanisms.

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