On stability of collisional coupling between relativistic electrons and ions in hot plasmas

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The collisional coupling of relativistic electrons and non-relativistic ions in hot plasmas has been analysed. It is found that relativistic effects produce a new feature: while the condition $T_e < 3T_i$ guarantees a stable collisional coupling between electrons and ions in low-temperature plasmas, relativistic effects shift the upper T_e/T_i boundary of stability to higher values. Moreover, for sufficiently high temperatures, $T_{e,i} > 75$ keV, collisional decoupling between relativistic electrons and ions becomes impossible.

I. INTRODUCTION

Relativistic plasma effects were recognized as important in astrophysics long ago and the necessary formalism for description of relativistic plasmas has been developed [1, 2]. Furthermore, the production and heating of plasmas by high-power laser pulses has been intensively studied [3], and for the interpretation of experimental results a relativistic treatment was recognized to be necessary [4, 5]. The progress in fusion, where high temperatures are mandatory [6–8], also makes the kinetics in hot plasmas a subject of importance, where relativistic effects sometimes need to be taken into account. So far, only a few problems related to relativistic effects in hot fusion plasmas have been considered: in particular, solving of the relativistic Spitzer problem for calculation of conductivity [9] and of current drive efficiency (see Ref. 10 and the references therein). Apart from this, some aspects of the transport theory in relativistic plasmas were considered in Ref. 11, where the covariant formulation with 4-vectors was applied.

There is a widespread opinion that relativistic effects in laboratory devices are important only with respect to the populations of highly energetic electrons (see, for example, Ref. 6). However, relativistic effects can appear also due to the macroscopic features of the relativistic thermodynamic equilibrium given by the Jüttner distribution function [2], also known as the relativistic Maxwellian [9]. In particular, contrary to the non-relativistic Maxwellian, the shape of the Jüttner-Maxwellian distribution function depends on the temperature, and is Gaussian only in the non-relativistic limit.

As is well known from experiments in toroidal devices [12–14], electrons and ions exchange energy through collisions, but if the electrons are much hotter than the ions the two species “decouple”. The collisional decoupling thus has not only the heating-power threshold, but also a threshold with respect to the ratio of electron and ion temperatures. Actually, the value of this temperature threshold depends on the transport phenomena (for example, the “electron root” in stellarators [15, 16] establishes a large positive radial electric field together with large T_e/T_i ratio, i.e. with $e-i$ collisional decoupling). Apart from this, the radiative losses (bremsstrahlung and cyclotron radiation) [17] become important for high temperatures in the energy balance of electrons. However, only the collisional channel of energy transfer from electrons to ions defines the minimal value of the temperature threshold of the decoupling, while the balance of heating and losses is responsible only for establishment of the steady state.

In this paper, the influence of relativistic effects on the temperature threshold with respect to collisional decoupling is studied. This investigation is applicable to the case when the heating of electrons by any external source (for example, high-power laser-pulses in implosive plasmas or radio-frequency wave-beams in toroidal fusion plasmas) is balanced predominantly by the collisional energy-exchange with the ions (which frequently is the most desirable case for experiments without direct heating of ions). The opposite case, when the collisional transfer of power is of minor importance and heating of electrons is balanced by transport and/or radiation losses, is usually undesirable and is not considered here.

II. ENERGY BALANCE IN RELATIVISTIC PLASMAS

Let us assume that electrons and ions have their own Maxwellians with the temperatures defined by the energy balance (this is true if the rate of thermalization within each of the plasma components is sufficiently high in comparison with the external heating). The Maxwellian for the ions with density n_i and temperature T_i is taken as the classical one,

$$f_{iM} = \frac{n_i}{\pi^{3/2} v_{\rm ti}^3} e^{-v^2/v_{\rm ti}^2}, \tag{1}$$

where $v_{\rm ti} = \sqrt{2T_i/m_i}$ is the ion thermal velocity, while the electrons with density n_e and temperature T_e are considered as relativistic with Jüttner-Maxwellian distribution function [2, 9], which it is convenient to represent

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as

$$f_{eJM} = \frac{n_e}{\pi^{3/2} u_{\rm te}^3} C(\mu_r)\, e^{-\mu_r(\gamma-1)}, \quad (2)$$

where $u_{\rm te} = p_{\rm te}/m_{e0}$. Here, $p_{\rm te} = \sqrt{2m_{e0}T_e}$ is the thermal momentum, m_{e0} is the rest-mass, $\mu_r = m_{e0}c^2/T_e$ and $\gamma = \sqrt{1+u^2/c^2}$ is the Lorentz factor with momentum per unit mass $u = v\gamma$. Since f_{eJM} is normalized by a density, $n_e = \int f_{eJM} d^3u$,

$$C(\mu_r) = \sqrt{\frac{\pi}{2\mu_r}} \frac{e^{-\mu_r}}{K_2(\mu_r)} \simeq 1 - \frac{15}{8\mu_r} + ... \quad (\mu_r \gg 1), \quad (3)$$

where $K_n(x)$ is the modified Bessel function of the second kind of the n-th order. Here, it is appropriate to recall that contrary to the classical Maxwellian, the shape of which is independent of the temperature, the relative "weight" of electrons with $u/u_{\rm te} \gg 1$ increases at high T_e, which shifts the "centre of mass" of the Jüttner-Maxwellian from the bulk to the tail.

The energy-balance equation in relativistic plasmas can be presented in the same form as in non-relativistic ones. By weighting the relativistic kinetic equation for electrons [9] with the energy $m_{e0}c^2(\gamma-1)$ and integrating over momentum, one can obtain the energy balance equation (for simplicity, all terms related to inhomogeneity are omitted),

$$\frac{\partial W_e}{\partial t} = P_{ei} + P_{\rm ext} - P_{\rm loss}, \quad (4)$$

where $W_e = \int m_{e0}c^2(\gamma-1) f_{eJM} d^3u$ is the energy enclosed in the relativistic Jüttner-Maxwellian, P_{ei} is the rate of energy exchange between relativistic electrons and classical ions, $P_{\rm ext}$ is the power of external heating and $P_{\rm loss}$ is the total loss including transport and radiative losses. A similar equation can be written also for the ions with $P_{ie} = -P_{ei}$.

Note, that the balance of $P_{\rm ext}$ and $P_{\rm loss}$ is important only for establishing the plasma temperature, and the temperature dependence of P_{ei} is the only factor responsible for appearance of the phenomenon known as "collisional decoupling", which is considered below. It can be mentioned here also that the losses due to bremsstrahlung [17], $P_{Bs} \propto \sqrt{T_e}$, which are increasing with T_e, do not produce any effect on the collisional energy transfer to the ions and formally it is assumed that P_{Bs} is included in $P_{\rm loss}$.

III. COLLISIONAL ENERGY EXCHANGE IN RELATIVISTIC PLASMAS

The rate of energy exchange between relativistic electrons and ions with Maxwellian distribution functions, $P_{ei} = m_{e0}c^2 \int (\gamma-1) C_{ei}[f_{eJM}, f_{iM}] d^3u$, can be written for the relativistic collision operator [9] as follows:

$$P_{ei} = 4\pi m_{e0} \frac{T_e - T_i}{T_e} \int_0^\infty \frac{u^3}{\gamma} F_u^{e/i}(u) f_{eJM}(u) du, \quad (5)$$

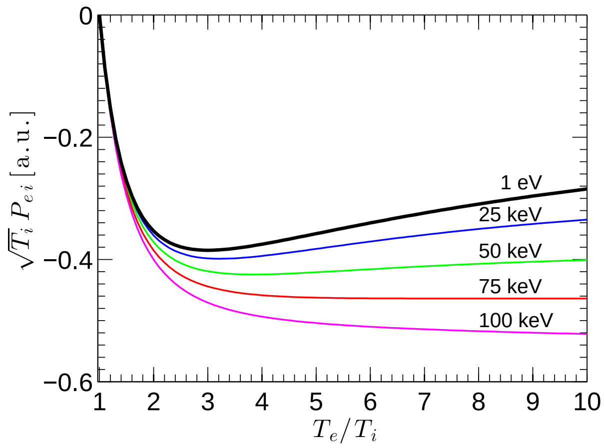

FIG. 1: (color online) The rate of energy exchange between the relativistic electrons and the ions, multiplied by $\sqrt{T_i}$, is shown as a function of T_e/T_i for different T_i. The case $T_i = 1$ eV is taken as the non-relativistic limit.

where $F_u^{e/i}(u)$ is the Coulomb drag of the relativistic electrons with ions. Since $m_i \gg m_{e0}$, the main contribution in the integral in Eq. (5) comes from the range $u \gg v_{ti}$ where $F_u^{e/i}(u)$ can be approximated [9] as

$$F_u^{e/i}(u) \simeq -\nu_{e0} u_{\rm th}^3 \frac{n_i Z_i^2}{n_e} \frac{m_e}{m_i} \frac{\gamma^2}{u^2} \quad (6)$$

with $\nu_{e0} = 4\pi n_e e^4 \ln\Lambda/(m_{e0}^2 u_{\rm th}^3)$. The final expression for the rate of $e-i$ energy exchange is [18]

$$P_{ei} = P_{ei}^{(cl)}\, C(\mu_r) \left(1 + \frac{2}{\mu_r} + \frac{2}{\mu_r^2}\right), \quad (7)$$

where $C(\mu_r)$ is defined by Eq. (3) and $P_{ei}^{(cl)}$ is the classical expression for the rate of collisional energy exchange between non-relativistic electrons and ions [19],

$$P_{ei}^{(cl)} = -4\sqrt{2\pi}\, e^4 n_e m_{e0}^{1/2} m_i^{-1} n_i Z_i^2 \ln\Lambda \frac{T_e - T_i}{T_e^{3/2}}. \quad (8)$$

From Eq. (7), one can obtain the ultra-relativistic limit [20], $1 \ll T_e/m_{e0}c^2 \ll \sqrt{137\ln\Lambda}$ (the upper boundary is defined by the validity of the small-angle scattering approximation [1]):

$$P_{ei}^{(ur)} \simeq -\frac{4\pi e^4 n_e n_i Z_i^2 \ln\Lambda}{m_{e0}^{1/2} m_i c} \frac{T_e - T_i}{T_e}. \quad (9)$$

In Fig. 1, $\sqrt{T_i} P_{ei}$ is plotted as a function of T_e/T_i with $T_e \geq T_i$ for different values of T_i starting from the non-relativistic limit with 1 eV (shown with the bold line). For convenience of graphical representation of P_{ei}, the scaling factor $\sqrt{T_i}$ is applied, which makes $\sqrt{T_i} P_{ei}^{(cl)}$ a function only of the temperature ratio T_e/T_i. One can see that for high temperatures the difference between P_{ei}

and $P_{ei}^{(cl)}$ becomes significant in the range $T_e/T_i > 2$, since the maximum of $|P_{ei}|$ shifts to the upper values (the shape of $P_{ei}^{(cl)}$ does not depend on the temperature). For sufficiently high temperatures, $T_{e,i} > 75$ keV, P_{ei} becomes a monotonic function of T_e/T_i, and the extrema disappears. The latter feature (absence of any extrema in P_{ei} for high temperatures) is in agreement with the ultra-relativistic limit Eq. (9), $P_{ei}^{(ur)} \propto -(T_e - T_i)/T_e$, which is monotonic and has the same sign of the slope as $P_{ei}^{(cl)}$ in the range $T_e/T_i < 3$. This is a qualitative difference from the non-relativistic limit and the consequences are considered in the next section.

IV. STABILITY OF THE COULOMB COUPLING

For any set of plasma parameters which corresponds to the steady state, $P_{ei} + P_{\text{ext}} - P_{\text{loss}} = 0$, the energy-balance equation Eq. (4) yields the following: if $P'_{ei} \equiv dP_{ei}/dT_e > 0$, this state is potentially unstable with respect to collisional decoupling, i.e. to growth of T_e/T_i with decreasing P_{ei}. Physically, this means that in plasmas where the ions are heated predominantly by the drag with hot electrons, positive feedback can appear [12–16] which leads to a rapid transition to a new steady state with $T_e \gg T_i$ defined by other factors. As a consequence, any further increasing of the electron heating power leads to further growth of T_e (but not T_i) and might even provoke a thermal collapse of the ions.

From the condition $P'_{ei} < 0$, one can see that stability of the collisional coupling in non-relativistic plasmas is guaranteed by a very simple relation: $T_e < 3T_i$. For arbitrary temperatures, when relativistic effects are non-negligible, this condition is more complicated,

$$T_e < \left(3 + \frac{2y}{1-y}\right) T_i, \qquad (10)$$

where

$$y(\mu_r) = -\mu_r \left(\frac{K_3(\mu_r)}{K_2(\mu_r)} - 1\right) + \frac{5}{2} + \frac{2(\mu_r + 2)}{\mu_r^2 + 2\mu_r + 2}. \qquad (11)$$

Since $y \simeq (8\mu_r)^{-1}$ for $\mu_r \gg 1$ and $y(c \to \infty) = 0$, Eq. (10), as expected, recovers the non-relativistic limit.

The analytical investigation of Eq. (10) is rather cumbersome, but can easily be done numerically. In Fig. 2, the solution of Eq. (10) is represented as the ratio $(T_e/T_i)^\star$ which corresponds to the extrema of P_{ei} plotted as a function of the ion temperature T_i. One can see that, starting from the low temperature limit $(T_e/T_i)^\star = 3$, the first extremum (maximum of $|P_{ei}|$) shifts towards higher values with increasing T_i and the extrema disappear for $T_i > 75$ keV, where $P_{ei}(xT_i; T_i)$ becomes a monotonic function of $x = T_e/T_i$. In the area where $P'_{ei} > 0$ (labelled "unstable" in Fig. 2), the collisional decoupling can easily appear since any increase of the electron temperature leads to a degradation of the collisional energy

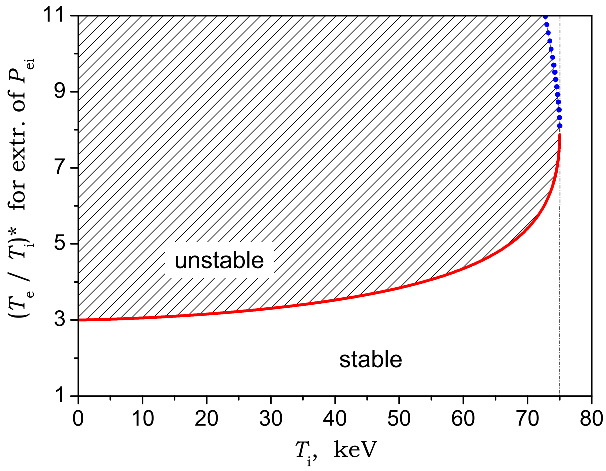

FIG. 2: (color online) Ratio $(T_e/T_i)^\star$ which corresponds to the maximum (full (red) line) and minimum of $|P_{ei}|$ (dotted (blue) line). The area where collisional decoupling is impossible is marked as "stable" and vice versa.

transfer to ions, creating positive feedback. If other factors do not prevent this feedback, a new steady state with $T_e \gg T_i$ must be established. In the opposite case, i.e. if $P'_{ei} < 0$, the present steady state is absolutely stable and no decoupling can arise. In Fig. 2 this area is labelled "stable".

V. SUMMARY AND DISCUSSION

In this paper, the condition for steady-state stability with respect to the collisional decoupling between electrons and ions in hot plasmas for a broad range of temperatures has been defined in the relativistic approach. It was shown that while the stability condition in non-relativistic plasmas is given by $T_e/T_i < 3$, relativistic effects make this threshold dependent on the temperature and shift its value to higher T_e/T_i with an increase of temperature. For temperatures $T_{e,i} > 75$ keV, the collisional coupling becomes absolutely stable for any temperature ratio. This result can be useful for interpretation of experiments with heating of plasmas in the electron channel, when the ion heating is caused exclusively by the collisional energy exchange with electrons.

Collisional decoupling appears only if the heating of electrons is sufficiently high and the temperature threshold can be reached. Physically, this means that the heating of electrons has a power threshold with respect to decoupling, but its value is defined by the concrete scenario and is not discussed here.

Since the consideration here is restricted to the collisional energy exchange between electrons and ions, the energy loss of electrons due to the radiation (in particular, the bremsstrahlung, which accompanies collisions), was excluded from consideration. In fully ionized low temperature plasmas, the contribution of radiation in the energy balance is negligible, but for higher temperatures this phenomena must be taken into account. Moreover, if conditions for the collisional decoupling appear for sufficiently high temperatures, the losses through radiation, $P_{loss} = P_{Bs} + P_{ce}$ (here, $P_{Bs} \propto \sqrt{T_e}$ is bremsstrahlung and $P_{ce} \propto T_e$ is cyclotron radiation), produce a stabilizing effect for electrons by counteracting the increase of T_e. However, the radiation does not have any direct influence on the power balance for ions. Thus, the radiation does not change the temperature threshold for the collisional decoupling even for high temperatures, but can be important for the power threshold.

Acknowledgments

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[1] S. T. Beliaev and G. I. Budker, DAN USSR **107**, 807 (1956), [Sov. Phys. Dokl. **1** 218 (1956)].
[2] S. R. de Groot, W. A. van Leewen, and Ch. G. van Weert, *Relativistic Kinetic Theory* (North-Holland Publishing Company, Amsterdam, 1980).
[3] M. Dunne, Nature Physics **2**, 2 (2006).
[4] M. Honda and K. Mima, J. Phys. Soc. Jpn. **67**, 3420 (1998).
[5] J. Robiche and J. M. Rax, Phys. Rev. E **70**, 046405 (2004).
[6] D. J. Ward, Plasma Phys. Controlled Fusion **52**, 124033 (2010).
[7] F. Wagner, A. Becoulet, R. Budny, V. Erckmann, D. Farina, G. Giruzzi, Y. Kamada, A. Kaye, F. Koechl, K. Lackner, et al., Plasma Phys. Controlled Fusion **52**, 124044 (2010).
[8] P. E. Stott, Plasma Phys. Controlled Fusion **47**, 1305 (2005).
[9] B. J. Braams and C. F. F. Karney, Phys. Fluids B **1**, 1355 (1989).
[10] N. B. Marushchenko, C. D. Beidler, S. V. Kasilov, W. Kernbichler, H. Maaßberg, R. Prater, and R. W. Harvey, Phys. Plasmas **18**, 032501 (2011).
[11] T. Mettens and R. Balescu, Phys. Fluids B **2**, 2076 (1990).
[12] D. Braun and G. Emmert, Nuclear Fusion **24**, 1393 (1984).
[13] F. Wagner and U. Stroth, Plasma Phys. Controlled Fusion **35**, 1321 (1993).
[14] X. M. Zhang, B. N. Wan, Z. W. Wu, X. Shen, and J. Fu, J. Plasma Phys. **76**, 229 (2010).
[15] H. Maassberg, C. D. Beidler, U. Gasparino, M. Romé, K. S. Dyabilin, N. B. Marushchenko, and S. Murakami, Phys. Plasmas **7**, 295 (2000).
[16] M. Romé, C. D. Beidler, H. Maaßberg, N. B. Marushchenko, Yu. A. Turkin, and the W7-AS Team, Plasma Phys. Controlled Fusion **48**, 353 (2006).
[17] G. Bekefi, *Radiative Processes in Plasmas* (John Wiley & Sohn, New York, 1966).
[18] Since the expression for the rate of energy exchange in relativistic plasmas calculated first by Beliaev and Budker (see Eq. (18) in Ref. [1]) was published with a few misprints, the correct formula is reproduced here.
[19] S. I. Braginskii, *Phenomenas of Transport in Plasmas*, in *Reviews of Plasma Physics* vol. 1 (Consultant Bureau, New York - London, 1965).
[20] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Pergamon, Oxford, 1981).