ITER relevant generalized Solovév equilibrium with parallel plasma flow

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Since plasma flow is common in astrophysical systems and play a role in the transition to improved confinement regimes of magnetic confinement devices there has been an increasing interest to self consistent equilibria with flow. In the static regime the most known and widely employed axisymmetric magnetohydrodynamic (MHD) analytic equilibrium is associated with the Solovév solution of the Grad-Sfafranov (GS) equation [1]. This solution, however has a limited number of free parameters which puts a restriction on the construction of realistic configurations, particularly ITER-like ones with a single lower x-point. This drawback was recently eliminated by an extension of the solution to contain arbitrary number of free parameters [2, 3]. Consequently, a variety of equilibria were constructed with boundary shaping pertinent to laboratory fusion plasmas and desirable values of confinement figures of merit. Aim of the present contribution is to extend further this Solovév solution to plasmas with incompressible flow parallel to the magnetic field on the basis of a generalized GS equation [Eq. (1) below]. Then, ITER-like equilibria are constructed. A stability consideration is also made by means of a sufficient condition for linear stability [8].

We start from the generalized GS equation for field aligned incompressible flows,

$$\Delta^\star u + \frac{1}{2}\frac{d}{du}\left(\frac{X^2}{1-M^2}\right) + \mu_0 R^2 \frac{dP_s}{du} = 0, \tag{1}$$

together with the Bernoulli relation for the pressure,

$$P = P_s(u) - \rho\frac{v^2}{2}. \tag{2}$$

Here, (z, R, ϕ) are cylindrical coordinates with z corresponding to the axis of symmetry; the function $u(R,z)$ labels the magnetic surfaces; $M(u)$ is the Mach function of the velocity with respect to the Alfvén velocity; $\rho(u)$ is the density and $X(u)$ relates to the toroidal magnetic field; for vanishing flow the surface function $P_s(u)$ coincides with the pressure; v is the velocity modulus which can be expressed in terms of surface functions and R; and $\Delta^\star = R^2\nabla\cdot(\nabla/R^2)$. Derivation of (1) and (2) in the general case of flows non-parallel to the magnetic field is provided in [4] and [5]. Note that (1) is identical in form with the usual (static) GS equation.

Therefore, any analytic solution to the GS equation can be smoothly extended to the parallel flow case. For convenience we will introduce dimensionless quantities by $\tilde{R} = R/R_0$, $\tilde{z} = z/R_0$, $\tilde{u} = u/(B_0 R_0^2)$, $\tilde{\rho} = \rho/\rho_0$, $\tilde{P} = P/(B_0^2/\mu_0)$; $\tilde{\mathbf{j}} = \mathbf{j}/(B_0/(\mu_0 R_0))$, where $\mathbf{j}$ is the current density, and $\tilde{\mathbf{v}} = \mathbf{v}/v_{A0}$, where $v_{A0} = B_0/\sqrt{\mu_0 \rho_0}$. The free parameters R_0 and B_0 are the radial coordinate of the geometric center of the configuration and the vacuum toroidal magnetic field thereon. Then, Eq. (1) remains identical in form for the tilted quantities by formally setting $\mu_0 = 1$. In the following the tilde will be dropped on the understanding of non-dimensionality.

By making the Solovév-like linearizing ansatz $P_s = P_1 u$ and $X^2/(1-M^2) = X_0^2 + 2X_1 u$, the resulting form of Eq. (1) admits the following solution which consists the basis of the present study:

$$u = u_p + u_h, \;\; u_p = \frac{P_1}{8}R^4 - \frac{X_1}{2}z^2, \tag{3}$$

$$\begin{aligned}
u_h \;=\;& c_1 + c_2\frac{R^2}{2} + c_3\left(z^2 + \frac{R^2}{2} - R^2\ln R\right) + c_4\left(\frac{z^2R^2}{2} - \frac{R^4}{8}\right) \\
&+ c_5\left(z^4 + 3z^2R^2 - \frac{15R^4}{8} - 6z^2R^2\ln R + \frac{3}{2}R^4\ln R\right) + c_6\left(\frac{z^4R^2}{2} - \frac{3z^2R^4}{4} + \frac{R^6}{16}\right) \\
&+ c_7\left(z^6 + \frac{15z^4R^2}{2} - \frac{255z^2R^4}{8} + \frac{25R^6}{8} - 15z^4R^2\ln R + \frac{45}{2}z^2R^4\ln R - \frac{15}{8}R^6\ln R\right) \\
&+ d_1 z + d_2\frac{zR^2}{2} + d_3\left(z^3 + \frac{3zR^2}{2} - 3zR^2\ln R\right) + d_4\left(\frac{z^3R^2}{2} - \frac{3zR^4}{8}\right) \\
&+ d_5\left(z^5 + 5z^3R^2 - \frac{75zR^4}{8} - 10z^3R^2\ln R + \frac{15}{2}zR^4\ln R\right),
\end{aligned} \tag{4}$$

Here, u_h is the solution of the homogeneous equation and u_p is a particular solution of the inhomogeneous equation. u_h consists of a symmetric in z part in connection with the coefficients c_i and an asymmetric in z part in connection with the coefficients d_j. The construction of this solution is based on an iterative algorithm which is explained in Sec. 2 of [3]. For $d_j = 0$ the equilibrium is up-down symmetric while an ITER-like equilibrium requires non symmetric terms. Note that $X_0 = 1$ because of the adopted normalization. It is emphasized that (3)-(4) hold for arbitrary Mach functions and densities. By exploiting the free parameters we have constructed ITER-like configurations with the following characteristics: major radius $R_0 = 6.2m$, minor radius $a = 2m$, elongation $\kappa = 1.33$, triangularity $\delta = 0.33$, safety factor on axis in the interval $1 \leq q_a \leq 2$ and average toroidal beta $\beta_t \approx 0.01$. The parametric values were fixed, similar to [2], by solving numerically the set of equations to follow in connection with the boundary shape and confinement figures of merit. First, without prescribing completely the boundary curve we impose u to vanish at four characteristic fixed points of the boundary, i.e. the inner point, $(R_{in}, z = 0)$, the outer point, $(R_{out}, z = 0)$, the higher point, (R_u, z_u), and the (lower) x-point, (R_x, z_x): $u(R_{in}, 0) = u(R_{out}, 0) = u(R_u, z_u) = u(R_x, z_x) = 0$. At the higher point and the x-point should hold the relations $\frac{\partial u}{\partial R}(R_u, z_u) = 0$, $\frac{\partial u}{\partial R}(R_x, z_x) = \frac{\partial u}{\partial z}(R_x, z_x) = 0$. Also, we require that the configuration is up-down symmetric near the plane $z=0$: $\frac{\partial u}{\partial z}(R_{in},0)=\frac{\partial u}{\partial z}(R_{out},0)=0$. Furthermore, it can be shown that the curvature of the bounding curve at the inner, outer and higher points should satisfy the relations [2] $\frac{\partial^2 u}{\partial z^2}(R_{in},0)=-N_1\frac{\partial u}{\partial R}(R_{in},0)$, $\frac{\partial^2 u}{\partial z^2}(R_{out},0)=-N_2\frac{\partial u}{\partial R}(R_{out},0)$, $\frac{\partial^2 u}{\partial R^2}(R_u,z_u)=-N_3\frac{\partial u}{\partial z}(R_u,z_u)$, where $N_1=\frac{(1-\alpha)^2}{\varepsilon\kappa^2}$, $N_2=-\frac{(1+\alpha)^2}{\varepsilon\kappa^2}$, $N_3=-\frac{\kappa}{\varepsilon}\cos\alpha^2$, $\alpha=\arcsin\delta$. In addition to the above boundary shaping equations we employ for the safety factor on axis and the toroidal beta the relations:

$$q_a=\frac{X}{R\sqrt{1-M^2}}\left(\frac{\partial^2 u}{\partial R^2}\frac{\partial^2 u}{\partial z^2}\right)^{-1/2}\Bigg|_{R=R_a,z=z_a}, \tag{5}$$

where $(R_a,\ z_a)$ is the position of the magnetic axis, and $\beta_t=\frac{\int_V P\,d\tau}{B_0^2/(2\mu_0)}$. Regarding the flow, we adopted the following alternative choices of M^2:

$$M^2=M_a^2\left(\frac{u}{u_a}\right)^n \text{ or } M^2=Cu^n(u_a-u)^m, \tag{6}$$

where $C=M_a^2\left[\frac{mu_a}{m+n}\right]^{-m}\left[\frac{nu_a}{m+n}\right]^{-n}$. Here, u_a refers to the magnetic axis, the free parameter M_a^2 corresponds to the maximum value of M^2 and m and n are related to the flow shear. In particular, for the former (latter) of (6) M^2 is peaked on- (off-)axis in connection with respective auxiliary heating of tokamaks. Typical value of M^2 for (large) tokamaks are of the order of 10^{-4} because of the experimental scaling $v\sim 10^{-1}v_s$, where $v_s=(\gamma P/\rho)^{1/2}$ is the sound velocity. An example of the equilibria constructed is shown in Fig. 1.

We now consider the important issue of the stability of (3)-(4), with respect to small linear MHD perturbations by applying a sufficient condition ([8]). This condition states that a general steady state of a plasma of constant density and incompressible flow parallel to **B** is linearly stable to small three-dimensional perturbations if the flow is sub-Alfvénic ($M^2<1$) and $A\geq 0$, where $A=A_1+A_2+A_3+A_4$ as given by Eqs. (15)-(18) of [7]. Consequently, we set $\rho=1$. The quantity A_1 is a destabilizing contribution ($A_1<0$) potentially related to current driven modes while A_2 relates to the variation of the magnetic field perpendicular to the magnetic surfaces. A_3 and A_4 are flow terms depending on the magnitude and the shear of the flow (in connection with the parameters M_a^2, n and m for the

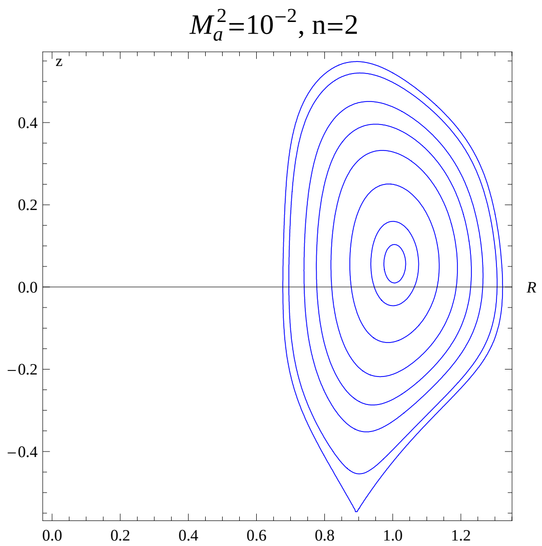

Figure 1: *An ITER-like equilibrium with parallel plasma flow in connection with solution (3)-(4) with $q_a=1.1$ and $\beta_t=0.01$.*

present study). The quantity A was calculated analytically by Mathematica. It turns out that the condition $A \geq 0$ is satisfied only in a small region close to the boundary irrespective of flow. As an example, the stability diagram showing the sign of A on the poloidal plane is given in Fig. 2 for the peaked Mach-function (6) and the nearly maximum permissible value $M_a^2 = 10^{-2}$ (in connection with the non-negativeness of the pressure). The condition is satisfied in the red-colored region. The diagram is nearly identical with the respective quasistatic one, i.e. $|\Delta A/A_{qs}| \leq 10^{-3}$, where A_{qs} are the quasistatic values of A and ΔA their differences from the stationary ones. Also, a similar result holds for the off-axis Mach function (6). Stability diagrams for the terms A_1, A_2, A_3 and A_4 show that A_2 is nearly everywhere stabilizing and of the same order of magnitude as A_1 but not large enough to overcome the destabilizing effect of A_1 . The flow term A_3 is stabilizing over a large part of the plasma but A_4 is destabilizing. Both flow contributions are at least three orders of magnitude lower than A_1. However taking into account the fact that the condition is sufficient, the above results do not necessarily imply that the equilibrium is unstable. Unlikely, the condition is satisfied in an appreciable plasma region in the cases of the *nonlinear* cat-eyes [6] and counter-rotating-vortices equilibria [8]. This result is in favor to the conjecture that the equilibrium nonlinearity may activate flow stabilization.

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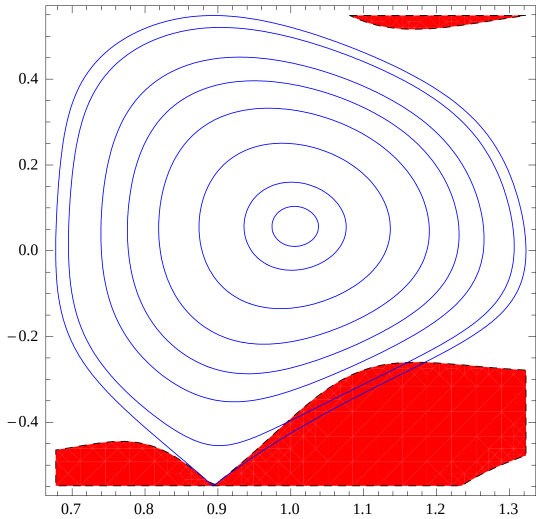

Figure 2: *Stability diagram showing the sign of the quantity A for the peaked Mach function (6) with* $M_a^2 = 10^{-2}$. *In the red-colored regions where* $A \geq 0$ *the stability condition is satisfied on the understanding that the physically relevant part lies within the plasma boundary.*