How smooth are particle trajectories in a ΛCDM Universe?

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ABSTRACT

Very. Indeed, it is shown here that in a flat, cold dark matter (CDM) dominated Universe with positive cosmological constant (Λ), modelled in terms of a Newtonian and collisionless fluid, particle trajectories are analytical in time (representable by a convergent Taylor series) until at least a finite time after decoupling. The time variable used for this statement is the cosmic scale factor, i.e., the "a-time", and not the cosmic time. For this, a Lagrangian-coordinates formulation of the Euler–Poisson equations is employed, originally used by Cauchy for 3-D incompressible flow. Temporal analyticity for ΛCDM is found to be a consequence of novel explicit all-order recursion relations for the a-time Taylor coefficients of the Lagrangian displacement field, from which we derive the convergence of the a-time Taylor series. A lower bound for the a-time where analyticity is guaranteed and shell-crossing is ruled out is obtained, whose value depends only on Λ and on the initial spatial smoothness of the density field. The largest time interval is achieved when Λ vanishes, i.e., for an Einstein–de Sitter universe. Analyticity holds also if, instead of the a-time, one uses the linear structure growth D-time, but no simple recursion relations are then obtained. The analyticity result also holds when a curvature term is included in the Friedmann equation for the background, but inclusion of a radiation term arising from the primordial era spoils analyticity.

Key words: cosmology: theory – large scale structure of Universe – dark matter – dark energy

1 INTRODUCTION

Recent observational results (Planck Collaboration *et al.* 2014) indicate that we live in a spatially flat ΛCDM Universe which is nowadays dominated by a cold dark matter (CDM) component and a cosmological constant, Λ > 0. In a primordial era, matter was tightly coupled to radiation via electroweak interactions. This tight coupling prevented matter to cluster significantly at early times since the radiation pressure acted as a counter force to the gravitational force. As the Universe expands its mean energy density decreases, which eventually lead to a freeze-out of these electroweak interactions. This freeze-out of the interactions (non-instantenous in space and time) happened roughly 380,000 years after the Big Bang, eventually enabling the matter to cluster. This epoch, usually called decoupling or recombination, marks the beginning of cosmological structure formation with initially smooth matter density fluctuations.

The observed large-scale structure of the Universe is mainly the result of the gravitational instability. To study the structure formation, it is usually assumed that the CDM component behaves as a Newtonian pressureless and curlfree fluid (so-called *cosmological dust*). Such a fluid is goverened by the Euler–Poisson equations with the addition of the Friedmann equation, where the latter describes the background evolution of the expanding Universe. Since that background evolution is by definition exactly homogeneous and isotropic, it is parametrised by a single function (of cosmic time t): the cosmic scale factor $a(t)$.

Analytical models for the cosmological structure formation can be formulated in either the Eulerian or Lagrangian frame of reference (for some reviews, see e.g. Bernardeau *et al.* 2002; Bernardeau 2013). The class of analytical models of the latter is dubbed the Lagrangian perturbation theory (LPT) (Buchert & Goetz 1987; Buchert 1989; Buchert 1992; Bouchet *et al.* 1992; Bouchet *et al.* 1995; Catelan 1995; Ehlers & Buchert 1997). Using this technique, the only dynamical variable is the displacement, which represents the gravitationally induced deviation of the particle trajectory field from the homogeneous background evolution of the Universe. To solve the Euler–Poisson equations

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in Lagrangian space, one usually requires some power series Ansatz for the displacement (for more information see also further below in the Introduction).

The LPT is widely applied in cosmology. Buchert (1992) showed that the Zel'dovich approximation (Zel'dovich 1970) is actually an instance of the first-order LPT. It simply and elegantly, quite well describes the gravitational evolution of CDM inhomogeneities, as long as the trajectories of the fluid particles do not cross. A Lagrangian approach has been also used in the so-called cosmological reconstruction problem in an expanding Universe (Frisch *et al.* 2002; Brenier *et al.* 2003), where it is shown that, in so far as the dynamics are governed by the Euler–Poisson equations, the knowledge of both the present highly non-uniform distribution of mass and of its primordial quasi-uniform distribution, uniquely determines the inverse Lagrangian map, defined as the transformation from present Eulerian positions to their respective initial positions. The LPT is also an important tool in numerical investigations. In recent years, second-order LPT (2LPT) has been successfully used to set up initial conditions for numerical N-body simulations (see e.g. Crocce, Pueblas & Scoccimarro 2006). Biasing models whose task is to find a relationship between the visible matter and the dark matter distribution, yield good predictions to cosmological observations when formulated in Lagrangian space (Mo & White 1996; Matsubara 2008b). Also, semi-analytical approaches to LPT deliver statistical estimators, such as the matter power spectrum and bispectrum, which compare favourably with results from N-body simulations (e.g., Matsubara 2008a; Rampf & Wong 2012; Vlah, Seljak & Baldauf 2014). These approaches make use of the LPT up to the fourth order (Rampf & Buchert 2012; Rampf 2012). Most of the LPT calculations are performed for a flat CDM universe with vanishing cosmological constant, i.e., for an Einstein–de Sitter (EdS) universe, and only few investigations, up to third order, are known for a ΛCDM Universe (see, e.g., Matsubara 1995; Lee 2014). The LPT has also been generalised to General Relativity, for a non-perturbative formulation see Buchert & Ostermann (2012) and Buchert, Nayet & Wiegand (2013). For a general relativistic treatment within the ΛCDM model see Rampf & Rigopoulos (2013), Rampf (2013), Rampf & Wiegand (2014).

Although LPT is widely applied in cosmology, very little is known about its convergence properties, which requires some knowledge about terms of arbitrarily high order. There are of course some exceptions, for example, Sahni & Shandarin (1996) investigated the case of an initial "top-hat underdensity" that is initially discontinuous, and found that low-order perturbation worked better than a higher-order one, which they regarded as a possible evidence for "semiconvergence" of the perturbation series (see also Moutarde *et al.* 1991). Recently, a novel Lagrangian-coordinates approach has been used to show that particle trajectories for an EdS universe are time analytical until a finite time (Zheligovsky & Frisch 2014). Their approach is based on a little-known Lagrangian formulation of ideal fluid flow derived by Cauchy in 1815 (Frisch & Villone 2014). In this paper we show that the approach of Zheligovsky & Frisch (2014) can be extended to a ΛCDM Universe (and even beyond, see below), derive novel recursion relations and prove time analyticity in the cosmic scale factor time, here used both as a time variable and as an expansion parameter.

In the LPT literature, analytical results for ΛCDM usually employ an expansion in powers of small displacements, which amounts to performing a (Taylor) expansion in powers of the initial peculiar gravitational potential, $\varphi_g^{(\text{init})} \sim 10^{-5}$. By this technique one seeks (reasonably well) *approximate* solutions of the nonlinear Euler–Poisson equations. As a consequence of such an approximation technique, one is forced to solve at each order in perturbation theory a second-order partial differential equation for the time coefficient of the displacement (the fastest growing solutions of these time coefficients are usually denoted with $D(t)$, $E(t)$, etc.). Because of that it is impossible to obtain recursion relations for ΛCDM by the use of such a "small displacement" expansion. In this paper, by contrast, we perform *a time-Taylor expansion* and seek for *exact* analytic solutions for the fully nonlinear Euler–Poisson equations in Lagrangian space. As a consequence of our expansion scheme, the displacement is represented in terms of a power series in the cosmic scale factor (even for a ΛCDM Universe!), and there is thus no need to explicitly solve partial differential equations for higher-order time coefficients.

This paper is organised as follows. In Section 2 we present various forms of the Euler–Poisson equations in a Eulerian-coordinate system and we show that regularity of the solution at short times requires certain *slaving constraints* on the initial conditions. In Section 3 we turn to the Euler–Poisson equations in Lagrangian coordinates, which take a particularly simple form when using the cosmic scale factor as the time parameter. Recursion relations are derived in Section 4, from which time-analyticity is derived in Section 5. Further results related to time-analyticity are discussed in Section 6. This includes the dependence of the time of guaranteed analyticity on the cosmological constant Λ (Section 6.1), the absence of shell-crossing during the time interval where analyticity can be proved (Section 6.2), the analyticity in the linear growth time variable D (Section 6.3) and the persistence of analyticity when curvature effects are included, as well as the problem that arises with radiation effects (Section 6.4). Finally, in Section 7, we summarise our main findings and highlight various challenging open problems, such as using analyticity to develop a semi-Lagrangian numerical approach to the cosmological reconstruction.

2   THE VARIOUS FORMS OF THE EULER–POISSON EQUATIONS AND THE SLAVING CONDITIONS

A flat, matter dominated Universe may be studied as a Newtonian and collisionless fluid whose governing equations are the Euler–Poisson equations. For a flat Universe with cosmological constant, the Euler–Poisson equations are (Peebles 1980)

$$\partial_t \tilde{\boldsymbol{U}} + \left(\tilde{\boldsymbol{U}} \cdot \boldsymbol{\nabla_r} \right) \tilde{\boldsymbol{U}} = -\boldsymbol{\nabla_r} \tilde{\phi}_g \,, \tag{1a}$$

$$\partial_t \tilde{\varrho} + \boldsymbol{\nabla_r} \cdot \left(\tilde{\varrho} \tilde{\boldsymbol{U}} \right) = 0 \,, \tag{1b}$$

$$\boldsymbol{\nabla}_r^2 \tilde{\phi}_g = 4\pi G \tilde{\varrho} - 3\Lambda \,. \tag{1c}$$

Here, r is the proper space coordinate, t the cosmic time, $\tilde{U}$ the velocity of the fluid, its density $\tilde{\varrho}$; the gravitational potential is denoted by $\tilde{\phi}_g$. As to the cosmological constant, it is here denoted by 3Λ. Observe that, as long as we use the cosmic time as independent time variable, the various dependent variables are surmounted by a tilde, which will be dropped when we change the time variable from cosmic time to cosmic scale factor a. As usual, we decompose the mass density, fluid velocity and gravitational field respectively in a sum of two terms, one describing the effect of the uniform background expansion, the other the fluctuations against its background,

$$\tilde{\varrho} = \bar{\tilde{\varrho}}(t) \left[1 + \tilde{\delta}\right], \qquad \tilde{U} = \frac{\dot{a}}{a} r + a\tilde{u}, \qquad \tilde{\phi}_g = \bar{\tilde{\phi}}_g + \tilde{\varphi}_g. \tag{2}$$

Here, $\tilde{\delta}$ is the density contrast and $a\tilde{u}$ the proper peculiar velocity. Furthermore, $a(t)$ is the cosmic scale factor which parametrises the global background expansion governed by the (first) Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = a^{-3} + \Lambda, \tag{3}$$

where we have used the fact that the background mass density is $\bar{\tilde{\varrho}}(t) \sim a^{-3}(t)$, and we have set $8\pi G\bar{\varrho}_0/3 = 1$ for convenience. The solution for the cosmic scale factor is easily obtained from the Friedmann equation, namely

$$a(t) = \Lambda^{-1/3}[\sinh(3\sqrt{\Lambda}t/2)]^{2/3}, \tag{4}$$

which is proportional to $t^{2/3}$ for small t (see e.g., Kofman, Gnedin & Bahcall 1993).

After performing the aforementioned decomposition, derived in Appendix A, we obtain the Euler–Poisson equations for the (comoving) peculiar velocity $\tilde{u}$

$$\partial_t \tilde{u} + (\tilde{u} \cdot \nabla_x) \tilde{u} = -2\frac{\dot{a}}{a}\tilde{u} - \frac{1}{a^2}\nabla_x\tilde{\varphi}_g, \tag{5a}$$

$$\partial_t \tilde{\delta} + \nabla_x \cdot \left[(1 + \tilde{\delta})\tilde{u}\right] = 0, \tag{5b}$$

$$\nabla_x^2 \tilde{\varphi}_g = \frac{3}{2a}\tilde{\delta}. \tag{5c}$$

Note that the peculiar Euler–Poisson equations depend on the comoving coordinate $x \equiv r/a$. Although these equations are indeed valid for a fluid model with cosmological constant, the latter does not explicitly appear in the peculiar approach, but only implicitly through Friedmann's background evolution equation (3).

Before considering the Euler–Poisson system (5a–5c) in Lagrangian space, let us briefly discuss the linearised system in Eulerian space. Formally linearising around the steady state $\tilde{u} = 0$, $\tilde{\delta} = 0$, we obtain the single differential equation for the density contrast,

$$\ddot{\tilde{\delta}} = -2\frac{\dot{a}}{a}\dot{\tilde{\delta}} + \frac{3}{2a^3}\tilde{\delta}. \tag{6}$$

The solution of (6) is most easily obtained by changing the time variable from cosmic time t to the cosmic scale factor a, here called a-time. We thus set $\tilde{\delta}(t) = \delta(a(t))$. We then have $\partial_t = \dot{a}\partial_a$. The above differential equation is then

$$(1 + \Lambda a^3) \left[\partial_{aa}\delta + \frac{3}{2a}\partial_a\delta\right] + \frac{3}{2}\Lambda a^2\partial_a\delta - \frac{3}{2a^2}\delta = 0, \tag{7}$$

where we have used the Friedmann equation (3). This equation has two solutions. One is called the decaying mode; it behaves as $a^{-3/2}$ for small a and thus blows up when $a \to 0$, thereby invalidating the linearisation. The other one, the growing mode, here called the linear growth function, is taken to be $D(a) = a\sqrt{1 + \Lambda a^3}\,_2F_1\left(3/2, 5/6, 11/6, -\Lambda a^3\right)$, where $_2F_1$ is the Gauss hypergeometric function (Demianski *et al.* 2005; Enqvist & Rigopoulos 2010; Hamber & Toriumi 2010). This solution is analytic around $a = 0$, has the small a expansion $D(a) = a - (2/11)\Lambda a^4 + $ h.o.t. and thus essentially reduces to the cosmic scale factor at short times. This is the physically appropriate solution, in which density fluctuations are growing linearly with a.

However, when studying the well-posedness of the nonlinear Euler–Poisson system (5a–5c) and, subsequently, when looking for analytic solutions in Lagrangian coordinates, a linear approach is, of course, no more appropriate. The latter just gives us a signal that the appropriate time variable might be proportional to $t^{2/3}$ or to a or D at short times. The nonlinear theory can be developed in all three such time variables but, for our purposes, it takes its simplest form when using the a-time (and not just at short ones). Indeed the Euler–Poisson system (5a–5c) can be rewritten in the following form:

$$\left(1 + \Lambda a^3\right) [\partial_a v + (v \cdot \nabla_x) v] = -\frac{3}{2a}(v + \nabla_x\varphi_g) - 3\Lambda a^2 v, \tag{8a}$$

$$\partial_a \delta + \nabla_x \cdot [(1 + \delta) v] = 0, \tag{8b}$$

$$\nabla_x^2 \varphi_g = \frac{\delta}{a}, \tag{8c}$$

where we used a new velocity variable v, the a-derivative of comoving particle positions, related to the old velocity by

$$\tilde{u}(x, t) = \dot{a} v(x, a(t)), \tag{9}$$

and we have set $\tilde{\varphi}_g(x, t) = (3/2)\varphi_g(x, a(t))$. Note that, with respect to Λ, the situation in (8a–8c) is the opposite of what we had with (5a–5c): Λ appears now explicitly, but the Friedmann equation can now be omitted as long as we do not want to revert to the cosmic time variable.

An important feature of (8a–8c) is the presence of cosmic scale factors in the denominators of the r.h.s. of (8a) and (8c). It indicates that the solution will be singular at $a = 0$ unless the corresponding numerators also vanish, that is, we have to satisfy the two following *slaving conditions* (Brenier 1987) at $a = 0$ (denoted by the superscript (init)):

$$\delta^{(\text{init})} = 0, \qquad \boldsymbol{v}^{(\text{init})} = -\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi_g^{(\text{init})}. \tag{10}$$

The second slaving condition immediately implies the curlfree character of the initial velocity, which persists by (8a) in Eulerian (but not in Lagrangian) coordinates. Traditionally, this irrotationality is often derived by linearising the Euler–Poisson equations and showing that the rotational component of the velocity decays in time (see e.g., Bernardeau *et al.* 2002). This linearisation would be justified if the velocity $\boldsymbol{v}$ would be small at short times as indeed the density contrast δ is. However, the velocity $\boldsymbol{v}$ and the gravitational force $\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi_g$ do not vanish as $a \to 0$. Thus, a linearisation-based argument is questionable.

The cosmic scale factor a, being dimensionless, is conveniently normalised to unity at the present-time epoch. At decoupling it then has a value of about 10^{-3}. Here, we let the cosmic scale factor start at the value zero, while using the Euler–Poisson model with slaving. As a consequence, the whole primordial (pre-decoupling) cosmology is just reduced to the prescription of the slaving conditions (10). We shall come back to this issue in the Concluding Remarks 7.

3   THE LAGRANGIAN FORMULATION OF ΛCDM

Time-Taylor expansions can be carried out in either Eulerian or in Lagrangian coordinates. For such expansions to be convergent, that is to have a non-vanishing radius of convergence, it is much preferable to work in Lagrangian coordinates. Indeed, if the initial conditions have spatial derivatives only up to some finite order, the Eulerian solutions will have time derivatives only up to the same order. The reason is that, when one takes a time derivative in Eulerian coordinates, this generates a space derivative, because of the $(\boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})\boldsymbol{v}$ term. Hence time-Taylor coefficients will not exist beyond that order. This does not happen in Lagrangian coordinates, where a limited amount of smoothness in the initial data suffices to ensure time-analyticity (Zheligovsky & Frisch 2014). Even when the initial data are spatially analytic, so that the solutions are time-analytic in both Eulerian and Lagrangian coordinates, the Lagrangian radius of convergence, being controlled by the largest strain in the initial data, is typically much larger than the Eulerian one, which depends on the largest initial velocity present (for more details on such matters, see Podvigina *et al.* 2015).

We now turn to the Lagrangian formulation of the Euler–Poisson equations (8a–8c). For this we use the (direct) Lagrangian map $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q}, a)$ from the initial $(a = 0)$ position $\boldsymbol{q}$ to the Eulerian position $\boldsymbol{x}$ at time a. The velocity $\boldsymbol{v}$, as it is apparent from its definition (9), is simply the Lagrangian a-time derivative of the Lagrangian position; more precisely, $\boldsymbol{v}(\boldsymbol{x}(\boldsymbol{q}, a), a) = \partial_a^{\text{L}}\boldsymbol{x}(\boldsymbol{q}, a)$, where ∂_a^{L} is the Lagrangian a-time derivative. At initial time, $a = 0$, the velocity is

$$\boldsymbol{v}^{(\text{init})}(\boldsymbol{q}) = \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{q}, 0), 0), \tag{11}$$

which coincides with the Eulerian one at initial time. We shall also use the Jacobian matrix $\nabla_j^{\text{L}} x_i(\boldsymbol{q}, a)$, where ∇_j^{L} denotes the Lagrangian space derivative with respect to q_j, and its determinant $J = \det\left(\nabla_j^{\text{L}} x_i\right)$, the Jacobian. Note that after shell-crossing, i.e., the first vanishing of J, the map ceases to have a unique inverse.

As an intermediate step to a fully Lagrangian formulation of the Euler–Poisson equations, we rewrite the momentum equation (8a) and the Poisson equation (8c) in a mixed form: some terms are more naturally expressed in Lagrangian coordinates and others in Eulerian coordinates; composition with the direct or inverse Lagrangian maps is here understood where appropriate:

$$\left(1 + \Lambda a^3\right) \partial_{aa}^{\text{L}}\boldsymbol{x} = -\frac{3}{2a}\left(\partial_a^{\text{L}}\boldsymbol{x} + \boldsymbol{\nabla}_{\boldsymbol{x}}\varphi_g\right) - 3\Lambda a^2 \partial_a^{\text{L}}\boldsymbol{x}, \tag{12a}$$

$$(1 + \delta)J = 1, \tag{12b}$$

$$\boldsymbol{\nabla}_{\boldsymbol{x}}^2 \varphi_g = \frac{\delta}{a}. \tag{12c}$$

Equation (12b) is a straightforward expression of mass conservation in Lagrangian coordinates, which need not be derived from its differential Eulerian counterpart. It is however invalid after shell-crossing. Due to the appearance of several velocities (multi-streaming), there are several branches of the inverse Lagrangian map. In that case (12b) must be replaced by

$$(1 + \delta) = \sum_n \frac{1}{|J_n|}, \tag{13}$$

where n labels the various Lagrangian locations having the same Eulerian location (Buchert 1995). It is likely that each branch of the multi-streaming system still satisfies the Euler–Poisson equations with the density given by (13), but this requires further investigation.

The derivation of the full Lagrangian formulation of the Euler–Poisson equations has two parts. The first part is almost identical to Cauchy's 1815 derivation of the Cauchy invariants equation for incompressible fluids (see Frisch & Villone 2014). Since it is already given in Zheligovsky & Frisch (2014) for the case of an EdS universe, we shall just recall the ideas briefly and give the final Cauchy equation. We observe that in the r.h.s. of (12a), because of the curlfree character of the velocity $\partial_a^{\text{L}}\boldsymbol{x}$, all the terms are Eulerian gradients. Multiplication by the Jacobian matrix changes these Eulerian gradients to Lagrangian gradients. These are then made to vanish by taking a Lagrangian curl. After this, a quadratically nonlinear equation is obtained for the Lagrangian map, which involves first and second time derivatives.

Cauchy observed that this equation can be exactly integrated in time. This "miracle" has been interpreted much later as stemming from the relabeling invariance of the variational formulation of the Euler equation and use of Noether's theorem (Frisch & Villone 2014, Section 5.2). The resulting Cauchy invariants equations have normally the initial vorticity in its r.h.s but, in the present case, the vorticity vanishes and the equations take the following form:

$$\varepsilon_{ijk}\nabla_j^{\mathrm{L}}x_l\,\partial_a^{\mathrm{L}}\nabla_k^{\mathrm{L}}x_l = 0\,, \qquad (i = 1, 2, 3) \tag{14}$$

where ε_{ijk} is the fundamental antisymmetric tensor, and summation over repeated indices is assumed. The l.h.s. of (14) is a Lagrangian curl.

The second part makes use of all three equations (12a)-(12c), and gives a third scalar equation:

$$\varepsilon_{ikl}\varepsilon_{jmn}\nabla_m^{\mathrm{L}}x_k\nabla_n^{\mathrm{L}}x_l\left[\left(1 + \Lambda a^3\right)\partial_{aa}^{\mathrm{L}} + 3\Lambda a^2\partial_a^{\mathrm{L}} + \frac{3}{2a}\partial_a^{\mathrm{L}}\right]\nabla_j^{\mathrm{L}}x_i = \frac{3}{a^2}\left(J - 1\right)\,. \tag{15}$$

This is done by taking the Eulerian divergence of (12a) to obtain the Eulerian Laplacian of the gravitational potential, which is then substituted in (12c). The density contrast δ is expressed in terms of the Jacobian, using (12b) and also substituted in the r.h.s. of (12c). Finally, the Eulerian divergence is expressed in terms of Lagrangian space derivatives and of the inverse Jacobian matrix. The latter is given in terms of the Jacobian matrix using the following identity: $\nabla_{x_i}q_j = \varepsilon_{ikl}\varepsilon_{jmn}\nabla_m^{\mathrm{L}}x_k\nabla_n^{\mathrm{L}}x_l/(2J)$, which follows from the observation that the Jacobian of the inverse Lagrangian map is the matrix inverse of the Jacobian of the direct map (see, e.g., Buchert & Goetz 1987). This derivation fails if the Jacobian J is not everywhere strictly positive in the space and time domain under consideration. In the latter case, the Lagrangian formulation (14)-(15) becomes invalid. We shall come back to this issue in Section 6.2.

4 THE FORMAL TAYLOR EXPANSION AND THE RECURSION RELATIONS

In the LPT literature, analytical derivations of results for ΛCDM usually rely on an expansion in powers of small displacements, which amounts to performing a power-series expansion in the initial peculiar gravitational potential (see e.g., Matsubara 1995; Bernardeau *et al.* 2002). In this paper we follow a different approach and perform an expansion in powers of a-time. Note that for the case of an EdS universe, one can obtain recursion relations for the displacement by both expansion techniques, and the resulting recursion relations are formally identical (Rampf 2012). For a ΛCDM Universe, however, explicit results for the displacement will generally differ depending on what expansion scheme is used.

We thus seek a solution to the Lagrangian equations (14)–(15) in the form of a power series in the a-time (i.e., a Taylor series) for the displacement $\boldsymbol{\xi} \equiv \boldsymbol{x} - \boldsymbol{q}$,

$$\boldsymbol{\xi}(\boldsymbol{q}, a) = \sum_{s=1}^{\infty}\boldsymbol{\xi}^{(s)}(\boldsymbol{q})\,a^s\,. \tag{16}$$

At first order, we have

$$\boldsymbol{\xi}^{(1)}(\boldsymbol{q}) = \boldsymbol{v}^{(\mathrm{init})}(\boldsymbol{q})\,, \tag{17}$$

as follows from $\boldsymbol{v} = \partial_a^{\mathrm{L}}\boldsymbol{x}$ and (11). The convergence of the series (16) will be examined in the next section. The Jacobian of the Lagrangian map $J = \det(I + \boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi})$, where I is the identity matrix, can be written as a sum of four terms. After substituting the expansion (16), the Jacobian becomes

$$J = 1 + \sum_{s=1}^{\infty}\mu_1^{(s)}a^s + \sum_{n_1+n_2=s}^{\infty}\mu_2^{(n_1,n_2)}a^s + \sum_{n_1+n_2+n_3=s}^{\infty}\mu_3^{(n_1,n_2,n_3)}a^s\,, \tag{18}$$

where the sums are restricted to values of n_1, n_2 and n_3 that are strictly positive and where the various quantities μ_1, μ_2 and μ_3 are expressions linear, quadratic and cubic, respectively, in the Lagrangian gradients of the Taylor coefficients, expressed as

$$\mu_1^{(n_1)} = \boldsymbol{\nabla}^{\mathrm{L}}\cdot\boldsymbol{\xi}^{(n_1)}\,, \tag{19}$$

$$\mu_2^{(n_1,n_2)} = \frac{1}{2}\left(\nabla_i^{\mathrm{L}}\xi_i^{(n_1)}\nabla_j^{\mathrm{L}}\xi_j^{(n_2)} - \nabla_j^{\mathrm{L}}\xi_i^{(n_1)}\nabla_i^{\mathrm{L}}\xi_j^{(n_2)}\right)\,, \tag{20}$$

$$\mu_3^{(n_1,n_2,n_3)} = \frac{1}{6}\varepsilon_{ikl}\varepsilon_{jmn}\nabla_m^{\mathrm{L}}\xi_k^{(n_1)}\,\nabla_n^{\mathrm{L}}\xi_l^{(n_2)}\,\nabla_j^{\mathrm{L}}\xi_i^{(n_3)}\,. \tag{21}$$

Substituting the expansion (16) into (15) and identifying the coefficients of the various powers of a, we find, for $s > 0$,

$$\left(s + \frac{3}{2}\right)(s - 1)\mu_1^{(s)} = \sum_{n_1+n_2=s}\left\{\left[\frac{3}{2} - 2n_2\left(n_2 + \frac{1}{2}\right)\right]\mu_2^{(n_1,n_2)} - 2\Lambda\left(n_2 - 3\right)\left(n_2 - 1\right)\mu_2^{(n_1,n_2-3)}\right\}$$

$$- \Lambda\left(s - 3\right)\left(s - 1\right)\mu_1^{(s-3)} + \sum_{n_1+n_2+n_3=s}\left\{\left[\frac{3}{2} - 3n_3\left(n_3 + \frac{1}{2}\right)\right]\mu_3^{(n_1,n_2,n_3)} - 3\Lambda\left(n_3 - 3\right)\left(n_3 - 1\right)\mu_3^{(n_1,n_2,n_3-3)}\right\}\,. \tag{22}$$

Here, by construction, all the μ coefficients vanish if one or several of their upper indices are zero or negative. We symmetrise the r.h.s. and

obtain a sequence of relations for $s \geq 1$:

$$\boldsymbol{\nabla}^{\mathrm{L}} \cdot \boldsymbol{\xi}^{(s)} \equiv \mu_1^{(s)} = \boldsymbol{\nabla}^{\mathrm{L}} \cdot \boldsymbol{v}^{(\mathrm{init})} \delta_1^s - \Lambda \frac{s-3}{s+3/2} \mu_1^{(s-3)}$$
$$+ \sum_{0<n<s} \left\{ \frac{(3-s)/2 - n^2 - (s-n)^2}{(s+3/2)(s-1)} \mu_2^{(n,s-n)} + \Lambda \frac{4s - n^2 - (s-n)^2 - 6}{(s+3/2)(s-1)} \mu_2^{(n,s-n-3)} \right\}$$
$$+ \sum_{n_1+n_2+n_3=s} \left\{ \frac{(3-s)/2 - n_1^2 - n_2^2 - n_3^2}{(s+3/2)(s-1)} \mu_3^{(n_1,n_2,n_3)} - \Lambda \frac{n_1^2 + n_2^2 + n_3^2 - 4s + 9}{(s+3/2)(s-1)} \mu_3^{(n_1,n_2,n_3-3)} \right\} , \tag{23}$$

where δ_1^s is the Kronecker symbol. Similarly, we obtain from the Cauchy invariants equation (14), for $s \geq 1$

$$\boldsymbol{\nabla}^{\mathrm{L}} \times \boldsymbol{\xi}^{(s)} \equiv \boldsymbol{T}^{(s)} = \frac{1}{2} \sum_{\substack{1 \leq k \leq 3, \\ 0 < n < s}} \frac{s - 2n}{s} \boldsymbol{\nabla}^{\mathrm{L}} \xi_k^{(n)} \times \boldsymbol{\nabla}^{\mathrm{L}} \xi_k^{(s-n)} . \tag{24}$$

As we see, (23) and (24) give the Lagrangian divergence and curl of the a-time-Taylor coefficient of order s in terms of lesser-order ones. We observe that the first explicit occurence of Λ in the recursion relations occurs at the fourth order, which is the same order at which the linear growth $D(a)$, at short a-times, differs from its EdS value: $D(a) = a - (2/11)\Lambda a^4 + O(a^7)$.

Eqs. (23) and (24) give a Helmholtz–Hodge decomposition of $\boldsymbol{\xi}^{(s)}$ into curl and divergence, from which it can be retrieved in a standard way (cf. e.g. Arfken & Weber 2005), namely

$$\boldsymbol{\xi}^{(s)} = \boldsymbol{\nabla}^{-2} \left(\boldsymbol{\nabla}^{\mathrm{L}} \mu_1^{(s)} - \boldsymbol{\nabla}^{\mathrm{L}} \times \boldsymbol{T}^{(s)} \right) , \tag{25}$$

where $\boldsymbol{\nabla}^{-2}$ is the inverse Laplacian in Lagrangian coordinates (taking into account the boundary conditions), and $\mu_1^{(s)}$ and $\boldsymbol{T}^{(s)}$ denote the r.h.s.'s of Eq. (23) and Eq. (24), respectively.

Using (25) and considering the first-order Lagrangian derivatives of $\xi_\nu^{(s)}$ we can combine the two recursion relations (23) and (24) into a single recursion relation for the gradient tensors $\nabla_\mu^{\mathrm{L}} \xi_\nu^{(s)}$ of the time-Taylor coefficients, which reads

$$\nabla_\mu^{\mathrm{L}} \xi_\nu^{(s)} = \nabla_\mu^{\mathrm{L}} v_\nu^{(\mathrm{init})} \delta_1^s - \Lambda \frac{s-3}{s+3/2} \mathcal{C}_{\mu\nu} \mu_1^{(s-3)} + \sum_{\substack{1 \leq j \leq 3, \\ j \neq \nu}} \mathcal{C}_{\mu j} \left(\sum_{\substack{1 \leq k \leq 3, \\ 0 < n < s}} \frac{2n - s}{s} \left(\nabla_\nu^{\mathrm{L}} \xi_k^{(n)} \right) \nabla_j^{\mathrm{L}} \xi_k^{(s-n)} \right)$$

$$+ \mathcal{C}_{\mu\nu} \left(\sum_{0<n<s} \left\{ \frac{(3-s)/2 - n^2 - (s-n)^2}{(s+3/2)(s-1)} \mu_2^{(n,s-n)} + \Lambda \frac{4s - n^2 - (s-n)^2 - 6}{(s+3/2)(s-1)} \mu_2^{(n,s-n-3)} \right\} \right.$$

$$\left. + \sum_{n_1+n_2+n_3=s} \left\{ \frac{(3-s)/2 - n_1^2 - n_2^2 - n_3^2}{(s+3/2)(s-1)} \mu_3^{(n_1,n_2,n_3)} + \Lambda \frac{4s - 9 - n_1^2 - n_2^2 - n_3^2}{(s+3/2)(s-1)} \mu_3^{(n_1,n_2,n_3-3)} \right\} \right) , \tag{26}$$

where $\mathcal{C}_{ij} \equiv \boldsymbol{\nabla}^{-2} \nabla_i^{\mathrm{L}} \nabla_j^{\mathrm{L}}$ is an operator of the Calderon–Zygmund type (cf. Zheligovsky & Frisch 2014). The recursion relations (26) are explicit but look somewhat lengthy. Actually, they enjoy an important property which allows us to prove time-analyticity in the next section: for $s > 1$, it is easily checked that all the coefficients involving s, n_1, ... are bounded (rational) functions. In the present case the bound is either unity or $\Lambda > 0$.

5   THE TIME-ANALYTICITY OF THE LAGRANGIAN SOLUTION

After having found explicit recursion relations for the time-Taylor series of the displacement field, it is natural to ask: is

$$\boldsymbol{\xi}(\boldsymbol{q}, a) = \sum_{s=1}^\infty \boldsymbol{\xi}^{(s)}(\boldsymbol{q}) a^s \tag{27}$$

a convergent series that thus defines a time-analytic function in the neighbourhood of the origin?

Let us give first some overall ideas of how the convergence results will be established. Given that the recursion relations take their most compact form (26) in terms of gradients, it is simpler to first prove the convergence for the gradient series of

$$\nabla_\mu^{\mathrm{L}} \xi_\nu(\boldsymbol{q}, a) = \sum_{s=1}^\infty \nabla_\mu^{\mathrm{L}} \xi_\nu^{(s)}(\boldsymbol{q}) a^s . \tag{28}$$

From the recursion relations (26) for the gradients of the Taylor coefficients, we shall be able to derive polynomial recursion *inequalities* for their norms. By reintroducing the a-time and summing over all orders (i.e., by using a generating function), we shall obtain a single inequality for an a-dependent cubic polynomial. The study of the evolution of its roots will give us the required bounds on the norms of the time-Taylor coefficients of the gradient series, from which the convergence of the series (28) and thus time-analyticity are established. The analyticity of the Lagrangian map and of the displacement field are then a consequence.

Let us proceed now with the details. First, we observe that the r.h.s.'s of the recursion relations (26) for the gradient tensors are themselves mostly polynomials (of degree not exceeding three) of the lesser-order gradient tensors. We write "mostly" because there are also the Calderon–Zygmund operators, $\mathcal{C}_{ij}$. In the present case, these operators stem from the nonlocal nature of the gravitational interaction. Technically speaking, these are pseudo-differential operators of degree zero (because an inverse Laplacian is compensated by two space derivatives). Essentially, such operators do not change the degree of differentiability of the functions they are applied to. But this is true only when using a suitable function space in which the Calderon–Zygmund operators are bounded.

For our purposes the function space in which such matters are simplest is the space ℓ_1 of spatially periodic functions, say, of periodicity 2π in all three space variables and such that their Fourier series is absolutely summable. Concretely, let a periodic function $\mathbf{f}(\boldsymbol{q})$ (of scalar, vector or tensor type) be expanded in a Fourier series:

$$\mathbf{f}(\boldsymbol{q}) = \sum_{\boldsymbol{k}} \hat{\mathbf{f}}_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}}\,, \tag{29}$$

where the summation is over all triplets $\boldsymbol{k}$ of signed integers. A periodic function is said to be in ℓ_1, if the norm

$$\|\mathbf{f}\| \equiv \sum_{\boldsymbol{k}} |\hat{\mathbf{f}}_{\boldsymbol{k}}| \tag{30}$$

is finite (for vector and tensor quantities, the modulus is defined as the maximum over all indices of the modulus of the various components). Since, for any i and j, one has $|k_i k_j / \boldsymbol{k}^2| \leq 1$, the Calderon–Zygmund operators are obviously bounded by unity in this space. It is also elementary to show that this space enjoys the algebra property:

$$\|\mathbf{f}\,\mathbf{g}\| \leq \|\mathbf{f}\|\|\mathbf{g}\|, \tag{31}$$

for any pair of functions $\mathbf{f}$ and $\mathbf{g}$ of finite ℓ_1 norm. Other function spaces with similar properties may be more realistic for cosmological applications in so far as periodicity is not required, such as Hölder spaces. For this we refer the reader to Section 2.5 of Zheligovsky & Frisch (2014).

Given the structure of the recursion relations (26) and using the boundedness of the Calderon–Zygmund operators and the algebra properties of the ℓ_1 norm, it is elementary to show that if $\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{v}^{(\mathrm{init})}$ is in the space ℓ_1, so will be the gradients of the Taylor coefficients $\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(s)}$. The assumption that the gradient of the initial velocity has absolutely summable Fourier coefficients is the key hypothesis of the present work.

Knowing that all the (Lagrangian) gradients of the Taylor coefficients are in ℓ_1 is not enough to ensure the convergence of the gradient Taylor series. For this, we need to obtain suitable bounds on the ℓ_1 norms of $\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(s)}$, and then we need to show that these bounds imply the convergence of the gradient time-Taylor series. Here we work with the time-Taylor series for the spatial gradient of the displacement and establish its time-analyticity, from which follows the time-analyticity of the displacement (and also, of course, of the full Lagrangian map). To obtain the bounds on the ℓ_1 norms, we shall mostly follow the approach presented in Zheligovsky & Frisch (2014). Of course, due to the presence of the cosmological constant, the third-order polynomials contain additional terms, and their study is more involved.

The first step is to bound the various $\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(s)}\|$, using the boundedness of the Calderon–Zygmund operators, the algebra property, and the boundedness of the rational coefficients, as explained at the end of section 4. We use (26) and thereby obtain, for any $s \geq 1$:

$$\left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(s)}\right\| \leq \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{v}^{(\mathrm{init})}\right\| \delta_1^s + 3\Lambda \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(s-3)}\right\| + 12 \sum_{i+j=s} \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(i)}\right\| \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(j)}\right\| + 6\Lambda \sum_{i+j=s} \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(i)}\right\| \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(j-3)}\right\|$$

$$+ 6 \sum_{i+j+k=s} \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(i)}\right\| \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(j)}\right\| \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(k)}\right\| + 6\Lambda \sum_{i+j+k=s} \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(i)}\right\| \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(j)}\right\| \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(k-3)}\right\|\,. \tag{32}$$

The second step is to introduce a generating function using the set of gradients of Taylor coefficients, namely

$$\tilde{\zeta}(a) \equiv \sum_{s=1}^{\infty} \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(s)}\right\| a^s\,. \tag{33}$$

It is easily checked that this generating function is an upper bound for the ℓ_1 (absolutely summable Fourier series) norm of the gradient of the displacement field at time a. Multiplying (32) by a^s and summing over s from one to infinity, we obtain the following inequality for the generating function:

$$\tilde{\zeta} \leq a \left\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{v}^{(\mathrm{init})}\right\| + 12\tilde{\zeta}^2 + 6\tilde{\zeta}^3 + 6\Lambda a^3 \tilde{\zeta}^2 + 6\Lambda a^3 \tilde{\zeta}^3 + 3\Lambda a^3 \tilde{\zeta}\,, \tag{34}$$

where $\tilde{\zeta}$ stands for $\tilde{\zeta}(a)$. To study this cubic polynomial inequality, it is convenient to introduce rescaled variables defined as follows:

$$\mathfrak{a} \equiv a\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{v}^{(\mathrm{init})}\|\,, \qquad \lambda \equiv \Lambda/\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{v}^{(\mathrm{init})}\|^3\,, \qquad \zeta(\mathfrak{a}) \equiv \tilde{\zeta}(a)\,. \tag{35}$$

The polynomial inequality (34) becomes then

$$p_\lambda(\mathfrak{a}, \zeta) \equiv 6\left(1 + \lambda\mathfrak{a}^3\right)\zeta^3 + 6\left(2 + \lambda\mathfrak{a}^3\right)\zeta^2 + \left(3\lambda\mathfrak{a}^3 - 1\right)\zeta + \mathfrak{a} \geq 0\,. \tag{36}$$

Sketches of the cubic polynomial $p_\lambda(\mathfrak{a}, \zeta)$ for $\lambda = 0$ and for $\lambda \neq 0$ are respectively given in Figs. 1 and 2.

The cubic polynomial $p_\lambda(\mathfrak{a}, \zeta)$ has at least one real root. For $\mathfrak{a} = 0$ it has three roots: zero, a positive root and a negative root. Since the displacement, and thus ζ, have to vanish at time $\mathfrak{a} = 0$, the only physically relevant root of the cubic polynomial is zero. Now, suppose we let the (rescaled scale-factor) time $\mathfrak{a}$ be small and positive, then the physically relevant root moves from zero to a small positive value
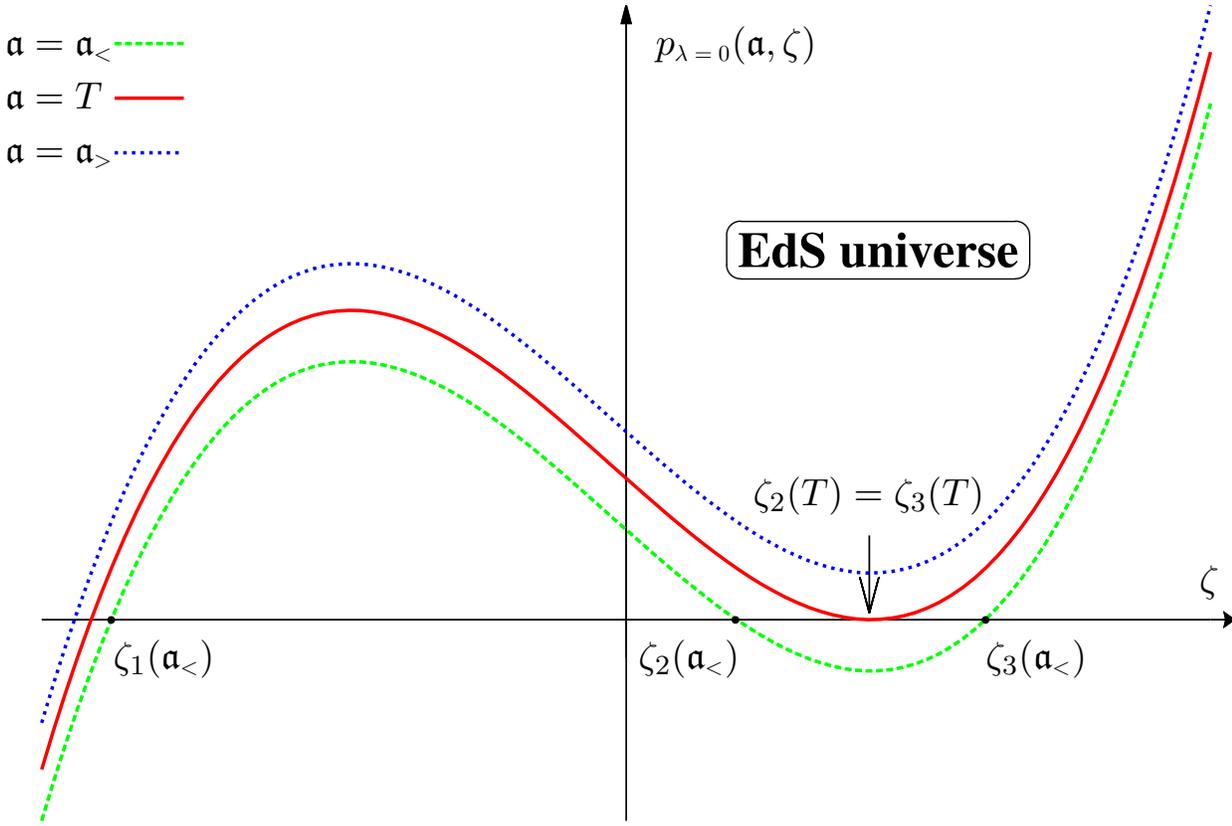
Figure 1. Sketch of the polynomial (36), for $\lambda = 0$, i.e., for the case of an EdS universe. Shown are three values of the rescaled time $\mathfrak{a}$ for which $0 < \mathfrak{a}_< < T(\lambda = 0) < \mathfrak{a}_>$, illustrating the behaviour of real roots of $p_0(\mathfrak{a}, \zeta)$ (the roots are shown as the points of intersection of the graph of $p_0(\mathfrak{a}_<, \zeta)$ and the horizontal axis). On increasing $\mathfrak{a}$, the graph slides up as a rigid curve. As a result, the roots ζ_1 and ζ_3 move to the left (i.e. become smaller), whereas ζ_2 moves to the right (i.e. becomes larger). At the critical value $T(0)$, when $\partial p_0/\partial \zeta = 0$ and the discriminant Δ vanishes, the two roots ζ_2 and ζ_3 collide and then disappear (with the emergence of a pair of complex conjugate roots).

$\zeta_2(\mathfrak{a}) \approx \mathfrak{a}$. The polynomial $p_\lambda(\zeta)$ is then strictly positive over the whole open interval $]0, \zeta_2(\mathfrak{a})[$, as required from the inequality (36), and this is the only physically relevant branch of positivity. Hence, within the interval $[0, \mathfrak{a}]$, the values of the generating function ζ are bounded by $\zeta_2(\mathfrak{a})$. As we increase the value of $\mathfrak{a}$, this boundedness property will hold as long as there are two positive roots in ζ for the polynomial $p_\lambda(\mathfrak{a}, \zeta)$. This is true until the two positive roots merge into a double root and then turn into complex roots. The value of the rescaled time for which (36) has a double root are the zeros of its discriminant, i.e.,

$$\Delta(\mathfrak{a}, \lambda) \equiv 12 \left(14 - 684\mathfrak{a} - 81\mathfrak{a}^2 - 76\mathfrak{a}^3\lambda - 702\mathfrak{a}^4\lambda - 162\mathfrak{a}^5\lambda + 75\mathfrak{a}^6\lambda^2 - 81\mathfrak{a}^8\lambda^2 + 90\mathfrak{a}^9\lambda^3 + 90\mathfrak{a}^{10}\lambda^3 - 27\mathfrak{a}^{12}\lambda^4 \right) = 0. \tag{37}$$

Because $\Delta(0, \lambda) > 0$ and the highest-order term, proportional to $\mathfrak{a}^{12}$ has a negative coefficient, it immediately follows by continuity that the discriminant equation has at least one positive root. For our purposes we need the *smallest* of such positive roots, since it is an upper bound for the physically relevant branch of $p_\lambda(\zeta) \geq 0$:

$$T(\lambda) \equiv \inf_i \mathfrak{a}_i \mid \Delta(\mathfrak{a}_i, \lambda) = 0, \tag{38}$$

called here the critical time. (In Section 6.1 we shall show how $T(\lambda)$ can be calculated.)

The importance of the critical time lies in the following result: *for any* $|\mathfrak{a}| < T(\lambda)$ *the time-Taylor series for the gradient of the displacement (now expressed in terms of the rescaled time) is convergent.* (Here $|\mathfrak{a}|$ denotes the modulus of the rescaled time $\mathfrak{a}$ which can now take also complex values.) Indeed, it follows from (33) and (35) that

$$\zeta(\mathfrak{a}) = \sum_{s=1}^{\infty} \zeta^{(s)} \mathfrak{a}^s, \tag{39}$$

where $\zeta^{(s)} = \|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(s)}\|/\|\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{v}^{(\mathrm{init})}\|^s \geq 0$. If $\zeta(T)$ is bounded by some constant C, we have

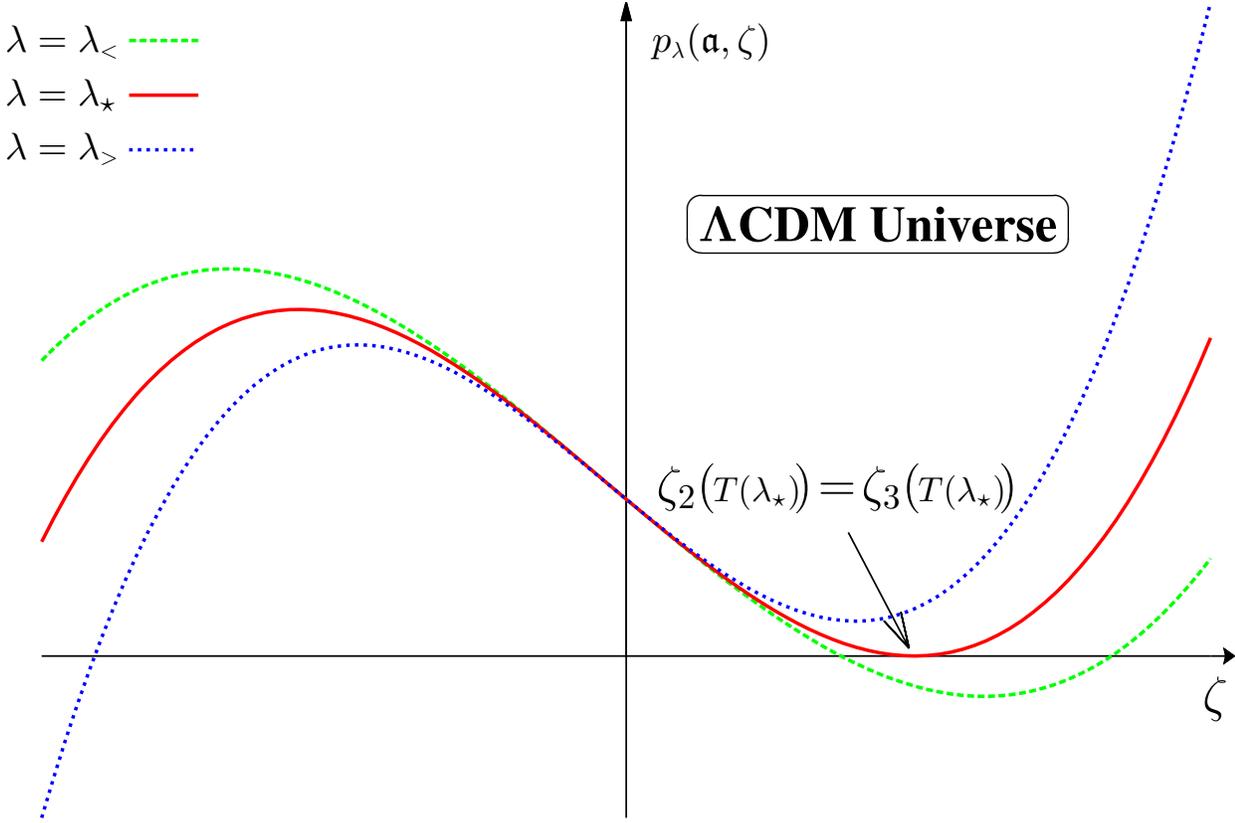$$\zeta(T) = \sum_{s=1}^{\infty} \zeta^{(s)} T^s \leq C. \tag{40}$$

Figure 2. Sketch of the polynomial (36) for any fixed rescaled time variable $\mathfrak{a} > 0$ and $\lambda \neq 0$ (i.e., the ΛCDM case). Three values of lambda are shown for which $\lambda_< < \lambda_\star < \lambda_>$. The value $\lambda_\star$ denotes the critical value for which the cubic polynomial p_λ develops a double root, i.e., where $\zeta_2(T(\lambda)) = \zeta_3(T(\lambda))$. In that case the fixed value of the rescaled time $\mathfrak{a}$ is precisely the critial value $T(\lambda_\star)$ for which the generating function ζ of the displacement is bounded. On increasing λ for any fixed rescaled time $\mathfrak{a} > 0$, the graph gets shifted into the upper left direction.

Since all the terms in the sum are non-negative, it follows that

$$\zeta^{(s)} \leq CT^{-s}, \tag{41}$$

and thus

$$\zeta(\mathfrak{a}) \leq C \sum_{s=1}^{\infty} \left(\frac{\mathfrak{a}}{T}\right)^s. \tag{42}$$

For $|\mathfrak{a}| < T$ the Taylor series (42) is bounded by a convergent geometric series. We have thus shown the convergence of the Taylor series in the complex time plane inside a disk, centered at the origin, $\mathfrak{a} = 0$ of radius at least $T(\lambda)$. The actual radius of convergence of the Taylor series, called the *radius of analyticity*, for which $T(\lambda)$ is only a lower bound, in general, can only be determined numerically.

6   FURTHER RESULTS ON ANALYTICTY

6.1   The lower bound on the radius of analyticity and its dependence on the cosmological constant

The lowest positive root $T(\lambda)$ of the discriminant equation (37) gives us a lower bound on the radius of analyticity of the time-Taylor series for the Lagrangian map. This discriminant equation is of twelfth degree in the rescaled time variable $\mathfrak{a} = a\|\boldsymbol{\nabla}^{\mathrm{L}} \boldsymbol{v}^{(\mathrm{init})}\|$ and we have not been able to solve it explicitly by radicals. We can however solve the discriminant equation perturbatively for small and large rescaled cosmological constant $\lambda = \Lambda/\|\boldsymbol{\nabla}^{\mathrm{L}} \boldsymbol{v}^{(\mathrm{init})}\|^3$ and numerically for other values.

For $\lambda = 0$, we recover the results for an EdS universe (Zheligovsky & Frisch 2014). In that case the discriminant equation reduces to $\Delta(\mathfrak{a}, 0) = 12(14 - 684\mathfrak{a} - 81\mathfrak{a}^2) = 0$, whose lowest positive root is

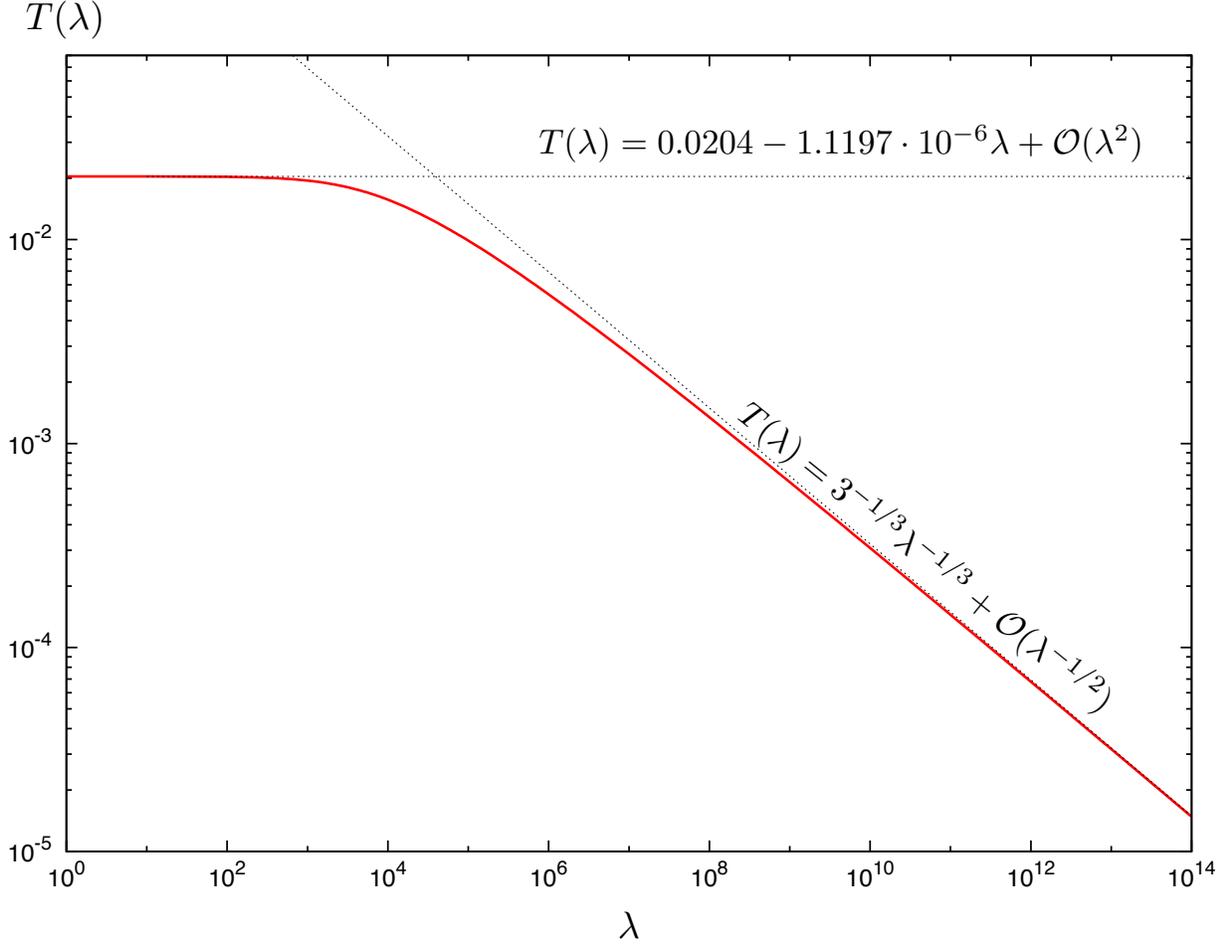$$T^{\mathrm{EdS}} \equiv T(0) = 3\sqrt{2} - 38/9 \simeq 0.0204. \tag{43}$$

Figure 3. Numerical solution of the discriminant equation (37). The results agree with the small- and large-λ expansion (44) and (47) in the appropriate ranges.

For small λ, it is then straightforward to obtain the λ-dependence of this root

$$T(\lambda) = T^{\rm EdS} - \left(T^{\rm EdS}\right)^3 \frac{52 - 333T^{\rm EdS}}{342 + 81T^{\rm EdS}} \lambda + O(\lambda^2) \simeq 0.0204 - 1.1197 \cdot 10^{-6}\lambda + O(\lambda^2) \,. \tag{44}$$

For large positive λ, we make the changes of variable $\tilde{\mathfrak{a}} \equiv \mathfrak{a}\lambda^{1/3}$ and $\tilde{\Delta}(\tilde{\mathfrak{a}}, \lambda) \equiv \Delta(\mathfrak{a}, \lambda)/12$, and rewrite the discriminant equation as

$$\tilde{\Delta}(\tilde{\mathfrak{a}}, \lambda) = P_4(\tilde{\mathfrak{a}}^3) + \lambda^{-1/3}P_{10}(\tilde{\mathfrak{a}}) + \lambda^{-2/3}P_8(\tilde{\mathfrak{a}}) = 0 \,, \tag{45}$$

where

$$P_4(\tilde{\mathfrak{a}}^3) = 14 - 76\tilde{\mathfrak{a}}^3 + 75\tilde{\mathfrak{a}}^6 + 90\tilde{\mathfrak{a}}^9 - 27\tilde{\mathfrak{a}}^{12} \,, \quad P_{10}(\tilde{\mathfrak{a}}) = -684\tilde{\mathfrak{a}} - 702\tilde{\mathfrak{a}}^4 + 90\tilde{\mathfrak{a}}^{10} \,, \quad P_8(\tilde{\mathfrak{a}}) = -81\tilde{\mathfrak{a}}^2 - 162\tilde{\mathfrak{a}}^5 - 81\tilde{\mathfrak{a}}^8 \,. \tag{46}$$

The polynomial $P_4(\tilde{\mathfrak{a}}^3)$ has a positive double root at $\tilde{\mathfrak{a}}_c = 3^{-1/3}$, from which follows that the dominant behaviour of $T(\lambda)$ is proportional to $3^{-1/3}\lambda^{-1/3}$. The first subdominant correction is obtained by Taylor expanding the polynomial P_4, P_{10} and P_8 around $\tilde{\mathfrak{a}}_c$. This eventually reveals the following large-λ behaviour:

$$T(\lambda) = 3^{-1/3}\lambda^{-1/3} + O(\lambda^{-1/2}) \,. \tag{47}$$

We have solved the discriminant equation (37) numerically for more than one hundred values of λ, suitably distributed between 10^{-2} and 10^{14}. The results are shown in Fig. 3 and agree with the small- and large-λ expansion (44) and (47) in the appropriate ranges. Note that $T(\lambda)$ is a monotonically decreasing function of λ.

Finally, we observe that, within the functional framework of the space ℓ_1 of absolutely summable Fourier series, the precise values of the constants appearing in the low-λ expansion (44) and the large-λ expansion (47) can most likely be improved. This will result in longer time intervals of guaranteed analyticity. For this one should adapt to the cosmological context the detailed Fourier-space estimates found, for incompressible flow, in Section 2.3 of Zheligovsky & Frisch (2014).

6.2 Shell-crossing and analyticity

In section 3, we pointed out that the Lagrangian formulation and the subsequent time-Taylor expansions as carried out in the present paper, are invalid if the Jacobian J vanishes during the interval of guaranteed analyticity. The vanishing of the Jacobian corresponds to the crossing of various fluid-particle trajectories. It is generally known as "shell-crossing". We now show that during the time interval of guaranteed analyticity, no shell-crossing can occur. Actually, denoting as usual the Jacobian of the Lagrangian map by J, we shall show that $|J - 1| \leq -37/9 + 3\sqrt{2} \approx 0.132$, which trivially implies the non-vanishing of the Jacobian (and not even a close call).

For this, we observe that the Jacobian can be written as follows in terms of the displacement field

$$J = \det(I + \boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}) = 1 + \boldsymbol{\nabla}^{\mathrm{L}} \cdot \boldsymbol{\xi} + \sum_{1 \leq i < j \leq 3} \left[(\nabla_i^{\mathrm{L}}\xi_i)\nabla_j^{\mathrm{L}}\xi_j - (\nabla_i^{\mathrm{L}}\xi_j)\nabla_j^{\mathrm{L}}\xi_i \right] + \det(\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}) , \tag{48}$$

where I denotes the identity matrix. Using the Taylor expansion (16) of the displacement in powers of the a-time and the generating function $\zeta = \sum_{s=1}^{\infty} |\boldsymbol{\nabla}^{\mathrm{L}}\boldsymbol{\xi}^{(s)}| a^s$, we obtain the following bound

$$|J - 1| \leq 6\zeta^3 + 6\zeta^2 + 3\zeta \equiv Q(\zeta) . \tag{49}$$

For convenience we have plotted the cubic polynomial $Q(\zeta)$ in Fig. 4.

As we have seen in Section 5, for any given rescaled cosmological constant λ and rescaled time $\mathfrak{a}$, the permitted values of the generating function ζ are between zero and the smallest positive root $\zeta_2(\mathfrak{a})$ of the cubic polynomial $p_\lambda(\zeta, \mathfrak{a})$, given by (36). We claim that this root is an increasing function of $\mathfrak{a}$ for fixed λ. Indeed, $p_\lambda(\mathfrak{a}, \zeta_2(\mathfrak{a})) = 0$, over the whole range of relevant $\mathfrak{a}$ values. By differentiating it with respect to $\mathfrak{a}$, we obtain:

$$\frac{\partial p_\lambda(\mathfrak{a}, \zeta_2(\mathfrak{a}))}{\partial \mathfrak{a}} = \frac{\partial \zeta_2(\mathfrak{a})}{\partial \mathfrak{a}} \left. \frac{\partial p_\lambda(\mathfrak{a}, \zeta)}{\partial \zeta} \right|_{\zeta = \zeta_2(\mathfrak{a})} + 1 + 3\lambda\mathfrak{a}^2 \left(6\zeta_2^3 + 6\zeta_2^2 + 3\zeta_2 \right) = 0 . \tag{50}$$

Since $[1 + 3\lambda\mathfrak{a}^2(6\zeta_2^3 + 6\zeta_2^2 + 3\zeta_2)] > 0$ for $\zeta_2 \geq 0$, we infer that

$$\left. \frac{\partial \zeta_2(\mathfrak{a})}{\partial \mathfrak{a}} \frac{\partial p_\lambda(\mathfrak{a}, \zeta)}{\partial \zeta} \right|_{\zeta = \zeta_2(\mathfrak{a})} < 0 . \tag{51}$$

The positivity of $\frac{\partial \zeta_2(\mathfrak{a})}{\partial \mathfrak{a}}$ follows from $\left. \frac{\partial p_\lambda(\mathfrak{a}, \zeta)}{\partial \zeta} \right|_{\zeta = \zeta_2(\mathfrak{a})} < 0$. The latter is a consequence of the fact that the polynomial $p_\lambda(\mathfrak{a}, \zeta)$ has three real roots, for fixed λ and fixed $\mathfrak{a}$ in the range of analyticity. Inspection of (36) shows that their product is negative, that one (ζ_1) is negative and that two (ζ_2 and ζ_3) are positive. The derivative $\frac{\partial p_\lambda(\mathfrak{a}, \zeta)}{\partial \zeta}$ is obviously positive for large $\zeta > 0$ and changes sign somewhere between ζ_2 and ζ_3. Since $\zeta_2 < \zeta_3$, this derivative is negative at ζ_2. This proves the monotonic increase of ζ_2 with $\mathfrak{a}$.

Next, we use (49) to bound $|J - 1|$ by $6\zeta_2^3 + 6\zeta_2^2 + 3\zeta_2$. The largest possible value of ζ_2 is obtained at the critical value $\mathfrak{a} = T(\lambda)$ where the polynomial $p_\lambda(\mathfrak{a}, \zeta)$ has a double root in ζ. This value can be obtained by demanding the simultaneous vanishing of $p_\lambda(\mathfrak{a}, \zeta)$ and of its derivative with respect to ζ, i.e., $p_\lambda(\zeta_2) = p'_\lambda(\zeta_2) = 0$ and $\zeta_2 \geq 0$, where a prime denotes a partial derivative with respect to ζ. From these conditions we obtain

$$\zeta_2(T(\lambda)) = \zeta_3(T(\lambda)) = \frac{-4 - 2\lambda T^3 + \sqrt{2}\sqrt{9 + \lambda T^3 (6 - \lambda T^3)}}{6 + 6\lambda T^3} . \tag{52}$$

Since this is a decreasing function of λ, it is bounded by its EdS value ($\lambda = 0$), where the latter is $\zeta_2 = -\frac{2}{3} + \frac{1}{6}\sqrt{18} \simeq 0.04044$. It follows that $|J - 1| < 0.132$, which is well below the value unity where shell-crossing might occur (see the vertical thick line in Fig. 4).

6.3 Analyticity in the linear growth function D

As already stated in Section 2, it could be of interest in cosmological studies to use as time variable, not the cosmic scale factor a, but the linear growth function $D(a)$, which is the growing solution of (7). With the normalisation $\lim_{a \to 0} D(a)/a = 1$, the linear growth function is given by

$$D(a) = a\sqrt{1 + \Lambda a^3} \, {}_2F_1\left(\frac{3}{2}, \frac{5}{6}, \frac{11}{6}, -\Lambda a^3 \right) , \tag{53}$$

where $_2F_1$ is the Gauss hypergeometric function. This solution is analytic around $a = 0$ and has the small-a expansion $D(a) = a - (2/11)\Lambda a^4 + O(a^7)$. (The full expression can be obtained from the Taylor expansion of the Gauss hypergeometric function, cf. Gradshteyn & Ryzhik (1965), pp. 1039–1045.) As $D(a)$ is analytic around $a = 0$ and also its derivative $\partial D/\partial a$ does not vanish at $a = 0$, it follows that $D(a)$ is invertible at around $a = 0$. The inverse linear growth function $a(D)$ is also analytic and has the following low-order expansion: $a(D) = D + (2/11)\Lambda D^4 + O(D^7)$.

We have shown in Section 5 that the Lagrangian map $\boldsymbol{x}(\boldsymbol{q}, a)$ is an analytic function of a, near $a = 0$. The composition $\boldsymbol{x}(\boldsymbol{q}, a(D))$ of the Lagrangian map with the inverse linear growth function is also analytic in D. (This follows basically from the observation that analytic functions are complex-differentiable functions and the use of the chain rule.) We remark that this argument is not valid when the cosmic time is chosen as time variable, instead of D, as the relation between a and t is not analytic near the origin.

Can we obtain simple recursion relations for the D-Taylor coefficients of the displacement $\boldsymbol{\xi}$, as we have done in Section 4? Recursion relations, yes, but simple, no. Indeed, from (14)–(15), we can derive a set of Lagrangian Euler–Poisson equations in the time variable D. However the D-dependence is then essentially not polynomial, but will involve the full inverse linear growth function $a(D)$, whose Taylor
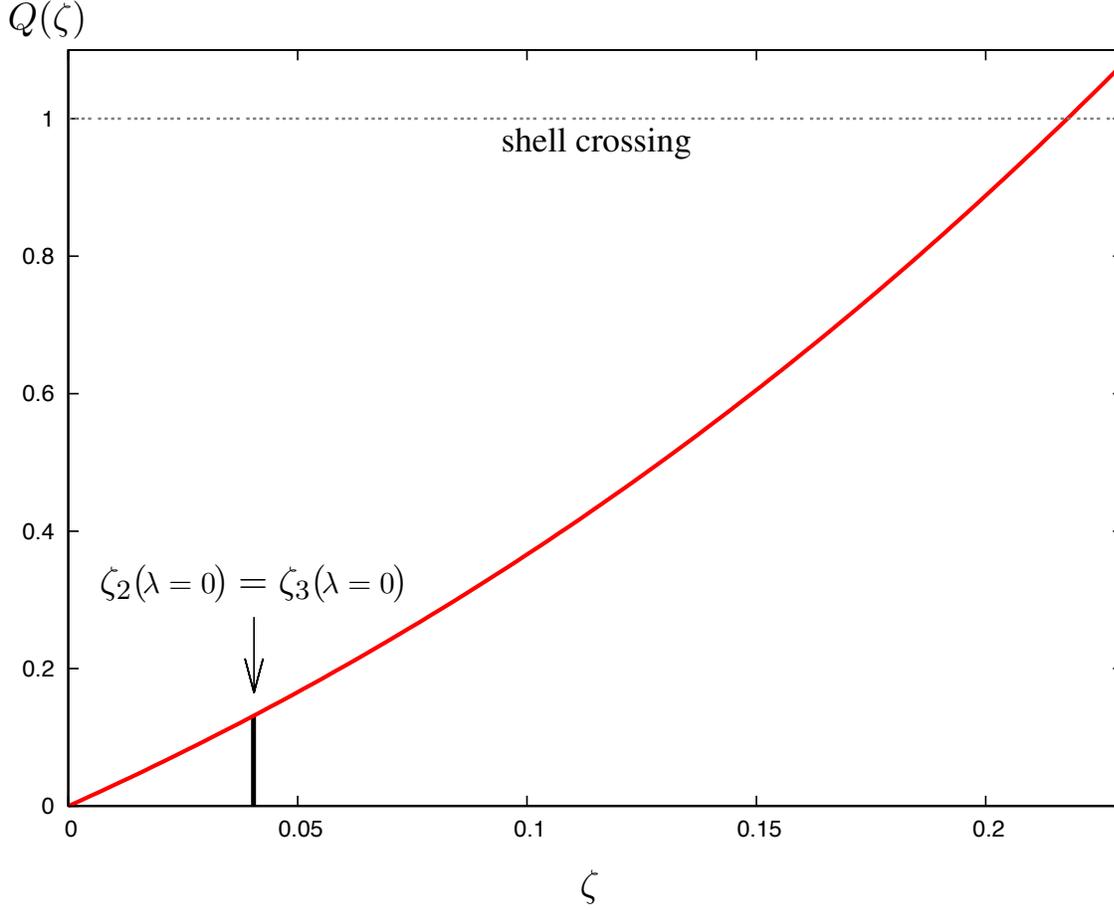
Figure 4. Lower bound on the norm of $|J - 1| \leq Q(\zeta) = 6\zeta^3 + 6\zeta^2 + 3\zeta$, see Eq. (49). The vertical thick line at roughly $\zeta = 0.04044$ indicates the largest possible value of the smallest positive root $\zeta_2(T(\lambda = 0))$ of the polynomial $p_\lambda(\mathfrak{a}, \zeta)$ for which analyticity is guaranteed. Since $\zeta_2(T(\lambda))$ is a decreasing function of λ (see (52)), it follows that $|J - 1| < 0.132$ for any value of $\Lambda > 0$, which is well below the value unity where shell-crossing might occur (the dotted line).

expansion has an infinite number of terms. As a consequence, the recursion relations will be very involved. If one really wants to reexpress the displacement as a power series in D, it is much simpler to just substitute the Taylor series for $a(D)$, obtained by inversion of the Taylor series of $D(a)$ into the expansion (16) in powers of a. Note that, with the latter strategy, there is no need to solve a succession of time differential equations for the nth order growth function (usually denoted with $E, F, \dots$), because the actual expansion parameter is precisely the cosmic scale factor, used as a time variable.

6.4   Effect on analyticity of the inclusion in the Friedmann equation of curvature and radiation terms

Inflation theory and observational evidence since the late nineties favour a flat Universe with zero curvature (see, e.g., Linde 1984 and Boomerang Collaboration *et al.* 2002). It is nevertheless of interest to point out that our analyticity result for a ΛCDM Universe survives when a small amount of curvature is introduced in the Friedmann equation (3). The latter then becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = a^{-3} + ka^{-2} + \Lambda\,, \tag{54}$$

where k is here not limited to being 0, ± 1 (although according to Mukhanov (2005), the only trivial background geometries which do not spoil exact homogeneity and isotropy are indeed the ones with curvature 0, ± 1). We have repeated the analysis of Sections 2–5, with this modified Friedmann equation and found that very little is changed. The slaving conditions (10) are unchanged. The recursion relations (26) are minimally modified. The polynomial inequality (36) in the generating function ζ takes now the following form in rescaled variables:

$$P_{\lambda,\mathfrak{K}}(\mathfrak{a}, \zeta) \equiv 6\left(1 + \lambda\mathfrak{a}^3 + 2\mathfrak{K}\mathfrak{a}\right)\zeta^3 + 6\left(2 + \lambda\mathfrak{a}^3 + \frac{8}{33}\mathfrak{K}\mathfrak{a}\right)\zeta^2 + \left(3\lambda\mathfrak{a}^3 + \frac{12}{7}\mathfrak{K}\mathfrak{a} - 1\right)\zeta + \mathfrak{a} \geq 0\,, \tag{55}$$

where $\mathfrak{K} = |k|/\|\boldsymbol{\nabla}^{\mathrm{L}} \boldsymbol{v}^{(\mathrm{init})}\|$, and $\mathfrak{a} = a\|\boldsymbol{\nabla}^{\mathrm{L}} \boldsymbol{v}^{(\mathrm{init})}\|$ and $\lambda = \Lambda/\|\boldsymbol{\nabla}^{\mathrm{L}} \boldsymbol{v}^{(\mathrm{init})}\|^3$ as before. Analyticity in the (rescaled) cosmic scale factor $\mathfrak{a}$ is proved as before, by studying the discriminant equation associated to the cubic polynomial $P_{\lambda,\mathfrak{K}}(\mathfrak{a}, \zeta)$. Indeed, this is again an equation of twelfth degree in $\mathfrak{a}$, whose lowest and highest degree terms, unchanged from the ΛCDM case, are respectively positive and negative, thus implying the existence of at least one positive root.

All this is not surprising: Actually, the curvature term in the Friedmann equation involves the inverse second power of the cosmic scale factor, which for $a \to 0$ is subdominant with respect to the matter term a^{-3} and can thus not bring about much change. Very different would be the inclusion in the background of *radiation effects*, stemming from a black-body distribution of photons. This adds a term in the Friedmann equation, proportional to the inverse fourth power of the scale factor, which drastically changes everything: slaving is lost, analyticity around $a = 0$ is lost, etc. The current favoured ΛCDM model has a background radiation density well below the matter density at decoupling and can actually be discarded after decoupling. When relating our results to numerical results or observations, it may be more convenient to use the cosmic time t rather than the a-time. Of course, when performing that change of variable, the radiation term in the Friedmann equation does not need to be omitted.

7 CONCLUDING REMARKS

In a Eulerian framework space and time are intertwined by Galilean transformations (in the Newtonian approximation). It follows that, with the limited spatial smoothness assumed in the present work – roughly, that the initial density fluctuations are slighly better than continuous in the spatial variables – one cannot obtain better than a corresponding limited temporal smoothness. In a Lagrangian framework, as we have seen, the situation is drastically different: provided we use an appropriate time variable, such as the cosmic scale factor a or the linear growth function D. Obtaining analyticity in time of the Lagrangian trajectories requires then only a limited initial spatial smoothness.

Even more surprising is that this analyticity is around the point $a = 0$, which seems to extend this utterly smooth birth of our structured Universe to the very origin of time, well before decoupling of matter from photons. Of course, this is just a property of the mathematical model used here to describe structure formation, namely the Euler–Poisson equations in a ΛCDM Universe. The interesting feature is that this model allows us to have a well-posed problem, devoid of catastrophic behaviour near decoupling, provided we use the *slaving conditions* (10) at $a = 0$. These slaving conditions resemble those used to start N-body simulations at early times, (i) small density fluctuations, (ii) an initial curlfree peculiar velocity field, related to the gravitational potential as in the Zel'dovich approximation. That these slaving conditions can be extended all the way to $a = 0$ means, from the point of view of boundary layer analysis (Cole 1968), that the matter-dominated era constitutes an outer solution for which, to leading order, the boundary matching to the inner solution (describing the primordial era) can be replaced by just an initial condition. This is true as long as, in the Friedmann equation for the background evolution in the matter-dominated era, we include only a matter term, a cosmological term and a possible curvature term, but no radiation term.

We remind the reader that our way of proving analyticity in the a-time is fully constructive. It rests indeed on a set of novel all-order recursion relations (26) for the time-Taylor coefficients of the displacement of (fluid) particles. Not only are these relations fully explicit, but they have a very specific mathematical structure, allowing us to obtain bounds for all the Taylor-coefficients and thus to establish analyticity by elementary means.

We now briefly indicate some of the possible future developments exploiting the analyticity result and the recursion relations. We shall mention two interlinked topics. One is about numerical integration of the Euler–Poisson equations by a multi-step technique and the other one about cosmological reconstruction.

The Taylor method as described in Sections 4 and 5 allows us to determine the solution of the Euler–Poisson equations for any time $0 < a < R(0)$, where $R(0)$ is the radius of convergence of the Taylor series around $a = 0$. (Here, it is best not to work with the rescaled time $\mathfrak{a}$.) A procedure, inspired from the Weierstrass analytic continuation technique for analytic functions, may allow us to obtain the solution beyond the time $R(0)$. For this, we select a sequence of times $0 < a_1 < a_2 < a_3 < \ldots$ in such a way that a_{n+1} is within the disk of convergence of the time-Taylor series around the time a_n, that is such that $|a_{n+1} - a_n| < R(a_n)$, where $R(a)$ is the radius of convergence of the Taylor expansion of the solution around time a. What we here call the solution is not just the Lagrangian map, but includes also the density which is controlled by the inverse of the Jacobian. Indeed, if the Jacobian vanishes, shell-crossing takes place and the Lagrangian map ceases to be invertible. Invertibility is essential, because at the end of each step, we must revert to a Eulerian description to be able to make a fresh start. Methods combining a Lagrangian step with a reversion to Eulerian coordinates are called semi-Lagrangian. A detailed description of a semi-Lagrangian method, called the Cauchy–Lagrangian method, which makes use of Cauchy's Lagrangian formulation of the ideal fluid dynamics can be found in Podvigina *et al.* (2015).

When dealing with the cosmological Euler–Poisson equations, the method of Podvigina *et al.* (2015) must be suitably adapted. One of the new difficulties stem from the non-autonomous character of our Euler–Poisson equations: the time appears explicitly in the equation. As a consequence, the Euler–Poisson equations are not invariant under a-time translations, but this problem can be circumvented.

In so far as the regime described by the Euler–Poisson equations does not include multi-streaming, a Cauchy–Lagrangian numerical method will not be able to address the same questions as N-body simulation techniques. There is however one important problem for which it appears well suited, namely the reconstruction of the dynamical history of the Universe from present-epoch galactic surveys (Frisch *et al.* 2002). It has been shown by Brenier *et al.* (2003) that the solution to the Euler–Poisson equations is uniquely determined by two boundary conditions, the slaving conditions (10) at the initial time and the density of matter at the current epoch. The latter is obtained from large

galactic surveys. Euler–Poisson reconstruction is an extension of the variational N-body reconstruction, introduced by Peebles (1989) for handling galaxy data on relatively small scales, such as that of the Local Group. Euler–Poisson reconstruction, which is meant for significantly larger scales where a continuum description is applicable and where multi-streaming can be ignored, does not suffer from the non-uniqueness problem of the variational N-body reconstruction. So far, Euler–Poisson reconstruction has not used the full solutions of the Euler–Poisson equations but just low-order Lagrangian perturbation approximations, such as the Zel'dovich approximation or the next order, denoted by 2LPT. Within such approximate frameworks, reconstruction becomes a Monge optimal transport problem (Frisch *et al.* 2002) with quadratic cost, which, after discretisation, can be solved by very efficient assignment algorithms such as the auction method of Bertsekas (1992). Alternatively, one can use Brenier's theorem (Brenier 1987) and solve an equivalent 3-D Monge–Ampère equation nonlinear PDE by iterative techniques (Zheligovsky, Podvigina & Frisch 2010). Reconstruction by such methods is known as Monge–Ampère–Kantorovich (MAK). We propose to use the MAK reconstruction as a starting point of full Euler–Poisson reconstruction, applying an iterative Newton-type method in which at each stage a Euler–Poisson initial value problem is solved by a Cauchy–Lagrangian method.

Finally, we remind the reader that our proofs were given entirely within a Newtonian framework. In that framework the time-analyticity results revealed a strong asymmetry between space and time in Lagrangian coordinates. The reason for that strong asymmetry is the "missing" convective term in the Newtonian fluid equations, when formulated in Lagrangian coordinates. (In the notation of our Eulerian approach in Section 2, the convective term is $\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\boldsymbol{v}$.) An interesting question is whether our time-analyticity results could be generalised to a Lagrangian coordinates formulation of a curlfree dust-fluid in General Relativity. It is well-known that the so-called synchronous-comoving coordinate system corresponds to a relativistic Lagrangian frame of reference, and, similar to the Lagrangian coordinates approach in the Newtonian framework, there is indeed no convective term in the relativistic equations of motion (in the so-called ADM approach; see e.g. Matarrese & Terranova 1996). These relativistic fluid equations are however decorated with Ricci-tensors that involve (single and double) spatial gradients acting on the metric (and on its inverse), and this could well imply a limited temporal smoothness for limited spatial smoothness. Besides that unsolved problem, the essential requirement for a time-analyticity result in General Relativity relies on having explicit all-order recursion relations, which are yet unkown. A possible starting point for such investigations could be the Lagrangian equations recently obtained by Alles *et al.* (2015).

REFERENCES

Alles A., Buchert T., Roumi F.A., Wiegand A., 2015, preprint (arXiv:1503.02566)
Arfken G.B., Weber H.J., 2005, Mathematical Methods for Physicists, Elsevier, Amsterdam
Bernardeau F., 2013, Les Houches lecture notes, volume 100, Oxford University Press, p. 17
Bernardeau F., Colombi S., Gaztanaga E., Scoccimarro R., 2002, Phys. Rep., 367,1
Bertsekas D.P., 1992, Comput. Optim. Appl., 1, 7
Boomerang Collaboration *et al.*, 2002, ApJ, 571, 604
Bouchet F.R., Juszkiewicz R., Colombi S., Pellat R., 1992, ApJ, L5–L8
Bouchet F.R., Colombi S., Hivon E., Juszkiewicz R., 1995, A&A, 296, 575
Brenier Y., 1987, C.R. Acad. Sci. Paris I, 305, 805
Brenier Y., Frisch U., Hénon M., Loeper G., Matarrese S., Mohayaee R., Sobolevski A., 2003, MNRAS, 346, 501
Buchert T., 1989, A&A, 223, 9
Buchert T., 1992, MNRAS, 254, 729
Buchert T., 1995, Proc. In Varenna 1995, Dark matter in the universe, 543
Buchert T., Goetz G., 1987, J. Math. Phys., 28, 2714
Buchert T., Nayet C., Wiegand A., 2013, Phys. Rev. D, 87, 12, 123503
Buchert T., Ostermann M., 2012, Phys. Rev. D, 86, 023520
Catelan P., 1995, MNRAS, 276, 115
Cole J.D., 1968, Perturbation Methods in Applied Mathematics, Ginn/Blaisdell, Waltham
Crocce M., Pueblas S., Scoccimarro R., 2006, MNRAS, 373, 369
Demianski M., Golda Z.A., Woszczyna A., 2005, Gen. Rel. Grav., 37, 2063
Ehlers J., Buchert T., 1997, Gen. Rel. Grav., 29, 733
Enqvist K., Rigopoulos G., 2010, J. Cosmol. Astropart. Phys., 1212, 005
Frisch U., Matarrese S., Mohayaee R., Sobolevski A., 2002, Nature, 417, 260
Frisch U., Villone B., 2014, Eur. Phys. J. H, 39, 325
Goroff M.H., Grinstein B., Rey S.J., Wise M.B., 1986, ApJ, 311, 6
Gradshteyn I.S., Ryzhik I.M., 1965, Table of integrals series and products, Academic Press (A. Jeffrey, ed.), fourth edition, New York, p 1039 - 1045
Hamber H.W., Toriumi R., 2010, Phys. Rev. D, 82, 043518
Kofman L.A., Gnedin N.Y., Bahcall N.A., 1993, ApJ., 413, 1

Lee S., 2014, Eur. Phys. J. C, 74, 11, 3146
Linde A.D., 1984, Rept. Prog. Phys., 47, 925
Matarrese S., Terranova D., 1996, MNRAS, 283, 400
Matsubara T., 1995, Prog. Theor. Phys., 94, 1151
Matsubara T., 2008a, Phys. Rev. D, 77, 063530
Matsubara T., 2008b, Phys. Rev. D, 78, 083519
Mo H.J., White S.D.M., 1996, MNRAS, 282, 347
Moutarde F., Alimi J.M., Bouchet F.R., Pellat R., Raman A., 1991, ApJ, 382, 377
Mukhanov V., 2005, Physical foundations of cosmology, Cambridge University Press, Cambridge, UK, 421 pp
Peebles P.J.E., 1980, The Large-Scale Structure of the Universe, Princeton Univ. Press, Princeton, NJ
Peebles P.J.E., 1989, ApJ, 344, L53
Planck Collaboration *et al.*, 2014, A&A, 571, A16
Podvigina O., Zheligovsky V., Frisch U., 2015, to be submitted to J. Comput. Phys.
Rampf C., 2012, J. Cosmol. Astropart. Phys., 1212, 004
Rampf C., 2013, Phys. Rev. D, 89, 063509
Rampf C., Buchert T., 2012, J. Cosmol. Astropart. Phys., 1206, 021
Rampf C., Rigopoulos G., 2013, MNRASL, 430, L54-L58
Rampf C., Wiegand A., 2014, Phys. Rev. D, 90, 12, 123503
Rampf C., Wong Y.Y.Y., 2012, J. Cosmol. Astropart. Phys., 1206, 018
Sahni S., Shandarin V., 1996, MNRAS, 282, 641
Vlah Z., Seljak U., Baldauf T., 2014, Phys. Rev. D, 91, 2, 023508
Zel'dovich Ya.B., 1970, A&A, 5, 84
Zheligovsky V., Frisch U., 2014, J. Fluid Mech., 749, 404
Zheligovsky V., Podvigina O., Frisch U., 2010, J. Comput. Phys., 229, 5043

APPENDIX A: DERIVATION OF THE PECULIAR EULER–POISSON EQUATIONS WITH COSMOLOGICAL CONSTANT

Here, although entirely elementary, we highlight the derivation of the peculiar Euler–Poisson equations (5a–5c) for the sake of completeness. We begin with the (full) Euler–Poisson equations (1a–1c), which we repeat here for convenience:

$$\partial_t \tilde{U} + \left(\tilde{U} \cdot \nabla_r \right) \tilde{U} = -\nabla_r \tilde{\phi}_g \,, \tag{A1}$$

$$\partial_t \tilde{\varrho} + \nabla_r \cdot \left(\tilde{\varrho} \tilde{U} \right) = 0 \,, \tag{A2}$$

$$\nabla_r^2 \tilde{\phi}_g = 4\pi G \tilde{\varrho} - 3\Lambda \,. \tag{A3}$$

Here $\tilde{U} = \tilde{U}(r, t)$ denotes the velocity field, $\tilde{\phi}_g = \tilde{\phi}_g(r, t)$ denotes the cosmological potential, $\tilde{\varrho} = \tilde{\varrho}(r, t)$ is the fluid density, and here the cosmological constant is 3Λ. Let us solve these equations for the *background variables* in an exactly homogeneous and isotropic Universe, where $\tilde{\varrho} = \bar{\tilde{\varrho}}(t)$, $\tilde{U} = \tilde{H}(t)\, r \equiv (\dot{a}/a)r$, and $\tilde{\phi}_g = \bar{\tilde{\phi}}_g$. We substitute these expressions into the above equations. For the divergence of Eq. (A1), we then have

$$3\dot{\tilde{H}} + 3\tilde{H}^2 = -\nabla_r^2 \bar{\tilde{\phi}}_g \,, \tag{A4}$$

whereas for Eq. (A3), we have at the background level

$$\nabla_r^2 \bar{\tilde{\phi}}_g = 4\pi G \bar{\tilde{\varrho}} - 3\Lambda \,. \tag{A5}$$

By substituting the last expression into Eq. (A4), we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \bar{\tilde{\rho}} + \Lambda \,. \tag{A6}$$

This is the usual form of the second Friedmann equation for a pressureless fluid. In the units after Eq.(3) in the main text, the above is $\ddot{a}/a = -a^{-3}/2 + \Lambda$, whose solution is $a(t) = \Lambda^{-1/3}[\sinh(3\sqrt{\Lambda}t/2)]^{2/3}$.

   Now we take the fluctuations of our variables into account, i.e., we demand

$$\tilde{\varrho} = \bar{\tilde{\varrho}}(t) \left[1 + \tilde{\delta} \right] \,, \qquad \tilde{U} = \frac{\dot{a}(t)}{a(t)} r + a(t)\tilde{u} \,, \qquad \tilde{\phi}_g = \bar{\tilde{\phi}}_g + \tilde{\varphi}_g \,, \tag{A7}$$

In order to write the Poisson equation for the *peculiar* potential $\tilde{\varphi}_g$, we use Eqs. (A3) and (A5), and obtain

$$\nabla_r^2 \tilde{\varphi}_g = \nabla_r^2 \tilde{\phi}_g - \nabla_r^2 \bar{\tilde{\phi}}_g = 4\pi G \left(\tilde{\rho} - \bar{\tilde{\rho}} \right) = 4\pi G \bar{\tilde{\varrho}} \tilde{\delta} \,. \tag{A8}$$

Observe that Λ has dropped out in the peculiar Poisson equation, i.e., Λ does only affect the background evolution of the Poisson equation.

   Let us now derive the peculiar evolution equation of Eq. (A1) by use of (A7). We have

$$\partial_t \left(\frac{\dot{a}}{a} r + a\tilde{u} \right) + \left[\left(\frac{\dot{a}}{a} r + a\tilde{u} \right) \cdot \nabla_r \right] \left(\frac{\dot{a}}{a} r + a\tilde{u} \right) = -\nabla_r \left(\bar{\tilde{\phi}}_g + \tilde{\varphi}_g \right) \,. \tag{A9}$$

Let us do the calculations in (A9) step by step. The l.h.s. of (A9) can be written as

$$\text{l.h.s.} = \frac{\ddot{a}}{a}\boldsymbol{r} - \frac{\dot{a}^2}{a^2}\boldsymbol{r} + \dot{a}\tilde{\boldsymbol{u}} + a\partial_t\tilde{\boldsymbol{u}} + \frac{\dot{a}^2}{a^2}(\boldsymbol{r}\cdot\boldsymbol{\nabla_r})\boldsymbol{r} + \dot{a}(\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla_r})\boldsymbol{r} + \dot{a}(\boldsymbol{r}\cdot\boldsymbol{\nabla_r})\tilde{\boldsymbol{u}} + a^2(\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla_r})\tilde{\boldsymbol{u}}$$

$$= \frac{\ddot{a}}{a}\boldsymbol{r} - \frac{\dot{a}^2}{a^2}\boldsymbol{r} + \dot{a}\tilde{\boldsymbol{u}} + a\partial_t\tilde{\boldsymbol{u}} + \frac{\dot{a}^2}{a^2}\boldsymbol{r} + \dot{a}\tilde{\boldsymbol{u}} + \dot{a}(\boldsymbol{r}\cdot\boldsymbol{\nabla_r})\tilde{\boldsymbol{u}} + a^2(\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla_r})\tilde{\boldsymbol{u}}\,. \tag{A10}$$

We thus obtain

$$a\partial_t\tilde{\boldsymbol{u}} + a^2(\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla_r})\tilde{\boldsymbol{u}} + 2\dot{a}\tilde{\boldsymbol{u}} + \dot{a}(\boldsymbol{r}\cdot\boldsymbol{\nabla_r})\tilde{\boldsymbol{u}} = -\boldsymbol{\nabla_r}\tilde{\varphi}_g. \tag{A11}$$

Now, we change the dependence of the above functions from $\boldsymbol{r}$ to $\boldsymbol{x} = \boldsymbol{r}/a$. For an arbitrary function $\tilde{f}(t,\boldsymbol{r}) = \tilde{f}(t, a(t)\boldsymbol{x})$, we have

$$\partial_t\tilde{f}|_{\boldsymbol{x}\text{ fixed}} = \partial_t\tilde{f}|_{\boldsymbol{r}\text{ fixed}} + \dot{a}(\boldsymbol{x}\cdot\boldsymbol{\nabla_r})\tilde{f}|_{t\text{ fixed}}\,. \tag{A12}$$

Doing the same for our Eqs. (A11), and note that $\boldsymbol{\nabla_r} = \frac{1}{a}\boldsymbol{\nabla_x}$, we obtain

$$a\partial_t\tilde{\boldsymbol{u}}|_{\boldsymbol{r}\text{ fixed}} - \dot{a}(\boldsymbol{r}\cdot\boldsymbol{\nabla_r})\tilde{\boldsymbol{u}} + a^2\left(\tilde{\boldsymbol{u}}\cdot\frac{1}{a}\boldsymbol{\nabla_x}\right)\tilde{\boldsymbol{u}} + 2\dot{a}\tilde{\boldsymbol{u}} + \dot{a}(\boldsymbol{r}\cdot\boldsymbol{\nabla_r})\tilde{\boldsymbol{u}} = -\frac{1}{a}\boldsymbol{\nabla_x}\tilde{\varphi}_g. \tag{A13}$$

Then, we divide by a and obtain

$$\partial_t\tilde{\boldsymbol{u}} + (\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla_x})\tilde{\boldsymbol{u}} + 2\frac{\dot{a}}{a}\tilde{\boldsymbol{u}} = -\frac{1}{a^2}\boldsymbol{\nabla_x}\tilde{\varphi}_g\,. \tag{A14}$$

The above equation, for an EdS universe was derived by Brenier *et al.* (2003). Note that we have corrected a typo in their equation (A12).