Computation of Stellarator Equilibria with the PIES Code Using Input of VMEC Results

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Introduction

Codes for solving the MHD equilibrium equations

$$\mathbf{J} \times \mathbf{B} = \nabla p, \ \nabla \times \mathbf{B} = \mathbf{J}, \ \nabla \cdot \mathbf{B} = 0 \tag{1}$$

for finite-aspect-ratio, high-beta, 3-D configurations are necessary for the design of fusion machines and analysis of experimental data.

Due to the absence of symmetries, which guarantee the existence of nested magnetic surfaces, in stellarators the treatment of magnetic fields with islands and/or stochastic regions is required.

The **VMEC** code [1] solves the equilibrium equations by a variational method in which the total plasma energy is minimized. The VMEC code presumes a nested toroidal flux surface geometry, and hence can not deal with the magnetic islands and stochastic regions mentioned above. However, it is generally believed that the solutions found by the VMEC code represent good approximations to the solutions including islands and stochastic regions. The VMEC code is robust and yields an accurate description of the flux surfaces with a minimum of computational effort and a minimum number of poloidal and toroidal harmonics. These features make the VMEC code the the most widely used of the 3-D equilibrium codes

The **PIES** code [2] solves the MHD equilibrium equations (1) by a Picard-like iteration scheme

$$\nabla \times \mathbf{B}^{(n+1)} = \mathbf{J}(\mathbf{B}^{(n)}). \tag{2}$$

$\mathbf{B}^{(n)}$ is the magnetic field at the start of the nth iteration. The computationally intensive part of the code is the calculation of the current density $\mathbf{J}(\mathbf{B}^{(n)}) = \mathbf{J}_\perp + \mathbf{J}_\parallel$ from the equilibrium equations (1) and $\nabla \cdot \mathbf{J} = 0$. Here, $\mathbf{J}_\perp$ is the diamagnetic part and $\mathbf{J}_\parallel$ is the parallel part of the current density. These current densities are found from

$$\mathbf{J}_\perp = -\frac{\nabla p \times \mathbf{B}}{B^2}, \tag{3}$$

$$\mathbf{B} \cdot \nabla \mu + \nabla \cdot \mathbf{J}_\perp = 0 \quad \text{with} \quad \mathbf{J}_\parallel = \mu \mathbf{B}. \tag{4}$$

Using this non-variational method, the PIES code is able to handle systems which do not have a nested toroidal flux surface geometry.

Due to the use of the Picard iteration scheme, the PIES code shows a very slow convergence rate, especially when large blending parameters (see below) have to be used to avoid numerical instabilities. In this work, finite-beta solutions obtained with the VMEC code are used as initial guesses for the PIES code in order to accelerate the convergence.

PIES convergence

Using magnetic coordinates (ρ, θ, ψ) and solving the equilibrium equations (1),

$$\mathbf{J} = \nabla\psi \times \nabla\theta \left(I'(\psi) + \frac{dp}{d\psi} \sum_{m,n}{}' \frac{m\mathcal{J}^{\rho}_{m,n}}{(n - \iota m)} \cos(n\phi - m\theta) \right)$$
$$+\nabla\phi \times \nabla\psi \left(-g'(\psi) + \frac{dp}{d\psi} \sum_{m,n}{}' \frac{n\mathcal{J}^{\rho}_{m,n}}{(n - \iota m)} \cos(n\phi - m\theta) \right). \qquad (5)$$

$I(\psi)$ and $g(\psi)$ are the net toroidal and net poloidal current, respectively, $\mathcal{J}^{\rho}_{m,n}$ are the Fourier coefficients of the Jacobian between laboratory and magnetic coordinates. m, n label the poloidal and toroidal mode number, respectively. As seen from equation (2) and depending on the shape of the local pressure profile, there may occur current density resonances near rational surfaces with $\iota = m/n$. The observed numerical instabilities mentioned often appear close to these resonances.

It was found to be useful or even necessary in this connection to blend the Fourier coefficients of the coordinates and fields with those of the previous iteration(s). This blending is accomplished with the algorithm, $A(n+1) = \bar{A}(n+1) + b_A \cdot (A(n) - A(n+1))$. A are the coefficients to be blended, e. g. coefficients of B^{ϕ}, x and ι, and b_A is the blending parameter for A. Particularly in the case of low shear and/or high values of $\langle\beta\rangle$, large values of blending parameters are necessary to avoid instability and to achieve convergence. The result is a very slow convergence rate. This slow convergence can be illustrated by the computation of W7-X equilibria (for W7-X, see e. g. [3]). For an equilibrium with islands, $\langle\beta\rangle = 3.75$ %, and using the vacuum field as initial guess, several hundred iteration are needed for convergence [4]. Generally, blending factors of 0.950 to 0.995 are found to be necessary for W7-X configurations.

It is possible to improve the convergence rate by using of 'Chebychev' periodic sequences of iterations with various blending parameters. Such a sequence allows the algorithm to perform one iteration step with a rather small blending parameter after some steps with large blending parameter without loss of stability.

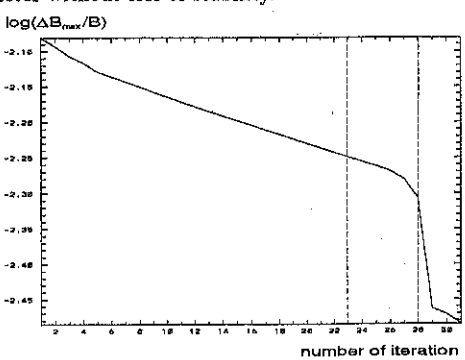

Figure 1: *max. correction $\Delta B_{max}/B$ versus the number of iterations. Iteration # 2 to # 31 with a Chebyshev sequence from # 24 to # 28 are shown. W7-X configuration, $\langle\beta\rangle = 4.1$ %.*

In fig. 1, the convergence of the magnetic field B is shown by plotting the change ΔB of B versus the number of iterations. One Chebychev sequence with a period of $N = 5$

from iteration No. 24 to iteration No. 28 is shown. Iteration No. 28 is the one with the smallest blending factor, the acceleration of the convergence is clearly visible.

Fixed-boundary results

Another method to improve the convergence rate is to improve the initial guess for the magnetic field. Here, VMEC results are used as initial guess. The field resulting from a VMEC run is transformed into PIES coordinates via an interface code. To illustrate this method of accelerated convergence, extensive computations of fixed-boundary W7-X equilibria with $\langle\beta\rangle$ from 3 to about 5 % with and without islands were done. The number of iterations necessary for a PIES run to converge with a VMEC field as initial guess was found to decrease by nearly one order of magnitude compared with corresponding PIES runs using the vacuum field as initial guess. For W7-X configurations without islands, about 30 to 40 iterations were found to be necessary to compute a equilibrium, in cases with islands, about 70 to 110 iterations were required depending on island size and possibilities to use Chebychev sequences. The number of toroidal and poloidal mode numbers used were $|n| \leq 8$ and $|m| \leq 10$ to 16. The numbers of radial points used were from 33 for configurations without islands up to 65 in cases with islands. For all considered W7-X configurations, the island widths were found to be very small compared with the plasma radius although the islands were located in the low-shear region of the ι profile, for example at $\iota = 5/6$. Typical values are in the region of a few percent.

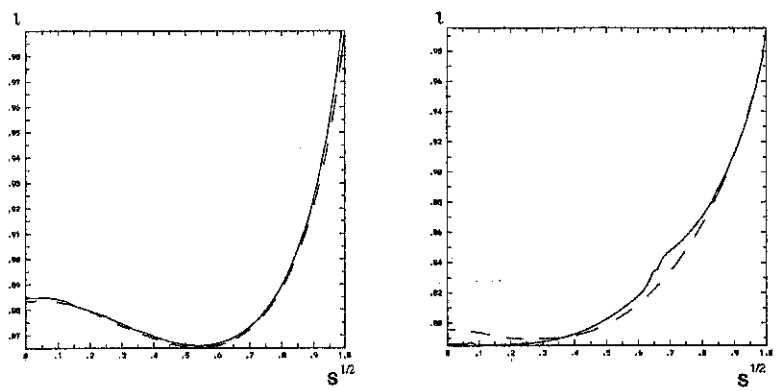

Figure 2: *Rotational transforms ι versus the normalized radius for W7-X equilibria with $\langle\beta\rangle$ = 3 % (left) and $\langle\beta\rangle$ = 5 % (right).*
Dashed line: VMEC calculations, full line: PIES calculations.

In fig. 2, the rotational transforms ι versus the radial flux label for a W7-X equilibrium with $\langle\beta\rangle = 3$ % and no islands (more specifically, with islands which size are below the discretization length) and for a case with $\langle\beta\rangle$ = about 5 % and (small) islands are shown. Both PIES and VMEC profiles are plotted in order to illustrate the practicality of using VMEC inputs. The profiles are quite similar for these configurations. In fig. 3, flux surfaces of a W7-X equilibrium with $\langle\beta\rangle = 5$ % obtained from PIES for three different iterations are shown. The left one is the field obtained from VMEC, the right one is the converged equilibrium field. The corresponding ι profile is on the right side of fig. 2. The island chain is associated with a transform of $\iota = 5/6$.

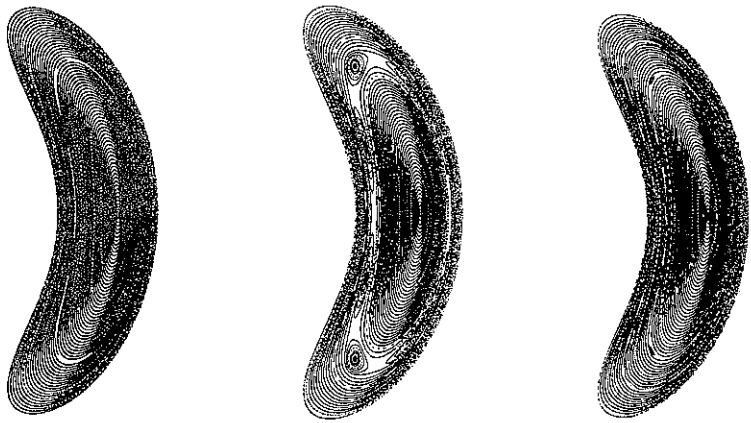

Figure 3: *Flux surfaces of a W7-X equilibrium with $\langle\beta\rangle = 5$ %, obtained from PIES after the 0th (left), the 11th (center) and the 96th (right) iteration.*
Number of poloidal, toroidal modes: $m \leq 10$, $|n| \leq 8$, 65 radial mesh points.

Conclusions

Using VMEC equilibria as PIES input, a considerable acceleration of the fixed-boundary PIES convergence has been realized. This makes the PIES code a suitable tool to perform equilibrium and, particularly, island studies for W7-X and other configurations.
Our next step, is the calculation of free-boundary equilibria. The **NEMEC** code combines the VMEC fixed boundary code and the **NESTOR** code [5]. The NESTOR code is used to calculate the Neumann problem in the vacuum region. The NEMEC code can thus be used to solve the free-boundary MHD equilibrium equations assuming 'good' flux surfaces in the plasma region. Combining the PIES code and the NESTOR vacuum code, one gets a **free-boundary PIES** code that solves the free-boundary equilibrium problem for a given external magnetic field.
The use of NEMEC W7-X results as input for the free-boundary PIES code is under development in order to accelerate convergence as in the fixed-boundary case.

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