Analysis of a kinetic energy principle for a 3D plasma equilibrium

Axel Könies

Max-Planck-Institut für Plasmaphysik, Teilinstitut Greifswald,

IPP-EURATOM-Association, Koitenhäger Landstr., D-17491 Greifswald

1. Introduction

Fast α-particles from fusion may destabilize toroidal Alfvén eigenmodes and ballooning modes in tokamaks, as was found e.g. in ref. [1]. In a 3D plasma confinement device there are reflected particles which are restricted to the period of the magnetic field. The latter have also been shown to be destabilizing if they are energetic [2].

Thus, a stability analysis of 3D plasma equilibria from the kinetic point of view is highly desirable.

2. Ideal MHD Energy Principle

The plasma potential energy of ideal MHD is given as [3,6]:

$$W_p = \frac{1}{2}\int\int\int d^3r \left[|C|^2 - \mathcal{A}(\vec{\xi}\cdot\nabla s)^2 + \gamma p(\nabla\cdot\vec{\xi})^2\right] \tag{1}$$

Clearly, the stabilizing terms are the fluid compression term proportional to γp and $\vec{C} = \nabla\times(\vec{\xi}\times\vec{B}) + \frac{\vec{j}\times\nabla s}{(\nabla s)^2}\vec{\xi}\cdot\nabla s$, with ξ the perturbation and s the flux label. The fluid compression term can be minimized to zero [6].

The only possibly destabilizing term $\mathcal{A}$ contains the curvature, local shear and parallel current density as destabilizing contributions.

3. Kinetic Energy Principle

The guiding center description of a plasma yields an energy principle for low frequency perturbations similar to the MHD one [4,5]. The kinetic energy principle is equivalent to the adiabatic conservation of the magnetic moment μ, the longitudinal action invariant $J = m\oint v_{\|}dl$, and - in the case of energetic particles - the flux through the drift orbit, the so called third adiabatic invariant $\Phi = \oint \alpha d\beta$ [5]. The triple (α, β, l) forms a coordinate system of flux, poloidal and field line coordinates, respectively.

This energy functional differs only in its kinetic term from the MHD result, i.e. the latter replaces the fluid compression term in eq. (1):

$$W_p = \frac{1}{2}\int\int\int d^3r \left[|C|^2 - \mathcal{A}(\vec{\xi}\cdot\nabla\alpha)^2 + W_k\right] \tag{2}$$

In the case of thermal plasmas where the particles can be assumed to be attached to the field lines the energy principle reduces to the well known Kruskal-Oberman result [4] in which the kinetic term is stabilizing. This kinetic term is given by:

$$W_k = -\frac{1}{2}\int d\alpha\, d\beta\, d\mu\, dJ \left(\frac{\partial F}{\partial\epsilon}\right)_{\alpha,\mu,J} \langle H\rangle^2 \tag{3}$$

The quantity $\langle H\rangle$ is the mean variation of the particle kinetic energy between the reflection points whereas F denotes the distribution function of the plasma with ϵ the particle energy.
Introducing the perturbation $\vec{\xi}$, the field line curvature vector $\vec{\kappa}$ and $\nu = \epsilon/\mu$ one obtains:

$$W_k = \frac{1}{2}\int d\alpha \frac{15}{8} p(\alpha) \sum_k \int d\beta \int_0^{1/B_{min}} d\nu \left(\oint \frac{dl}{\sqrt{1-\nu B}}\right)^{-1} \times$$
$$\times \left(\oint dl \left[\sqrt{1-\nu B}(2\vec{\kappa}\cdot\vec{\xi}_\perp + \vec{\nabla}\cdot\vec{\xi}_\perp) - \frac{\vec{\nabla}\cdot\vec{\xi}_\perp + \nu B\vec{\kappa}\cdot\vec{\xi}_\perp}{\sqrt{1-\nu B}}\right]\right)^2 \qquad (4)$$

This kinetic energy principle (eq.(2)) is investigated for a sequence (with sequence parameter t) of 3D magnetic field configurations interpolating between an unstable l=1,2 stellarator ($t = 0$) and the stable Wendelstein 7-X ($t = 1$). The details of the interpolation are given in ref. [7].
The stability analysis has been done with a kinetic generalization of the CAS3D code [6], the numerical field line integration scheme has been adopted from NOVA-K [8]. The eigenvalue problem corresponding to eq. (2) has been solved.

4. Results

For the computation of eq.(4) the possible particle trajectories along the field lines have to be catalogued. The torodial periodicity and the special magnetic field structure of W7-X like configurations allow a categorization of reflected particles on surfaces of constant s and ν in not more than three groups (Fig. 1).
The by far most important contributions to the kinetic energy term stem from the reflected particles. The passing particles average over the whole flux surface and are therefore almost negligible, see Fig.2. The eigenvalues of the kinetic energy principle

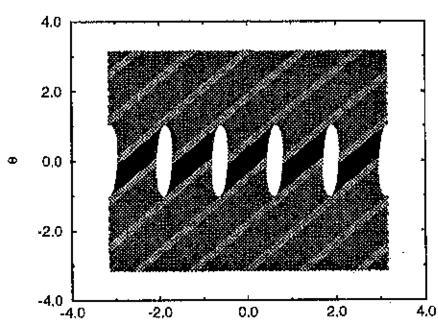
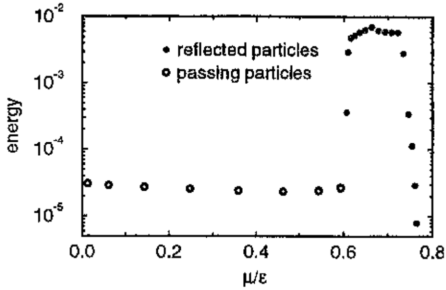


Fig.1: *shows a cut through the magnetic field strength at the $s = 0.343$ surface at $1/\nu = B_{ref} = 1.6080$. The mean B is $\langle B\rangle = 1.4752$. At the white spots $B > B_{ref}$ is valid. Their edges form the reflection points of the trajectories. It can be seen that 3 groups of particles exist: reflected within 1/5 (black), 4/5 (dark gray), 5/5 (light gray) of the torus*

Fig.2: *The main contributions to the kinetic energy term arise from the reflected particles which are characterized by an ϵ/μ value between $1/B_{max}$ and $1/B_{min}$ on a flux surface. (here, for example the contribution from the $s = 0.34$ surface is given).*



have been calculated extending the CAS3D code with eq. (2).

It can be shown that although the structure of the functional of the kinetic term is completely different from the MHD structure there exist so called mode families [6] as in the MHD analysis. The toroidal Fourier indices within a mode family differ by an integral multiple of the number of field periods. A coupling between modes of different families does not exist. Also, the so called phase factor transformation [6,7] to deal with high mode numbers can be performed as in ideal MHD.

The inclusion of the kinetic energy term into the eigenvalue problem tends to shift the main contributions from the perturbation to the plasma edge and, simultaneously, to lower them (Figs. 3,4).

Fig.3: *Contributions to the plasma energy for the unstable $t = 0.4$ configuration and for both the ideal MHD case (broken lines) and the kinetic energy principle (full lines).*

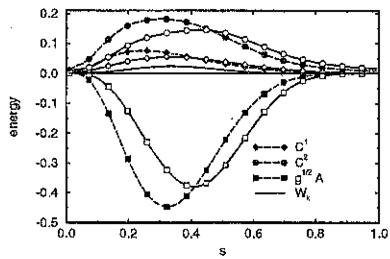
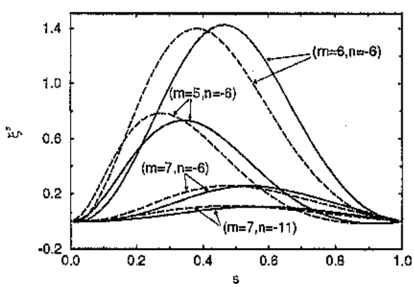
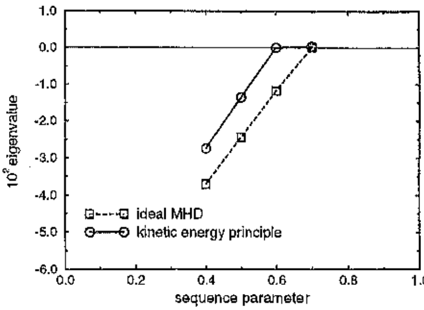


Fig.4: *The radial dependence of the four most prominent ξ^s perturbation modes (out of 12) for the sequence parameter $t = 0.4$. The ideal MHD modes are those with broken lines*



The point of marginal stability is shifted by approximately 0.1 towards lower values of the sequence parameter t compared to the ideal MHD due to the stabilizing kinetic term (Fig. 5).
A perturbative treatment of W_k with ideal eigenfunctions yields an eigenvalue which is approximately 10% larger than that from the exact solution.

Fig.5: *Eigenvalues calculated with the* CAS3D-K *code for a sequence of 3D plasma equilibria.*



5. Conclusions

A kinetic energy principle for thermal plasmas (neglecting drifts) has been investigated in a 3D magnetic field and the according eigenvalue problem has been solved.
In comparison to ideal MHD the kinetic energy principle is more stable leading to a small shift of the point of marginal stability in a sequence between stable and unstable equilibria. The reflected particles have been found to be the most important contributors to the kinetic energy term in the functional.
The energy functional will serve as starting point for the investigation of hot particle instabilities. For the latter particle drifts have to be included as outlined in [5].

6. References

[1] D. Spong, et al. Fusion Technology, **18**, 496 (1990).

[2] J. Nührenberg and L.-J. Zheng, *Proc. 20th EPS Conf. Contr. Fusion Plasma Phys.* ECA17c I-413 (1993).

[3] I.B. Bernstein, E.A. Frieman, M.D. Kruskal, and R.M. Kulsrud, Proc. R. Soc. London Ser. A **244**, 17(1958).

[4] M.D. Kruskal and C.R. Oberman, Phys. Fluids **1**, 275 (1958).

[5] J.W. Van Dam, M.N. Rosenbluth, and Y.C. Lee, Phys. Fluids **25**, 1349 (1982).

[6] C. Nührenberg, Phys. Fluids B **5**, 3195 (1993).

[7] C. Nührenberg, Phys. Plasmas **3**, 2401 (1996).

[8] C.Z. Cheng, Phys. Fluids B **2**, 1427 (1990).

[9] A. Könies, DPG-Frühjahrstagung, Mainz 1997.