Growth Rates of Resistive Ballooning Modes in ASDEX Upgrade and W7-AS

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Introduction

Being pressure gradient driven, resistive ballooning modes limit the achievable beta values in toroidal devices. In particular, the pressure gradient formation will depend on their growth rates so that it is of interest to calculate the latter in regions with large gradients. This paper presents the results of such calculations for the tokamak ASDEX Upgrade and for the low shear stellarator W7-AS using a general treatment which merges approaches described elsewhere [1, 2].

For ASDEX Upgrade we are interested in the high confinement phases where the plasma equilibrium is characterized by steep pressure gradients at the plasma edge. We concentrate on the immediate neighbourhood of the separatrix with a plasma pressure profile characteristic for ASDEX Upgrade.

For W7-AS we consider a sequence of finite β-equilibria. We look at the dependence of the real growthrate on the Lundquist-Reynolds number and discuss the marginality of the equilibria at different β-values.

Theory Background

The investigation of the stability behaviour of a toroidal plasma against resistive ballooning modes [3] requires the solution of a Sturm-Liouville problem in two complex functions $\mathbf{w} = (\mathrm{w}_1, \mathrm{w}_2)$ (the modes) with nonlinear dependence on the complex growthrate γ. The equations – written in a form suitable for stellarators as well as for tokamaks – are

$$\mathbf{B}\cdot\nabla\left\{\frac{(1+\mathrm{S}^2)}{\mathrm{D}|\nabla\Psi|^2}\mathbf{B}\cdot\nabla \mathrm{w}_1\right\} - \left\{2\mu_0\frac{\mathrm{dp}}{\mathrm{d}\Psi}\frac{\kappa_\mathrm{n} - \mathrm{S}\kappa_\mathrm{g}}{|\nabla\Psi|} + \mu_0\rho\gamma^2\frac{1+\mathrm{S}^2}{|\nabla\Psi|^2}\right\}\mathrm{w}_1 - 2\mu_0\frac{\mathrm{dp}}{\mathrm{d}\Psi}\frac{\kappa_\mathrm{n} - \mathrm{S}\kappa_\mathrm{g}}{|\nabla\Psi|}\mathrm{w}_2 = 0$$

$$\mathbf{B}\cdot\nabla\left\{\frac{1}{\mathrm{B}^2}\mathbf{B}\cdot\nabla \mathrm{w}_2\right\} + 2\mu_0\frac{\mathrm{dp}}{\mathrm{d}\Psi}\left(\frac{4\pi^2\mathrm{n}^2\eta}{\mu_0\gamma} + \frac{\rho\gamma^2}{\mu_0(\mathrm{dp}/\mathrm{d}\Psi)^2}\right)\frac{\kappa_\mathrm{n} - \mathrm{S}\kappa_\mathrm{g}}{|\nabla\Psi|}\mathrm{w}_1 +$$

$$+\left\{\frac{4\pi^2\mathrm{n}^2\eta}{\mu_0\gamma}\left(2\mu_0\frac{\mathrm{dp}}{\mathrm{d}\Psi}\frac{\kappa_\mathrm{n} - \mathrm{S}\kappa_\mathrm{g}}{|\nabla\Psi|} - \mu_0\rho\gamma^2\frac{1+\mathrm{S}^2}{|\nabla\Psi|^2}\right) - \frac{\mu_0\rho\gamma^2}{\mathrm{B}^2}\frac{1+\beta}{\beta}\right\}\mathrm{w}_2 = 0$$

where we have used the following abbreviations and definitions

$$\mathrm{D} \equiv 1 + \frac{4\pi^2\mathrm{n}^2\eta}{\mu_0\gamma}\frac{\mathrm{B}^2(1+\mathrm{S}^2)}{|\nabla\Psi|^2} \tag{3}$$

$$\mathrm{S} = \mathrm{S}(\mathbf{x}_0, \mathbf{x}) \equiv \frac{|\nabla \mathrm{V}|^2}{\mathrm{B}}\int_{\mathbf{x}_0}^{\mathbf{x}}\frac{(\nabla \mathrm{V}\times\mathbf{B})\cdot\mathbf{rot}(\nabla \mathrm{V}\times\mathbf{B})}{|\nabla \mathrm{V}|^4}\frac{\mathbf{B}\cdot\mathrm{d}\mathbf{x}'}{\mathrm{B}^2} \tag{4}$$

$$\kappa_\mathrm{g} \equiv \frac{\underline{\kappa}\cdot(\nabla \mathrm{V}\times\mathbf{B})}{|\nabla \mathrm{V}|\mathrm{B}}, \quad \kappa_\mathrm{n} \equiv \frac{\underline{\kappa}\cdot\nabla \mathrm{V}}{|\nabla \mathrm{V}|}, \quad \underline{\kappa} \equiv \frac{1}{2}\left\{\frac{2\mu_0\nabla \mathrm{p}}{\mathrm{B}^2} + \frac{1}{\mathrm{B}^2}\nabla_\perp \mathrm{B}^2\right\}. \tag{5}$$

They must be considered along the field line (described by the position vector $\mathbf{x}$) passing through the mode-localization point $\mathbf{x}_0$. V is the volume enclosed by the magnetic surface containing$\mathbf{x}_0$, Ψ the poloidal flux, n the toroidal mode number and ρ the mass density. Further, $\beta = \mu_0 \gamma_T p / B^2$ with $\gamma_T = c_p/c_v$. All other quantities have their usual meaning. Note that no assumptions on axisymmetry have to be made.

We define small and large radii r_s and R_s, respectively, the poloidal Alfvén transit time τ_A used for the normalization of γ, the resistive time τ_R and an effective Lundquist-Reynolds number according to

$$\tau_A = \frac{(\mu_0 \rho)^{1/2} r_s}{B_s}, \quad \tau_R = \frac{r_s^2 \mu_0}{\eta}, \quad L_{eff} = \frac{\tau_R}{4\pi^2 n^2 \tau_A} \tag{6}$$

where $B_s = 2\pi R_s \Phi' \simeq B_T$. Further, we transform to 4 real modes $\mathbf{u}$ by $w \mapsto \mathbf{u} = \{\mathrm{real}(w_1), -\mathrm{imag}(w_1), \mathrm{real}(w_2), -\mathrm{imag}(w_2)\}$ and introduce a parameter t along the localization field line by $\Phi' \partial/\partial t \equiv \mathbf{B} \cdot \nabla$, where Φ is the toroidal flux and $\Phi' = d\Phi/dV$. The problem to be solved can then be shown to be equivalent to a consideration of the quadratic functional in $\mathbf{u}$

$$F(L_{eff}, \gamma, \mathbf{x}_0) = \int_{-\infty}^{+\infty} \mathcal{L}(L_{eff}, \gamma, \mathbf{x}_0, t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) dt \tag{7}$$

with the Lagrange density $\mathcal{L} = \frac{1}{2}(\dot{\mathbf{u}} \cdot \mathbf{P} \cdot \dot{\mathbf{u}} - \mathbf{u} \cdot \mathbf{Q} \cdot \mathbf{u})$: Satisfying the stationarity conditions of F with respect to $\mathbf{u}$ is equivalent to the solution of the transformed ballooning mode equations

$$(\mathbf{P} \cdot \dot{\mathbf{u}})^{\cdot} + \mathbf{Q} \cdot \mathbf{u} = 0 \tag{8}$$

with the boundary conditions $\mathbf{u}(t = \mp\infty) = 0$. For a given equilibrium the 4x4-matrices $\mathbf{P}$ and $\mathbf{Q}$ have the parametric dependence $\mathbf{P} = \mathbf{P}(L_{eff}, \gamma, \mathbf{x}_0)$ and $\mathbf{Q} = \mathbf{Q}(L_{eff}, \gamma, \mathbf{x}_0)$. Their dependence on $\mathbf{x}_0$ is due to the accumulated shear S (4). Using a magnetic coordinate system (V, θ, φ) S can be written as

$$S = -\frac{|\nabla V|^2}{B} \left[X + \sqrt{g} \frac{B^{\varphi} g^{\theta V} - B^{\theta} g^{\varphi V}}{|\nabla V|^2} \right]_{\mathbf{x}_0}^{\mathbf{x}} \tag{9}$$

where X is determined solving the 3 ordinary differential equations

$$\frac{d\theta}{dt} = \frac{B^{\theta}(\theta, \varphi)}{\Phi'}, \qquad \frac{d\varphi}{dt} = \frac{B^{\varphi}(\theta, \varphi)}{\Phi'}, \qquad \frac{dX}{dt} = \frac{\sqrt{g}}{\Phi'} \left\{ B^{\theta} \frac{\partial B^{\varphi}}{\partial V} - B^{\varphi} \frac{\partial B^{\theta}}{\partial V} \right\} \tag{10}$$

with the initial conditions $\theta_0 = \theta(\mathbf{x}_0)$, $\varphi_0 = \varphi(\mathbf{x}_0)$ and $X_0 = 0$.

Computational Setup

Limiting the size of the integration region of the Lagrangian to a sufficiently large value and discretizing on M intervals leads to an approximation of (7) of the form $F_M = \mathbf{y}^T \cdot \mathbf{G}_M \cdot \mathbf{y}$. Here $\mathbf{y}$ comprises function values of $\mathbf{u}$ on the localization field line. This way the stationarity conditions for the Lagrangian make up a system of $4(M-1)$ homogeneous equations for $\mathbf{y}$: $\mathbf{G}_M \cdot \mathbf{y} = 0$, where $\mathbf{G}_M = \mathbf{G}_M(L_{eff}, \gamma, \mathbf{x}_0)$. In order to find – with L_{eff} and the localization point $\mathbf{x}_0$ as parameters – those values of γ for which there are non-trivial solutions we have used the method of inverse vector iteration. This way, for a given plasma equilibrium, the growthrate of resistive ballooning modes can be obtained in the form $\gamma = \gamma(L_{eff}, \mathbf{x}_0)$.

Application to Tokamaks: ASDEX Ugprade

For a particular equilibrium with a steep pressure profile near the plasma boundary a localization point (LP) of the resistive ballooning modes with a distance of 0.44 cm from the outer side of the separatrix was chosen.

In this region of high shear ($d(\ln q)/d(\ln\rho) \simeq 16$, $\rho = ((\Psi_{LP} - \Psi_A)/(\Psi_B - \Psi_A))^{1/2}$, Ψ_A – flux on axis, Ψ_B – at plasma boundary) we started with a study of the ideal ballooning limit $L_{eff} \to \infty$ and $\gamma \to 0$. In this case the half-bandwidth of the symmetric system matrix G_M reduces to one and the stability boundaries correspond to vanishing eigenvalues in the method of inverse vector iteration. Fig.1 shows that the plasma is stable for a value of the pressure gradient of 238 kPa/m. Scaling the explicitly appearing pressure derivative in the ideal ballooning mode equation with a factor of about 1.52 causes transition into the first unstable regime. This result was confirmed by calculations using equilibria with correspondingly higher edge pressure gradients.

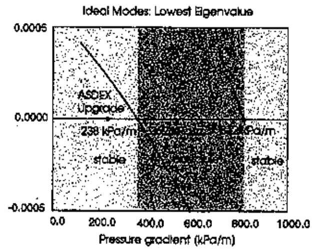

Figure 1: Actual pressure gradient at x_0 and its values at the stability boundaries of ideal modes.

Another confirmation was found in full resistive calculations (Fig.2). With an effective Lundquist-Reynolds number of $L_{eff} = 1.2 \times 10^5$ the plasma is nearly ideal and ballooning-stable for an unscaled pressure.

Estimates of the ideal ballooning limit using the simple $\alpha - s$ model – depending still on how α and s are generalized to be acceptable for a toroidal noncircular plasma – indicate lower critical values for the edge pressure gradient. This fact must be ascribed the combined effects of finite aspect-ratio and ellipticity of the magnetic surfaces, which are significant for ASDEX Upgrade. For an axisymmetric configuration the accumulated shear S (4) has the form

$$S = -\frac{|\nabla V|^2}{B}\left[X - \frac{B_T}{|\nabla V|^2}\frac{\Psi_R \sin\theta + \Psi_z \cos\theta}{\Psi_R \cos\theta - \Psi_z \sin\theta}\right]_{\theta_{LP}}^{\theta}$$

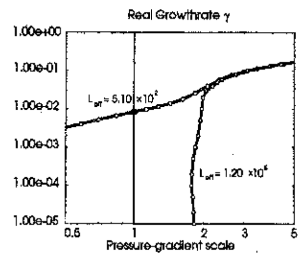

Figure 2: Resistive growth rates for scaled pressure derivatives and different values of L_{eff}.

It secular part is

$$S_s = -s(\theta - \theta_{LP}), \qquad s = -\frac{\nabla\Psi \cdot \nabla q}{2\pi B} \qquad (11)$$

It can be shown that particularly the second term in (11) describing the effects of toroidal shift and ellipticity and the shear parameter s in S_s have considerable influence on the ballooning stability of ASDEX Upgrade.

Application to Stellarators: W7-AS

From the variety of magnetic configurations allowed by the coil system of W7-AS we chose a typical one at $\iota_{vac} \approx 0.34$ with no vertical field applied. We investigate a sequence of equilibria with increasing plasma pressure ($\beta_o \approx 0.1 - 1\%$) calculated with the equilibrium code NEMEC [4]. The pressure profile used is $p = p_o(1 - \Phi/\Phi_B)$, the toroidal current vanishes within each flux surface. A stability analysis with respect to ideal and resistive interchange modes shows stability for the considered β-range. The following investigations is limited to real γ.

First we consider the dependence of γ on L_{eff}. Analytic theory predicts $\gamma \sim L_{eff}^{-1/3}$. For a standard extension of the field line of 24 times the long way around the torus the behaviour of γ with L_{eff} is shown in Fig. 3 (thick curve). For values of $L_{eff} > 260$ we find no solution, although analytic theory predicts solutions for finite resistivity in any case. However, deviations from the simple power law for small L_{eff} have been reported earlier [1], but, it was also noticed that for small growthrates the mode's extension along the field line grows. Increasing the field line range by a factor of 8 ($24 \rightarrow 192$), we find solutions with γ approaching the scaling law for $L_{eff} > 60$ and, additionally, there are now solutions for $L_{eff} > 260$. This means that for low resistivity the modes need a considerable range along the field line to be excited. In the case of low L_{eff} no change in γ appears with increasing extension of the field line range. Thus, the deviation from the power law for high resistivity is real.

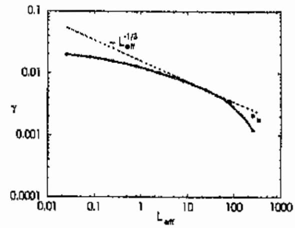

Figure 3: Growthrate of resistive ballooning modes as function of L_{eff} in W7-AS.

Next, we discuss an intuitive pressure scaling in the resistive ballooning equations. To predict behaviour of γ with increasing pressure gradient we considered a sequence of equilibria. In order to keep the number of equilibria small, one would like to predict the behaviour at higher gradients from equilibria with lower ones or vice versa by scaling the explicitly appearing pressure gradient in the equations (1,2). To check the applicability of this scaling we investigated the appearence of resistive ballooning solutions in 6 equilibria with different central pressures for $L_{eff} = 253$, the same field line as above with an extension of 24. Fig.4 shows that in our case equilibria with pressure gradients lower than the marginal one overestimate the pressure where the first solutions appear whereas equilibria with higher gradients stay "unstable".

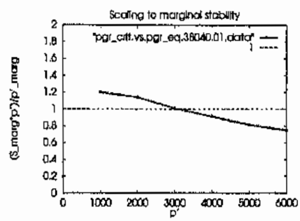

Figure 4: Scaling to marginal: $p'_{marg} = 3200$

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