

EXTENSION OF THE RAY EQUATIONS OF GEOMETRIC OPTICS TO INCLUDE DIFFRACTION EFFECTS

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The propagation of waves for which the wave length is small compared to the length scale over which the medium changes is usually treated within geometric optics. In this formalism the wave beam is represented by a number of rays which are independently traced through the medium. In the case of a focussed beam the rays can cross. The geometric optics approximation then leads to caustics, where the electric field amplitude is infinite. In the region where a caustic is formed the geometric optics approximation breaks down. For a correct description of the wave propagation, diffraction effects must then be taken into account.

In the literature several methods to include diffraction effects in the description of wave propagation in a magnetized plasma are given. A generalization of the method of S. Choudary and L.B. Felsen [1,2] was developed by E. Mazzucato [3]. This method uses a complex eikonal function, where the imaginary part describes the electric field profile of the beam. This method will be investigated in detail in this paper.

COMPLEX EIKONAL METHOD

In the eikonal method the electric field E is written in the form

$$E = e(x)E_0(x)\exp(iS(x)), \quad (1)$$

where S is the eikonal function. Normally, this function is taken to be real and, consequently, the wave vector and frequency, defined by $k \equiv \partial S / \partial x$, $\omega \equiv -\partial S / \partial t$, are real. The functions $E_0(x)$ and $e(x)$ give the amplitude and polarization of the wave. If the length scale over which the plasma parameters change is much larger than the wave length these functions change only slightly over a wave length. In the dispersion relation, which is derived through the substitution of Eq. (1) in the wave equation the terms which contain derivatives of these functions with respect to position can be neglected in comparison with the terms which contain derivatives of the eikonal function S . After deviding by $E_0(x)$, the wave equation and the dispersion relation do not contain any information on the profile of the wave amplitude. Because the ray equations of geometric optics are derived from the dispersion relation, no profile effects are retained in this description.

The method that uses a complex eikonal function intends to include effects of the profile in the description of the wave propagation. For this purpose an imaginary part of the function S is introduced $\text{Im}(S(x)) = -\ln(E_0(x))$. The complex eikonal function leads to a complex wave vector. The imaginary part $k_i \equiv d\text{Im}[S]/dx$ contains the profile information. The medium in which the wave propagates is assumed to be in steady state. The wave source is assumed to have constant frequency and strength. These assumptions make that the wave frequency in the medium is real and constant.

The same derivation that gave the ray equations of geometric optics can now be repeated. A dispersion relation is obtained which depends on the electric field

profile. From this dispersion relation ray equations, which include diffraction effects, are obtained.

Below the equations will be expanded in a small parameter to obtain a tractable set of equations. To be able to expand various quantities in the same dimensionless small parameter δ , the velocity of light c and a frequency ω_0 close to the wave frequency will be used to make all length and time scales dimensionless.

Three different length scales are discerned, the wave length λ , the width w of the beam, and the length scale L over which the plasma parameters change. It is assumed that these length scales can be ordered in the following way

$$L : w : \lambda \sim \delta^{-2} : \delta^{-1} : 1, \quad (2)$$

where δ is a small parameter.

Because of the ordering (2) the imaginary part of the wave vector k_i which describes the electric field profile, is of the order δ compared with the real part k_r . Consistency with $w_0 \approx \delta^{-1}$, where w_0 is the width of the beam, demands that the beam focussing is moderate. It turns out that the gradient dk_r/dx , which is related to the curvature of the phase front, must be taken of the order δ^2 or smaller to satisfy this demand. The imaginary part of the wave vector k_i , its gradient dk_i/dx , and dk_r/dx are allowed to vary over the width of the beam, i.e. $d^n k_i/dx^n = \mathcal{O}(\delta^{n+1})$ with $n \geq 0$, and $d^n k_r/dx^n = \mathcal{O}(\delta^{n+1})$ with $n \geq 1$.

DISPERSION RELATION

It will be shown below that the leading order description of diffraction effects requires an expansion of the dispersion relation up to and including second order terms in the small quantity $\delta = |k_i|/|k_r|$. It is clear that all terms of the order δ^2 in the dispersion relation can generate effects of the same order. The usual derivation of this relation assumes a homogeneous plasma and a constant wave vector. This assumption can no longer be made if the dispersion relation is to be derived including all terms of the order δ^2 , because a consistent ordering of terms requires that both dk_r/dx and dk_i/dx are of the order δ^2 .

A method to derive the dispersion relation including all second order terms in the small parameter δ is given in Ref. [4]. The final result can be written in the general form

$$\omega = \omega_0(\mathbf{k}, \mathbf{x}) + iF_{\alpha\beta}(\omega_0(\mathbf{k}, \mathbf{x}), \mathbf{k}, \mathbf{x}) \frac{\partial k_\alpha}{\partial x_\beta}, \quad (3)$$

where $\omega = \omega_0(\mathbf{k}, \mathbf{x})$ is the usual dispersion relation of the homogeneous medium with constant wave vector. This wave vector, however, now is complex and the dispersion relation must be evaluated as $\omega_0(\mathbf{k}_r + i\mathbf{k}_i, \mathbf{x})$. The tensor $F_{\alpha\beta}$ is, like ω_0 , of the order δ^0 . Therefore, the second term of the right hand side of Eq. (3) is a correction of the order δ^2 to the usual dispersion relation.

THE RAY EQUATIONS

The dispersion relation can be written in the form $\omega = \omega(\mathbf{k}, \mathbf{x}, \partial\mathbf{k}/\partial\mathbf{x})$. The ray equations are derived from the demand that the dispersion relation is satisfied for all positions

$$\frac{d\omega}{dx} = \frac{\partial\omega}{\partial\mathbf{x}} \bigg|_{\mathbf{k}, d\mathbf{k}/d\mathbf{x}} + \frac{\partial\omega}{\partial\mathbf{k}} \bigg|_{\mathbf{x}, d\mathbf{k}/d\mathbf{x}} \frac{d\mathbf{k}}{dx} + \frac{\partial\omega}{\partial(\partial\mathbf{k}/\partial\mathbf{x})} \bigg|_{\mathbf{k}, \mathbf{x}} \frac{d^2\mathbf{k}}{dx^2} = 0. \quad (4)$$

The ray equations can be derived from this equation by taking the real and imaginary part, taking the ray to propagate in the direction

$$\frac{dx}{d\tau} = \mathbf{v} = \text{Re} \left(\frac{\partial \omega}{\partial \mathbf{k}} \bigg|_{\mathbf{x}, d\mathbf{k}/d\mathbf{x}} \right), \quad (5)$$

and expanding the equations in the small parameter $\delta = |\mathbf{k}_i|/|\mathbf{k}_r|$. The final result can be written in the form [4]

$$\frac{dx_\alpha}{d\tau} = \frac{\partial \omega_r}{\partial k_{r,\alpha}} \quad (6a)$$

$$\frac{dk_{r,\alpha}}{d\tau} = - \frac{\partial \omega_r}{\partial x_\alpha} + k_{i,\gamma} \frac{\partial^2 \omega_r}{\partial k_{r,\gamma} \partial k_{r,\eta}} \frac{dk_{i,\eta}}{dx_\alpha} + F_{r,\gamma\eta} \frac{d^2 k_{i,\gamma}}{dx_\eta dx_\alpha} \quad (6b)$$

$$\frac{dk_{i,\alpha}}{d\tau} = - k_{i,\gamma} \frac{\partial^2 \omega_r}{\partial k_{r,\gamma} \partial x_\alpha} - k_{i,\gamma} \frac{\partial^2 \omega_r}{\partial k_{r,\gamma} \partial k_{r,\eta}} \frac{dk_{r,\eta}}{dx_\alpha} - F_{r,\gamma\eta} \frac{d^2 k_{r,\gamma}}{dx_\eta dx_\alpha}, \quad (6c)$$

where

$$\omega_r = \omega_0(\mathbf{k}_r, \mathbf{x})$$

$$\bar{\mathbf{F}}_r = \bar{\mathbf{F}}(\omega_0(\mathbf{k}_r, \mathbf{x}), \mathbf{k}_r, \mathbf{x})$$

are real functions which do not depend on the imaginary part of the wave vector. To derive Eqs. (6) the dispersion relation (3) had to be expanded up to second order in the small quantity δ .

The first term on the right hand side of Eq. (6a), and the first term on the right hand side of Eq. (6b), are the terms of geometric optics. The other terms in these equations are at least one order in magnitude smaller. The direction of propagation of a ray is equal to that of geometric optics. The diffraction effects appear indirectly through a modification of the real part of the wave vector which is driven by the electric field profile through k_i , dk_i/dx , and $d^2 k_i/dx dx$. If the width of the beam decreases the gradient of the electric field profile increases and the second and third terms on the right hand side of Eq. (6b) become larger, causing more bending of the rays away from each other. The rays will, therefore, never cross and no caustics will be formed.

RELATION TO OTHER WORK

Using the complex eikonal method Mazzucato [3] derived a set of ray equations which include diffraction. It can be shown that his equations differ from the ray equations (Eqs. (6)) in this paper through the terms containing the tensor $\bar{\mathbf{F}}$. This tensor entered our equations through the correction on the dispersion relation that contains the gradients of the wave vector. In Ref. [3] it was assumed that the Altair-Appleton-Hartree dispersion relation is satisfied.

The statement that all terms of the order δ^2 have to be retained in the dispersion relation would be too strong. In principle, one could treat the effects of the gradient

of the wave vector in the transport of amplitude equation. This solution, however, is unnatural because effects of the same order are then treated in a different way. It also goes against the philosophy of the complex eikonal method, where the beam profile effects are introduced through the dispersion relation to derive ray equations that properly describe the diffraction effects. If the transport of amplitude equation has to be considered separately one can just as well treat all diffraction effects in this equation.

It must be noted, however, that Ref. [3], applies the formalism to the treatment of a Gaussian beam. An example of the electric field of such a beam would be

$$\mathbf{E} = \frac{1}{w(z)} \exp \left[-\frac{x^2 + y^2}{w^2(z)} + ikz + ik \frac{x^2 + y^2}{2R(z)} - i\omega t \right], \quad (7)$$

where the beam is assumed to propagate along the z -axis, $w(z)$ is the width of the beam and $R(z)$ is the radius of curvature. Both w and $1/R$ are assumed to vary on the long length scale $L = \delta^{-2}$. It can be verified that in this case $d^2 k_i / dx^2 = O(\delta^4)$, and $d^2 \mathbf{k}_r / dx^2 = O(\delta^4)$. Therefore, all terms including the tensor $\bar{\mathbf{F}}_r$ can be neglected in Eqs. (6). In the set of Eqs. (6) then the function $\omega_r(\mathbf{k}_r, \mathbf{x})$ appears, and it is equivalent to that of Mazzucato.

In the case of a Gaussian beam the beam can be described entirely with a few parameters (width and radius of curvature of the phase front). In this case a reduced set of equations can be derived that gives the evolution of these parameters. Such a set is discussed in Refs. [4-6].

CONCLUSIONS

In this paper an extension of the ray equations of geometric optics to include diffraction effects for a wave beam propagating in a dispersive anisotropic medium is derived. The results of this paper extend the previous results to more general beams, and shed light on the conditions under which the assumptions made in previous work break down.

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