Higher order rectifiability of rectifiable sets via averaged discrete curvatures

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June 2, 2015

Abstract

For an m dimensional $\mathcal{H}^m$ measurable set Σ we define, axiomatically, a class of Menger like curvatures $\kappa : \Sigma^{m+2} \to [0,\infty)$ which imitate, in the limiting sense, the classical curvature if Σ is of class $\mathscr{C}^2$. With each κ we associate an averaged curvature $\mathcal{K}^{l,p}_{\kappa}[\Sigma] : \Sigma \to [0,\infty]$ by integrating κ^p with respect to $l-1$ parameters and taking supremum with respect to $m+2-l$ parameters. We prove that if Σ is a priori $(\mathcal{H}^m, m)$ rectifiable (of class $\mathscr{C}^1$) with $\mathcal{H}^m(\Sigma) < \infty$ and $\mathcal{K}^{l,p}_{\kappa}[\Sigma](a) < \infty$ for $\mathcal{H}^m$ almost all $a \in \Sigma$, then Σ is in fact $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$, where $\alpha = 1 - m(l-1)/p$. We also prove an analogous result for the tangent-point curvature and we show that α is sharp.

1 Introduction

Whenever $T = (p_0, \dots, p_k) \in (\mathbb{R}^n)^{k+1}$ is a $(k+1)$ tuple of points in $\mathbb{R}^n$ let $\triangle T$ be the convex hull of the set $\{p_0, \dots, p_k\} \subseteq \mathbb{R}^n$ and let $\mathcal{H}^m$ denote the m dimensional Hausdorff measure over $\mathbb{R}^n$. Fix $\gamma \in (0,\infty)$ and define $\kappa^{\gamma}_{\mathrm{vol}} : (\mathbb{R}^n)^{m+2} \to [0,\infty)$ by

$$\kappa^{\gamma}_{\mathrm{vol}}(T) = \left(\frac{\mathcal{H}^{m+1}(\triangle T)}{\operatorname{diam}(\triangle T)^{m+1}} \right)^{\gamma} \frac{1}{\operatorname{diam}(\triangle T)} \quad \text{whenever } \mathcal{H}^{m+1}(T) > 0$$

and $\kappa^{\gamma}_{\mathrm{vol}}(T) = 0$ if $\mathcal{H}^{m+1}(T) = 0$. In [Kol15] we have studied regularity properties of compact m dimensional sets $\Sigma \subseteq \mathbb{R}^n$ which are AD regular, i.e.,

$$C^{-1} r^m \leqslant \mathcal{H}^m(\Sigma \cap \mathbf{B}(a,r)) < C r^m \quad \text{for some } C \in (1,\infty) \text{ and all } a \in \Sigma,\ r \in (0, \operatorname{diam}\Sigma)\,,$$

have finite curvature energy $\mathcal{M}^p$, i.e.,

$$\mathcal{M}^p(\Sigma) = \int_{\Sigma^{m+2}} \kappa^{1}_{\mathrm{vol}}(q_0, \dots, q_{m+1})^p \, d\mathcal{H}^m(q_0) \cdots d\mathcal{H}^m(q_{m+1}) \quad \text{for some } p > m(m+2)\,,$$

and, roughly speaking, do not have "holes" which was expressed be kind of a local Reifenberg flatness condition. In this setting we obtained $\mathscr{C}^{1,\alpha}$ regularity, where $\alpha = 1 - m(m+2)/p$, which can be seen as an analogue of the classical Morrey-Sobolev embedding $W^{2,p}(\mathbb{R}^{m(m+2)}) \subseteq \mathscr{C}^{1,\alpha}(\mathbb{R}^{m(m+2)})$. Concerning this result two questions arise.

(A) *What kind of regularity can be deduced from finiteness of $\mathcal{M}^p(\Sigma)$ if one does not assume neither AD regularity nor any structural condition modeling the lack of "holes"?*

(B) *Assuming only that Σ is $\mathcal{H}^m$ measurable with $\mathcal{H}^m(\Sigma) < \infty$, does the condition $\mathcal{M}^p(\Sigma) < \infty$ imply, for some $p \in (0,\infty)$, that Σ is $(\mathcal{H}^m, m)$ rectifiable, i.e., $\mathcal{H}^m(\Sigma) < \infty$ and Σ can be covered, up to a set of $\mathcal{H}^m$ measure zero, by a countably many Lipschitz graphs?*

Question (B) has been answered very recently in the thesis of Meurer [Meu15], who proved that if $\gamma = 2/(m(m+1))$, $p = m(m+1)$, and

$$\int_{\Sigma^{m+2}} \kappa_{\mathrm{vol}}^{\gamma}(q_0, \dots, q_{m+1})^p \, d\mathcal{H}^m(q_0) \cdots d\mathcal{H}^m(q_{m+1}) < \infty \,,$$

then Σ is indeed $(\mathcal{H}^m, m)$ rectifiable. Actually he proved this result for a whole class of discrete curvatures $\kappa : (\mathbb{R}^n)^{m+2} \to [0,\infty)$ which he calls *proper integrands* and which includes $\kappa_{\mathrm{vol}}^{\gamma}$ when $\gamma = 2/(m(m+1))$.

Our main result involves answering question (A) assuming that Σ is $(\mathcal{H}^m, m)$ rectifiable with $\mathcal{H}^m(\Sigma) < \infty$. To state the theorem we need to define the class of discrete curvatures to which it applies. For $k \in \mathbb{N} \sim \{0\}$ let

$$\mathcal{D}_k = \left\{ T \in (\mathbb{R}^n)^{k+1} : \mathcal{H}^k(\triangle T) > 0 \right\}$$

be the set of tuples of vertexes of non-degenerate k dimensional simplices in $\mathbb{R}^n$. We also write $\operatorname{span}\{v_1, \dots, v_k\}$ for the linear span of the vectors $v_1, \dots, v_k \in \mathbb{R}^n$.

1.1 Definition. Let $m, n \in \mathbb{N} \sim \{0\}$, $0 < m < n$, $\gamma \in (0,\infty)$. We call $\kappa : (\mathbb{R}^n)^{m+2} \to [0,\infty)$ an m *dimensional Menger like curvature with exponent* $\gamma \in (0,\infty)$ (or just a *Menger like curvature*) if the following conditions hold

(a) *continuity:*

$$\kappa|_{\mathcal{D}_{m+1}} : \mathcal{D}_{m+1} \to [0,\infty) \quad \text{is continuous}\,;$$

(b) *degenerate simplices have zero curvature:*

$$\kappa(T) = 0 \quad \text{whenever } T \in (\mathbb{R}^n)^{m+2} \text{ and } \mathcal{H}^{m+1}(\triangle T) = 0\,;$$

(c) *boundedness on unit diameter simplices:*

$$\sup\left\{ \kappa(T) : T \in \mathcal{D}_{m+1} ,\, \operatorname{diam}(\triangle T) = 1 \right\} < \infty\,;$$

(d) *curvature scaling:*

$$\kappa(\lambda T) = \lambda^{-1} \kappa(T) \quad \text{whenever } \lambda \in (0,\infty) \text{ and } T \in \mathcal{D}_{m+1}\,;$$

(e) *repulsiveness:* if

$$d, \delta \in (0,\infty)\,, \quad T = (a, b_1, \dots, b_m, c) \in \mathcal{D}_{m+1}\,, \quad \operatorname{diam}(\triangle T) \leqslant d\,,$$
$$m! \mathcal{H}^m\big(\triangle(a, b_1, \dots, b_m)\big) \geqslant \delta d^m\,, \quad \text{and} \quad P = \operatorname{span}\left\{ b_1 - a, \dots, b_m - a \right\},$$

then there exists $\Lambda = \Lambda(\delta, \kappa) \in (0,\infty)$ such that

$$\kappa(T) \geqslant \left(\frac{\Lambda \operatorname{dist}(c - a, P)}{d} \right)^{\gamma} \frac{1}{d}\,.$$

We say that κ is *tame* if additionally

(f) there exists $\Gamma = \Gamma(\kappa) \in (0, \infty)$ such that for any $\alpha \in (0, 1]$ the following holds: if $\Sigma \subseteq \mathbb{R}^n$ is a graph of some function $f \in \mathscr{C}^{1,\alpha}(\mathbb{R}^m, \mathbb{R}^{n-m})$, and $T = (p_0, \ldots, p_{m+1}) \in \Sigma^{m+2}$ satisfies $\mathcal{H}^{m+1}(\triangle T) > 0$, and

$$K = \sup\left\{ \frac{\|Df(\mathfrak{p}(q)) - Df(\mathfrak{p}(p_0))\|}{|\mathfrak{p}(q) - \mathfrak{p}(p_0)|^\alpha} : 0 < |q - p_0| \leqslant \operatorname{diam}(\triangle T) \right\},$$

where $\mathfrak{p} : \mathbb{R}^n \to \mathbb{R}^m$ is given by $\mathfrak{p}(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$, then

$$\kappa(T) \leqslant (\Gamma K)^\gamma \operatorname{diam}(\triangle T)^{\alpha\gamma - 1}.$$

For $k \in \mathbb{N} \sim \{0\}$ let μ_Σ^k be the product of k copies of $\mathcal{H}^m \llcorner \Sigma$, i.e.,

$$\mu_\Sigma^k = (\mathcal{H}^m \llcorner \Sigma)^k. \tag{1}$$

Given $p \in (0, 1)$, $l \in \{1, \ldots, m + 2\}$, $p_0 \in \Sigma$ and a Menger like curvature κ we define the *averaged curvatures*

$$\mathcal{K}_\kappa^{l,p}[\Sigma](p_0) = \left(\int (\mu_\Sigma^{m+2-l}) \operatorname*{ess\,sup}_{p_l, \ldots, p_{m+1} \in \Sigma} \kappa(p_0, \ldots, p_{m+1})^p \, d\mu_\Sigma^{l-1}(p_1, \ldots, p_{l-1}) \right)^{1/p} \tag{2}$$

with the understanding that there is no essential supremum in case $l = m + 2$ and there is no integral in case $l = 1$. With the above definitions we can now state our main higher order rectifiability result.

1.2 Theorem. *Assume*

$$m, n \in \mathbb{N}, \quad 0 < m < n, \quad \Sigma \subseteq \mathbb{R}^n \textit{ be } (\mathcal{H}^m, m) \textit{ rectifiable}, \quad \mathcal{H}^m(\Sigma) < \infty,$$
$$p, \gamma \in (0, \infty), \quad l \in \{1, \ldots, m + 2\}, \quad p > m(l - 1), \quad \alpha = \tfrac{1}{\gamma}\left(1 - \tfrac{m(l-1)}{p}\right) \in (0, 1],$$
$$\kappa \textit{ is a Menger like curvature with exponent } \gamma,$$
$$\mathcal{K}_\kappa^{l,p}[\Sigma](p_0) < \infty \textit{ for } \mathcal{H}^m \textit{ almost all } p_0 \in \Sigma.$$

Then Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$, i.e., there exists a countable family $\mathcal{M}$ of m dimensional submanifolds of $\mathbb{R}^n$ of class $\mathscr{C}^{1,\alpha}$ such that $\mathcal{H}^m(\Sigma \sim \bigcup \mathcal{M}) = 0$ (cf. 3.1).

Moreover, in case κ is tame and either $\gamma = 1$ or $\alpha < 1$ the exponent α is optimal, i.e.,

(a) if $\alpha = 1$ and $\gamma = 1$, then for each $\varepsilon \in (0, 1)$ there exists Σ which is not $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{2,\varepsilon}$ but $\mathcal{K}_\kappa^{l,p}[\Sigma](p_0) < \infty$ for $\mathcal{H}^m$ almost all $p_0 \in \Sigma$.

(b) if $\alpha < 1$, then for each $\varepsilon \in (0, 1)$ such that $\alpha + \varepsilon < 1$ there exists Σ which is not $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$ but $\mathcal{K}_\kappa^{l,p}[\Sigma](p_0) < \infty$ for $\mathcal{H}^m$ almost all $p_0 \in \Sigma$.

Combining 1.2 with [Meu15, Theorem 3.4] we can, at least if $\gamma = 2/(m(m + 1))$, essentially drop the assumption that Σ is $(\mathcal{H}^m, m)$ rectifiable. First define

$$\mathcal{M}_\kappa^{l,p}(\Sigma) = \int_\Sigma \mathcal{K}_\kappa^{l,p}[\Sigma](p_0)^p \, d\mathcal{H}^m(p_0). \tag{3}$$

We obtain

1.3 Corollary. *Let $\Sigma \subseteq \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(\Sigma) < \infty$, and κ be a translation invariant Menger like curvature with exponent $\gamma = 2/(m(m+1))$, and $p \geqslant m(m+1)$. If $\mathcal{M}_\kappa^{m+2,p}(\Sigma) < \infty$, then Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$, where $\alpha = 1 - m(m+1)/p$.*

The axioms of 1.1 are meant to reflect, in case $\gamma = 1$, properties of the so called *tangent-point curvature* $r_{tp}[\Sigma]^{-1}$ given by

$$r_{tp}[\Sigma](a,b)^{-1} = \frac{2\operatorname{dist}\big(b-a, \operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)\big)}{|b-a|^2} \quad \text{for } a,b \in \Sigma \text{ with } b \neq a\,, \tag{4}$$

where $\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)$ denotes the approximate tangent cone to Σ at a; see 3.7. For each $a \in \Sigma$ such that $\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)$ is an m dimensional plane, $r_{tp}[\Sigma](a,b)^{-1}$ equals exactly the inverse of the radius of the unique m dimensional sphere S passing through a and b and such that $\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a) = \operatorname{Tan}(S, a)$. Moreover, if Σ is a graph of some $\mathscr{C}^2$ function, then $\limsup_{\Sigma \ni b \to a} r_{tp}[\Sigma](a,b)^{-1}$ equals the operator norm of the second fundamental form of $\Sigma \subseteq \mathbb{R}^n$ at a; see 4.1. Thus, $r_{tp}[\Sigma](a,b)^{-1}$ can be seen as a relaxed version of the classical curvature and condition 1.1 (e), where the plane spanned by the first $m+1$ points of T plays the role of a "tangent plane", is meant to imitate the behavior of $r_{tp}[\Sigma]^{-1}$. We added the parameter γ to make it possible to include in our class all the discrete curvatures defined in [LW09, §1.2 , §6.1.1] and [LW11, §10]. Menger like curvatures with exponent $2/(m(m+1))$ which are translation invariant are also proper integrands in the sense of [Meu15, Definition 3.1].

The additional condition 1.1 (f) says roughly that if κ is tame, it imitates closely some classical notion of curvature; see 6.1. In particular, it implies that if Σ is a compact submanifold of $\mathbb{R}^n$ of class $\mathscr{C}^2$, then κ is bounded on Σ^{m+2}. In 6.2 we show that $\kappa_{\mathrm{vol}}^\gamma$ as well as

$$\kappa_h^\gamma(T) = \left(\frac{h_{\min}(T)}{\operatorname{diam}(\triangle T)}\right)^\gamma \frac{1}{\operatorname{diam}(\triangle T)}\,,$$

where $h_{\min}(T)$ denotes the minimal height of the simplex $\triangle T$, are tame. This implies that whenever $\kappa \lesssim \kappa_h^\gamma$, then κ is also tame; see 6.3. We use this observation to check that some of the discrete curvatures of [LW09, LW11] are tame; see A.2. Although we are mainly interested in tame Menger like curvatures we do not impose the condition 1.1 (f) by default since not all known examples satisfy it. In particular, $\kappa_{\mathbb{S}}$ defined for $T \in \mathcal{D}_{m+1}$ as the inverse of the radius of the unique m dimensional sphere containing all the $m+2$ points of T is not tame unless $m = 1$; in this case $\gamma = 1$ (see A.1 for an algebraic expression defining $\kappa_{\mathbb{S}}$). Observe that if $m = 1$, then $\kappa_{\mathbb{S}}$ is the original Menger–Melnikov curvature (cf. [Men30, II.1] and [Mel95, Definition 2]).

Let $a, b \in \Sigma$ be such that $a \neq b$ and the approximate tangent cone $\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)$ is an m dimensional plane. For $p \in (0, \infty)$ set

$$\begin{aligned} \tau^{1,p}[\Sigma](a) &= (\mathcal{H}^m \llcorner \Sigma) \operatorname{ess\,sup}\big(\Sigma \ni b \mapsto r_{tp}[\Sigma](a,b)^{-1}\big) \\ \text{and} \quad \tau^{2,p}[\Sigma](a) &= \left(\int_\Sigma r_{tp}[\Sigma](a,b)^{-p} \, d\mathcal{H}^m(b)\right)^{1/p}. \end{aligned} \tag{5}$$

Our second result, which is proven first and can be seen as a model case for 1.2, allows to deduce higher rectifiability of Σ from finiteness $\mathcal{H}^m$ almost everywhere of one of $\tau^{1,p}[\Sigma]$ or $\tau^{2,p}[\Sigma]$.

1.4 Theorem. *Let* $m \in \mathbb{N} \sim \{0\}$, $\Sigma \subseteq \mathbb{R}^n$ *be* $(\mathcal{H}^m, m)$ *rectifiable of class* $\mathscr{C}^1$, $l \in \{1,2\}$, $p \in \mathbb{R}$, $p > m(l-1)$. *Assume* $\tau^{l,p}[\Sigma](a) < \infty$ *for* $\mathcal{H}^m$ *almost all* $a \in \Sigma$. *Then* Σ *is* $(\mathcal{H}^m, m)$ *rectifiable of class* $\mathscr{C}^{1,\alpha}$, *where* $\alpha = 1 - m(l-1)/p$.

Moreover, the exponent α *is optimal, i.e.,*

(a) *if* $l = 1$, *then for each* $\varepsilon \in (0,1)$ *there exists* Σ *which is not* $(\mathcal{H}^m, m)$ *rectifiable of class* $\mathscr{C}^{2,\varepsilon}$ *but* $\tau^{1,p}[\Sigma](a) < \infty$ *for* $\mathcal{H}^m$ *almost all* $a \in \Sigma$.

(b) *if* $l = 2$, *then for each* $\varepsilon \in (0,1)$ *such that* $\alpha + \varepsilon < 1$ *there exists* Σ *which is not* $(\mathcal{H}^m, m)$ *rectifiable of class* $\mathscr{C}^{1,\alpha+\varepsilon}$ *but* $\tau^{2,p}[\Sigma](a) < \infty$ *for* $\mathcal{H}^m$ *almost all* $a \in \Sigma$.

Theorems 1.4 and 1.2 fit into a whole series of regularity results proven in the context of discrete curvatures and associated energies in the recent years. Motivated by questions arising in knot theory and variational problems emerging there Strzelecki and von der Mosel [SvdM07] and Strzelecki, von der Mosel, and Szumańska [SSvdM09, SSvdM10] studied the functionals $\mathcal{M}^{l,p}_{\kappa}$ in the case $m = 1$, $l \in \{1,2,3\}$, $\kappa = \kappa_{\mathbb{S}}$, and Σ is a rectifiable closed curve, i.e., $\Sigma = \operatorname{im}\Gamma$ for a Lipschitz map $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ satisfying $|\Gamma'(t)| = L$ for $\mathcal{H}^1$ almost all $t \in \mathbb{R}/\mathbb{Z}$ and some fixed $L \in (0,\infty)$. They proved, among other things, that finiteness of $\mathcal{M}^{l,p}_{\kappa_{\mathbb{S}}}(\Sigma)$ for some $p > l$ implies $\mathscr{C}^{1,1-l/p}$ regularity of Γ and, in case $l = 1$, that finiteness of $\mathcal{M}^{l,p}_{\kappa_{\mathbb{S}}}(\Sigma)$ is equivalent to Γ being of Sobolev class $W^{2,p}$. This has been further extended by Blatt [Bla13b] who showed, for $l \in 2,3$ and $p > l$, that the condition $\mathcal{M}^{l,p}_{\kappa_{\mathbb{S}}}(\Sigma) < \infty$ characterizes curves for which Γ is in the Sobolev–Slobodeckij space $W^{2-(l-1)/p,p}$. Similar results were also proven for the tangent-point energies

$$\mathcal{T}^{l,p}(\Sigma) = \int \tau^{l,p}[\Sigma](a)^p \, d\mu^1_{\Sigma}(a) \quad \text{for } l = 1,2\,.$$

In [SvdM12] the authors show $\mathscr{C}^{1,1-2/p}$ regularity for curves having finite $\mathcal{T}^{2,p}$ energy with $p > 2$.

Going into higher dimensions, in a pioneering work [SvdM11], Strzelecki and von der Mosel initiated the study of discrete curvatures for surfaces in $\mathbb{R}^3$. They defined $\kappa_{\mathrm{SvdM}}(T) = \mathcal{H}^3(\triangle T)\mathcal{H}^2(\partial \triangle T)^{-1} \operatorname{diam}(\triangle T)^{-2}$ whenever $T \in \mathcal{D}_3$ and proved that if $\Sigma \subseteq \mathbb{R}^3$ is a closed connected compact Lipschitz surface and $\mathcal{M}^{4,p}_{\kappa_{\mathrm{SvdM}}}(\Sigma) = E < \infty$ for some $p > 8$, then Σ must be a submanifold of $\mathbb{R}^3$ of class $\mathscr{C}^{1,1-8/p}$ whose local graph representation is controlled in terms of E and p only. The same authors established in [SvdM13] a similar result for a wide class of m dimensional subsets of $\mathbb{R}^n$ (called δ-*admissible* there) having $\mathcal{T}^{2,p}$ energy finite for some $p > 2m$; in this case one obtains $\mathscr{C}^{1,1-2m/p}$ regularity. Building on their ideas, the author studied a different class of m dimensional compact subsets of $\mathbb{R}^n$ described briefly at the beginning of this introduction (they are called m-*fine* in [Kol15]). We proved another geometric Sobolev–Morrey type embedding theorem for the $\mathcal{M}^{l,p}_{\kappa}$ energies : if Σ is m-fine, $l \in \{1,2,\ldots,m+2\}$, $\kappa = \kappa^1_{\mathrm{vol}}$, and $\mathcal{M}^{l,p}_{\kappa}(\Sigma) < \infty$ for some $p > ml$, then Σ is a $\mathscr{C}^{1,1-ml/p}$ submanifold of $\mathbb{R}^n$. This has been further extended by Blatt [Bla13a], Blatt and the author [BK12] and the author, Strzelecki, and von der Mosel [KSvdM13] showing that finiteness of $\mathcal{T}^{l,p}$ and $\mathcal{M}^{l,p}_{\kappa}$ with $\kappa = \kappa^1_{\mathrm{vol}}$ actually *characterizes* submanifolds which are locally graphs of $W^{2-m(l-1)/p,p} \cap \mathscr{C}^1$ maps.

Each of the embedding theorems mentioned above comes with a very rigid control of the local graph representation of Σ in terms of the energy $\mathcal{M}^{l,p}_{\kappa}$ or $\mathcal{T}^{l,p}$. This allowed to prove [KSvdM15] finiteness of ambient $\mathscr{C}^1$ isotopy types of submanifolds having energy bounded by some fixed value and made it possible to find minimizers of the energy in each isotopy class.

Definition 1.1 implies that $\kappa(T) \leqslant C \operatorname{diam}(\triangle T)^{-1}$ for any Menger like curvature κ and some constant $C = C(\kappa)$. Thus, it is not hard to deduce (see 6.4) that if $p < m(l-1)$ and Σ is $\mathcal{H}^m$ measurable with $\mathcal{H}^m(\Sigma) < \infty$, then $\mathcal{K}^{l,p}_{\kappa}[\Sigma](a) < \infty$ and $\tau^{l,p}[\Sigma](a) < \infty$ for $\mathcal{H}^m$ almost all $a \in \Sigma$, and if Σ is additionally AD regular, then $\mathcal{M}^{l,p}_{\kappa}(\Sigma) < \infty$ and $\mathcal{T}^{l,p}(\Sigma) < \infty$; see 6.5. Therefore, the interesting case is when $m(l-1) \leqslant p < \infty$.

If $p = m(l-1)$, and $\Sigma \subseteq \mathbb{R}^n$ is $\mathcal{H}^m$ measurable with $\mathcal{H}^m(\Sigma) < \infty$, and κ is a Menger like curvature with exponent $2/(m(m+1))$, then one might expect that finiteness $\mathcal{H}^m$ almost everywhere of $\mathcal{K}^{l,p}_{\kappa}[\Sigma]$ implies that Σ is $(\mathcal{H}^m, m)$ rectifiable.

In case $m = 1$, $l = 3$, and $\kappa = \kappa_{\mathbb{S}}$ is the Menger curvature, finiteness of $\mathcal{M}^{l,p}_{\kappa}(\Sigma)$ implies $(\mathcal{H}^1, 1)$ rectifiability. This is the famous theorem of David [Dav98] (see also Léger [Lég99]) proven in connection with the Vitushkin conjecture characterizing removable sets for bounded analytic functions. The result of Meurer [Meu15] constitutes a generalization of David's theorem and has been proven using Léger's approach. In this context, Farag [Far99] showed that there is no equivalent of the Menger–Melnikov curvature in higher dimensions which would relate to the Riesz transform in the way $\kappa_{\mathbb{S}}$ relates to the Cauchy transform (cf. [Mel95]). Moreover, an analogue of the Vitushkin conjecture in $\mathbb{R}^n$ has already been proven by Nazarov, Tolsa, and Volberg [NTV14] without employing any discrete curvature. Nonetheless, Menger like curvatures can still be used to study rectifiability of sets or measures.

By the results of Lerman and Whitehouse [LW09, LW11], if Σ is AD regular, $l = m+2$, $\gamma = 2/(m(m+1))$, $\kappa = \kappa^{\gamma}_{\mathrm{vol}}$, and $B \subseteq \mathbb{R}^n$ is a ball, then $\mathcal{M}^{l,p}_{\kappa}(\Sigma \cap B)$ is comparable to the *Jones' square function*

$$J_2(\Sigma \cap \tilde{B}) = \int_{\Sigma \cap \tilde{B}} \int_0^{\operatorname{diam} \tilde{B}} \beta^m_{\Sigma,2}(x,r)^2 \frac{dr}{r} \, d\mathcal{H}^m(x) \,,$$

where $\tilde{B}$ is a ball with the same center as B and radius scaled by a factor and the number $\beta^m_{\Sigma,2}(x,r)$ is defined by

$$\beta^m_{\Sigma,2}(x,r) = \inf_L \left(r^{-m} \int_{\Sigma \cap \mathbf{B}(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^2 d\mathcal{H}^m(y) \right)^{1/2} ,$$

where the infimum is taken with respect to all affine m dimensional planes L in $\mathbb{R}^n$. Having in mind the various characterizations of uniformly rectifiable sets given by David and Semmes [DS93, Theorem 1.57] the results of [LW09, LW11] yield yet another characterization. In connection with that result it is adequate to mention one more development concerning rectifiability. Tolsa [Tol15] and Azzam and Tolsa [AT15] proved that an $\mathcal{H}^m$ measurable set Σ with $\mathcal{H}^m(\Sigma) < \infty$ is $(\mathcal{H}^m, m)$ rectifiable *if and only if*

$$\int_0^1 \beta^m_{\Sigma,2}(x,r)^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^m \text{ almost all } x \in \Sigma \,. \tag{6}$$

Consulting [LW09, LW11] and [Meu15, Theorem 5.6] which give estimates for $\beta^m_{\Sigma,2}$ in terms of the curvature energies one might acquire an impression that the integral in (6) and $\mathcal{K}^{l,p}_{\kappa}[\Sigma]$ might be mutually comparable if $\gamma = 2/(m(m+1))$. However, by now it is not clear whether the results of [AT15] imply [Meu15] or vice versa. Certainly the two results were proven in parallel and using different techniques.

The fact remains that $\gamma = 2/(m(m+1))$ is exactly the right exponent in the context of characterizing $(\mathcal{H}^m, m)$ rectifiability. This is surprising since $\gamma = 1$ seems more natural and the curvatures with $\gamma = 1$ were successfully used to characterize Sobolev and fractional Sobolev submanifolds in [Bla13a, BK12, KSvdM13].

Organization of the paper

In section 2 we introduce the notation and in section 3 we recall various facts from geometric measure theory. In 3.17 we prepare the basic setup for the proofs and in 3.22 we formulate a corollary of [Sch09, Lemma A.1] which will be our main tool in the proofs. Next in section 4 we prove 1.4. After that in section 5 we prove some auxiliary results which will allow to show that κ_h^γ is tame. In section 6 we show that κ_h^γ and κ_{vol}^γ are tame and study some basic properties of Menger like curvatures which allow to show sharpness of our results (cf. 6.6). In section 7 we prove theorem 1.2. And in the appendix A we show that all the curvatures of [LW09, LW11] as well as $\kappa_{\mathbb{S}}$ satisfy definition 1.1.

2 Notation

In principle we shall use the book of Federer [Fed69] as our main reference and source of definitions, and we shall adopt some, but not all, of its notation. In particular we shall write $\{x \in X : P(x)\}$, in contrast to $X \cap \{x : P(x)\}$, for the set of those $x \in X$ which satisfy some predicate P. We also prefer to say that a function is "injective" rather than "univalent". The symbols $\mathbb{R}$ and $\mathbb{N}$ shall be used for the set of real and natural numbers including zero respectively. The closure of a subset A of some topological space X shall be denoted by $\overline{A}$ and the interior by $\operatorname{Int} A$. Moreover, whenever $s, t \in \mathbb{R} \cup \{-\infty, \infty\}$ and $s < t$, we shall write (s,t), $[s,t]$ for the open and closed intervals in $\mathbb{R}$ and also $(s,t]$ and $[s,t)$ with the usual meaning. If A and B are sets, we write $A \sim B$ for the set of these $a \in A$ which do *not* belong to B (set theoretic difference). If X is a vector space, $A, B \subseteq X$, $c \in X$ and $r \in (0,\infty)$ we adopt the notation $c + A = \{c + a : a \in A\}$, $rA = \{ra : a \in A\}$ and $A + B = \{a + b : a \in A\,,\ b \in B\}$. When we write $\mathbb{R}^n$ we always mean the n dimensional Euclidean space with the standard scalar product denoted $u \bullet v$ for $u, v \in \mathbb{R}^n$. We shall write $\mathbf{U}(a,r)$ and $\mathbf{B}(a,r)$ for the open and closed ball centered at a and of radius r in the metric space to which a belongs to. We adopt the definition of a *measure* from [Fed69, 2.1.2], which is sometimes called an *outer measure* in the literature. The symbols $\mathcal{H}^m$ and $\mathcal{L}^m$ stand for the m dimensional Hausdorff and Lebesgue (outer) measures as defined in [Fed69, 2.10.2(1) and 2.6.5]. Whenever $m \in \mathbb{N} \sim \{0\}$ we use the symbol $\boldsymbol{\alpha}(m)$ for the Lebesgue measure of the unit ball in $\mathbb{R}^m$. If X and Y are normed vector spaces, $U \subseteq X$ is open, $k \in \mathbb{N}$, and $\alpha \in [0,1]$, then a function $f : U \to Y$ is said to be of class $\mathscr{C}^{k,\alpha}$ if f is continuous, has continuous derivatives up to order k (cf. [Fed69, 3.1.1 , 3.1.11]), and the k^{th} order derivative $D^k f$ satisfies the Hölder condition with exponent α (cf. [Fed69, 5.2.1]); in this case we write $f \in \mathscr{C}^{k,\alpha}(U, Y)$. The image of a set $A \subseteq X$ under a mapping $f : X \to Y$ is denoted $f[A]$ and similarly $f^{-1}[B]$ denotes the preimage of a set $B \subseteq Y$. We write id_X for the identity function on X. Whenever X is a metric space, $A \subseteq X$, and $x \in X$, we use the notation $\operatorname{dist}(x, A)$ for the distance of x from A. We write A^l to denote the Cartesian product of $l \in \mathbb{N} \sim \{0\}$ copies of a set A and if $f : A \to B$, then $f^l : A^l \to B^l$ is the Cartesian product of l copies of f, i.e. $f^l(a_1, \dots, a_l) = (f(a_1), \dots, f(a_l))$. For the Grassmannian of m dimensional planes in $\mathbb{R}^n$ we write $\mathbf{G}(n, m)$ (cf. [Fed69, 1.6.2]). With each $P \in \mathbf{G}(n, m)$ we associate the orthogonal projection

$$P_\natural : \mathbb{R}^n \to P \subseteq \mathbb{R}^n \quad \text{onto } P$$

and the orthogonal complement

$$P^\perp = \left\{ u \in \mathbb{R}^n : u \bullet v = 0 \text{ for all } v \in P \right\} = \ker P_\natural \,.$$

Whenever $v_1, \dots, v_k$ are vectors in some vector space X, we write $\operatorname{span}\{v_1, \dots, v_l\}$ for the linear span of these vectors, and if $p_0, \dots, p_k \in X$ the *convex hull* of the set $\{p_0, \dots, p_m\}$ is denoted $\triangle(p_0, \dots, p_m) \subseteq X$. If μ measures some set X, $f : X \to \mathbb{R}$ is μ-measurable and $A \subseteq X$ is μ-measurable, we write $⨍_A f \, d\mu = \mu(A)^{-1} \int_A f \, d\mu$ for the *mean value* of f on A. By $X \ni x \mapsto f(x)$ we mean an *unnamed function* with domain X mapping $x \in X$ to $f(x)$. For the *essential supremum* of a function $f : X \to \mathbb{R}$ with respect to a measure μ over X we write $(\mu) \operatorname{ess\,sup}(f)$, which is defined to be equal to $(\mu)_{(\infty)}(f)$ in the notation of [Fed69, 2.4.12]. To optimize space we shall sometimes write $(\mu) \operatorname{ess\,sup}_{x \in X}(f(x))$ instead of $(\mu) \operatorname{ess\,sup}(X \ni x \mapsto f(x))$.

The reader might also want to recall the definitions of the space of orthogonal projections $\mathbf{O}^*(n, m)$ (cf. [Fed69, 1.7.4]) and of the exterior algebra $\bigwedge_* X$ of a vector space X (cf. [Fed69, 1.3]) with its associated wedge product $\wedge$.

We do not reserve any special symbol for constants. By a "constant" we mean a positive real valued function depending on some parameters which will not always be displayed; to express the dependencies explicitly we shall write $C = C(a, b, c)$ whenever C depends *only* on a, b, and c.

3 Preliminaries

Rectifiability, densities, and tangent cones

3.1 Definition (cf. [AS94, Definition 2.1]). Let $\alpha \in [0, 1]$, $m, k \in \mathbb{N} \sim \{0\}$. A set $\Sigma \subseteq \mathbb{R}^n$ is said to be $(\mathcal{H}^m, m)$ *rectifiable of class* $\mathscr{C}^{k,\alpha}$ if and only if $\mathcal{H}^m(\Sigma) < \infty$ and there exists a countable family $\mathcal{M}$ of m dimensional submanifolds of $\mathbb{R}^n$ of class $\mathscr{C}^{k,\alpha}$ (cf. [Fed69, 3.1.19]) such that $\mathcal{H}^m(\Sigma \sim \bigcup \mathcal{M}) = 0$.

3.2 Remark. In case $\alpha = 0$ and $k = 1$ the above definition coincides with [Fed69, 3.2.14(4)].

3.3 Definition (cf. [Fed69, 2.10.19]). Let μ be a measure over $\mathbb{R}^n$, $m \in \mathbb{N}$ and $a \in \mathbb{R}^n$. The *lower and upper* m*-densities of* μ *at* a are given by

$$\begin{aligned} \boldsymbol{\Theta}^m_*(\mu, a) &= \liminf_{r \downarrow 0} \boldsymbol{\alpha}(m)^{-1} r^{-m} \mu(\mathbf{B}(a, r)) \,, \\ \boldsymbol{\Theta}^{*m}(\mu, a) &= \limsup_{r \downarrow 0} \boldsymbol{\alpha}(m)^{-1} r^{-m} \mu(\mathbf{B}(a, r)) \,. \end{aligned}$$

If both quantities agree, then we write

$$\boldsymbol{\Theta}^m(\mu, a) = \boldsymbol{\Theta}^{*m}(\mu, a) = \boldsymbol{\Theta}^m_*(\mu, a) \,.$$

3.4 Definition (cf. [Fed69, 3.1.21]). Assume $S \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$. The *tangent cone of* S *at* a is the set

$$\operatorname{Tan}(S, a) = \big\{ v \in \mathbb{R}^n : \forall \varepsilon > 0 \; \exists\, b \in S \; \exists\, t > 0 \quad |b - a| \leqslant \varepsilon \text{ and } |t(b-a) - v| \leqslant \varepsilon \big\} .$$

3.5 Remark. If $u, v \in \mathbb{R}^n \sim \{0\}$, then for any $t \in (0, \infty)$ there holds

$$|(u \bullet v)|u|^{-2}u - v| \leqslant |tu - v| .$$

Therefore, if $v \in \operatorname{Tan}(S, a) \sim \{0\}$ and $0 < \varepsilon < |v|$, then there exists $b \in S \sim \{a\}$ with $|b - a| \leqslant \varepsilon$ and $(b - a) \bullet v > 0$ and one can take $t = (b - a) \bullet v|b - a|^{-2}$ in 3.4.

3.6 Proposition. *Let* $S \subseteq \mathbb{R}^n$, $T \in \mathbf{G}(n, m)$ *and* $a \in \mathbb{R}^n$. *Assume*

$$\lim_{\substack{b \to a \\ b \in S}} \frac{|T_\natural^\perp(b-a)|}{|b-a|} = \lim_{r \downarrow 0} \sup \left\{ \frac{|T_\natural^\perp(b-a)|}{|b-a|} : b \in S \cap \mathbf{B}(a, r) \sim \{a\} \right\} = 0 .$$

Then $\operatorname{Tan}(S, a) \subseteq T$.

Proof. If $\operatorname{Tan}(S, a) \sim \{0\} = \varnothing$, the conclusion is evident. Suppose, then, that there exists $v \in \operatorname{Tan}(S, a) \sim \{0\}$. Using 3.4 and 3.5, we see that for each $\varepsilon \in (0, |v|)$ there exists $b \in S \sim \{a\}$ with $|b - a| \leqslant \varepsilon$ and $|v - t(b - a)| \leqslant \varepsilon$, where $t = (b - a) \bullet v|b - a|^{-2}$. Hence,

$$\begin{aligned} |T_\natural^\perp v| &\leqslant |T_\natural^\perp(v - t(b-a))| + |t T_\natural^\perp(b-a)| \\ &\leqslant \varepsilon + |v| \sup \left\{ \frac{|T_\natural^\perp(c-a)|}{|c-a|} : c \in S \cap \mathbf{B}(a, \varepsilon) \sim \{a\} \right\} \end{aligned}$$

holds for all $\varepsilon \in (0, |v|)$. Letting $\varepsilon \to 0$, we obtain $v \in T$; thus, $\operatorname{Tan}(S, x) \subseteq T$. □

3.7 Definition (cf. [Fed69, 3.2.16]). Let μ be a measure over $\mathbb{R}^n$, $m \in \mathbb{N}$ and $a \in \mathbb{R}^n$. The m-*approximate tangent cone of* μ *at* a is

$$\operatorname{Tan}^m(\mu, a) = \bigcap \big\{ \operatorname{Tan}(S, a) : S \subseteq \mathbb{R}^n \,, \boldsymbol{\Theta}^m(\mu \llcorner \mathbb{R}^n \sim S, a) = 0 \big\} .$$

3.8 Remark. For $a, v \in \mathbb{R}^n$, $\varepsilon \in (0, \infty)$ define the cone

$$\mathbf{E}(a, v, \varepsilon) = \big\{ b \in \mathbb{R}^n : \exists\, t \in (0, \infty) \; |t(b-a) - v| < \varepsilon \big\} .$$

Then (cf. [Fed69, 3.2.16]) $v \in \operatorname{Tan}^m(\mu, a)$ if and only if

$$\boldsymbol{\Theta}^{*m}(\mu \llcorner \mathbf{E}(a, v, \varepsilon)) > 0 \quad \text{for all } \varepsilon \in (0, \infty).$$

Note that, as in 3.5, if $0 < \varepsilon < |v|$, then $b \in \mathbf{E}(a, v, \varepsilon)$ if and only if

$$b \neq a \quad \text{and} \quad \frac{b-a}{|b-a|} \bullet \frac{v}{|v|} > \left(1 - \frac{\varepsilon^2}{|v|^2} \right)^{1/2} .$$

3.9 Remark. If $\Sigma \subseteq \mathbb{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^1$, then by [Fed69, 3.2.19] for $\mathcal{H}^m$ almost all $a \in \Sigma$

$$\boldsymbol{\Theta}^m(\mathcal{H}^m \llcorner \Sigma, a) = 1 \quad \text{and} \quad \operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a) \in \mathbf{G}(n, m) .$$

3.10 Proposition. *Let* $S \subseteq \mathbb{R}^n$, $T \in \mathbf{G}(n, m)$ *and* $a \in \mathbb{R}^n$. *Assume*

$$\lim_{r \downarrow 0} r^{-m} \int_{S \cap \mathbf{B}(a,r)} \frac{|T_\natural^\perp(b-a)|}{|b-a|} \, d\mathcal{H}^m(b) = 0 \,. \tag{7}$$

Then $\operatorname{Tan}^m(\mathcal{H}^m \llcorner S, a) \subseteq T$.

Proof. If $\operatorname{Tan}^m(\mathcal{H}^m \llcorner S, a) \sim \{0\} = \varnothing$, the conclusion is evident. In all other cases we shall prove the proposition by contradiction. If there existed $v \in \operatorname{Tan}^m(\mathcal{H}^m \llcorner S, a) \sim T$; then $|T_\natural^\perp v| > 0$. Recalling 3.8, if $\varepsilon \in (0, \infty)$ satisfied $\varepsilon < \frac{1}{4}|T_\natural^\perp v|$, then for each $b \in \mathbf{E}(a, v, \varepsilon)$ setting $t = (b-a) \bullet v |b-a|^{-2}$, we would have

$$\begin{aligned} \left| \frac{b-a}{|b-a|} \bullet \frac{v}{|v|} \right| \cdot \left| T_\natural^\perp \frac{b-a}{|b-a|} \right| = \frac{|T_\natural^\perp t(b-a)|}{|v|} &\geqslant \frac{|T_\natural^\perp v| - |T_\natural^\perp (v - t(b-a))|}{|v|} \\ &\geqslant \frac{|T_\natural^\perp v| - \varepsilon}{|v|} \geqslant \frac{3}{4} \frac{|T_\natural^\perp v|}{|v|} > 0 \,. \end{aligned}$$

Hence, for any $r > 0$ and $\varepsilon \in \left(0, \frac{1}{4}|T_\natural^\perp v|\right)$, we would obtain

$$\begin{aligned} r^{-m} \int_{S \cap \mathbf{B}(a,r)} \frac{|T_\natural^\perp(b-a)|}{|b-a|} \, d\mathcal{H}^m(b) &\geqslant r^{-m} \int_{S \cap \mathbf{E}(a,v,\varepsilon) \cap \mathbf{B}(a,r)} \frac{|T_\natural^\perp(b-a)|}{|b-a|} \, d\mathcal{H}^m(b) \\ &\geqslant \frac{3}{4} \frac{|T(x)_\natural^\perp v|}{|v|} \left(1 - \frac{\varepsilon^2}{|v|^2}\right)^{-1/2} \frac{\mathcal{H}^m(S \cap \mathbf{E}(a, v, \varepsilon) \cap \mathbf{B}(a, r))}{r^m} \,. \end{aligned} \tag{8}$$

Since we assumed $v \in \operatorname{Tan}^m(\mathcal{H}^m \llcorner S, a)$, we could argue that $\Theta^{*m}(\mathcal{H}^m \llcorner (S \cap \mathbf{E}(a, v, \varepsilon))) > 0$ for all $\varepsilon \in (0, \infty)$. Then, for $\varepsilon \in (0, \frac{1}{4}|T_\natural^\perp v|)$, taking $\limsup_{r \downarrow 0}$ on both sides of (8), we would get

$$\limsup_{r \downarrow 0} r^{-m} \int_{S \cap \mathbf{B}(a,r)} \frac{|T_\natural^\perp(b-a)|}{|b-a|} \, d\mathcal{H}^m(b) > 0 \,,$$

which is impossible due to the assumption (7). Thereby, we conclude that it was not possible to choose $v \in \operatorname{Tan}^m(\mathcal{H}^m \llcorner S, a) \sim T$; thus $\operatorname{Tan}^m(\mathcal{H}^m \llcorner S, a) \subseteq T$. $\square$

3.11 Remark. Observe that the condition

$$\lim_{r \downarrow 0} r^{-m-1} \int_{S \cap \mathbf{B}(a,r)} |T_\natural^\perp(b-a)| \, d\mathcal{H}^m(b) = 0 \tag{9}$$

implies

$$\lim_{r \downarrow 0} r^{-m} \int_{S \cap \mathbf{B}(a,r)} \frac{|T_\natural^\perp(b-a)|}{|b-a|} \, d\mathcal{H}^m(b) = 0 \,, \tag{10}$$

which can be verified by representing the integral over $S \cap \mathbf{B}(a, r)$ by a series of integrals over "annuli" $S \cap \mathbf{B}(a, 2^{-k} r) \sim \mathbf{U}(a, 2^{-k-1} r)$ for $k \in \mathbb{N}$. Hence, the conclusion of 3.10 holds also with assumption (10) replaced by (9).

Product measures

Recall that whenever $\Sigma \subseteq \mathbb{R}^n$ the measure μ^l_Σ was defined by (1).

3.12 Remark. If ϕ measures X and $M \subseteq X$ is ϕ measurable, then $(\phi \times \phi)\llcorner(M \times M) = (\phi\llcorner M) \times (\phi\llcorner M)$. This follows from the definition of the product measure (cf. [Fed69, 2.6.1]) using the fact that, if M is ϕ measurable and $A \subseteq M$, then A is ϕ measurable if and only if A is $\phi\llcorner M$ measurable. In particular, if $n = m$ and $A \subseteq \mathbb{R}^m$ is $\mathcal{L}^m$ measurable, then $\mathcal{H}^m = \mathcal{L}^m$ and $\mathcal{H}^{ml}\llcorner A^l = \mu^l_A$ since, in this case, $\mathcal{H}^{ml} = \mathcal{L}^{ml}$ is the Cartesian product of l copies of $\mathcal{L}^m$; see [Fed69, 2.6.5].

In general, when $m < n$, the measure $\mathcal{H}^{ml}$ over $(\mathbb{R}^n)^l$ is *not* the Cartesian product of l copies of $\mathcal{H}^m$ over $\mathbb{R}^n$ and μ^l_Σ might not be the same as the measure $\mathcal{H}^{ml}\llcorner\Sigma^l$; see [Fed69, 3.2.24].

3.13 Proposition. *Let* $m, n, k \in \mathbb{N}$, $m \leqslant \min\{n, k\}$, $\Sigma \subseteq \mathbb{R}^n$ *be* $\mathcal{H}^m$ *measurable and* $g : \mathbb{R}^n \to \mathbb{R}^k$ *be Lipschitz such that* $g|_\Sigma$ *is injective. For* $l \in \mathbb{N} \sim \{0\}$ *define*

$$\nu^l_\Sigma = \underbrace{(\mathcal{H}^m\llcorner g[\Sigma]) \times \cdots \times (\mathcal{H}^m\llcorner g[\Sigma])}_{l \text{ times}} \quad \textit{and} \quad g^l : (\mathbb{R}^n)^l \to (\mathbb{R}^k)^l ,$$

$$g^l(x_1, \dots, x_l) = (g(x_1), \dots, g(x_l)) \quad \textit{for } (x_1, \dots, x_l) \in (\mathbb{R}^n)^l.$$

Then for any μ^l_Σ *measurable set* $S \subseteq (\mathbb{R}^n)^l$ *there holds*

$$\nu^l_\Sigma(g^l[S]) \leqslant \operatorname{Lip}(g)^{ml}\mu^l_\Sigma(g) .$$

Proof. For $l = 1$ the conclusion follows directly from the construction of the Hausdorff measure; see [Fed69, 2.10.2]. Assume $l > 1$ and let $S \subseteq (\mathbb{R}^n)^l$ be μ^l_Σ measurable set. Define $\mathcal{B}$ to be the set of sequences of l-tuples of ν^1_Σ measurable subsets of $\mathbb{R}^k$ which cover $g^l[S]$, i.e.

$$\mathcal{B} = \Big\{(B_{1,i}, \dots, B_{l,i})_{i\in\mathbb{N}} : B_{j,i} \subseteq \mathbb{R}^k \text{ are } \nu^1_\Sigma \text{ measurable} ,\ g^l[S] \subseteq \bigcup_{i\in\mathbb{N}} B_{1,i} \times \cdots \times B_{l,i}\Big\} .$$

In a similar manner define

$$\mathcal{A} = \Big\{(A_{1,i}, \dots, A_{l,i})_{i\in\mathbb{N}} : A_{j,i} \subseteq \mathbb{R}^n \text{ are Borel} ,\ S \subseteq \bigcup_{i\in\mathbb{N}} A_{1,i} \times \cdots \times A_{l,i}\Big\} .$$

Observe that if $(A_{1,i}, \dots, A_{l,i})_{i\in\mathbb{N}} \in \mathcal{A}$, then $(g[A_{1,i}], \dots, g[A_{l,i}])_{i\in\mathbb{N}} \in \mathcal{B}$, by [Fed69, 2.2.13]. Therefore,

$$\mathcal{A} \subseteq \big\{(g^{-1}[B_{1,i}], \dots, g^{-1}[B_{l,i}])_{i\in\mathbb{N}} : (B_{1,i}, \dots, B_{l,i})_{i\in\mathbb{N}} \in \mathcal{B}\big\} \tag{11}$$

Since $g|_\Sigma$ is injective, we have

$$g\big[g^{-1}[B] \cap \Sigma\big] = B \cap g[\Sigma] \quad \text{for any } B \subseteq \mathbb{R}^k . \tag{12}$$

Using the definition of the product measure [Fed69, 2.6.1], then the conclusion for $l = 1$, and after that (11) together with (12), we obtain

$$\begin{aligned}
\nu_\Sigma^l(g[S]) &= \inf\left\{ \sum_{i=0}^{\infty} \prod_{j=1}^{l} \mathcal{H}^m(B_{j,i} \cap g[\Sigma]) : (B_{1,i}, \ldots, B_{l,i})_{i\in\mathbb{N}} \in \mathcal{B} \right\} \\
&\leqslant \mathbf{Lip}(g)^{ml} \inf\left\{ \sum_{i=0}^{\infty} \prod_{j=1}^{l} \mathcal{H}^m(g^{-1}[B_{j,i}] \cap \Sigma) : (B_{1,i}, \ldots, B_{l,i})_{i\in\mathbb{N}} \in \mathcal{B} \right\} \\
&\leqslant \mathbf{Lip}(g)^{ml} \inf\left\{ \sum_{i=0}^{\infty} \prod_{j=1}^{l} \mathcal{H}^m(A_{j,i} \cap \Sigma) : (A_{1,i}, \ldots, A_{l,i})_{i\in\mathbb{N}} \in \mathcal{A} \right\} = \mathbf{Lip}(g)^{ml} \mu_\Sigma^l(S) \,,
\end{aligned}$$

where the last equality holds because $\mu_\Sigma^1 = \mathcal{H}^m \llcorner \Sigma$ is Borel regular. □

Graphs of functions and the slope of the tangent plane to a graph

A convenient way to work with graphs of functions defined on some $T \in \mathbf{G}(n,m)$ and with values in $T^\perp$ is to express the function using orthonormal bases for T and $T^\perp$. To do that one can choose orthogonal projections $\mathfrak{p} \in \mathbf{O}^*(n,m)$ and $\mathfrak{q} \in \mathbf{O}^*(n,n-m)$ (cf. [Fed69, 1.7.4]) such that $\operatorname{im} \mathfrak{p}^* = T$ and $\operatorname{im} \mathfrak{q}^* = T^\perp$. Since this is going to be done frequently, let us summarize the procedure in the following way.

3.14. Let

$$\begin{gathered}
n, m \in \mathbb{N} \sim \{0\}\,, \quad m \leqslant n\,, \quad \mathfrak{p} \in \mathbf{O}^*(n,m)\,, \quad \mathfrak{q} \in \mathbf{O}^*(n, n-m)\,, \\
T \in \mathbf{G}(n,m)\,, \quad A \subseteq \mathbb{R}^m\,, \quad f : A \to \mathbb{R}^{n-m}\,, \quad F : A \to \mathbb{R}^n
\end{gathered}$$

be such that

$$\operatorname{im} \mathfrak{p}^* = T\,, \quad \operatorname{im} \mathfrak{q}^* = T^\perp\,, \quad F = \mathfrak{p}^* + \mathfrak{q}^* \circ f\,.$$

Then

$$\begin{gathered}
\mathfrak{q}|_{T^\perp} : T^\perp \to \mathbb{R}^{n-m} \quad \text{and} \quad \mathfrak{p}|_T : T \to \mathbb{R}^m \quad \text{are a linear isometries,} \\
F \text{ is injective,} \quad F^{-1} : \operatorname{im} F \to \mathbb{R}^m\,, \quad F^{-1} = \mathfrak{p}|_{\operatorname{im} F}\,, \quad \operatorname{im} F = \operatorname{graph}(f)\,.
\end{gathered}$$

Moreover, $\bigwedge_k \mathfrak{p}$ is an orthogonal projection (cf. [Fed69, 1.7.6]) for any $k \in \mathbb{N}$, so if f is differentiable at some $x \in \operatorname{Int} A$, then

(13)
$$\begin{gathered}
\|DF(x)\|^m \geqslant \|\textstyle\bigwedge_m DF(x)\| \geqslant \|\bigwedge_m \mathfrak{p} \circ \bigwedge_m DF(x))\| = \|\bigwedge_m \mathbf{id}_{\mathbb{R}^m}\| = 1\,, \\
Df(x) = 0 \quad \text{if and only if} \quad DF(x) = \mathfrak{p}^* \quad \text{if and only if} \quad \operatorname{Tan}(\operatorname{im} F, F(x)) = T\,.
\end{gathered}$$

Finally, if $\mathbf{Lip}(f) \leqslant L$, then F is bilipschitz and

$$\mathbf{Lip}(F) \leqslant (1 + L^2)^{1/2} \quad \text{and} \quad \mathbf{Lip}(F^{-1}) = \mathbf{Lip}(\mathfrak{p}|_{\operatorname{im} F}) \leqslant 1\,.$$

The following remark, made in the spirit of [All72, 8.9(5)], allows to express the "slope" of the tangent plane to a graph by the norm of the derivative of the function; see 3.16.

3.15 Remark. Assume $T \in \mathbf{G}(n,m)$ and $\eta \in \operatorname{Hom}(T,T^\perp)$. Set $S = \{v+\eta(v) : v \in T\} \in \mathbf{G}(n,m)$. Observe that the function $[0,\infty) \ni t \mapsto t^2(1+t^2)^{-1}$ is increasing; hence, using [All72, 8.9(3)],

$$\begin{aligned}
\|S_\natural - T_\natural\|^2 = \|T_\natural^\perp \circ S_\natural\|^2 &= \sup\left\{|T_\natural^\perp u|^2|u|^{-2} : u \in S \sim \{0\}\right\} \\
&= \sup\left\{|\eta(w)|^2|w+\eta(w)|^{-2} : w \in T \sim \{0\}\right\} \\
&= \sup\left\{|\eta(w)|^2(1+|\eta(w)|^2)^{-1} : w \in T\,,\ |w| = 1\right\} = \frac{\|\eta\|^2}{1+\|\eta\|^2}\,.
\end{aligned}$$

3.16 Corollary. *Let* m, n, $\mathfrak{p}$, $\mathfrak{q}$, T, A, f, *and* F *be as in 3.14. Assume that* $\Sigma \subseteq \operatorname{im} F$, $a \in \Sigma$ *is such that* $\operatorname{Tan}(\Sigma,a) \in \mathbf{G}(n,m)$ *and that* f *is differentiable at* $x = \mathfrak{p}(a)$. *Then, employing 3.15 with* $\mathrm{D}f(x)$ *in place of* η, *we obtain*

$$\|\operatorname{Tan}(\Sigma,a)_\natural - T_\natural\|^2 = \frac{\|\mathrm{D}f(x)\|^2}{1+\|\mathrm{D}f(x)\|^2} \tag{14}$$

Let $b \in \Sigma$ *and set* $y = \mathfrak{p}(b)$. *Then* $b = F(y)$ *and* $\mathrm{D}F(x)(y-x) \in \operatorname{Tan}(\Sigma,a)$. *Define*

$$\begin{aligned}
&u = \mathfrak{q}^*(f(y) - f(x) - \mathrm{D}f(x)(y-x)) = F(x) - F(y) - \mathrm{D}F(x)(y-x) \in T^\perp \\
\textit{and}\quad & v = \operatorname{Tan}(\Sigma,a)_\natural^\perp(b-a) = \operatorname{Tan}(\Sigma,a)_\natural^\perp u\,.
\end{aligned}$$

Then, by (14), *we get*

$$|u-v| = |\operatorname{Tan}(\Sigma,a)_\natural u| = |\operatorname{Tan}(\Sigma,a)_\natural T_\natural^\perp u| \leqslant \|\operatorname{Tan}(\Sigma,a)_\natural - T_\natural\||u| = \frac{\|\mathrm{D}f(x)\|\,|u|}{(1+\|\mathrm{D}f(x)\|^2)^{1/2}}\,.$$

In consequence,

$$\begin{aligned}
&|\operatorname{Tan}(\Sigma,a)_\natural^\perp(b-a)| \leqslant |f(y) - f(x) - \mathrm{D}f(x)(y-x)|\,, \\
&|\operatorname{Tan}(\Sigma,a)_\natural^\perp(b-a)| \geqslant \left(1 - \frac{\|\mathrm{D}f(x)\|}{(1+\|\mathrm{D}f(x)\|^2)^{1/2}}\right)|f(y) - f(x) - \mathrm{D}f(x)(y-x)|\,.
\end{aligned} \tag{15}$$

In particular, if f *is Lipschitz with* $\operatorname{Lip}(f) \leqslant \frac{1}{2}$, *recalling that the function* $[0,\infty) \ni t \mapsto t^2(1+t^2)^{-1}$ *is increasing, we obtain*

$$\operatorname{Tan}(\Sigma,a)_\natural^\perp(b-a) \leqslant |f(y) - f(x) - \mathrm{D}f(x)(y-x)| \leqslant 2\operatorname{Tan}(\Sigma,a)_\natural^\perp(b-a)\,.$$

3.17 Remark. If $\Sigma \subseteq \mathbb{R}^n$ is $\mathcal{H}^m$ measurable and $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^1$, then there exists a countable collection $\mathcal{A} = \{M_i : i \in \mathbb{N}\}$ of $\mathscr{C}^1$ submanifolds of $\mathbb{R}^n$ such that $\mathcal{H}^m(\Sigma \sim \bigcup \mathcal{A}) = 0$. Given $\alpha \in (0,1]$, to prove that Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$, it suffices to prove that $\Sigma \cap M_i$ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$ for each $i \in \mathbb{N}$ separately. Therefore, assume $\Sigma \subseteq M$ for some $\mathscr{C}^1$ submanifold of $\mathbb{R}^n$.

Employing the definition of a submanifold [Fed69, 3.1.19(5)], we can represent M locally, around any $a \in M$, as a graph over the tangent plane $\operatorname{Tan}(M,a)$ of some $\mathscr{C}^1$ function, i.e. we can find a neighborhood U_a of a in $\mathbb{R}^n$ and projections $\mathfrak{p}_a \in \mathbf{O}^*(n,m)$, $\mathfrak{q}_a \in \mathbf{O}^*(n,n-m)$ such that

$$\begin{gathered}
\operatorname{im}\mathfrak{p}_a^* = \operatorname{Tan}(M,a)\,,\quad \operatorname{im}\mathfrak{q}_a^* = \operatorname{Tan}(M,a)^\perp\,,\quad \mathfrak{p}_a|_{M\cap U_a} \text{ is injective}\,, \\
(\mathfrak{p}_a|_{M\cap U_a})^{-1} : \mathfrak{p}_a[U_a] \to \mathbb{R}^n \text{ is of class } \mathscr{C}^1\,,\quad \mathrm{D}\big((\mathfrak{p}_a|_{M\cap U_a})^{-1}\big)(\mathfrak{p}_a(a)) = \mathfrak{p}_a^*\,.
\end{gathered}$$

Set $F_a = (\mathfrak{p}_a|_{M \cap U_a})^{-1}$ and $f_a = \mathfrak{q}_a \circ F_a$; then

$$F_a = \mathfrak{p}_a^* + \mathfrak{q}_a^* \circ f_a\,, \quad f_a(\mathfrak{p}_a(a)) = 0\,, \quad Df_a(\mathfrak{p}_a(a)) = 0\,.$$

Define an open "cuboid" adjusted to $\operatorname{Tan}(M, a)$ of radius $r \in (0, \infty)$ by the formula

$$\begin{aligned}\mathbf{C}(a, r) &= a + \mathfrak{p}_a^*\big[\mathbf{U}(0, r)\big] + \mathfrak{q}_a^*\big[\mathbf{U}(0, r)\big] \\ &= \big\{y \in \mathbb{R}^n : |\mathfrak{p}_a(y) - \mathfrak{p}_a(a)| < r \text{ and } |\mathfrak{q}_a(y) - \mathfrak{q}_a(a)| < r\big\}\,.\end{aligned}$$

Recall U_a is a neighborhood of a in $\mathbb{R}^n$ so $a \in \operatorname{Int} U_a$. Thus, given any $L \in (1, 2]$ for all $a \in M$ there exists a radius $r_a > 0$ such that

$$\begin{gathered} M \cap \mathbf{C}(a, r_a) = F_a[\mathbf{U}(\mathfrak{p}_a(a), r_a)]\,, \\ \text{and} \quad 1 + \|Df_a(x)\| \leqslant L \quad \text{for } x \in \mathbf{B}(\mathfrak{p}_a(a), r_a). \end{gathered}$$

Next, observe that Σ is a second-countable space as a subspace of a second-countable space $\mathbb{R}^n$; hence, it has the Lindelöf property (cf. [Mun00, Theorem 30.3]). Thus, from the open covering $\big\{\mathbf{C}(a, r_a) \cap \Sigma : a \in \Sigma\big\}$ of Σ, one can choose a countable subcovering $\big\{\mathbf{C}(a_j, r_{a_j}) \cap \Sigma : j \in \mathbb{N}\big\}$ of Σ. Now, to prove that Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$, it suffices to prove that $\Sigma \cap \mathbf{C}(a_j, r_{a_j})$ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$ for each $j \in \mathbb{N}$ separately.

Therefore, in the sequel we shall usually assume that the following conditions hold

$$(16) \quad \left\{\begin{gathered} m, n, \mathfrak{p}, \mathfrak{q}, A, f, F \text{ are as in 3.14}\,, \quad L \in (1, 2]\,, \quad r_0 \in (0, \infty)\,, \quad x_0 \in \mathbb{R}^m\,, \\ A \subseteq \mathbf{U}(x_0, r_0) \text{ is } \mathcal{L}^m \text{ measurable}\,, \quad f = g|_A \text{ for some } g \in \mathscr{C}^1(\mathbf{U}(x_0, r_0), \mathbb{R}^{n-m})\,, \\ g(x_0) = 0\,, \quad Dg(x_0) = 0\,, \quad 1 + \operatorname{Lip}(g) \leqslant L\,, \quad \Sigma = F[A]\,. \end{gathered}\right.$$

3.18 Remark. Let Σ be a compact m dimensional submanifold of $\mathbb{R}^n$ of class $\mathscr{C}^2$ and $a \in \Sigma$. Proceeding as in 3.17, there exists a neighborhood U of a in $\mathbb{R}^n$ such that we can represent $\Sigma \cap U$ as the graph of a function f over the tangent plane $\operatorname{Tan}(\Sigma, a)$, i.e., we can find $\mathfrak{p} \in \mathbf{O}^*(n, m)$, $\mathfrak{q} \in \mathbf{O}^*(n, n - m)$ such that $\operatorname{im} \mathfrak{p}^* = \operatorname{Tan}(\Sigma, a)$, and $\operatorname{im} \mathfrak{q}^* = \operatorname{Tan}(\Sigma, a)^\perp$, and $\mathfrak{p}|_{\Sigma \cap U}$ is injective, and $f = \mathfrak{q} \circ (\mathfrak{p}|_{\Sigma \cap U})^{-1} : \mathfrak{p}[U] \to \mathbb{R}^{n-m}$ is of class $\mathscr{C}^2$, and $Df(\mathfrak{p}(a)) = 0$. Then

$$\begin{gathered} \mathfrak{q}^*(D^2 f(\mathfrak{p}(a))(\mathfrak{p}(u), \mathfrak{p}(v))) = \mathbf{b}_\Sigma(a)(u, v) \quad \text{whenever } u, v \in \operatorname{Tan}(\Sigma, a)\,, \\ \|\mathbf{b}_\Sigma(a)\| = \|D^2 f(\mathfrak{p}(a))\| = \lim_{s \downarrow 0} \sup \left\{ \frac{\|Df(y) - Df(\mathfrak{p}(a))\|}{|y - \mathfrak{p}(a)|} : y \in \mathbf{B}(\mathfrak{p}(a), s) \sim \big\{\mathfrak{p}(a)\big\} \right\}, \end{gathered}$$

where $\mathbf{b}_\Sigma$ denotes the second fundamental form of $\Sigma \subseteq \mathbb{R}^n$; cf. [Sim83, 7.3].

Main higher order rectifiability criterion for graphs

To talk about approximate features of functions (limits, continuity, differentiability; cf. [Fed69, 2.9.2 , 3.1.2 , 3.2.16]) one needs to provide two parameters: a measure and a Vitali relation (cf. [Fed69, 2.8.16]). It will be convenient to define a standard family of Vitali relations.

3.19 Definition. For $k \in \mathbb{N} \sim \{0\}$, we set

$$\mathcal{V}_k = \big\{(x, \mathbf{B}(x, r)) : x \in \mathbb{R}^k\,,\ r \in (0, \infty)\big\}\,.$$

3.20 Remark. If $k \in \mathbb{N} \sim \{0\}$ and ϕ is a measure over $\mathbb{R}^k$ such that all open sets are ϕ measurable and $\phi(A) < \infty$ for all bounded sets $A \subseteq \mathbb{R}^k$, then due to [Fed69, 2.8.18] the family $\mathcal{V}_k$ is a ϕ Vitali relation.

In the following proposition, whenever we write $\operatorname{ap} \mathrm{D} f$ we mean the approximate differential with respect to $(\mathcal{L}^m, \mathcal{V}_m)$.

3.21 Proposition. *Let $\alpha \in (0,1]$. Suppose $A \subseteq \mathbb{R}^m$ is $\mathcal{L}^m$-measurable and such that $\boldsymbol{\Theta}^m(\mathcal{L}^m \llcorner A, a) = 1$ for all $a \in A$. Let $f : A \to \mathbb{R}^{n-m}$ be $(\mathcal{L}^m, \mathcal{V}_m)$ approximately differentiable on A and satisfy one of the following conditions*

$$\limsup_{r \downarrow 0} r^{-m} \int_{A \cap \mathbf{B}(y,r)} \frac{|f(z) - f(y) - \operatorname{ap} \mathrm{D} f(y)(z-y)|}{|z-y|^{1+\alpha}} \, \mathrm{d}\mathcal{L}^m(z) < \infty \quad \textit{for all } y \in A \,,$$

or

$$(\mathcal{L}^m, \mathcal{V}_m) \operatorname{ap} \limsup_{z \to y} \frac{|f(z) - f(y) - \operatorname{ap} \mathrm{D} f(y)(z-y)|}{|z-y|^{1+\alpha}} < \infty \quad \textit{for all } y \in A \,.$$

Then there exist functions $f_k \in \mathscr{C}^{1,\alpha}(\mathbb{R}^m, \mathbb{R}^{n-m})$, such that

$$\mathcal{L}^m \left(A \sim \textstyle\bigcup_{k=1}^{\infty} \left\{ x \in A : f(x) = f_k(x) \textit{ and } \operatorname{ap} \mathrm{D} f(x) = \mathrm{D} f_k(x) \right\} \right) = 0.$$

In particular, if $\mathcal{L}^m(A) < \infty$, then the graph of f is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$.

Proof. The proof can be found in [Sch09, Lemma A.1] for the case $\alpha = 1$. If $0 < \alpha < 1$, exactly the same proof, with relevant occurrences of 2 replaced by $1 + \alpha$, establishes the assertion. $\square$

3.22 Corollary. *Let m, n, $\mathfrak{p}$, $\mathfrak{q}$, T, A, f, and F be as in 3.14. Suppose $\alpha \in (0,1]$, A is $\mathcal{L}^m$ measurable, $\mathcal{L}^m(A) < \infty$, and f is $(\mathcal{L}^m, \mathcal{V}_m)$ approximately differentiable on A. Set $\Sigma = F[A]$. Assume that one of the following conditions is satisfied for $\mathcal{H}^m$ almost all $a \in \Sigma$*

$$\limsup_{r \downarrow 0} r^{-m} \int_{\Sigma \cap \mathbf{B}(a,r)} \frac{|\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)^{\perp}_{\natural}(b-a)|}{|b-a|^{1+\alpha}} \, \mathrm{d}\mathcal{H}^m(b) < \infty \tag{17}$$

or

$$(\mathcal{H}^m \llcorner \Sigma, \mathcal{V}_n) \operatorname{ap} \limsup_{b \to a} \frac{|\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)^{\perp}_{\natural}(b-a)|}{|b-a|^{1+\alpha}} < \infty \,. \tag{18}$$

Then Σ is $\mathcal{H}^m$ measurable and $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$.

Proof. Employing [Fed69, 3.1.8] we can divide A into a countable family of $\mathcal{L}^m$ measurable sets $\{A_i : i \in \mathbb{N}\}$ such that f restricted to each of A_i is Lipschitz and $\bigcup_{i \in \mathbb{N}} A_i = A$. Then $F|_{A_i}$ is bilipschitz and, since $\mathcal{H}^m$ and $\mathcal{L}^m$ are Borel regular, $\Sigma_i = F[A_i]$ is $\mathcal{H}^m$ measurable for each $i \in \mathbb{N}$. Hence, $\Sigma = \bigcup_{i \in \mathbb{N}} \Sigma_i$ is also $\mathcal{H}^m$ measurable. Moreover, if one of the conditions (17) or (18) is satisfied for $\mathcal{H}^m$ almost all $a \in \Sigma$, then the same condition holds for $\mathcal{H}^m$ almost all $a \in \Sigma_i$ for each $i \in \mathbb{N}$. Hence, it suffices to prove the Corollary separately for each A_i and Σ_i in place of A and Σ. In the sequel we will assume this replacement has been done and that f has been extended to the whole of $\mathbb{R}^m$ by means of the Kirszbraun's theorem [Fed69, 2.10.43], so that we have

$$\begin{gathered} f : \mathbb{R}^m \to \mathbb{R}^{n-m} \quad \text{satisfies} \quad L = \operatorname{Lip}(f) < \infty \\ \text{and} \quad \operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a) = \operatorname{Tan}(\Sigma, a) \quad \text{for } \mathcal{H}^m \text{ almost all } a \in \Sigma. \end{gathered}$$

Define the set $\Sigma' \subseteq \Sigma$ in the following way:

- if (17) holds for $\mathcal{H}^m$ almost all $a \in \Sigma$, set

$$\Sigma' = \left\{ a \in \Sigma : \limsup_{r \downarrow 0} r^{-m} \int_{\Sigma \cap \mathbf{B}(a,r)} \frac{|\operatorname{Tan}(\Sigma, a)^{\perp}_{\natural}(b-a)|}{|b-a|^{1+\alpha}} \, d\mathcal{H}^m(b) < \infty \right\};$$

- if (18) holds for $\mathcal{H}^m$ almost all $a \in \Sigma$, set

$$\Sigma' = \left\{ a \in \Sigma : (\mathcal{H}^m \llcorner \Sigma, \mathcal{V}_n) \operatorname{ap}\limsup_{b \to a} \frac{|\operatorname{Tan}(\Sigma, a)^{\perp}_{\natural}(b-a)|}{|b-a|^{1+\alpha}} < \infty \right\}.$$

Since $\mathcal{H}^m(\Sigma \sim \Sigma') = 0$, we know Σ' is $\mathcal{H}^m$ measurable. Recall the definitions of $\mathfrak{p}$, $\mathfrak{q}$, and F from 3.14. Set $B' = \mathfrak{p}[\Sigma']$ and note that $B' = F^{-1}[\Sigma']$ so it is $\mathcal{L}^m$ measurable. Next, set $\tilde{B} = \{x \in B' : Df(x) \text{ exists}\}$. Then $\mathcal{L}^m(B' \sim \tilde{B}) = 0$ due to the Rademacher's theorem (cf. [Fed69, 3.1.6]); hence, $\tilde{B}$ is also $\mathcal{L}^m$ measurable. Define $B = \{x \in \tilde{B} : \boldsymbol{\Theta}^m(\mathcal{L}^m \llcorner \tilde{B}, x) = 1\}$. Then, by [Fed69, 2.9.11], B is $\mathcal{L}^m$ measurable, $\mathcal{L}^m(\tilde{B} \sim B) = 0$ and $\boldsymbol{\Theta}^m(\mathcal{L}^m \llcorner B, x) = 1$ for *all* $x \in B$. Observe

(19) $$\mathcal{H}^m(\Sigma \sim F[B]) = \mathcal{H}^m(\Sigma \sim \Sigma') + \mathcal{H}^m(F[B' \sim B]) = 0 \quad \text{because } F \text{ is Lipschitz;}$$

hence, it suffices to check that 3.21 applies to $f|_B$.

Set $\lambda = (1 + L^2)^{-1/2} \in (0, 1]$ and note that $\operatorname{Lip}(F) \leqslant \lambda^{-1}$; hence,

(20) $$F[\mathbf{B}(\mathfrak{p}(a), \lambda r) \cap B] \subseteq \mathbf{B}(a, r) \cap F[B] \quad \text{for each } a \in \Sigma \text{ and } r \in (0, \infty);$$

Employing (19) combined with 3.16 (15) and then applying the area formula [Fed69, 3.2.3] together with (20) and 3.14 (13), we obtain

$$\begin{aligned} & r^{-m} \int_{\mathbf{B}(a,r) \cap \Sigma} \frac{|\operatorname{Tan}(\Sigma, a)^{\perp}_{\natural}(b-a)|}{|b-a|^{1+\alpha}} \, d\mathcal{H}^m(b) \\ & \quad \geqslant \lambda^{1+\alpha}(1 - \lambda L) r^{-m} \int_{\mathbf{B}(a,r) \cap F[B]} \frac{|f(\mathfrak{p}(b)) - f(\mathfrak{p}(a)) - Df(\mathfrak{p}(a))(\mathfrak{p}(b) - \mathfrak{p}(a))|}{|\mathfrak{p}(b) - \mathfrak{p}(a)|^{1+\alpha}} \, d\mathcal{H}^m(b) \\ & \quad \quad \geqslant \lambda^{1+\alpha+m}(1 - \lambda L)(\lambda r)^{-m} \int_{\mathbf{B}(x, \lambda r) \cap B} \frac{|f(y) - f(x) - Df(x)(y-x)|}{|y-x|^{1+\alpha}} \, d\mathcal{L}^m(y) \end{aligned}$$

for $r \in (0, \infty)$, $x \in B$, and $a = F(x)$. Hence, if (17) holds, then one can employ 3.21 to see that $F[B]$ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$ and, due to (19), so is Σ.

Fix $a \in F[B]$ and set $x = \mathfrak{p}(a)$. For $y \in B$ and $b \in F[B]$ define

$$g(b) = \frac{|\operatorname{Tan}(\Sigma, a)^{\perp}_{\natural}(b-a)|}{|b-a|^{1+\alpha}}, \quad h(y) = \frac{|f(y) - f(x) - Df(x)(y-x)|}{|y-x|^{1+\alpha}},$$
$$\text{and} \quad \phi = \mathcal{H}^m \llcorner F[B] = \mathcal{H}^m \llcorner \Sigma,$$

Setting $\Delta = \lambda^{1+\alpha}(1-\lambda L)$ we obtain, by 3.16 (15) and the area formula [Fed69, 3.2.3],

$$\Delta h(\mathfrak{p}(b)) \leqslant g(b) \quad \text{and} \quad \mathcal{L}^m(S) \leqslant \phi(F[S]) \leqslant \lambda^{-m}\mathcal{L}^m(S)$$

whenever $b \in F[B]$ and $S \subseteq B$ is $\mathcal{L}^m$ measurable. Hence, for each $r, t \in (0,\infty)$

$$\{y \in B : h(y) > t\} \subseteq \mathfrak{p}\big[\{b \in F[B] : g(b) > \Delta t\}\big]\,,$$
$$\frac{\mathcal{L}^m(\mathbf{B}(x,\lambda r) \cap \{y \in B : h(y) > t\})}{\mathcal{L}^m(\mathbf{B}(x,r)\cap B)} \leqslant \frac{\phi(\mathbf{B}(a,r) \cap \{b \in F[B] : g(b) > \Delta t\})}{\lambda^m \phi(\mathbf{B}(a,r))}\,.$$

Therefore,

$$\text{(21)} \quad \inf\left\{t \in \mathbb{R} : \lim_{r\downarrow 0} \frac{\mathcal{L}^m(\mathbf{B}(x,\lambda r) \cap \{y \in B : h(y) > t\})}{\mathcal{L}^m(\mathbf{B}(x,r)\cap B)} = 0\right\}$$
$$\leqslant \inf\left\{t \in \mathbb{R} : \lim_{r\downarrow 0} \frac{\phi(\mathbf{B}(a,r) \cap \{b \in F[B] : g(b) > \Delta t\})}{\lambda^m \phi(\mathbf{B}(a,r))} = 0\right\}\,.$$

For any $x \in B$ we have $\boldsymbol{\Theta}^m(\mathcal{L}^m \llcorner B, x) = 1$ so it follows that

$$\text{(22)} \qquad \lim_{r\downarrow 0} \frac{\mathcal{L}^m(\mathbf{B}(x,\lambda r))}{\mathcal{L}^m(\mathbf{B}(x,r)\cap B)} = \lambda^m < \infty\,.$$

Recalling $x = \mathfrak{p}(a) \in B$ was chosen arbitrarily and combining (21) with (22) yields

$$(\mathcal{L}^m, \mathcal{V}_m)\operatorname{ap}\limsup_{y\to x} h(y) \leqslant (\phi, \mathcal{V}_n)\operatorname{ap}\limsup_{b\to a} g(b)$$

for all $x \in B$ and $a = F(x)$. Consequently, if (18) holds, then one can employ 3.21 to see that $F[B]$ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$ and, because of (19), so is Σ. □

4 Higher order rectifiability via the tangent-point curvature

Recall that if $\Sigma \subseteq \mathbb{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^1$ and $a, b \in \Sigma$ are such that $\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a) \in \mathbf{G}(n,m)$, then $r_{tp}[\Sigma](a,b)$ was defined by (4) and $\tau^{1,p}[\Sigma]$ and $\tau^{2,p}[\Sigma]$ were defined by (5).

4.1 Remark. If Σ is an m dimensional submanifold of $\mathbb{R}^n$ of class $\mathscr{C}^2$ and $a \in \Sigma$, then

$$\lim_{r\downarrow 0} \sup\big\{r_{tp}[\Sigma](a,b)^{-1} : b \in \Sigma \cap \mathbf{B}(a,r)\big\} = \|\mathbf{b}_\Sigma(a)\|\,,$$

where $\mathbf{b}_\Sigma$ denotes the second fundamental form of $\Sigma \subseteq \mathbb{R}^n$ which can be verified by means of 3.16 (15) and 3.18.

4.2 Lemma. *Let* $\alpha \in (0,1]$, $l \in \{1,2\}$, $p > m(l-1)$, $f \in \mathscr{C}^{1,\alpha}(\mathbb{R}^m, \mathbb{R}^{n-m})$*, and* $\Sigma \subseteq \mathbb{R}^n$ *be the graph of* $f|_{\mathbf{B}(0,1)}$. *Assume one of the following holds*

(a) $l = 2$ *and* $\alpha > 1 - m(l-1)/p$,

(b) or $l = 1$ *and* $\alpha = 1$,

then $\tau^{l,p}[\Sigma](a) < \infty$ *for all* $a \in \Sigma$.

Proof. In both cases (a) and (b) for any $a, b \in \Sigma$ with $a \neq b$ we have, by 3.16 (15),

$$r_{\mathrm{tp}}[\Sigma](a,b)^{-1} \leqslant 2K|b-a|^{\alpha-1},$$

where $K = \sup\{\|Df(\mathfrak{p}(c)) - Df(\mathfrak{p}(a))\| |\mathfrak{p}(c) - \mathfrak{p}(a)|^{-\alpha} : c \in \Sigma,\ 0 < |c-a| \leqslant |b-a|\}$ and $\mathfrak{p} \in \mathbf{O}^*(n,m)$ is given by $\mathfrak{p}(x_1,\dots,x_n) = (x_1,\dots,x_m)$. Hence, if $l = 1$, then $\alpha = 1$ and $\tau^{1,p}[\Sigma](a) < 2K$ for all $a \in \Sigma$. Otherwise, $l = 2$ and $p(\alpha - 1) > -m$; thus, $\tau^{2,p}[\Sigma](a) < \infty$ for all $a \in \Sigma$ by the area formula. □

4.3 Theorem. *Consider the situation as in 3.17* (16)*. Let $l \in \{1,2\}$, $p \in \mathbb{R}$ satisfy $p > m(l-1)$, and $\alpha = 1 - \frac{m(l-1)}{p}$. Assume $\tau^{l,p}[\Sigma](a) < \infty$ for $\mathcal{H}^m$ almost all $a \in \Sigma$. Then Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$.*

Proof. In case $l = 1$ one applies 3.22 directly to see that Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,1}$.

Assume now that $l = 2$. For brevity of the notation let us set

$$d(a,b) = |\mathrm{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)^{\perp}_{\natural}(b-a)|$$

whenever $a, b \in \Sigma$ are such that $\mathrm{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a) \in \mathbf{G}(n,m)$. Using Hölder's inequality one gets

$$\text{(23)} \quad \int_{\mathbf{B}(a,r)} \frac{d(a,b)}{|b-a|^{2-m/p}}\, d\mu^1_\Sigma(b) \\ \leqslant \left(\int_{\mathbf{B}(a,r)} \frac{d(a,b)^p}{|b-a|^{2p}}\, d\mu^1_\Sigma(b)\right)^{1/p} \left(\int_{\mathbf{B}(a,r)} |b-a|^{m/(p-1)}\, d\mu^1_\Sigma(b)\right)^{1-1/p}.$$

for $\mathcal{H}^m$ almost all $a \in \Sigma$ and all $r \in (0,\infty)$. Recalling 3.17 (16), in particular $\mathrm{Lip}(F) \leqslant L$, and employing the area formula [Fed69, 3.2.3], we get

$$\text{(24)} \quad \left(\int_{\mathbf{B}(a,r)} |b-a|^{m/(p-1)}\, d\mu^1_\Sigma(b)\right)^{1-1/p} \\ \leqslant \left(L^m \int_{\mathbf{B}(p(a),r)\cap A} |y-x|^{m/(p-1)}\, d\mathcal{L}^m(y)\right)^{1-1/p} \leqslant \left(L^m \boldsymbol{\alpha}(m) \frac{p-1}{p}\right)^{1-1/p} r^m$$

for $\mathcal{H}^m$ almost all $a \in \Sigma$ and all $r \in \mathbb{R}$ with $0 < r < r_0 - |\mathfrak{p}(a - x_0)|$. Combining (23) with (24) gives

$$\lim_{r\downarrow 0} r^{-m} \int_{\mathbf{B}(a,r)} \frac{d(a,b)}{|b-a|^{1+\alpha}}\, d\mu^1_\Sigma(b) \leqslant C \lim_{r\downarrow 0} \left(\int_{\mathbf{B}(a,r)} \frac{d(a,b)^p}{|b-a|^{2p}}\, d\mu^1_\Sigma(b)\right)^{1/p} = 0$$

for $\mathcal{H}^m$ almost all $a \in \Sigma$, where $C = (L^m \boldsymbol{\alpha}(m)(1-1/p))^{1-1/p}$. Applying 3.22 we see that Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$. □

Now we are in position to prove our second main theorem.

Proof of Theorem 1.4. The first part of theorem 1.4 follows immediately by combining 4.3 with 3.17. Thus, it remains to show that α is indeed optimal.

In case $l = 2$, choose $\varepsilon \in (0,1)$ such that $\alpha + 2\varepsilon < 1$ and consider a function $f : [0,1] \to \mathbb{R}$ of class $\mathscr{C}^{1,\alpha+2\varepsilon}$ which graph is not $(\mathcal{H}^1, 1)$ rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$ (a construction of such function can be found, e.g., in [AS94, Appendix]). Then, setting $\Sigma = \operatorname{graph}(f)$, we obtain $\tau^{1,p}[\Sigma](a) < \infty$ for all $a \in \Sigma$, by 4.2, but Σ is not $(\mathcal{H}^1, 1)$ rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$.

In case $l = 1$ we proceed similarly. We chose $\varepsilon \in (0,1)$ and a function $f : [0,1] \to \mathbb{R}$ of class $\mathscr{C}^2$ which graph is not $(\mathcal{H}^1, 1)$ rectifiable of class $\mathscr{C}^{2,\varepsilon}$. Setting $\Sigma = \operatorname{graph}(f)$ and applying 4.2 shows sharpness of α. □

4.4 Remark. Rectifiability of class $\mathscr{C}^{1,1}$ implies rectifiability of class $\mathscr{C}^2$ (cf. [AS94, Proposition 3.2]) so if $l = 1$ in 1.4, then Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^2$.

5 Estimate on the minimal height of a simplex inside a slab

Recall that whenever we write $T \in (\mathbb{R}^n)^{k+1}$ for some $k \in \mathbb{N} \sim \{0\}$ we mean that $T = (p_0, \dots, p_k)$ for some $k+1$ points $p_0, \dots, p_k \in \mathbb{R}^n$. If $\mathcal{H}^k(\triangle T) > 0$, then $\triangle T$ is a k dimensional simplex with vertexes $p_0, \dots, p_k$.

5.1 Definition. The *minimum height* of $T = (p_0, \dots, p_k) \in (\mathbb{R}^n)^{k+1}$ is defined as

$$h_{\min}(T) = \min\left\{ |(P_i)_\natural^\perp(p_i - p_j)| : i, j \in \{0, 1, \dots, k\},\ i \neq j \right\},$$

where $P_i = \operatorname{span}\left\{ p_j - p_l : j, l \in \{0, 1, \dots, k\} \sim \{i\} \right\}$.

Remark. Note that $h_{\min}(T) = 0$ if and only if $\mathcal{H}^k(\triangle T) = 0$.

Remark. If $T = (p_0, \dots, p_k)$, $S = (q_0, \dots, q_j)$ for some $j, k \in \mathbb{N} \sim \{0\}$ satisfy $j \leqslant k$, $h_{\min}(T) > 0$, and $\{q_0, \dots, q_j\} \subseteq \{p_0, \dots, p_k\}$, then $h_{\min}(S) \geqslant h_{\min}(T)$. In other words, the minimal height of $\triangle T$ is shorter then any height of any of the faces of $\triangle T$.

Now we shall estimate $h_{\min}(T)$ in case we know $\triangle T$ lies inside a thin slab as is the case when the vertexes of $\triangle T$ lie on a smooth submanifold of $\mathbb{R}^n$ and $\operatorname{diam}(\triangle T)$ is very small. First we need to show that the minimal height of $\triangle T$ is realized by a line segment which might not be contained inside $\triangle T$ but at least lies close to $\triangle T$ – this is proven in 5.2. Next, we show how to estimate $h_{\min}(T)$ in terms of the thickness of the slab; see 5.3.

5.2 Lemma. *Let* $m \in \mathbb{N}$*, and* $T = (p_0, \dots, p_{m+1}) \in (\mathbb{R}^n)^{m+2}$*, and* $P = \{0\}$ *if* $m = 0$ *or* $P = \operatorname{span}\{p_1 - p_0, \dots, p_m - p_0\}$ *if* $m > 0$*, and* $q = p_0 + P_\natural(p_{m+1} - p_0)$*. Assume*

$$h_{\min}(T) = |P_\natural^\perp(p_{m+1} - p_0)| = |p_{m+1} - q| > 0$$

$$\textit{and} \quad q - p_0 = \sum_{i=1}^m t_i(p_i - p_0) \quad \textit{for some } t_1, \dots, t_m \in \mathbb{R}.$$

Then $t_i \in (-1, 1)$ *for each* $i = 1, 2, \dots, m$.

Proof. Define $v_i = p_i - p_0$ for $i = 1, \dots, m+1$ and $w = P_\natural^\perp(p_{m+1} - p_0)|P_\natural^\perp(p_{m+1} - p_0)|^{-1}$. Note that for each $i = 0, 1, \dots, m+1$ the quantity $(m+1)\mathcal{H}^{m+1}(\triangle T)$, which does not depend on i, is equal to the product of the height of $\triangle T$ lowered from p_i and the $\mathcal{H}^m$ measure of the face of $\triangle T$ which does not contain p_i. Hence, since the height lowered from p_{m+1} is minimal, the face $\triangle(p_0,\dots,p_m)$ must have maximal $\mathcal{H}^m$ measure among the faces of $\triangle T$. Recall $m!\mathcal{H}^m(\triangle(p_0,\dots,p_m)) = |v_1\wedge\dots\wedge v_m|$. Using [Fed69, 1.7.5] and the fact that w is orthogonal to all of $v_1,\dots,v_m$ we obtain

$$\begin{aligned}|v_2\wedge\dots\wedge v_{m+1}|^2 = t_1^2|v_1\wedge v_2\wedge\dots\wedge v_m|^2 + h_{\min}(T)^2|w\wedge v_2\wedge\dots\wedge v_m|^2 \\ > t_1^2|v_1\wedge v_2\wedge\dots\wedge v_m|^2 .\end{aligned}$$

Since $|v_2\wedge\dots\wedge v_{m+1}|^2$ cannot be larger that $|v_1\wedge v_2\wedge\dots\wedge v_m|^2$, which is maximal, we see that $t_1^2 < 1$. Considering $|v_1\wedge\dots\wedge v_{i-1}\wedge v_{i+1}\wedge\dots\wedge v_{m+1}|^2$ in place of $|v_2\wedge\dots\wedge v_{m+1}|^2$ and estimating as above we see that also $t_i^2 < 1$ for $i = 2,\dots,m$. □

5.3 Lemma. *Let* $m\in\mathbb{N}$, $h\in[0,\infty)$, $T\in(\mathbb{R}^n)^{m+2}$, $a\in\mathbb{R}^n$, *and* $S\in\mathbf{G}(n,m)$. *Assume*

$$\triangle T \subseteq \left\{ b\in\mathbb{R}^n : |S_\natural^\perp(b-a)| \leqslant h \right\} .$$

Then $h_{\min}(T) \leqslant m(m+1)h$.

Proof. Without loss of generality we may assume $h_{\min}(T) > 0$. Let $p_0,\dots,p_{m+1}\in\mathbb{R}^n$ be such that $T = (p_0,\dots,p_{m+1})$. Set $P_{m+2} = \operatorname{span}\{p_1-p_0,\dots,p_{m+1}-p_0\}$. Since $h_{\min}(T) > 0$, we know $\dim P_{m+2} = m+1$. For $i = 1,\dots,m+1$ define

$$T_i = (p_0,\dots,p_i) \quad\text{and}\quad P_i = \operatorname{span}\{p_1-p_0,\dots,p_{i-1}-p_0\}\in\mathbf{G}(n,i-1) .$$

Possibly permuting the tuple T we can assume

$$h_{\min}(T_i) = |P_{i\natural}^{\perp}(p_i-p_0)| > 0 \quad\text{for } i = 1,\dots,m+1. \tag{25}$$

We shall first prove by induction the following *claim*:

if $i\in\{1,2,\dots,m+1\}$ *and there exist* $S_i\in\mathbf{G}(n,j-1)$ *and* $h_i\in(0,\infty)$ *satisfying*

$$\triangle T_i \subseteq \left\{ b\in\mathbb{R}^n : |S_{i\natural}^{\perp}(b-p_0)| \leqslant h_i \right\} \quad\textit{and}\quad S_i\subseteq P_{j+1} , \tag{26}$$

then $h_{\min}(T_i) \leqslant \frac{1}{2}i(i+1)h_i$.

If $i = 1$ and (26) holds for some $h_1\in(0,\infty)$ and $S_1\in\mathbf{G}(n,0)$, then $T_1 = (p_0,p_1)$ and $S_1 = \{0\}$. Hence, $\triangle T_1\subseteq\mathbf{B}(p_0,h_1)$, which immediately gives $h_{\min}(T_1) = |p_1-p_0| \leqslant h_1$.

Assume now $j\in\{2,\dots,m+1\}$, our claim is true for $i = 1,2,\dots,j-1$, and (26) holds for some $S_j\in\mathbf{G}(n,j-1)$ and $h_j\in(0,\infty)$. Consider the minimal height of $\triangle T_j$, which, due to the way we ordered vertexes of $\triangle T$, cf. (25), is realized by the height lowered from p_j. Setting $q = p_0 + P_{j\natural}(p_j-p_0)$ we have $h_{\min}(T_j) = |p_j-q|$. Let $u = P_{j\natural}^{\perp}(p_j-p_0) = p_j-q$ and $\zeta = |u|/h_j$ so that $|u| = h_{\min}(T_j) = \zeta h_j$. We claim that $\zeta \leqslant \frac{1}{2}j(j+1)$. Assume the contrary, i.e., $\zeta > \frac{1}{2}j(j+1) > j$. If this was the case, we could employ 5.2 to see that $q = \sum_{l=1}^{j-1} t_l(p_l-p_0)$ for some $t_1,\dots,t_{j-1}\in(-1,1)$ and then

$$|S_{j\natural}P_{j\natural}^{\perp}u| = |S_{j\natural}u| \geqslant |u| - |S_{j\natural}^{\perp}(p_j-p_0) - S_{j\natural}^{\perp}(q-p_0))| \geqslant \zeta h_j - jh_j = (\zeta-j)h_j .$$

Using [All72, 8.9(3)], we could then write

$$\|P_{j\natural} - S_{j\natural}\| = \|S_{j\natural}\circ P_{j\natural}^{\perp}\| \geqslant \frac{|S_{j\natural}P_{j\natural}^{\perp}u|}{|u|} \geqslant \frac{(\zeta-j)h_j}{\zeta h_j} = 1 - \frac{j}{\zeta} > 0 . \tag{27}$$

Recalling $\|S_{j\natural} - P_{j\natural}\| = \|S_{j\natural}^{\perp} \circ P_{j\natural}\|$, by [All72, 8.9(3)], we could find $v \in P_j \cap (S_j \cap P_j)^{\perp}$ such that $|v| = 1$ and $|S_{j\natural}^{\perp} v| = \|S_{j\natural} - P_{j\natural}\|$. Next, we would define $S_{j-1} = S_j \cap P_j$ and, recalling $\dim(S_j + P_j) = \dim(P_{j+1}) = j$, we would observe that

$$\dim S_{j-1} = \dim(S_j \cap P_j) = \dim S_j + \dim P_j - \dim(S_j + P_j) = j - 2\,;$$
$$\text{hence,} \quad S_{j-1} = \{\sigma \in P_j : \sigma \bullet v = 0\}\,,$$

and for any $b \in \triangle T_{j-1}$, since $S_{j-1} \subseteq S_j$ and $b - p_0 \in P_j$, employing (27),

$$h_j \geqslant |S_{j\natural}^{\perp}(b - p_0)| = |S_{j\natural}^{\perp} S_{j-1\natural}^{\perp}(b - p_0)| = \big|\big((b - p_0) \bullet v\big) S_{j\natural}^{\perp} v\big| \geqslant \frac{\zeta - j}{\zeta} \big|S_{j-1\natural}^{\perp}(b - p_0)\big|\,,$$

Hence, setting $h_{j-1} = \frac{\zeta}{\zeta - j} h_j$, we would obtain

$$\triangle T_{j-1} \subseteq \big\{b \in \mathbb{R}^n : |S_{j-1\natural}^{\perp}(b - p_0)| \leqslant h_{j-1}\big\}\,, \quad S_{j-1} \subseteq P_j\,,$$
$$\text{and} \quad h_{\min}(T_{j-1}) \geqslant h_{\min}(T_j) = \zeta h_j = (\zeta - j) h_{j-1} > \tfrac{1}{2} j(j-1) h_{j-1}\,,$$

which contradicts the inductive hypothesis.

Now it suffices to show that the claim can be applied to $T = T_{m+1}$ with $S_{m+1} = S$ and $h_{m+1} = h$. If it happens that $\dim S \cap P_{m+2} = l < m$, then $S^{\perp} \cap P_{m+2} = \operatorname{span}\{e_1, \dots, e_{m+1-l}\}$ for some orthonormal vectors $e_1, \dots, e_{m+1-l} \in \mathbb{R}^n$ and setting $\tilde{S} = (S \cap P_{m+2}) + \operatorname{span}\{e_1, \dots, e_{m-l}\}$ we obtain $\triangle T \subseteq \big\{b \in \mathbb{R}^n : |\tilde{S}_{\natural}^{\perp}(b - a)| \leqslant h\big\}$. Therefore, possibly considering $\tilde{S}$ in place of S, we can assume $S \subseteq P_{m+2}$. Furthermore, by triangle inequality,

$$\triangle T \subseteq \big\{b \in \mathbb{R}^n : |S_{\natural}^{\perp}(b - p_0)| \leqslant 2h = 2h_{m+1}\big\}\,.$$

Therefore, by the proven above claim, $h_{\min}(T) \leqslant m(m+1)h$. □

6 Basic properties of Menger like curvatures

Recall that if $\Sigma \subseteq \mathbb{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^1$ and κ is a Menger like curvature, then $\mathcal{K}_{\kappa}^{l,p}[\Sigma]$ was defined by (2) and $\mathcal{M}_{\kappa}^{l,p}$ by (3).

6.1 Remark. If Σ is a compact m dimensional submanifold of $\mathbb{R}^n$ of class $\mathscr{C}^2$, $p_0 \in \Sigma$, and κ is a tame Menger like curvature with exponent 1, then it follows from 3.18 that

$$\lim_{d \downarrow 0} \sup\big\{\kappa(T) : T = (p_0, \dots, p_{m+1}) \in \Sigma^{m+2} \cap \mathcal{D}_{m+1}\,, \operatorname{diam}(\triangle T) \leqslant d\big\} \leqslant \Gamma \|\mathbf{b}_{\Sigma}(p_0)\|\,,$$

where $\Gamma = \Gamma(\kappa)$ is a constant and $\mathbf{b}_{\Sigma}$ denotes the second fundamental form of $\Sigma \subseteq \mathbb{R}^n$.

Next, we give two principal examples of Menger like curvatures. A few other examples, including all the discrete curvatures of [LW09, LW11], can be found in Appendix A.

6.2 Example. Let $\gamma \in (0, \infty)$. For $T \in \mathcal{D}_{m+1}$ we set

$$\kappa_{\mathrm{vol}}^{\gamma}(T) = \left(\frac{(m+1)!\mathcal{H}^{m+1}(\triangle T)}{\operatorname{diam}(\triangle T)^{m+1}}\right)^{\gamma} \frac{1}{\operatorname{diam}(\triangle T)}\,,$$
$$\text{and} \quad \kappa_{h}^{\gamma}(T) = \left(\frac{h_{\min}(T)}{\operatorname{diam}(\triangle T)}\right)^{\gamma} \frac{1}{\operatorname{diam}(\triangle T)}\,,$$

and we set $\kappa^\gamma_{\mathrm{vol}}(T) = 0 = \kappa^\gamma_h(T)$ whenever $h_{\min}(T) = 0$ in accordance with 1.1 (b). Obviously $\kappa^\gamma_{\mathrm{vol}}$ and $\kappa^\gamma_h(T)$ satisfy 1.1 (a) (c) (d).

Assume now that $d, \delta \in (0,\infty)$, $T = (a, b_1, \dots, b_m, c)$ and P are as in 1.1(e). Note that

$$(m+1)!\mathcal{H}^{m+1}(\triangle T) = \big|(b_1 - a) \wedge \dots \wedge (b_m - a)\big| \operatorname{dist}(c - a, P)\,; \tag{28}$$

hence

$$\left(\frac{\operatorname{dist}(c-a,P)}{\operatorname{diam}(\triangle T)}\right)^\gamma \operatorname{diam}(\triangle T)^{-1} \geqslant \kappa^\gamma_{\mathrm{vol}}(T) \geqslant \left(\frac{\delta \operatorname{dist}(c-a,P)}{d}\right)^\gamma d^{-1}\,, \tag{29}$$

where the first inequality follows from the simple estimate $\big|(b_1 - a) \wedge \dots \wedge (b_m - a)\big| \leqslant \operatorname{diam}(\triangle T)^m$. Therefore, $\kappa^\gamma_{\mathrm{vol}}$ is a Menger like curvature with exponent γ and $\Lambda(\delta,\kappa) = \delta$.

To see that κ^γ_h satisfies 1.1 (e) define $\pi_i : (\mathbb{R}^n)^{m+2} \to (\mathbb{R}^n)^{m+1}$ to be the projection forgetting the i^{th} coordinate, i.e.

$$\pi_i(q_0, \dots, q_{m+1}) = (q_0, \dots, q_{i-1}, q_{i+1}, \dots, q_{m+1}) \quad \text{for } q_0, \dots, q_{m+1} \in \mathbb{R}^n\,.$$

Then, for $i = 0, \dots, m+1$, the height of $\triangle T$ lowered from the i^{th} vertex equals

$$h_i(T) = \frac{(m+1)!\mathcal{H}^{m+1}(\triangle T)}{m!\mathcal{H}^m(\triangle \pi_i(T))}\,.$$

Recalling $\operatorname{diam}(\triangle T) < d$, we have $m!\mathcal{H}^m(\triangle\pi_i(T)) \leqslant d^m$ and $|(b_1-a)\wedge\dots\wedge(b_m-a)|d^{-m} \geqslant \delta$, and using (28) we obtain

$$h_i(T) \geqslant \delta \operatorname{dist}(c-a,P) \quad \text{for } i = 0, \dots, m+1\,.$$

Hence, $h_{\min}(T) = \min\big\{h_i(T) : i = 0, 1, \dots, m+1\big\} \geqslant \delta \operatorname{dist}(c-a,P)$ and κ^γ_h is a Menger like curvature with exponent γ and $\Lambda(\delta,\kappa) = \delta$.

To see that κ^γ_h is tame assume $\Sigma \subseteq \mathbb{R}^n$ is a graph of some function $f : \mathbb{R}^m \to \mathbb{R}^{n-m}$ of class $\mathscr{C}^{1,\alpha}$, where $\alpha \in (0,1]$. Let $T = (p_0, \dots, p_{m+1}) \in \Sigma^{m+2}$ satisfy $h_{\min}(\triangle T) > 0$ and let $\mathfrak{p} \in \mathbf{O}^*(n,m)$ be such that $\mathfrak{p}(x_1, \dots, x_n) = (x_1, \dots, x_m)$. Set $d = \operatorname{diam}(\triangle T)$ and

$$K = \sup\big\{\|Df(\mathfrak{p}(q) - Df(\mathfrak{p}(p_0))\| |\mathfrak{p}(q) - \mathfrak{p}(p_0)|^{-\alpha} : q \in \Sigma\,,\, 0 < |q - p_0| \leqslant d\big\}\,.$$

Note $|f(\mathfrak{p}(q) - f(\mathfrak{p}(p_0)) - Df(\mathfrak{p}(p_0))(\mathfrak{p}(q) - \mathfrak{p}(p_0))| \leqslant K|\mathfrak{p}(q) - \mathfrak{p}(p_0)|^{1+\alpha} \leqslant K|q - p_0|^{1+\alpha}$ for all $q \in \Sigma$ with $0 < |q - p_0| \leqslant d$. Hence, employing 3.16 (15), we obtain $\triangle T \subseteq \big\{q \in \mathbb{R}^n : |\operatorname{Tan}(\Sigma, p_0)^\perp_\natural(q - p_0)| \leqslant Kd^{1+\alpha}\big\}$. Thus, 5.3 yields $h_{\min}(T) \leqslant Km(m+1)d^{1+\alpha}$; consequently κ^γ_h satisfies 1.1 (f).

Observe that since $\kappa^\gamma_{\mathrm{vol}}$ is invariant under permutations of its parameters, (29) shows that $\kappa^\gamma_{\mathrm{vol}}(T) \leqslant \kappa^\gamma_h(T)$, so $\kappa^\gamma_{\mathrm{vol}}$ is also tame.

6.3 Corollary. *A sufficient condition for a Menger like curvature κ with exponent γ to be tame is that $\kappa \leqslant \Gamma(\kappa)\kappa^\gamma_h$ for some constant $\Gamma = \Gamma(\kappa)$.*

Remark. If $\gamma = 1$, then $\kappa^\gamma_{\mathrm{vol}}$ is, up to a constant, the same as one of the discrete curvatures studied in the series of articles [BK12, KS13, KSvdM13, Kol15, KSvdM15]. If $\gamma = 2/(m(m+1))$, then κ^γ_h coincides with the curvature defined in [LW11, §10].

The following lemma demonstrates, that $p = m(l-1)$ is the minimal exponent for which finiteness μ_Σ^1 almost everywhere of $\mathcal{K}_\kappa^{l,p}[\Sigma]$ may imply higher regularity of Σ.

6.4 Lemma. *Let κ be a Menger like curvature and $\Sigma \subseteq \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(\Sigma) < \infty$. Assume $l \in \{2, 3, \dots, m+2\}$ and $p \in \big(0, m(l-1)\big)$. Then $\mathcal{K}_\kappa^{l,p}[\Sigma](p_0) < \infty$ for $\mathcal{H}^m$ almost all $p_0 \in \Sigma$.*

Proof. Recall that $\Theta^{*m}(\mu_\Sigma^1, p_0) \leqslant 1$ for $\mathcal{H}^m$ almost all $p_0 \in \Sigma$ by [Fed69, 2.10.19(5) and 2.10.6]. Fix $p_0 \in \Sigma$ such that $\Theta^{*m}(\mu_\Sigma^1, p_0) \leqslant 1$. Let $C \in (1, \infty)$ and choose $r_0 \in (0, \infty)$ such that $\mu_\Sigma^1(\mathbf{B}(p_0, r)) \leqslant C\boldsymbol{\alpha}(m) r^m$ for all $r \in (0, r_0)$. For $s, r \in [0, \infty]$ with $s < r$ define

$$A(s, r) = \big\{(p_1, \dots, p_{l-1}) \in (\mathbb{R}^n)^{l-1} : s < \operatorname{diam}(\{p_0, p_1, \dots, p_{l-1}\}) \leqslant r\big\}$$

and note $A(s, r) \subseteq \mathbf{B}(p_0, r)^{l-1}$. Set $L = \sup\big\{\kappa(T) : T \in (\mathbb{R}^n)^{m+2}, \operatorname{diam}(\triangle T) = 1\big\}$; then $L < \infty$ by 1.1 (b) (c). Directly from the definition 1.1 it follows that for $T \in \mathcal{D}_{m+1}$

$$\begin{aligned} &\kappa(T) = \operatorname{diam}(\triangle T)^{-1} \kappa(\operatorname{diam}(\triangle T)^{-1} T) \leqslant L \operatorname{diam}(\triangle T)^{-1}\,; \\ &\text{hence,} \quad \kappa_l[\Sigma](p_0, \dots, p_{l-1}) \leqslant L \operatorname{diam}\big(\{p_0, \dots, p_{l-1}\}\big)^{-1}. \end{aligned}$$

Thus, we can estimate

$$\begin{aligned} (30) \quad \mathcal{K}_\kappa^{l,p}[\Sigma](p_0) &\leqslant L \int \operatorname{diam}(\{p_0, p_1, \dots, p_{l-1}\})^{-p} \, d\mu_\Sigma^{l-1}(p_1, \dots, p_{l-1}) \\ &= L \int_{A(1,\infty)} \operatorname{diam}(\{p_0, p_1, \dots, p_{l-1}\})^{-p} \, d\mu_\Sigma^{l-1}(p_1, \dots, p_{l-1}) \\ &\quad + L \sum_{j=0}^{\infty} \int_{A(2^{-j-1}, 2^{-j})} \operatorname{diam}(\{p_0, p_1, \dots, p_{l-1}\})^{-p} \, d\mu_\Sigma^{l-1}(p_1, \dots, p_{l-1}) \\ &\leqslant L\big(\mathcal{H}^m(\Sigma)\big)^{l-1} + L\Big(\sum_{j=0}^{\infty} (C\boldsymbol{\alpha}(m) 2^{-jm})^{l-1} 2^{p(j+1)}\Big) \\ &= L\big(\mathcal{H}^m(\Sigma)\big)^{l-1} + L(C\boldsymbol{\alpha}(m))^{l-1} \Big(\frac{2^p}{1 - 2^{p - m(l-1)}}\Big) < \infty. \quad \square \end{aligned}$$

If we assume additionally a uniform lower bound on the density ratios for $\Sigma \subseteq \mathbb{R}^n$ (see below), then we also obtain finiteness of the full energy $\mathcal{M}_\kappa^{l,p}(\Sigma)$ for $p < m(l-1)$.

6.5 Corollary. *Let κ be a Menger like curvature, $\Sigma \subseteq \mathbb{R}^n$ be $\mathcal{H}^m$ measurable and such that $\mathcal{H}^m(\Sigma) < \infty$. Assume that there exist $r_0 \in (0, \infty)$ and $C \in (0, \infty)$ such that for all $x_0 \in \Sigma$ and all $r \in (0, r_0]$ there holds*

$$\mathcal{H}^m(\Sigma \cap \mathbf{B}(x_0, r)) \leqslant C r^m.$$

Then $\mathcal{M}_\kappa^{l,p}(\Sigma) < \infty$ for all $l \in \{2, 3, \dots, m+2\}$ and $p \in \big(0, m(l-1)\big)$.

Proof. Since the estimate (30) of the proof of 6.4 holds in the present case for all $p_0 \in \Sigma$ and $\mathcal{H}^m(\Sigma) < \infty$, the assertion is evident. $\square$

Next, we show that if $p > m(l-1)$, κ is tame, and $\Sigma \subseteq \mathbb{R}^n$ is a graph of a smooth enough function, then the energy $\mathcal{M}_\kappa^{l,p}(\Sigma)$ is finite.

6.6 Lemma. *Let* $m, n \in \mathbb{N}$, $p, \gamma \in (0,\infty)$, $l \in \{2,\dots,m+2\}$, $\alpha, \beta \in [0,1)$. *Assume*

$$\kappa \text{ is a tame Menger like curvature with exponent } \gamma,$$
$$1 \leqslant m < n\,, \quad p \geqslant m(l-1)\,, \quad \alpha = \tfrac{1}{\gamma}\big(1 - \tfrac{m(l-1)}{p}\big) < 1\,, \quad \alpha < \beta\,, \quad \beta\gamma < 1\,,$$
$$f \in \mathscr{C}^{1,\beta}(\mathbb{R}^m, \mathbb{R}^{n-m})\,, \quad \Sigma = \operatorname{graph}(f|_{\mathbf{B}(0,1)})\,.$$

Then $\mathcal{M}^{l,p}_{\kappa}(\Sigma) < \infty$.

Proof. For $p_0 \in \Sigma$ and $i \in \mathbb{N}$ define

$$A_i(p_0) = \big\{(p_1,\dots,p_{l-1}) \in (\mathbb{R}^n)^{l-1} : 2^{-i} < \operatorname{diam}(\triangle(p_0,\dots,p_{l-1})) \leqslant 2^{-i+1}\big\}\,.$$

Set $L = 1 + \operatorname{Lip}(f)$. Since κ is tame we can find $\Gamma = \Gamma(\kappa) \in (0,\infty)$ and $K = K(f) \in (0,\infty)$ as in 1.1 (f). Setting $C = (\Gamma K)^{p\gamma}$, using 1.1 (f), and noting $m(l-1) + p(\beta\gamma - 1) > 0$ we estimate

$$\begin{aligned}
\mathcal{M}^{l,p}_{\kappa}(\Sigma) &\leqslant C \int \sum_{i=0}^{\infty} \int_{A_i(p_0)} \operatorname{diam}(\triangle(p_0,\dots,p_{l-1}))^{p(\beta\gamma-1)}\, d\mu^{l-1}_{\Sigma}(p_1,\dots,p_{l-1})\, d\mu^{1}_{\Sigma}(p_0) \\
&\leqslant C\mathcal{H}^m(\Sigma)\boldsymbol{\alpha}(m)^{l-1}(2L)^{m(l-1)} \sum_{i=0}^{\infty} 2^{-i(m(l-1)+p(\beta\gamma-1))} < \infty\,. \quad \square
\end{aligned}$$

7 Higher order rectifiability via averaged Menger like curvatures

7.1. In this section we shall consider the following situation:

$$l, m, n \in \mathbb{N} \sim \{0\}\,, \quad p \in [1,\infty)\,, \quad \alpha, \gamma \in (0,\infty)\,,$$
$$m \leqslant n\,, \quad 1 \leqslant l \leqslant m+2\,, \quad m(l-1) < p\,, \quad \alpha = \gamma^{-1}\big(1 - \tfrac{m(l-1)}{p}\big) \leqslant 1\,,$$
$$\Sigma \subseteq \mathbb{R}^n \text{ is } \mathcal{H}^m \text{ measurable}\,, \quad \mu^j_{\Sigma} \text{ is defined by (1) for each } j \in \mathbb{N} \sim \{0\}\,,$$
$$\kappa \text{ is a Menger like curvature with exponent } \gamma \text{ (see 1.1)}\,,$$
$$\mathcal{K}^{l,p}_{\kappa}[\Sigma] \text{ is defined by (2)}\,,$$
$$\kappa_l[\Sigma](p_0,\dots,p_{l-1}) = \big(\mu^{m+2-l}_{\Sigma}\big) \operatorname*{ess\,sup}_{p_l,\dots,p_{m+1} \in \Sigma} \kappa(p_0,\dots,p_{m+1})\,,$$

with the understanding that $\kappa_{m+2}[\Sigma] = \kappa$, and for $\delta \in [0,1]$, $a \in \mathbb{R}^n$ and $r \in (0,\infty)$

$$X_\delta(a,r) = \big\{(b_1,\dots,b_m) \in (\mathbf{B}(a,r) \cap \Sigma)^m : |(b_1 - a) \wedge \dots \wedge (b_m - a)| \geqslant \delta r^m\big\}\,,$$
$$E(a,r) = \mathcal{K}^{l,p}_{\kappa}[\Sigma \cap \mathbf{B}(a,r)](a)^p\,.$$

7.2 Remark. By [Fed69, 2.10.19(2)], if $S \subseteq \mathbb{R}^n$ and $\mathcal{H}^m(S) < \infty$, then $\Theta^{m*}(\mathcal{H}^m \llcorner S, a) > 0$ for $\mathcal{H}^m$ almost all $a \in S$. In consequence, S is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$ if and only if $\big\{a \in S : \Theta^{m*}(\mathcal{H}^m \llcorner S, a) > 0\big\}$ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$. Therefore, being interested in rectifiability of Σ, we do not loose any generality assuming $\Theta^{m*}(\mathcal{H}^m \llcorner \Sigma, a) > 0$ for all $a \in \Sigma$.

7.3 Remark. If κ is a Menger like curvature, then $\kappa|_{\mathcal{D}_{m+1}} : \mathcal{D}_{m+1} \to [0,\infty)$ is continuous. Assume $\Theta^{m*}(\mathcal{H}^m \llcorner \Sigma, a) > 0$ for all $a \in \Sigma$. Then, for $(p_0,\dots,p_{l-1}) \in \Sigma^l$, we have

$$\kappa_l[\Sigma](p_0,\dots,p_{l-1}) = \sup\big\{\kappa(p_0,\dots,p_{m+1}) : p_l,\dots,p_{m+1} \in \Sigma\big\}\,.$$

7.4 Lemma. *Let* $r, \delta \in (0,\infty)$*,* $\varepsilon \in (0,1)$*,* $P, Q \in \mathbf{G}(n,m)$*,* $v_1, \ldots, v_m \in \mathbb{R}^n$ *satisfy*

$$Q = \operatorname{span}\{v_1, \ldots, v_m\}, \quad |v_1 \wedge \cdots \wedge v_m| \geqslant \delta r^m, \quad |v_i| \leqslant r, \quad \textit{and} \quad |P_\natural^\perp v_i| \leqslant \varepsilon r$$

for $i = 1, \ldots, m$*. Then* $\|P_\natural - Q_\natural\| \leqslant m\delta^{-1}\varepsilon$*.*

Proof. By [All72, 8.9(3)], there exists $u \in Q$ such that

$$|u| = 1 \quad \text{and} \quad \|P_\natural - Q_\natural\| = \|P_\natural^\perp \circ Q_\natural\| = |P_\natural^\perp u| .$$

Choose $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that $u = \sum_{i=1}^m \alpha_i v_i$. For each $i = 1, \ldots, m$ we have

$$\begin{aligned} \alpha_i &= \big(v_1 \wedge \cdots \wedge v_{i-1} \wedge u \wedge v_{i+1} \wedge \cdots \wedge v_m\big) \bullet \frac{v_1 \wedge \cdots \wedge v_m}{|v_1 \wedge \cdots \wedge v_m|^2} \\ \text{and} \quad |\alpha_i| &= \frac{|v_1 \wedge \cdots \wedge v_{i-1} \wedge u \wedge v_{i+1} \wedge \cdots \wedge v_m|}{|v_1 \wedge \cdots \wedge v_m|} \leqslant \frac{1}{\delta r}; \\ \text{hence,} \quad \|P_\natural - Q_\natural\| &= |P_\natural^\perp u| \leqslant \sum_{i=1}^m |\alpha_i| |P_\natural^\perp v_i| \leqslant m\delta^{-1}\varepsilon . \end{aligned}$$

□

7.5 Remark. We shall frequently use the Chebyshev's inequality in the following form. Whenever μ measures some set X, $f : X \to \mathbb{R}$ is a μ measurable function, $t \in (0,\infty)$ and $A \subseteq X$ is μ measurable, then

$$\int_A |f| \, d\mu \geqslant \int_{\{x \in A : |f(x)| > t\}} |f| \, d\mu \geqslant t\mu(\{x \in A : |f(x)| > t\}) .$$

For any $K \in (0,\infty)$, setting $t = K \fint_A |f| \, d\mu$ one obtains

$$\mu\left(\left\{x \in A : |f(x)| > K \fint_A |f| \, d\mu\right\}\right) \leqslant K^{-1}\mu(A) .$$

7.6 Lemma. *Consider the situation as in 7.1 and assume* $\Theta^{m*}(\mu_\Sigma^1, a) > 0$ *for all* $a \in \Sigma$*. Let* $\delta, \sigma \in (0,1)$*,* $A \in [1,\infty)$ *and* $r_0 \in (0,\infty)$*. Define* S *to be the set of those* $a \in \Sigma$ *for which* $E(a, 4r_0) < \infty$*, and*

(31) $A^{-1}\boldsymbol{\alpha}(m) r^m \leqslant \mu_\Sigma^1(\mathbf{B}(a,r)) \leqslant A\boldsymbol{\alpha}(m) r^m , \quad \textit{and} \quad \mu_\Sigma^m(X_\delta(a,r)) \geqslant \sigma\mu_\Sigma^m(\mathbf{B}(a,r)^m) ,$

whenever $r \in (0, r_0]$*. Then there exists a constant* $C = C(m, l, p, \sigma, \gamma, \delta, A)$ *and for each* $a \in S$ *there exists* $T(a) \in \mathbf{G}(n,m)$ *such that*

(a) *in case* $l < m + 2$*: for all* $b \in \Sigma \cap \mathbf{B}(a, r_0)$

$$|T(a)_\natural^\perp(b - a)| \leqslant C E(a, |b-a|)^{1/(\gamma p)} |b-a|^{1+\alpha}$$

and, whenever $b \in S \cap \mathbf{B}(a, \frac{1}{2}r_0)$*,*

$$\|T(a)_\natural - T(b)_\natural\| \leqslant C E(a, |b-a|)^{1/(\gamma p)} |b-a|^{\alpha} ;$$

in particular $\operatorname{Tan}(\Sigma, a) \subseteq T(a)$*, by 3.6;*

(b) *in case* $l = m+2$*: for any* $r \in (0, r_0]$

$$\left(\fint_{\mathbf{B}(a,r)} \operatorname{dist}(c-a, T(a))^p \, d\mu_\Sigma^1(c) \right)^{1/p} \leqslant C E(a, 4r)^{1/(\gamma p)} r^{1+\alpha} \,;$$

in particular $\operatorname{Tan}^m(\mu_\Sigma^1, a) \subseteq T(a)$*, by 3.10 and 3.11.*

Proof. Obviously we can assume S is not empty – otherwise there is nothing to prove. Set $K = 2^{4m^2} A^{2m} \sigma^{-1}$. For $a \in \mathbb{R}^n$ and $r \in (0, \infty)$ define $\Sigma(a,r) = \Sigma \cap \mathbf{B}(a,r)$ and, if $2 \leqslant l \leqslant m+1$, set

$$Y(a,r) = \left\{ (b_1, \dots, b_m) \in \Sigma(a,r)^m : \kappa_l[\Sigma(a,r)](a, b_1, \dots, b_{l-1})^p > \frac{K E(a,r)}{\mu_\Sigma^{l-1}(\mathbf{B}(a,r)^{l-1})} \right\} ,$$

if $l = m+2$, set

$$Y(a,r) = \left\{ (b_1, \dots, b_m) \in \Sigma(a,r)^m : \int_{\mathbf{B}(a,r)} \kappa(a, b_1, \dots, b_m, c)^p \, d\mu_\Sigma^1(c) > \frac{K E(a,r)}{\mu_\Sigma^m(\mathbf{B}(a,r)^m)} \right\} ,$$

and if $l = 1$, set $Y(a,r) = \varnothing$. Employing Chebyshev's inequality 7.5 we obtain

$$\mu_\Sigma^m(Y(a,r)) \leqslant K^{-1} \mu_\Sigma^m(\mathbf{B}(a,r)^m) \tag{32}$$

for all $l \in \{1, \dots, m+2\}$, $a \in \mathbb{R}^n$ and $0 < r < \infty$. Since $K > \sigma^{-1}$, using (31), we get for each $a \in S$ and $0 < r \leqslant r_0$

$$\mu_\Sigma^m(X_\delta(a,r) \sim Y(a,r)) \geqslant \left(\sigma - \tfrac{1}{K}\right) \mu_\Sigma^m(\mathbf{B}(a,r)^m) > 0 \,. \tag{33}$$

Recall that $\Theta^{m*}(\mu_\Sigma^1, a) > 0$ for all $a \in \Sigma$; thus, employing remark 7.3, we can replace the "(μ_Σ^{m+2-l}) ess sup" in the definition of $\kappa_l[\Sigma]$ by the usual "sup". Let $\Lambda = \Lambda(2^{-m}\delta, \kappa)$ be the number defined in 1.1(e) for κ. For any $a \in S$, $0 < r \leqslant r_0$, and $(g_1, \dots, g_m) \in X_\delta(a,r) \sim Y(a,r)$ if $P = \operatorname{span}\{g_1 - a, \dots, g_m - a\}$ and $1 \leqslant l \leqslant m+1$, then by (31) and 1.1(e)

$$\begin{aligned} \frac{K E(a,r)}{\left(A^{-1} \boldsymbol{\alpha}(m) r^m\right)^{l-1}} &\geqslant \frac{K E(a,r)}{\mu_\Sigma^{l-1}(\mathbf{B}(a,r)^{l-1})} \geqslant \kappa_l[\Sigma(a,r)](a, g_1, \dots, g_{l-1})^p \\ &\geqslant \sup_{b \in \Sigma(a,r)} \kappa(a, g_1, \dots, g_m, b)^p \geqslant \sup_{b \in \Sigma(a,r)} \left[\left(\frac{\Lambda \operatorname{dist}(b-a, P)}{2r} \right)^\gamma \frac{1}{2r} \right]^p \end{aligned}$$

which implies, recalling $\alpha = \gamma^{-1}(1 - m(l-1)/p)$,

$$\begin{gathered} \sup_{b \in \Sigma(a,r)} \operatorname{dist}(b-a, P) \leqslant C_1 E(a,r)^{1/(\gamma p)} r^{1+\alpha} \,, \\ \text{where} \quad C_1 = A^{(l-1)/(\gamma p)} K^{1/(\gamma p)} \boldsymbol{\alpha}(m)^{(1-l)/(\gamma p)} 2^{1+1/\gamma} \Lambda^{-1} \,. \end{gathered} \tag{34}$$

An analogous computation shows that in case $l = m+2$, for any $a \in S$, $0 < r \leqslant r_0$, and $(g_1, \dots, g_m) \in X_\delta(a,r) \sim Y(a,r)$ if $P = \operatorname{span}\{g_1 - a, \dots, g_m - a\}$, then

$$\left(\fint_{\mathbf{B}(a,r)} \operatorname{dist}(c-a, P)^p \, d\mu_\Sigma^1(c) \right)^{1/p} \leqslant C_1 E(a,r)^{1/(\gamma p)} r^{1+\alpha} \,. \tag{35}$$

Now, we shall prove the lemma in case $1 \leqslant l \leqslant m+1$. Due to (33), for each $a \in S$ and $0 < r \leqslant r_0$ there exists an m-tuple

$$\big(g_1(a,r),\dots,g_m(a,r)\big) \in X_\delta(a,r) \sim Y(a,r)$$

and we can define

$$P(a,r) = \operatorname{span}\big\{(g_1(a,r)-a),\dots,(g_m(a,r)-a)\big\} \in \mathbf{G}(n,m)\ .$$

Whenever $a \in S$ and $0 \leqslant s \leqslant r \leqslant r_0$, noting $g_i(a,s) \in \Sigma(a,r)$ for $i = 1,\dots,m$, we may employ (34) together with 7.4 to obtain

$$\|P(a,r)_\natural - P(a,s)_\natural\| \leqslant m\delta^{-1} C_1 E(a,r)^{1/(\gamma p)} r^\alpha\,.$$

Therefore, for each $a \in S$, the spaces $P(a,r)$ converge as $r \to 0$ to some $T(a) \in \mathbf{G}(n,m)$ and

$$\|P(a,r)_\natural - T(a)_\natural\| \leqslant C_2 E(a,r)^{1/(\gamma p)} r^\alpha\,, \quad \text{where } C_2 = m\delta^{-1}C_1\,.$$

Moreover, by (34) and the triangle inequality, for any $a \in S$ and $b \in \Sigma \cap \mathbf{B}(a,r_0)$

$$|T(a)_\natural^\perp(b-a)| \leqslant (C_1 + C_2) E(a,|b-a|)^{1/(\gamma p)} |b-a|^{1+\alpha}\,.$$

Assume $a \in S$, $r \in (0,r_0]$ and $b \in S \sim \{a\}$ are such that $|b-a| = \frac{1}{2}r$. Then for each $i = 1,\dots,m$ there holds $|g_i(b,\frac12 r) - a| \leqslant r$ and it follows from (34) that

$$\Big|P(a,r)_\natural\big(g_i(b,\tfrac12 r) - a\big)\Big| \leqslant 2C_1 E(a,r)^{1/(\gamma p)} r^{1+\alpha}\,;$$

hence, employing 7.4, we get

$$\big\|P(a,r)_\natural - P\big(b,\tfrac12 r\big)_\natural\big\| \leqslant 2^{1+\alpha} C_2 E(a,r)^{1/(\gamma p)} |b-a|^\alpha\,.$$

In consequence, for all $a, b \in S$, $r \in (0,\infty)$ with $|a-b| = \frac12 r \leqslant \frac12 r_0$

$$\begin{aligned}\|T(a)_\natural - T(b)_\natural\| &\leqslant \|T(a)_\natural - P(a,r)_\natural\| + \|P(a,r)_\natural - P\left(b,\tfrac{r}{2}\right)_\natural\| + \|P\left(b,\tfrac{r}{2}\right)_\natural - T(b)_\natural\| \\ &\leqslant C_3 E(a,r)^{1/(\gamma p)} |b-a|^\alpha\,, \qquad \text{where } C_3 = C_2(2 + 2^{1+\alpha}).\end{aligned}$$

This finishes the proof in case $1 \leqslant l \leqslant m+1$.

Next, we shall consider the case $l = m+2$. For $a \in S$ and $i = \mathbb{N}$ define inductively

$$\begin{gathered}\rho_i = 2^{-i} r_0\,, \quad Q_0(a) = P(a,\rho_0)\,, \\ Z_i(a) = \left\{ c \in \Sigma(a,\rho_i) : \operatorname{dist}(c-a, Q_i(a))^p > K \fint_{\mathbf{B}(a,\rho_i)} \operatorname{dist}(z-a, Q_i(a))^p \, d\mu_\Sigma^1(z) \right\}\,, \\ W_i(a) = \big\{ (c_1,\dots,c_m) \in \Sigma(a,\rho_i)^m : \exists j \in \{1,\dots,m\}\ c_j \in Z_i(a) \big\}\ ,\end{gathered}$$

and, whenever $i \geqslant 1$,

$$\begin{gathered}\big(h_{i,1}(a),\dots,h_{i,m}(a)\big) \in X(a,\rho_i) \sim (Y(a,\rho_i) \cup W_{i-1}(a))\ , \\ Q_i(a) = \operatorname{span}\big\{h_{i,1}(a) - a,\dots,h_{i,m}(a) - a\big\}\,.\end{gathered}$$

Note that $(h_{i,1}(a), \dots, h_{i,m}(a))$ exists for all $i \in \mathbb{N}$ and $a \in S$. Indeed, for $i \in \mathbb{N}$ and $a \in S$ Chebyshev's inequality 7.5 yields

$$\begin{aligned} \mu_\Sigma^1(Z_i(a)) &\leqslant \tfrac{1}{K}\mu_\Sigma^1(\mathbf{B}(a,\rho_i))\,; \\ \text{hence} \quad \mu_\Sigma^m(W_i(a)) &\leqslant \big(1 - \big(1 - \tfrac{1}{K}\big)^m\big)\mu_\Sigma^m(\mathbf{B}(a,\rho_i)^m)\,, \end{aligned}$$

which implies for $i \in \mathbb{N} \sim \{0\}$, combining (31) with (32) and noting $(1-(1-K^{-1})^m) \leqslant 2^m K^{-1}$ and $K > A^{2m}\sigma^{-1}(1+2^{m^2+m})$,

$$\mu_\Sigma^m\big(X(a,\rho_i) \sim (Y(a,\rho_i) \cup W_{i-1}(a))\big) \geqslant \big(\boldsymbol{\alpha}(m)\rho_i^m\big)^m \Big(\frac{\sigma}{A^m} - \frac{A^m}{K}\big(1+2^{m^2+m}\big)\Big) > 0\,.$$

Observe that for $a \in S$, $i = \mathbb{N} \sim \{0\}$ and $j = 1, 2, \dots, m$, employing (35),

$$\begin{aligned} \operatorname{dist}(h_{i,j}(a) - a, Q_{i-1}(a)) &\leqslant \bigg(K \fint_{\mathbf{B}(a,\rho_{i-1})} \operatorname{dist}(z-a, Q_{i-1}(a))^p \,d\mu_\Sigma^1(z)\bigg)^{1/p} \\ &\leqslant 2^{1+\alpha}K^{1/p}C_1 E(a,\rho_{i-1})^{1/(\gamma p)}\rho_i^{1+\alpha}\,. \end{aligned}$$

Therefore, lemma 7.4 yields for $a \in S$ and $i \in \mathbb{N} \sim \{0\}$

$$\|Q_i(a)_\natural - Q_{i-1}(a)_\natural\| \leqslant C_4 E(a,\rho_{i-1})^{1/(\gamma p)}\rho_i^\alpha\,, \quad \text{where } C_4 = m\delta^{-1}2^{1+\alpha}K^{1/p}C_1\,.$$

Summing up a geometric series we see that for $a \in S$ the spaces $Q_i(a)$ converge as $i \to \infty$ to some $T(a) \in \mathbf{G}(n,m)$ satisfying

$$\|Q_i(a)_\natural - T(a)_\natural\| \leqslant C_5 E(a,2\rho_i)^{1/(\gamma p)}\rho_i^\alpha\,, \quad \text{where } C_5 = (1-2^{-\alpha})^{-1}C_4\,.$$

Let $a \in S$, $\rho \in (0,\infty)$ and $i \in \mathbb{N}$ be such that $\rho_{i+1} < \rho \leqslant \rho_i \leqslant r_0$. Then

$$\begin{aligned} \Big(\fint_{\mathbf{B}(a,\rho)} \operatorname{dist}(c-a, T(a))^p \,d\mu_\Sigma^1(c)\Big)^{1/p} &\leqslant \Big(\fint_{\mathbf{B}(a,\rho)} \operatorname{dist}(c-a, Q_i(a))^p \,d\mu_\Sigma^1(c)\Big)^{1/p} \\ &\quad + \Big(\fint_{\mathbf{B}(a,\rho)} \|Q_i(a) - T(a)\|^p |c-a|^p \,d\mu_\Sigma^1(c)\Big)^{1/p} \\ &\leqslant (C_1 + C_5)E(a,2\rho_i)^{1/(\gamma p)}\rho_i^{1+\alpha} \leqslant C_6 E(a,4\rho)^{1/(\gamma p)}\rho^{1+\alpha}\,, \end{aligned}$$

where $C_6 = 2^{1+\alpha}(C_1 + C_5)$. □

7.7 Lemma. *Consider the situation as in 3.17* (16) *and let* X_δ *be defined as in 7.1. Given* $\sigma \in (0,1)$ *there exist* $L_0 = L_0(m,\sigma) \in \big(1, \frac{5}{4}\big)$, $A = A(m,\sigma) \in (1,2]$, *and* $\delta = \delta(m) \in \big(0, \frac{1}{2}\big)$ *such that if* $L \leqslant L_0$ *and* $a \in \Sigma$ *satisfy for some* $R_0 \in (0,\infty)$

$$A^{-1}\boldsymbol{\alpha}(m)r^m \leqslant \mu_\Sigma^1(\mathbf{B}(x,r)) \leqslant A\boldsymbol{\alpha}(m)r^m \quad \textit{for } r \in (0, LR_0]\,,$$

then

$$\mu_\Sigma^m(X_\delta(a,r)) \geqslant \sigma\mu_\Sigma^m(\mathbf{B}(x,r)^m) \quad \textit{for } r \in (0, R_0]\,.$$

Proof. Choose $\varepsilon \in (0,1)$ such that $\sigma + 2\varepsilon < 1$ and $\varepsilon \leqslant 3\sigma$. For $\lambda \in [0,1]$ set

$$Z_\lambda = \left\{(u_1, \dots, u_m) \in \mathbb{R}^{m \cdot m} : |u_i| \leqslant 1 \text{ for } i = 1, \dots, m \text{ and } |u_1 \wedge \cdots \wedge u_m| \geqslant \lambda\right\}.$$

Due to continuity of the exterior multiplication (i.e. $\wedge$) and continuity of the Radon measure $\mathcal{L}^{m \cdot m}$ the mapping $[0,1] \ni \lambda \mapsto \mathcal{L}^{m \cdot m}(Z_\lambda)$ is continuous. Moreover, $\mathcal{L}^{m \cdot m}(Z_0) = \mathcal{L}^{m \cdot m}(\mathbf{B}(0,1)^m)$ and $\mathcal{L}^{m \cdot m}(Z_1) = 0$. Hence, there exists $\delta \in (0, \frac{1}{2})$ such that

$$\mathcal{L}^{m \cdot m}(Z_{2\delta}) = (\sigma + 2\varepsilon)\mathcal{L}^{m \cdot m}(\mathbf{B}(0,1)^m). \tag{36}$$

Set

$$A = \min\left\{\left(1 - \frac{\varepsilon}{m}\right)^{-1}, \left(1 + \frac{\varepsilon}{3\sigma}\right)^{1/m}\right\}, \tag{37}$$

$$L_0 = \min\left\{\left(\frac{4}{3}\right)^{1/m}, 1 + \frac{\delta}{2m}, \left(\left(1 - \frac{\varepsilon}{m}\right)A\right)^{-1/m}, \left(1 + \frac{\varepsilon}{3\sigma}\right)^{1/m^2}\right\}. \tag{38}$$

If $r \in (0, LR_0]$, employing (38) and noting that $\Sigma \cap \mathbf{B}(a,r) = F[\mathfrak{p}[\Sigma \cap \mathbf{B}(a,r)]]$, we get

$$\mathcal{L}^m(\mathfrak{p}[\Sigma \cap \mathbf{B}(a,r)]) \geqslant L^{-m}\mathcal{H}^m(\Sigma \cap \mathbf{B}(a,r)) > \left(1 - \frac{\varepsilon}{m}\right)\boldsymbol{\alpha}(m)r^m. \tag{39}$$

Recall that $(1+t)^m \geqslant 1 + mt$ for all $t \in [-1, \infty)$ by convexity of the function $[-1,\infty) \ni t \mapsto (1+t)^m$; hence, $1 - \left(1 - \frac{\varepsilon}{m}\right)^m \leqslant \varepsilon$. We compute using (39)

$$\mathcal{L}^{m \cdot m}\big(\mathbf{B}(p(a), r)^m \sim \mathfrak{p}[\Sigma \cap \mathbf{B}(a,r)]^m\big) \leqslant \varepsilon\,(\boldsymbol{\alpha}(m)r^m)^m. \tag{40}$$

Combining (36) and (40), we obtain

$$\mathcal{L}^{m \cdot m}((\mathfrak{p}(a) + rZ_{2\delta}) \cap \mathfrak{p}[\Sigma \cap \mathbf{B}(a,r)]^m) \geqslant (\sigma + \varepsilon)(\boldsymbol{\alpha}(m)r^m)^m \quad \text{for } r \in (0, LR_0]. \tag{41}$$

Fix $r \in (0, LR_0]$ and let $(u_1, \dots, u_m) \in rZ_{2\delta}$. Set

$$\eta_i = F(\mathfrak{p}(a) + u_i) - F(\mathfrak{p}(a)) = \mathfrak{p}^*(u_i) + \mathfrak{q}^*(f(\mathfrak{p}(a) + u_i) - f(\mathfrak{p}(a))) \quad \text{for } i = 1, \dots, m.$$

Note, recalling (38), that $|\mathfrak{q}^*(f(\mathfrak{p}(a) + u_i) - f(\mathfrak{p}(a))| \leqslant \operatorname{Lip}(f|_{\mathbf{B}(p(a),r)})|u_i| \leqslant \frac{\delta}{2m}r$. Hence, using the estimates $(1+t)^m \leqslant 1 + \frac{mt}{1-mt}$ for $t \in (-\infty, \frac{1}{m})$ and $\delta < \frac{1}{2}$, we get

$$\begin{aligned}
|\eta_1 \wedge \cdots \wedge \eta_m| &\geqslant |\mathfrak{p}^*(u_1) \wedge \cdots \wedge \mathfrak{p}^*(u_m)| - r^m \sum_{i=1}^m \binom{m}{i}\left(\frac{\delta}{2m}\right)^i \\
&= |u_1 \wedge \cdots \wedge u_m| - r^m\left(\left(1 + \frac{\delta}{2m}\right)^m - 1\right) \geqslant \left(2\delta - \tfrac{2}{3}\delta\right) r^m \geqslant \tfrac{4}{3}\delta r^m,
\end{aligned}$$

which shows, recalling $L^m \leqslant \frac{4}{3}$ by (38),

$$F^m\left[(\mathfrak{p}(a) + rZ_{2\delta}) \cap \mathfrak{p}[\Sigma \cap \mathbf{B}(a,r)]^m\right] \subseteq X_{3\delta/(2L^m)}(a, Lr) \subseteq X_\delta(a, Lr). \tag{42}$$

Let $r \in (0, R_0]$, set $s = Lr$ and note that $s \leqslant LR_0$. Employing (42), then using 3.13 with $g = \mathfrak{p}$, after that recalling 3.12 to argue that $(\mathscr{L}^m \llcorner \mathfrak{p}[\Sigma])^m = \mathscr{L}^{m \cdot m} \llcorner \mathfrak{p}[\Sigma]^m$, and finally applying (41), we obtain

$$
\begin{aligned}
\mu_\Sigma^m(X_\delta(a,s)) &\geqslant \mu_\Sigma^m\big(F^m\big[(\mathfrak{p}(a) + rZ_{2\delta}) \cap \mathfrak{p}[\Sigma \cap \mathbf{B}(a,r)]^m\big]\big) \\
&\geqslant \mathscr{L}^{m \cdot m}\big((\mathfrak{p}(a) + rZ_{2\delta}) \cap \mathfrak{p}[\Sigma \cap \mathbf{B}(a,r)]^m\big) \\
&\geqslant (\sigma + \varepsilon)(\boldsymbol{\alpha}(m) r^m)^m = \frac{\sigma+\varepsilon}{L^{m\cdot m}}(\boldsymbol{\alpha}(m)s^m)^m \\
&\geqslant \frac{\sigma+\varepsilon}{A^m L^{m\cdot m}} \mathcal{H}^m(\Sigma \cap \mathbf{B}(a,s))^m\,.
\end{aligned} \tag{43}
$$

Now observe that due to (37) and (38) we have $A^m \leqslant 1 + \frac{\varepsilon}{3\sigma}$ and $L^{m \cdot m} \leqslant 1 + \frac{\varepsilon}{3\sigma}$; hence, recalling $\varepsilon \leqslant 3\sigma$,

$$
\sigma + \varepsilon \geqslant \sigma + \left(\frac{2}{3} + \frac{\varepsilon}{9\sigma}\right)\varepsilon \geqslant A^m L^{m\cdot m}\sigma\,. \tag{44}
$$

Plugging (44) into (43) finishes the proof. □

Now we can prove our first main theorem.

Proof of Theorem 1.2. Recalling 7.2 we can assume $\boldsymbol{\Theta}^{m*}(\mathcal{H}^m \llcorner \Sigma, a) > 0$ for all $a \in \Sigma$. Set $\sigma = \frac{1}{2}$ and let $A \in (1,2]$, $L_0 \in \left(1, \frac{5}{4}\right)$ and $\delta \in (0, \frac{1}{2})$ be given by 7.7. Employing 3.17, we can further assume that Σ satisfies the conditions of 3.17 (16) with $L = L_0$. Let $\mathfrak{p}$, $\mathfrak{q}$, g, x_0, and r_0 be as in 3.17 (16). Set $M = (\mathfrak{p}^* + \mathfrak{q}^* \circ g)[\mathbf{U}(x_0, r_0)]$ so that $\Sigma \subseteq M$ and M is an m dimensional submanifold of $\mathbb{R}^n$ of class $\mathscr{C}^1$. For each $a \in M$ and $r \in (0, \infty)$ let $\mathfrak{p}_a$, $\mathfrak{q}_a$, f_a, F_a, $\mathbf{C}(a,r)$, r_a be defined as in 3.17. For $j \in \mathbb{N} \sim \{0\}$ define the sets

$$
\Sigma_j = \left\{ a \in \Sigma : r_a \geqslant \frac{L}{j} \text{ and } \frac{1}{A}\boldsymbol{\alpha}(m)s^m \leqslant \mu_\Sigma^1(\mathbf{B}(a,s)) \leqslant A\boldsymbol{\alpha}(m)s^m \text{ for } 0 < s \leqslant \frac{L}{j} \right\}
$$

and observe, recalling 3.9, that $\mathcal{H}^m(\Sigma \sim \bigcup_{j=1}^\infty \Sigma_j) = 0$. Applying 7.7, with Σ replaced by $\Sigma \cap \mathbf{C}(a, L/j)$, we see that for all $j \in \mathbb{N} \sim \{0\}$ and $a \in \Sigma_j$

$$
\mu_\Sigma^m\big(X_\delta(a, 1/j)\big) \geqslant \sigma \mu_\Sigma^m\big(\mathbf{B}(a, 1/j)^m\big)\,.
$$

Next, for each $j \in \mathbb{N} \sim \{0\}$ and $a \in \Sigma_j$, setting $R_j = \frac{1}{j}$, we can employ 7.6 to find a constant $C = C(m, l, p, \sigma, \gamma, \delta, A)$ and an m-plane $T(a) \in \mathbf{G}(n,m)$ such that

- either $1 \leqslant l \leqslant m+1$, and $\mathrm{Tan}(\Sigma, a) \subseteq T(a)$, and for all $b \in \Sigma \cap \mathbf{B}(a, R_j)$

$$
|T(a)_\natural^\perp(b-a)| \leqslant CE(a, |b-a|)^{1/(\gamma p)}|b-a|^{1+\alpha}
$$

- or $l = m+2$, $\mathrm{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a) \subseteq T(a)$ and for all $r \in (0, R_j]$

$$
\left(⨍_{\mathbf{B}(a,r)} \mathrm{dist}(b-a, T(a))^p \, d\mu_\Sigma^1(b) \right)^{1/p} \leqslant CE(a, 4r)^{1/(\gamma p)} r^{1+\alpha}\,.
$$

Recalling $\Sigma \subseteq M$ and 3.9 we see that $\mathrm{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a) = \mathrm{Tan}(M, a) \in \mathbf{G}(n,m)$ for $\mathcal{H}^m$ almost all $a \in \Sigma$. Thus, for each $j \in \mathbb{N} \sim \{0\}$ we actually have $T(a) = \mathrm{Tan}(\Sigma, a)$ for $\mathcal{H}^m$ almost all $a \in \Sigma_j$ because $\mathbf{G}(n,m) \ni \mathrm{Tan}(\Sigma, a) \subseteq T(a) \in \mathbf{G}(n,m)$. In consequence, employing $\mathcal{H}^m(\Sigma \sim \bigcup_{j=1}^\infty \Sigma_j) = 0$, we obtain

(a) either $1 \leqslant l \leqslant m+1$ and for $\mathcal{H}^m$ almost all $a \in \Sigma$

$$\lim_{\substack{b \to a \\ b \in \Sigma}} \frac{|\mathrm{Tan}(\Sigma, a)_{\natural}^{\perp}(b-a)|}{|b-a|^{1+\alpha}} = 0$$

(b) or $l = m+2$ and for $\mathcal{H}^m$ almost all $a \in \Sigma$

$$\lim_{r \downarrow 0} r^{-(1+\alpha)} \left(r^{-m} \int_{\mathbf{B}(a,r)} \mathrm{dist}(b-a, \mathrm{Tan}(\Sigma, a))^p \, d\mu_{\Sigma}^1(b) \right)^{1/p} = 0 \,.$$

If (a) holds then, by 3.22, Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$. If (b) holds, then by Hölder's inequality

$$⨍_{\mathbf{B}(a,r)} \mathrm{dist}(b-a, \mathrm{Tan}(\Sigma, a)) \, d\mu_{\Sigma}^1(b) \leqslant \left(⨍_{\mathbf{B}(a,r)} \mathrm{dist}(b-a, \mathrm{Tan}(\Sigma, a))^p \, d\mu_{\Sigma}^1(b) \right)^{1/p}$$

and by a similar argument as in 3.11 we obtain

$$\lim_{r \downarrow 0} r^{-m} \int_{\mathbf{B}(a,r)} \frac{|\mathrm{Tan}(\Sigma, a)(b-a)|_{\natural}^{\perp}}{|b-a|^{1+\alpha}} \, d\mu_{\Sigma}^1(b) = 0 \,.$$

Hence, one can employ 3.22 once more and see that in this case Σ is also $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$.

If $\alpha = \gamma^{-1}(1 - m(l-1)/p) < 1$ and κ is tame, then the exponent α is sharp. To see that assume $m = 1$, $n = 2$, and $\varepsilon \in (0,1)$ is such that $\alpha + 2\varepsilon < 1$. Consider a function $f : [0,1] \to \mathbb{R}$ of class $\mathscr{C}^{1,\alpha+2\varepsilon}$ which graph is not $(\mathcal{H}^1, 1)$ rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$ (a construction of such function can be found, e.g., in [AS94, Appendix]). Set $\Sigma = \mathrm{graph}(f)$. Then $\mathcal{M}_{\kappa}^{l,p}(\Sigma) < 0$, by 6.6, but Σ is not $(\mathcal{H}^1, 1)$ rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$.

If $\alpha = 1$, and $\gamma = 1$, and κ is tame, then $l = 1$ and it follows directly from 1.1 (f) that $\mathcal{K}_{\kappa}^{l,p}[\Sigma]$ is bounded whenever Σ is a compact subset of a graph of a $\mathscr{C}^{1,1}$ function. Thus, proceeding as before, for each $\varepsilon \in (0, 1/2)$ one can find a function $f \in \mathscr{C}^{2,2\varepsilon}([0,1], \mathbb{R})$ such that $\mathrm{graph}(f)$ is not $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{2,\varepsilon}$ but $\mathcal{K}_{\kappa}^{l,p}[\Sigma](a) < \infty$ for all $a \in \Sigma$. $\square$

7.8 Remark. If $\alpha = \gamma^{-1}(1 - m(l-1)/p) = 1$, then Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^2$.

7.9 Remark. Suppose $\gamma = 1$, $l = m+2$, $\kappa = \kappa_{\mathrm{vol}}^1$, $p > m(m-1)$, $\alpha = 1 - m(m-1)/p$. In this case one cannot expect to have an implication in the reverse direction, i.e. it is not possible to prove that if Σ is $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$, then $\mathcal{K}_{\kappa}^{l,p}[\Sigma](p_0) < \infty$ for $\mathcal{H}^m$ almost all $p_0 \in \Sigma$. This can be seen by finding a function f of class $\mathscr{C}^{1,\alpha}$ such that for $\Sigma = \mathrm{graph}(f)$ one has $\mathcal{K}_{\kappa}^{l,p}[\Sigma](p_0) = \infty$ for all $p_0 \in \Sigma$ – such example is provided in [KS13, §6].

A Examples of Menger like curvatures

A.1 Example. Let $m, n \in \mathbb{N}$ satisfy $1 \leqslant m < n$, $T = (p_0, \ldots, p_{m+1}) \in \mathcal{D}_{m+1}$. Let $S(T)$ be the unique m dimensional sphere containing all the vertexes of $\triangle T$ and $r(T)$ be the radius of $S(T)$. We define

$$\kappa_{\mathbb{S}}(T) = r(T)^{-1} \,.$$

If $T \in (\mathbb{R}^n)^{m+2}$ and $h_{\min}(T) = 0$, then we set $\kappa_{\mathbb{S}}(T) = 0$ in accordance with 1.1 (b). To prove that $\kappa_{\mathbb{S}}$ is a Menger like curvature we shall derive an analytic expression for $r(T)$. In this example we shall treat $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$, and $\mathbb{R}^{n+1}$ as a subspace of $\mathbb{R}^{n+2}$ with the standard inclusions.

Define $X = \operatorname{span}\{p_i - p_0 : i = 1, \ldots, m+1\}$ to be the $m+1$ dimensional linear subspace of $\mathbb{R}^n$ containing $(-p_0) + \triangle T$ and let $e_1, \ldots, e_{n+2}$ be an orthonormal basis of $\mathbb{R}^{n+2}$ such that $e_1, \ldots, e_{m+1}$ span X. Assume $S(T)$ has radius $r \in (0, \infty)$ and center $c \in \mathbb{R}^n$. If $p_{m+2} \in S(T)$, then the vector $(c, |c|^2) \in \mathbb{R}^{n+1}$ must be the unique solution to the following system of *linear* equations for $x \in \mathbb{R}^{n+1}$

$$-2 \sum_{j=1}^{m+1} (p_i \bullet e_j)(x \bullet e_j) + x \bullet e_{n+1} = r^2 - |p_i|^2 \quad \text{for } i = 0, 1, \ldots, m+2,$$
$$x \bullet e_j = 0 \quad \text{for } j \in \mathbb{N} \text{ with } m+2 \leqslant j \leqslant n\,.$$

This can only happen if the vector $(r^2 - |p_0|^2, \ldots, r^2 - |p_{m+2}|^2) \in \mathbb{R}^{m+3}$ is a linear combination of the vectors $(p_0 \bullet e_j, \ldots, p_{m+2} \bullet e_j) \in \mathbb{R}^{m+3}$ for $j = 1, \ldots, m+1$ and the vector $(1, 1, \ldots, 1) \in \mathbb{R}^{m+3}$, which, in turn, is equivalent to the condition

$$\zeta(p_0) \wedge \cdots \wedge \zeta(p_{m+2}) = 0 \quad \text{where } \zeta : \mathbb{R}^n \to \mathbb{R}^{n+2} \text{ is given by } \zeta(x) = (x, |x|^2, 1). \tag{45}$$

Thus, $S(T)$ consists exactly of these points $p_{m+2} \in \mathbb{R}^n$, which satisfy (45). Define the map $\xi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ by $\xi(x) = (x, |x|^2)$ and set $q_i = p_i - p_0$ for $i = 1, \ldots, m+1$. Then

$$(-p_0) + S(T) = \big\{ q \in \mathbb{R}^n : \xi(q) \wedge \xi(q_1) \wedge \cdots \wedge \xi(q_{m+1}) = 0 \big\}\,.$$

Note that $(-p_0) + S(T)$ still has radius r and has center $\tilde{c} = c - p_0$. For $k = 1, 2, \ldots, n+1$ let $* : \bigwedge_k \mathbb{R}^{n+1} \to \bigwedge_{n-k} \mathbb{R}^{n+1}$ be the Hodge star operator with respect to $e_1 \wedge \cdots \wedge e_{n+1}$; see [Fed69, 1.7.8]. Set

$$\psi = e_{m+2} \wedge \cdots \wedge e_n \in \textstyle\bigwedge_{n-(m+1)} \mathbb{R}^{n+1}$$
$$\text{and} \quad \omega = \xi(q_1) \wedge \cdots \wedge \xi(q_{m+1}) \in \textstyle\bigwedge_{m+1} \mathbb{R}^{n+1}\,.$$

For each $q = \sum_{j=1}^n x_j e_j \in (-p_0) + S(T)$ we have $x_j = 0$ for $j = m+2, \ldots, n$ and

$$*\big(\xi(q) \wedge \omega \wedge \psi\big) = 0 = |q - \tilde{c}|^2 - r^2\,. \tag{46}$$

The left-hand side and the right-hand side of (46) are polynomials of degree 2 in the variables $x_1, \ldots, x_n$. Comparing the coefficients of these two polynomials we obtain

$$\tilde{c} \bullet e_j = \frac{-*(e_j \wedge \omega \wedge \psi)}{2*(e_{n+1} \wedge \omega \wedge \psi)} \quad \text{for } j = 1, \ldots, m+1,$$
$$\tilde{c} \bullet e_j = 0 \quad \text{for } j = m+2, \ldots, n, \quad \text{and} \quad r^2 = |\tilde{c}|^2\,.$$

Let $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the injection map, i.e. $\iota(x) = (x, 0)$. Observe

$$\omega = \iota(q_1) \wedge \cdots \wedge \iota(q_{m+1}) + \sum_{i=1}^{m+1} \iota(q_1) \wedge \cdots \wedge |q_i|^2 e_{n+1} \wedge \cdots \wedge \iota(q_{m+1})\,,$$

$$e_j \wedge \omega = \sum_{i=1}^{m+1} e_j \wedge \iota(q_1) \wedge \cdots \wedge |q_i|^2 e_{n+1} \wedge \cdots \wedge \iota(q_{m+1}) \quad \text{for } j = 1, \ldots, m+1,$$

$$\big|*(e_{n+1} \wedge \omega \wedge \psi)\big| = |q_1 \wedge \cdots \wedge q_{m+1}|\,,$$
$$\big|*(e_j \wedge \omega \wedge \psi)\big| = \big|e_j \bullet *(\omega \wedge \psi)\big| = \big|\omega \bullet *(e_j \wedge \psi)\big| \quad \text{for } j = 1, \ldots, n+1.$$

Thus, $\left|*(e_j \wedge \omega \wedge \psi)\right| = 0$ for $j = m+2, \dots, n$ and we get

$$\sum_{j=1}^{m+1} \bigl(-*(e_j \wedge \omega \wedge \psi)\bigr)^2 = |*(\omega \wedge \psi)|^2 - |*(e_{n+1} \wedge \omega \wedge \psi)|^2 \tag{47}$$
$$= |\omega|^2 - |q_1 \wedge \cdots \wedge q_{m+1}|^2 .$$

Therefore, we can write

$$\kappa_{\mathbb{S}}(T) = r^{-1} = \frac{2\bigl|(p_1 - p_0) \wedge \cdots \wedge (p_{m+1} - p_0)\bigr|}{\bigl(|\xi(p_1 - p_0) \wedge \cdots \wedge \xi(p_{m+1} - p_0)|^2 - |(p_1 - p_0) \wedge \cdots \wedge (p_{m+1} - p_0)|^2\bigr)^{1/2}} .$$

Since $\kappa_{\mathbb{S}}(T)$ is the inverse of the radius of $S(T)$ conditions 1.1 (a) (c) (d) are immediately satisfied. To check 1.1 (e) assume d, δ, $T = (a, b_1, \dots, b_m, c)$ and P are as in 1.1(e). Note that $|*(e_j \wedge \omega \wedge \psi)| = |e_j \wedge \omega| \leqslant d^{m+2}$ for $j = 1, \dots, m+1$, so recalling (47)

$$\kappa_{\mathbb{S}}(T) \geqslant \frac{2\delta \operatorname{dist}(c - a, P)}{d^2}$$

and we see that $\kappa_{\mathbb{S}}$ is a Menger like curvature with $\gamma = 1$ and $\Lambda(\delta, \kappa) = 2\delta$.

Remark. If $m = 1$, then $\kappa_{\mathbb{S}}(a, b, c) = 4\mathcal{H}^2(\triangle(a, b, c))/(|a - b||b - c||c - a|)$ is the original Menger curvature. If $m \geqslant 2$, then it is not hard to check that $\kappa_{\mathbb{S}}$ is not tame. This feature was already observed in the last paragraph of [SvdM11, Appendix B].

A.2 Example. Let $\gamma \in (0, \infty)$. For $T = (q_0, q_1, \dots, q_{m+1}) \in \mathcal{D}_{m+1}$ and $i \in \{0, 1, \dots, m+1\}$ define

$$p_m \sin_i(T) = \frac{(m+1)!\mathcal{H}^{m+1}(\triangle T)}{\prod_{j=0, j \neq i}^{m+1} |q_j - q_i|} ,$$
$$P_i(T) = \operatorname{span}\bigl\{q_k - q_l : k, l \in \{0, 1, \dots, m+1\} \sim \{i\}\bigr\} \in \mathbf{G}(n, m) .$$
$$\mathfrak{c}_d(T) = \operatorname{diam}(\triangle T)^{-1}\bigl((m+2)^{-1} \textstyle\sum_{i=0}^{m+1} p_m \sin_i^2(T)\bigr)^{\gamma/2} ,$$
$$\mathfrak{c}_{\min}(T) = \operatorname{diam}(\triangle T)^{-1}\bigl(\min\{p_m \sin_i(T) : i = 0, 1, \dots, m+1\}\bigr)^{\gamma} ,$$
$$\mathfrak{c}_{\max}(T) = \operatorname{diam}(\triangle T)^{-1}\bigl(\max\{p_m \sin_i(T) : i = 0, 1, \dots, m+1\}\bigr)^{\gamma} ,$$
$$\text{and} \quad \mathfrak{c}_{\mathrm{alg}}(T) = \left(\frac{p_m \sin_0(T)}{\prod_{1 \leqslant i < j \leqslant m+1} |q_i - q_j|}\right)^{\gamma} .$$

If $T \in (\mathbb{R}^n)^{m+2}$ and $h_{\min}(T) = 0$, then we set $\mathfrak{c}_{\mathrm{alg}}(T) = \mathfrak{c}_{\max}(T) = \mathfrak{c}_{\min}(T) = \mathfrak{c}_d(T) = 0$ in accordance with 1.1 (b).

Observe that $p_m \sin_i(\lambda T) = p_m \sin_i(T) \leqslant 1$ for $T \in \mathcal{D}_{m+1}$. Hence, whenever $\kappa \in \{\mathfrak{c}_d , \mathfrak{c}_{\min} , \mathfrak{c}_{\max} , \mathfrak{c}_{\mathrm{alg}}\}$, it follows that $\kappa(\lambda T) = \lambda^{-1}\kappa(T)$; thus, κ satisfies 1.1 (a) (c) (d).

To check 1.1 (e) assume d, δ, $T = (a, b_1, \dots, b_m, c)$ and P are as in 1.1(e). Employing 6.2 (28) we obtain

$$p_m \sin_i(T) \geqslant \frac{\delta \operatorname{dist}(c - a, P)}{d} \quad \text{for } i = 0, 1, \dots, m+1 ,$$
$$\text{so} \quad \kappa(T) \geqslant \left(\frac{\delta \operatorname{dist}(c - a, P)}{d}\right)^{\gamma} \frac{1}{d} \quad \text{whenever } \kappa \in \{\mathfrak{c}_d , \mathfrak{c}_{\min} , \mathfrak{c}_{\max} , \mathfrak{c}_{\mathrm{alg}}\} .$$

This shows that all of $\mathfrak{c}_d$, $\mathfrak{c}_{\min}$, $\mathfrak{c}_{\max}$, $\mathfrak{c}_{alg}$ are Menger like curvatures with $\gamma = 2/(m(m+1))$ and $\Lambda(\delta, \kappa) = \delta$.

To check that $\mathfrak{c}_{\min}$ is also tame take $T = (q_0, \dots, q_{m+1}) \in \mathcal{D}_{m+1}$. Permuting the tuple T we can assume $h_{\min}(T) = \operatorname{dist}(q_{m+1} - q_0, P)$, where $P = \operatorname{span}\{q_1 - q_0, \dots, q_m - q_0\}$. Using the triangle inequality we can find $i \in \{0, 1, \dots, m\}$ such that $2|q_{m+1} - q_i| \geqslant \operatorname{diam}(\triangle T)$. Then we get

$$\begin{aligned} p_m \sin_i(T) &= \frac{m! \mathcal{H}^m\big(\triangle(\{q_0, \dots, q_{m+1}\} \sim \{q_{m+1}\})\big) \operatorname{dist}(q_{m+1} - q_i, P)}{\prod_{k \in \{0,1,\dots,m\} \sim \{i\}} |q_k - q_i| \cdot |q_{m+1} - q_i|} \\ &\leqslant \frac{2 \operatorname{dist}(q_{m+1} - q_i, P)}{\operatorname{diam}(\triangle T)} = \frac{2 h_{\min}(T)}{\operatorname{diam}(\triangle T)} . \end{aligned}$$

Thus, $\mathfrak{c}_{\min}$ is tame, by 6.3.

Remark. If one takes $\gamma = 2/(m(m+1))$, then the discrete curvatures of A.2 coincide with the curvatures defined in [LW09, §1.2 , §6.1.1] and [LW11, §10].

Remark. We do not know whether any of $\mathfrak{c}_d$, $\mathfrak{c}_{\max}$, $\mathfrak{c}_{alg}$ is tame or not.

Acknowledgements

The author was partially supported by the Foundation for Polish Science and partially by NCN Grant no. 2013/10/M/ST1/00416 while he was on leave from Institute of Mathematics of the University of Warsaw.

The author is also indebted to Ulrich Menne for many fruitful discussions.

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