Prepared for submission to JCAP

Next-to-next-to-leading order gravitational spin-orbit coupling via the effective field theory for spinning objects in the post-Newtonian scheme

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Abstract. We implement the effective field theory for gravitating spinning objects in the post-Newtonian scheme at the next-to-next-to-leading order level to derive the gravitational spin-orbit interaction potential at the third and a half post-Newtonian order for rapidly rotating compact objects. From the next-to-next-to-leading order interaction potential, which we obtain here in a Lagrangian form for the first time, we derive straightforwardly the corresponding Hamiltonian. The spin-orbit sector constitutes the most elaborate spin dependent sector at each order, and accordingly we encounter a proliferation of the relevant Feynman diagrams, and a significant increase of the computational complexity. We present in detail the evaluation of the interaction potential, going over all contributing Feynman diagrams. The computation is carried out in terms of the nonrelativistic gravitational fields, which are advantageous also in spin dependent sectors, together with the various gauge choices included in the effective field theory for gravitating spinning objects, which also optimize the calculation. In addition, we automatize the effective field theory computations, and carry out the automated computations in parallel. Such automated effective field theory computations would be most useful to obtain higher order post-Newtonian corrections. We compare our Hamiltonian to the ADM Hamiltonian, and arrive at a complete agreement between the ADM and effective field theory results. The derivation presented here is essential to obtain further higher order post-Newtonian corrections, and to reach the accuracy level required for the successful detection of gravitational radiation.

Contents

1 Introduction

In light of the upcoming operation of second-generation ground-based interferometers worldwide, such as Advanced LIGO in the US [1], Advanced Virgo in Europe [2], and KAGRA in Japan [3], we may witness a direct detection of gravitational waves (GW) within the end of the decade, which will open a new era of observational gravitational wave astronomy. Inspiralling binaries of compact objects are of the most promising sources in the accessible frequency band of these experiments, where they can be treated analytically within the post-Newtonian (PN) approximation of General Relativity [4]. As the search for the GW signals employs the matched filtering technique, there is a pressing need to obtain accurate waveform templates, using the Effective-One-Body (EOB) formulation to model the continuous signal [5].

Recently the fourth PN (4PN) order correction has been completed for the non spinning case [6], and it is necessary to reach a similar accuracy level for the spin dependent case, as such compact objects are expected to be rapidly rotating. Of the spin dependent sectors the spin-orbit sector contributes the leading order (LO) spin-dependent PN correction, and represents the most dominant spin effects. The LO spin-orbit correction at the 1.5PN order has been obtained by Tulczyjew already at 1959 [7]. Yet, the next-to-leading order (NLO) spin-orbit interaction has been approached much later, first at the level of the equations of motion (EOM) in [8], and [9], and then within the ADM Hamiltonian formalism [10]. The next-to-next-to-leading order (NNLO) spin-orbit interaction was first derived in Hamiltonian form in [11, 12], building on [13, 14], and then at the level of the EOM in [15, 16].

In this work we apply the effective field theory (EFT) for gravitating spinning objects in the PN scheme, formulated in [17], at the NNLO level, which was first treated in [18]. We derive the NNLO gravitational spin-orbit interaction potential at the 3.5PN order for rapidly rotating compact objects. The EFT for gravitating spinning objects [17], builds on the novel, self-contained EFT approach for the binary inspiral, which was introduced in [19, 20] for the non spinning case, and where an extension to spinning objects was first approached in [21], considering the seminal works in [22, 23] for spin in flat and curved spacetime, respectively. The EFT for gravitating spinning objects enables to directly obtain the EOM via a proper variation of the action [17, 24]. Furthermore, it also enables to obtain the corresponding Hamiltonians in a straightforward manner from the potentials derived via this formulation [17]. Indeed, from the potential, which we obtain here in a Lagrangian form for the first time, we derive the corresponding NNLO spin-orbit Hamiltonian, and then compare our result to the ADM Hamiltonian in [11, 12]. We arrive at a complete agreement between the ADM and EFT results.

The spin-orbit sector constitutes the most elaborate spin dependent sector at each order, see [17] for the LO and NLO levels, and [18] for other sectors at the NNLO level. Accordingly, we encounter here a proliferation of the relevant Feynman diagrams, where there are 132 diagrams contributing to this sector, and a significant increase of the computational complexity, e.g. there are 32 two-loop diagrams here. We also recall that as the spin is derivative-coupled, higher-order tensor expressions are required for all integrals involved in the calculations, compared to the non spinning case. Yet, the computation is carried out in terms of the nonrelativistic gravitational (NRG) fields [25, 26], which are advantageous also in spin dependent sectors, as was first shown in [27], and later also in [17, 18, 28, 29]. We also apply the various gauge choices included in the EFT for gravitating spinning objects [17], which also optimize the calculation. In addition, we automatize the EFT computations here, and carry out the automated computations in parallel, where we have used the suite of free packages xAct with the Mathematica software [30, 31]. Such automated EFT computations would be most useful to obtain higher order PN corrections. It should be stressed that in order to obtain further higher order results, all lower order results are required, consistently within one formalism, and so also in that respect the derivation presented in this work is essential.

The outline of the paper is as follows. In section 2 we briefly review the EFT for gravitating spinning objects in the PN scheme, and present the Feynman rules required for the EFT computation. In section 3 we present the evaluation of the NNLO spin-orbit interaction potential, going over all contributing Feynman diagrams, and giving the value of each diagram. In section 4 we present the NNLO spin-orbit potential EFT result, and from it we obtain the corresponding EFT Hamiltonian. We then compare our result to the ADM Hamiltonian, where we resolve the difference between the Hamiltonians, using higher order PN canonical transformations, and arrive at a complete agreement between the ADM and EFT results. In section 5 we summarize our main conclusions. Finally, in appendix A we provide the additional irreducible two-loop tensor integrals required for this work.

2 The EFT for gravitating spinning objects in the PN scheme

In this section we present the effective action, and the Feynman rules, which are derived from it, and are required for the EFT computation of the NNLO spin-orbit interaction. We employ here the NRG fields, as applied with spin in [17, 18, 27–29]. Here, we briefly review and build on [17, 18, 24, 27, 28], following similar notations and conventions as those that were used there. Hence we use $c \equiv 1$, $\eta_{\mu\nu} \equiv \text{Diag}[1,-1,-1,-1]$, and the convention for the Riemann tensor is $R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\lambda\alpha}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\mu}_{\lambda\beta}\Gamma^{\lambda}_{\nu\alpha}$. The scalar triple product appears here with no brackets, i.e. $\vec{a}\times\vec{b}\cdot\vec{c} \equiv (\vec{a}\times\vec{b})\cdot\vec{c}$. The notation $\int_{\vec{k}} \equiv \int \frac{d^d\vec{k}}{(2\pi)^d}$ is used for abbreviation. In fact, the generic d-dimensional dependence can be and is suppressed in what follows, and d can be set to 3, except for computations which involve loops, where only the d dependence from the generic d-dimensional Feynman integrals, see appendix A in [18], should be considered.

First, in terms of the NRG fields the metric reads

$$
\begin{aligned}
g_{\mu\nu} &= \begin{pmatrix} e^{2\phi} & -e^{2\phi}A_j \\ -e^{2\phi}A_i & -e^{-2\phi}\gamma_{ij} + e^{2\phi}A_iA_j \end{pmatrix} \\
&\simeq \begin{pmatrix} 1+2\phi+2\phi^2+\frac{4}{3}\phi^3 & -A_j-2A_j\phi-2A_j\phi^2 \\ -A_i-2A_i\phi-2A_j\phi^2 & -\delta_{ij}+2\phi\delta_{ij}-\sigma_{ij}-2\phi^2\delta_{ij}+2\phi\sigma_{ij}+A_iA_j+\frac{4}{3}\phi^3\delta_{ij} \end{pmatrix},
\end{aligned}
\tag{2.1}
$$

where we have written the approximate metric in the weak-field limit up to the orders in the fields that are required for this sector.

We recall that the effective action, describing the binary system, is given by

$$
S = S_{\text{g}} + \sum_{I=1}^{2} S_{(\text{I})\text{pp}}, \tag{2.2}
$$

where S_{g} is the pure gravitational action, and $S_{(\text{I})\text{pp}}$ is the worldline point particle action for each of the two particles in the binary. The gravitational action is the usual Einstein-Hilbert action plus a gauge-fixing term, which we choose as the fully harmonic gauge, such that we have

$$
S_{\text{g}} = S_{\text{EH}} + S_{\text{GF}} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R + \frac{1}{32\pi G}\int d^4x\sqrt{g}\,g_{\mu\nu}\Gamma^{\mu}\Gamma^{\nu}, \tag{2.3}
$$

where $\Gamma^{\mu} \equiv \Gamma^{\mu}_{\rho\sigma}g^{\rho\sigma}$.

From the gravitational action we derive the propagators, and the self-gravitational vertices. The NRG scalar, vector, and tensor field propagators in the harmonic gauge are then given by

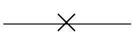
$$
= \langle\, \phi(x_1) \;\; \phi(x_2)\,\rangle = \quad 4\pi G \quad \int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{k^2}\,\delta(t_1-t_2), \tag{2.4}
$$

$$
= \langle A_i(x_1)\;A_j(x_2)\rangle = -16\pi G\;\delta_{ij}\int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{k^2}\,\delta(t_1-t_2), \tag{2.5}
$$

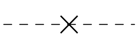
$$
= \langle\sigma_{ij}(x_1)\sigma_{kl}(x_2)\rangle = \quad 32\pi G\;P_{ij;kl}\int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{k^2}\,\delta(t_1-t_2), \tag{2.6}
$$

where $P_{ij;kl} \equiv \frac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-2\delta_{ij}\delta_{kl}\right)$.

The Feynman rules for the propagator correction vertices are given by



$$
= \quad \frac{1}{8\pi G}\quad\int d^4x\;\left(\partial_t\phi\right)^2, \tag{2.7}
$$

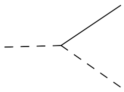


$$
= -\frac{1}{32\pi G}\int d^4x\;\left(\partial_t A_i\right)^2, \tag{2.8}
$$

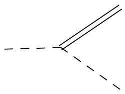
$$= \frac{1}{128\pi G}\int d^4x \;\left[2(\partial_t\sigma_{ij})^2-(\partial_t\sigma_{ii})^2\right], \tag{2.9}$$

where the crosses represent the self-gravitational quadratic vertices, which contain two time derivatives.

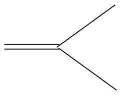
The Feynman rules for the three-graviton vertices required for the NNLO of the spin-orbit interaction are given by



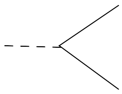
$$= \frac{1}{8\pi G}\int d^4x \;\phi\left(\partial_iA_j\left(\partial_iA_j-\partial_jA_i\right)+\left(\partial_iA_i\right)^2\right), \tag{2.10}$$



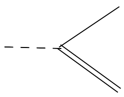
$$= -\frac{1}{64\pi G}\int d^4x\Big[2\sigma_{ij}\left(\partial_iA_k\partial_jA_k+\partial_kA_i\partial_kA_j-2\partial_kA_i\partial_jA_k+2\partial_iA_j\partial_kA_k\right) -\sigma_{kk}\left(\partial_iA_j\left(\partial_iA_j-\partial_jA_i\right)+\left(\partial_iA_i\right)^2\right)\Big], \tag{2.11}$$



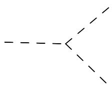
$$= \frac{1}{16\pi G}\int d^4x \;\left[\left(2\sigma_{ij}\partial_i\phi\partial_j\phi-\sigma_{jj}\partial_i\phi\partial_i\phi\right)+\left(\sigma_{ii}\left(\partial_t\phi\right)^2-16\partial_t\sigma_{ii}\phi\partial_t\phi\right)\right], \tag{2.12}$$



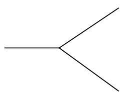
$$= -\frac{1}{4\pi G}\int d^4x \;\left(A_i\partial_i\phi\partial_t\phi\right), \tag{2.13}$$



$$= \frac{1}{8\pi G}\int d^4x \;\left[2\sigma_{ij}\left(\partial_i\phi\partial_tA_j-\partial_t\phi\partial_iA_j\right)-\sigma_{jj}\left(\partial_i\phi\partial_tA_i-\partial_t\phi\partial_iA_i\right)\right], \tag{2.14}$$



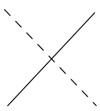
$$= \frac{1}{16\pi G}\int d^4x \;\left(A_i\partial_iA_j\partial_tA_j\right), \tag{2.15}$$



$$= -\frac{1}{2\pi G}\int d^4x \;\left[\phi\left(\partial_t\phi\right)^2\right], \tag{2.16}$$

where the first three vertices are stationary, and can be read off from the stationary Kaluza-Klein part of the gravitational action. The next four vertices are time dependent, and contain up to two time derivatives.

The Feynman rule for the four-graviton vertex required to the order considered here is given by



$$= \frac{1}{4\pi G}\int d^4x \;\phi^2\left(\partial_iA_j\left(\partial_iA_j-\partial_jA_i\right)+\left(\partial_iA_i\right)^2\right), \tag{2.17}$$

where this vertex is stationary.

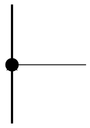
Next, we recall that the minimal coupling part of the point particle action of each of the particles with spins is given by

$$S_{\rm pp} = \int d\lambda\left[-m\sqrt{u^2}-\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu}\right], \tag{2.18}$$

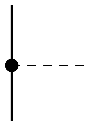
where λ is the affine parameter, $u^\mu\equiv dx^\mu/d\lambda$ is the 4-velocity, and $\Omega^{\mu\nu}$, $S_{\mu\nu}$ are the angular velocity and spin tensors of the particle, respectively [17]. We parametrize the worldline using the coordinate time $t=x^0$, i.e. $\lambda=t$, so that we have for $u_\mu\equiv dx^\mu/d\lambda$: $u^0=1$,

$u^i = dx^i/dt \equiv v^i$. Since the spin-orbit interaction is linear in the spins, only the minimal coupling part of the action, i.e. that which appears in eq. (2.18) is required. We stress that in the spin-orbit sector both mass and spin couplings play central roles in the interaction.

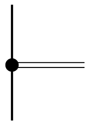
Let us then present first the mass couplings required for this sector. The Feynman rules of the one-graviton couplings to the worldline mass required for the NNLO spin-orbit interaction are given by



$$= -\, m \int dt\ \phi\ \left[1 + \frac{3}{2}v^2 + \frac{7}{8}v^4 + \cdots\right], \tag{2.19}$$



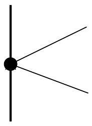
$$= m \int dt\ A_i v^i\ \left[1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \cdots\right], \tag{2.20}$$



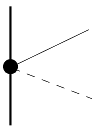
$$= \frac{1}{2} m \int dt\ \sigma_{ij} v^i v^j\ \left[1 + \frac{1}{2}v^2 + \cdots\right], \tag{2.21}$$

where the heavy solid lines represent the worldlines, and the spherical black blobs represent the masses on the worldline. The ellipsis denotes higher orders in v, beyond the order considered here.

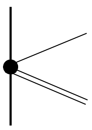
For the two-graviton couplings to the worldline mass required here, we have the following Feynman rules:



$$= -\frac{1}{2} m \int dt\ \phi^2\ \left[1 - \frac{9}{2}v^2 + \cdots\right], \tag{2.22}$$

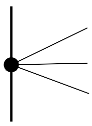


$$= m \int dt\ \phi A_i v^i\ \left[1 - \frac{3}{2}v^2 + \cdots\right], \tag{2.23}$$

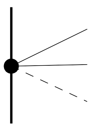


$$= -\frac{3}{2} m \int dt\ \phi \sigma_{ij} v^i v^j\ \left[1 + \cdots\right]. \tag{2.24}$$

Finally, for the three-graviton couplings to the worldline mass required here, we have the following Feynman rules:



$$= -\frac{1}{6} m \int dt\ \phi^3\ \left[1 + \cdots\right], \tag{2.25}$$



$$= \frac{1}{2} m \int dt\ \phi^2 A_i v^i\ \left[1 + \cdots\right]. \tag{2.26}$$

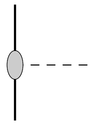
Let us go on to the spin couplings required for this sector. These are given here in terms of the physical spatial components of the local spin variable in the canonical gauge

[17]. First, we have contributions from kinematic terms involving spin without field coupling [17]. To the order we are considering, these are given by

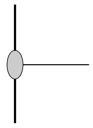
$$L_{\rm kin} = -\vec{S}\cdot\vec{\Omega} + \frac{1}{2}\vec{S}\cdot\vec{v}\times\vec{a}\left(1+\frac{3}{4}v^2+\frac{5}{8}v^4\right), \tag{2.27}$$

where $S_{ij} = \epsilon_{ijk}S_k$, $\Omega_{ij} = \epsilon_{ijk}\Omega_k$, ϵ_{ijk} is the 3-dimensional Levi-Civita symbol, and $a^i \equiv \dot{v}^i$. We recall that all indices are Euclidean.

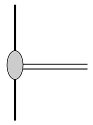
The required Feynman rules of the one-graviton couplings to the worldline spin are thus



$$= \int dt\, \left[\epsilon_{ijk}S_k\left(\frac{1}{2}\partial_i A_j + \frac{1}{4}v^i v^l\left(\partial_l A_j - \partial_j A_l\right)\left(3+\frac{7}{4}v^2\right) + v^i\partial_t A_j\left(1+\frac{1}{2}v^2\right) + v^i a^j A_l v^l\right)\right], \tag{2.28}$$



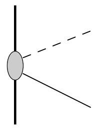
$$= \int dt\, \left[\epsilon_{ijk}S_k v^i\left(\partial_j\phi\left(2+v^2+\frac{3}{4}v^4\right) - a^j\phi\left(2+3v^2\right)\right)\right], \tag{2.29}$$



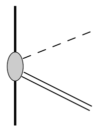
$$= \int dt\, \left[\frac{1}{2}\epsilon_{ijk}S_k\left(\partial_i\sigma_{jl}\,v^l + \left(\frac{1}{2}\partial_i\sigma_{lm} - \partial_l\sigma_{im}\right)v^j v^l v^m - \frac{3}{2}\partial_t\sigma_{il}\,v^j v^l - \frac{1}{2}\sigma_{il}\left(v^j a^l - v^l a^j\right)\right)\right], \tag{2.30}$$

where the (gray) oval blobs represent the spins on the worldlines.

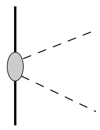
For the two-graviton couplings to the worldline spin, the Feynman rules required here are:



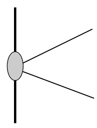
$$= \int dt\, \left[2\epsilon_{ijk}S_k\left(\partial_i A_j\,\phi + A_j\,\partial_t\phi\,v^i\right)\right], \tag{2.31}$$



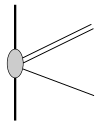
$$= \int dt\, \left[\frac{1}{4}\epsilon_{ijk}S_k\sigma_{il}\left(\partial_j A_l - \partial_l A_j\right)\right], \tag{2.32}$$



$$= \int dt\, \left[\frac{1}{2}\epsilon_{ijk}S_k\left(\partial_j A_i\,A_l\,v^l + A_i\,\partial_t A_j\right)\right], \tag{2.33}$$

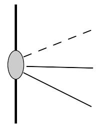


$$= \int dt\, \left[4\epsilon_{ijk}S_k\phi\left(\partial_i\phi\,v^j + v^i a^j\phi\right)\right], \tag{2.34}$$



$$= \int dt\, \left[\epsilon_{ijk}S_k\sigma_{il}\left(\partial_j\phi\,v^l + \partial_l\phi\,v^j\right)\right]. \tag{2.35}$$

Finally, we also have to include three-graviton spin couplings. For three-graviton couplings to the worldline spin, the only Feynman rule required here is

 $$= \int dt \left[4\epsilon_{ijk} S_k \partial_i A_j \, \phi^2\right]. \tag{2.36}$$

Note that similarly to the NLO, where the two-scalar spin coupling is absent, here at NNLO the three-scalar spin coupling vanishes due to the use of the NRG fields, and our gauge of the rotational spin variables. Its appearance is then deferred to higher PN orders, which is advantageous.

3 Next-to-next-to-leading order spin-orbit interaction

In this section we evaluate the relevant two-body effective action by its diagrammatic expansion. As explained in [18], in the NNLO spin-orbit potential, which is evaluated at 3.5PN order, we have diagram contributions up to order G^3, coming from all 12 possible topologies appearing at these orders, as displayed in figures 1-6 below: one topology at $O(G)$, two at $O(G^2)$, and nine topologies at $O(G^3)$. For the construction of the Feynman diagrams we use the Feynman rules presented in the previous section, see [18] for more detail. We have a total of 132 diagrams contributing to the NNLO spin-orbit interaction. Eventually, we will also have contribution from terms with higher order time derivatives coming from the point-mass 2PN order [32], and up to NLO spin-orbit [17] sectors.

We denote $\vec{r} \equiv \vec{x}_1 - \vec{x}_2$, $r \equiv |\vec{r}|$, and $\vec{n} \equiv \frac{\vec{r}}{r}$. Labels 1 and 2 are used for the left and right worldlines in the figures, respectively. All of the diagrams should be included together with their mirror images. Accordingly, the $(1 \leftrightarrow 2)$ notation stands for a term, whose value is obtained under the interchange of particles labels. Finally, a multiplicative factor of $\int dt$ is omitted from all diagram values.

3.1 One-graviton exchange

For the NNLO spin-orbit interaction we have 8 one-graviton exchange diagrams as shown in figure 1. In addition to the one-graviton exchange diagrams, which already appeared in the NLO spin-orbit sector [17, 28], new diagrams are added here by inserting further propagator correction vertices. Tensor Fourier integrals of up to order 5 are required here, due to the derivative-coupling of spin, which makes the computations heavier, see [18], and appendix A there.

At the NNLO level we inevitably obtain terms with higher order time derivative, i.e. with accelerations and precessions, and these are all kept until they are treated rigorously in the resulting action [17, 24], see section 4 below. Finally, we recall that there are several ways to evaluate diagrams with time derivatives, which differ only by total time derivatives. Our convention for their evaluation is, that time derivatives from the spin couplings are taken on their respective worldlines, whereas those from the propagator correction vertices are taken symmetrically on the worldlines.

The values of the one-graviton exchange diagrams are given as follows:

$$\begin{aligned}\text{Fig. 1(a)} =& \frac{2Gm_2}{r^2}\vec{S}_1\times\vec{n}\cdot\vec{v}_2 + \frac{Gm_2}{r^2}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2 + \vec{S}_1\times\vec{n}\cdot\vec{v}_2 v_2^2 - \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]\\ &+\frac{4Gm_2}{r}\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2 + \frac{4Gm_2}{r}\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2 + \frac{Gm_2}{4r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(7v_1^2\vec{v}_1\cdot\vec{v}_2\end{aligned}$$

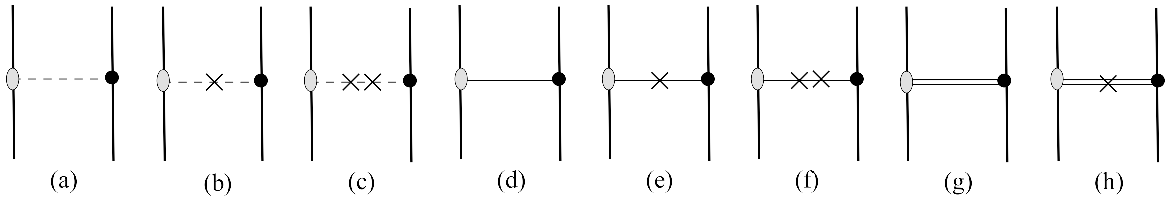


Figure 1. NNLO spin-orbit Feynman diagrams of one-graviton exchange.

$$
\begin{aligned}
&+6\vec{v}_1\cdot\vec{v}_2 v_2^2\big)+3\vec{S}_1\times\vec{n}\cdot\vec{v}_2 v_2^4-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{n}v_1^2+2\vec{v}_1\cdot\vec{n}v_2^2\big)\Big]\\
&-\frac{2Gm_2}{r}\Big[2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\vec{v}_1\cdot\vec{v}_2-2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{a}_1-\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\big(v_1^2\\
&+v_2^2\big)\Big]+\frac{2Gm_2}{r}\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\big(v_1^2+v_2^2\big), \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 1(b)} =&\frac{Gm_2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2-3\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)+\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big]\\
&-\frac{Gm_2}{r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big]+\frac{Gm_2}{r}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\\
&+Gm_2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2+\frac{Gm_2}{2r^2}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big((\vec{v}_1\cdot\vec{v}_2)^2-3\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\big)\\
&+\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2v_2^2-3\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(v_1^2\vec{v}_2\cdot\vec{n}\\
&+5\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-\vec{v}_2\cdot\vec{n}v_2^2-3\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\big)\Big]+\frac{Gm_2}{2r}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2\\
&-\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2\big)+3\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{a}_2\\
&+\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(4\vec{a}_1\cdot\vec{v}_2+2\vec{v}_2\cdot\vec{a}_2-\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\big(8\vec{v}_1\cdot\vec{v}_2\\
&-5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)-\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}v_2^2-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\big(v_1^2-v_2^2-(\vec{v}_1\cdot\vec{n})^2\big)\Big]\\
&+\frac{Gm_2}{2r}\Big[3\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}v_2^2\\
&+\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\big(8\vec{v}_1\cdot\vec{v}_2-5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)\Big]+\frac{1}{2}Gm_2\Big[4\vec{S}_1\times\dot{\vec{a}}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\\
&+4\ddot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}+3\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_1\cdot\vec{a}_2+3\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_1\cdot\vec{a}_2\\
&-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\vec{a}_1\cdot\vec{n}-5\vec{S}_1\times\vec{a}_1\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}+3\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_1\cdot\vec{a}_2\\
&+2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_2\cdot\vec{a}_2+8\dot{\vec{S}}_1\times\vec{a}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}+\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2v_2^2\\
&-5\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}\Big]-2Gm_2r\Big[\vec{S}_1\times\dot{\vec{a}}_1\cdot\vec{a}_2+\ddot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_2\\
&+2\dot{\vec{S}}_1\times\vec{a}_1\cdot\vec{a}_2\Big], \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 1(c)} =&\frac{Gm_2}{4r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_1^2v_2^2+2(\vec{v}_1\cdot\vec{v}_2)^2-12\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+3v_2^2(\vec{v}_1\cdot\vec{n})^2\\
&+3v_1^2(\vec{v}_2\cdot\vec{n})^2+15(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\big)+2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+\vec{v}_1\cdot\vec{n}v_2^2\\
&-3\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)\Big]+\frac{Gm_2}{4r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(2\vec{v}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2+\vec{a}_1\cdot\vec{n}v_2^2\\
&-v_1^2\vec{a}_2\cdot\vec{n}-2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2+3\vec{a}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2-3\vec{a}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&+2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{a}_2-\vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)-\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\big(v_2^2-(\vec{v}_2\cdot\vec{n})^2\big)\\
&-2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\big(v_1^2\vec{v}_2\cdot\vec{n}+2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-3\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\big)\\
&+4\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\big(\vec{v}_1\cdot\vec{v}_2-\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)+\vec{S}_1\times\vec{v}_2\cdot\vec{a}_2\big(v_1^2-(\vec{v}_1\cdot\vec{n})^2\big)\Big]\\
&+\frac{Gm_2}{2r}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\big(2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+\vec{v}_1\cdot\vec{n}v_2^2-3\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)\\
&-\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\big(v_2^2-(\vec{v}_2\cdot\vec{n})^2\big)\Big]-\frac{1}{4}Gm_2\Big[\vec{S}_1\times\vec{n}\cdot\dot{\vec{a}}_2\big(v_1^2-(\vec{v}_1\cdot\vec{n})^2\big)\\
&+2\vec{S}_1\times\vec{v}_1\cdot\dot{\vec{a}}_2\vec{v}_1\cdot\vec{n}-\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{a}_1\cdot\vec{a}_2-\vec{a}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)\\
&+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}+2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\big(\vec{a}_1\cdot\vec{v}_2-\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\times\vec{a}_1\cdot\vec{a}_2\vec{v}_2\cdot\vec{n}\\
&+\vec{S}_1\times\vec{v}_2\cdot\vec{a}_2\vec{a}_1\cdot\vec{n}-2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{a}_2-\vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)\\
&+2\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}+4\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2\big(\vec{v}_1\cdot\vec{v}_2-\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)\\
&+4\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_2\vec{v}_2\cdot\vec{n}+2\dot{\vec{S}}_1\times\vec{v}_2\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}+\ddot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\big(v_2^2\\
&-(\vec{v}_2\cdot\vec{n})^2\big)\Big]-\frac{1}{4}Gm_2r\Big[\vec{S}_1\times\vec{n}\cdot\dot{\vec{a}}_2\vec{a}_1\cdot\vec{n}+\vec{S}_1\times\vec{a}_1\cdot\dot{\vec{a}}_2\\
&+2\dot{\vec{S}}_1\times\vec{n}\cdot\dot{\vec{a}}_2\vec{v}_1\cdot\vec{n}+2\dot{\vec{S}}_1\times\vec{v}_1\cdot\dot{\vec{a}}_2-\ddot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}\\
&+2\ddot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_2\cdot\vec{n}+\ddot{\vec{S}}_1\times\vec{v}_2\cdot\vec{a}_2\Big]-\frac{1}{4}Gm_2r^2\ddot{\vec{S}}_1\times\vec{n}\cdot\dot{\vec{a}}_2, \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 1(d)}=&-\frac{2Gm_2}{r^2}\vec{S}_1\times\vec{n}\cdot\vec{v}_1-\frac{Gm_2}{r^2}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2+3v_2^2\big)+\frac{2Gm_2}{r}\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\\
&-\frac{Gm_2}{4r^2}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(6v_1^2v_2^2+3v_1^4+7v_2^4\big)+\frac{3Gm_2}{r}\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\big(v_1^2+v_2^2\big), \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 1(e)}=&-\frac{Gm_2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2-3\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]\\
&-\frac{Gm_2}{r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big]-\frac{Gm_2}{r}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\\
&+\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big]-\frac{Gm_2}{2r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2\vec{v}_1\cdot\vec{v}_2+3\vec{v}_1\cdot\vec{v}_2v_2^2\\
&-3\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+9\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{n}v_1^2+3\vec{v}_1\cdot\vec{n}v_2^2\big)\Big]\\
&-\frac{Gm_2}{2r}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}-3\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{a}_2\big)+\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(v_1^2\vec{v}_2\cdot\vec{n}\\
&+3\vec{v}_2\cdot\vec{n}v_2^2\big)-2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\big(\vec{v}_1\cdot\vec{v}_2-\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{a}_1\\
&+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\big(v_1^2+3v_2^2\big)\Big]-\frac{Gm_2}{2r}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2\vec{v}_2\cdot\vec{n}+3\vec{v}_2\cdot\vec{n}v_2^2\big)\\
&+\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\big(v_1^2+3v_2^2\big)\Big]-Gm_2\Big[3\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{a}_2+3\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{a}_2\\
&-\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}-\vec{S}_1\times\vec{v}_1\cdot\dot{\vec{a}}_1\vec{v}_2\cdot\vec{n}\Big], \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 1(f)}=&-\frac{Gm_2}{4r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2v_2^2+2(\vec{v}_1\cdot\vec{v}_2)^2-12\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+3v_2^2(\vec{v}_1\cdot\vec{n})^2\\
&+3v_1^2(\vec{v}_2\cdot\vec{n})^2+15(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\big)-2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(v_1^2\vec{v}_2\cdot\vec{n}+2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-3\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\big)\Big]-\frac{Gm_2}{4r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(2\vec{v}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2+\vec{a}_1\cdot\vec{n}v_2^2
\end{aligned}
$$

$$
\begin{aligned}
&- v_1^2\vec{a}_2\cdot\vec{n} - 2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2 + 3\vec{a}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2 - 3\vec{a}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big) \\
&+ 2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 + \vec{v}_1\cdot\vec{n}v_2^2 - 3\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\big(v_2^2 \\
&- (\vec{v}_2\cdot\vec{n})^2\big) + 2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{a}_1\cdot\vec{v}_2 - \vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big) + 4\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 \\
&- \vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\big(v_1^2 - (\vec{v}_1\cdot\vec{n})^2\big)\Big] - \frac{Gm_2}{2r}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 \\
&+ \vec{v}_1\cdot\vec{n}v_2^2 - 3\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big) + 2\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - \vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)\Big] \\
&+ \frac{1}{4}Gm_2\Big[\vec{S}_1\times\vec{n}\cdot\dot{\vec{a}}_1\big(v_2^2 - (\vec{v}_2\cdot\vec{n})^2\big) - 2\vec{S}_1\times\dot{\vec{a}}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n} \\
&+ \ddot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(v_2^2 - (\vec{v}_2\cdot\vec{n})^2\big) - 2\ddot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n} - \vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{a}_1\cdot\vec{a}_2 \\
&- \vec{a}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big) + 2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(\vec{v}_1\cdot\vec{a}_2 - \vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\vec{a}_2\cdot\vec{n} \\
&- \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\vec{a}_1\cdot\vec{n} - 2\vec{S}_1\times\vec{a}_1\cdot\vec{a}_2\vec{v}_1\cdot\vec{n} - 2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{a}_2 \\
&- \vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big) - 2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_1\big(v_2^2 - (\vec{v}_2\cdot\vec{n})^2\big) + 4\dot{\vec{S}}_1\times\vec{a}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n} \\
&- 2\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}\Big] - \frac{1}{4}Gm_2r\Big[\vec{S}_1\times\vec{n}\cdot\dot{\vec{a}}_1\vec{a}_2\cdot\vec{n} - \vec{S}_1\times\dot{\vec{a}}_1\cdot\vec{a}_2 \\
&+ \ddot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_2\cdot\vec{n} - \ddot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_2 + 2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_1\vec{a}_2\cdot\vec{n} \\
&- 2\dot{\vec{S}}_1\times\vec{a}_1\cdot\vec{a}_2\Big], \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 1(g)} =& \frac{2Gm_2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1v_2^2 - \vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2\Big] + \frac{Gm_2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2v_2^2 \\
&- (\vec{v}_1\cdot\vec{v}_2)^2 + v_2^4\big) - \vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2v_2^2 + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\Big] \\
&- \frac{2Gm_2}{r}\Big[\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1v_2^2 + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{a}_1\cdot\vec{v}_2 + 2\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2\Big] \\
&- \frac{3Gm_2}{r}\vec{v}_1\cdot\vec{v}_2\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 1(h)} =& \frac{Gm_2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2v_2^2 - 3\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\big) - \vec{S}_1\times\vec{n}\cdot\vec{v}_2\big((\vec{v}_1\cdot\vec{v}_2)^2 \\
&- 3\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 + \vec{v}_1\cdot\vec{n}v_2^2\big)\Big] \\
&- \frac{Gm_2}{r}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{a}_2 - \vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}v_2^2 \\
&+ \vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2 - \vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{a}_2 \\
&- \vec{S}_1\times\vec{a}_1\cdot\vec{v}_2v_2^2 - \vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 + \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\vec{v}_1\cdot\vec{v}_2\Big] \\
&+ \frac{Gm_2}{r}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}v_2^2 - \dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 \\
&+ \dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2v_2^2\Big] + Gm_2\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{a}_2 - \vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{a}_1\cdot\vec{a}_2 \\
&- \vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{a}_1\cdot\vec{v}_2 + 2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{a}_2 - \dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_1\cdot\vec{a}_2 \\
&- \dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{v}_2\Big]. \qquad (3.8)
\end{aligned}
$$

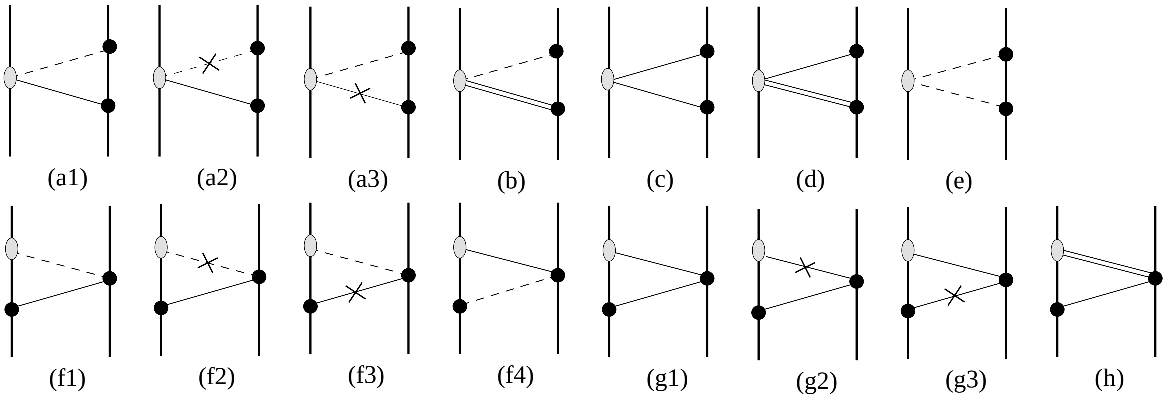


Figure 2. NNLO spin-orbit Feynman diagrams of two-graviton exchange.

3.2 Two-graviton exchange and cubic self-interaction

3.2.1 Two-graviton exchange

For the NNLO spin-orbit interaction we have 15 two-graviton exchange diagrams, as shown in figure 2, where they either contain a two-graviton spin or mass coupling.

The two-graviton exchange diagrams also require only tensor Fourier integrals. We encounter here two-graviton exchange diagrams, which involve time derivatives, either from the spin couplings or from propagator correction vertices.

The values of the two-graviton exchange diagrams are given in the following:

$$\text{Fig. 2(a1)} = -\frac{8G^2m_2^2}{r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_2 - \frac{8G^2m_2^2}{r^3}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_2 v_2^2 - \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big], \tag{3.9}$$

$$\begin{aligned}\text{Fig. 2(a2)} = &-\frac{4G^2m_2^2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - (\vec{v}_2\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big]\\ &+\frac{4G^2m_2^2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{a}_2\big(2\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n}\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big]\\ &-\frac{4G^2m_2^2}{r^2}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2 - \frac{4G^2m_2^2}{r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\text{Fig. 2(a3)} = &-\frac{4G^2m_2^2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - 3(\vec{v}_2\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big]\\ &-\frac{4G^2m_2^2}{r^2}\vec{v}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{a}_2 - \frac{4G^2m_2^2}{r^2}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2,\end{aligned} \tag{3.11}$$

$$\text{Fig. 2(b)} = \frac{4G^2m_2^2}{r^3}v_2^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2, \tag{3.12}$$

$$\text{Fig. 2(c)} = -\frac{4G^2m_2^2}{r^3}v_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1 + \frac{4G^2m_2^2}{r^2}\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1, \tag{3.13}$$

$$\text{Fig. 2(d)} = -\frac{4G^2m_2^2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2 + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big], \tag{3.14}$$

$$\text{Fig. 2(e)} = \frac{8G^2m_2^2}{r^3}\vec{v}_1\cdot\vec{v}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_2 + \frac{8G^2m_2^2}{r^2}\vec{S}_1\times\vec{v}_2\cdot\vec{a}_2, \tag{3.15}$$

$$\text{Fig. 2(f1)} = -\frac{2G^2m_1m_2}{r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_2 - \frac{G^2m_1m_2}{r^3}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2 + 3\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_1^2$$

$$- v_2^2\big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n}\Big] - \frac{4G^2 m_1 m_2}{r^2} \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 - \frac{4G^2 m_1 m_2}{r^2} \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2, \tag{3.16}$$

$$\begin{aligned}\text{Fig. 2(f2)} =& -\frac{G^2 m_1 m_2}{r^3}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \big(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - (\vec{v}_1 \cdot \vec{n})^2\big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \big(\vec{v}_1 \cdot \vec{n} \\ & - 2\vec{v}_2 \cdot \vec{n}\big)\Big] + \frac{G^2 m_1 m_2}{r^2}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2\Big] \\ & + \frac{G^2 m_1 m_2}{r^2} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 \big(\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n}\big) - \frac{G^2 m_1 m_2}{r} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2, \end{aligned} \tag{3.17}$$

$$\begin{aligned}\text{Fig. 2(f3)} =& -\frac{G^2 m_1 m_2}{r^3}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \big(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 3(\vec{v}_1 \cdot \vec{n})^2\big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n}\Big] \\ & + \frac{G^2 m_1 m_2}{r^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 + \frac{G^2 m_1 m_2}{r^2} \vec{v}_1 \cdot \vec{n} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2, \end{aligned} \tag{3.18}$$

$$\text{Fig. 2(f4)} = -\frac{8G^2 m_1 m_2}{r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{v}_2, \tag{3.19}$$

$$\begin{aligned}\text{Fig. 2(g1)} =& \frac{2G^2 m_1 m_2}{r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 + \frac{G^2 m_1 m_2}{r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \big(4v_1^2 - 9v_2^2\big) \\ & - \frac{2G^2 m_1 m_2}{r^2} \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1, \end{aligned} \tag{3.20}$$

$$\begin{aligned}\text{Fig. 2(g2)} =& \frac{G^2 m_1 m_2}{r^3}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \big(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - (\vec{v}_1 \cdot \vec{n})^2\big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n}\Big] \\ & - \frac{G^2 m_1 m_2}{r^2}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \big(\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n}\big) - \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2\Big] \\ & - \frac{G^2 m_1 m_2}{r^2}\Big[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 \big(\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n}\big) - \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2\Big], \end{aligned} \tag{3.21}$$

$$\begin{aligned}\text{Fig. 2(g3)} =& \frac{G^2 m_1 m_2}{r^3}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \big(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 3(\vec{v}_1 \cdot \vec{n})^2\big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n}\Big] \\ & - \frac{G^2 m_1 m_2}{r^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 - \frac{G^2 m_1 m_2}{r^2} \vec{v}_1 \cdot \vec{n} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1, \end{aligned} \tag{3.22}$$

$$\text{Fig. 2(h)} = \frac{6G^2 m_1 m_2}{r^3}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 v_2^2 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{v}_2\Big]. \tag{3.23}$$

3.2.2 Cubic self-interaction

For the NNLO spin-orbit interaction we have 49 cubic self-interaction diagrams, as shown in figure 3, where the cubic vertices contain up to two time derivatives.

The cubic self-gravitational interaction diagrams require first the application of one-loop tensor integrals up to order 3, in addition to the Fourier tensor integrals, see appendix A in [18]. Here, we encounter time derivatives from the spin couplings, the propagator correction vertices, and the time dependent cubic self-gravitational vertices.

The values of the cubic self-interaction diagrams are given as follows:

$$\begin{aligned}\text{Fig. 3(a1)} =& \frac{8G^2 m_2^2}{r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 + \frac{4G^2 m_2^2}{r^3}\Big[3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{v}_2 + 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 v_2^2 \\ & - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n}\Big] + \frac{8G^2 m_2^2}{r^2} \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 + \frac{8G^2 m_2^2}{r^2} \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2, \end{aligned} \tag{3.24}$$

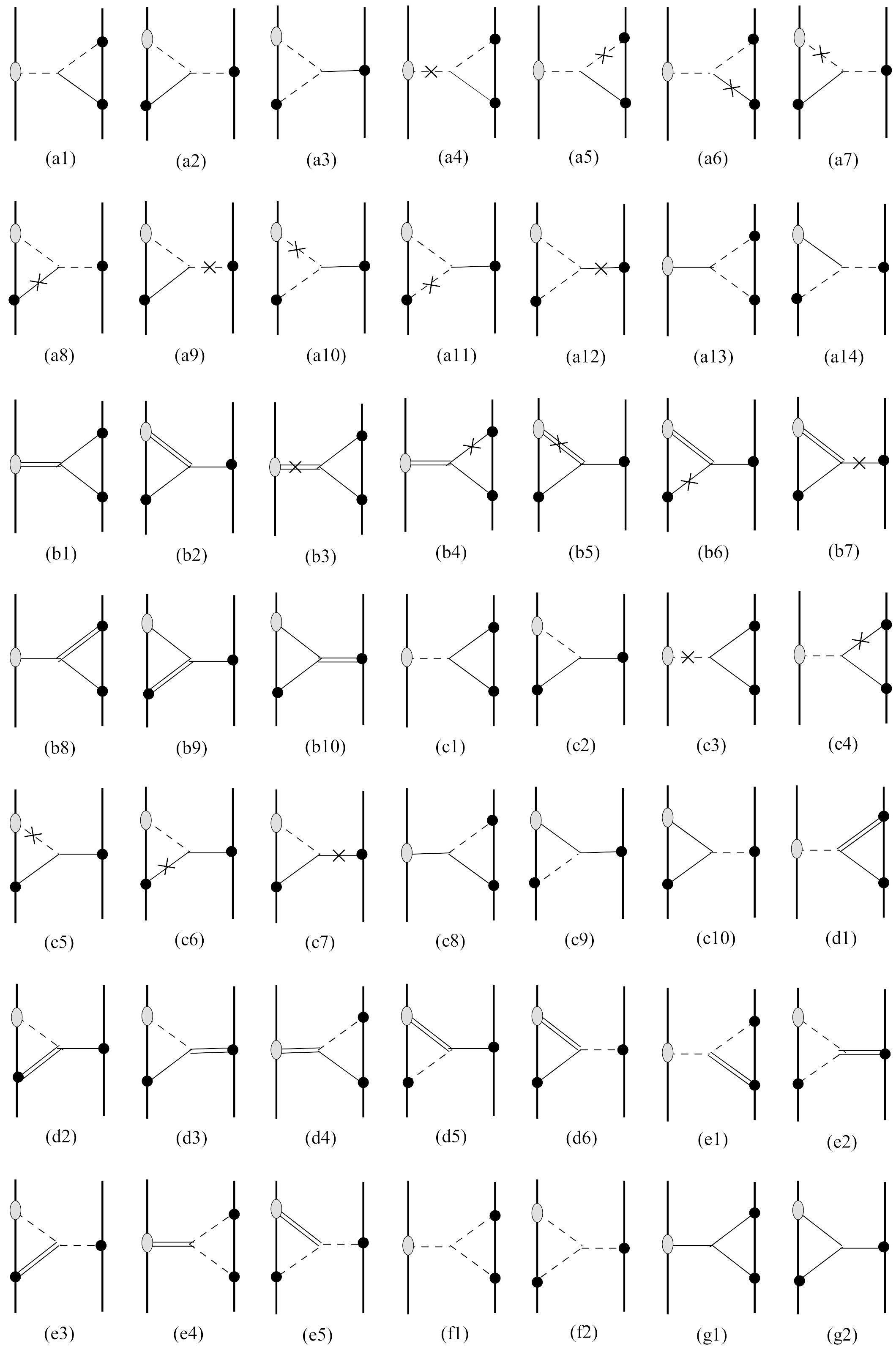


Figure 3. NNLO spin-orbit Feynman diagrams of cubic self-interaction.

$$\text{Fig. 3(a2)} = \frac{2G^2m_1m_2}{r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_2 + \frac{G^2m_1m_2}{r^3}\Big[7\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2 + \vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(3v_1^2 + v_2^2\big) - 5\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big] + \frac{8G^2m_1m_2}{r^2}\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2 + \frac{8G^2m_1m_2}{r^2}\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2, \tag{3.25}$$

$$\text{Fig. 3(a3)} = -\frac{2G^2m_1m_2}{r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_1 - \frac{G^2m_1m_2}{r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(8v_1^2+3v_2^2\big) + \frac{8G^2m_1m_2}{r^2}\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1, \tag{3.26}$$

$$\text{Fig. 3(a4)} = \frac{8G^2m_2^2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big] - \frac{4G^2m_2^2}{r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n} - \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big] + \frac{8G^2m_2^2}{r^2}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2 + \frac{4G^2m_2^2}{r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2, \tag{3.27}$$

$$\text{Fig. 3(a5)} = \frac{4G^2m_2^2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - (\vec{v}_2\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big] + \frac{2G^2m_2^2}{r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{a}_2\cdot\vec{n} - 4\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n} + 2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2 + \vec{S}_1\times\vec{v}_2\cdot\vec{a}_2\Big] + \frac{4G^2m_2^2}{r^2}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2 + \frac{4G^2m_2^2}{r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2, \tag{3.28}$$

$$\text{Fig. 3(a6)} = \frac{4G^2m_2^2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - 3(\vec{v}_2\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big] + \frac{2G^2m_2^2}{r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{a}_2\cdot\vec{n} + \vec{S}_1\times\vec{v}_2\cdot\vec{a}_2\Big] + \frac{4G^2m_2^2}{r^2}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2, \tag{3.29}$$

$$\text{Fig. 3(a7)} = -\frac{G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2 - v_2^2 + 4(\vec{v}_2\cdot\vec{n})^2\big) + 2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_1^2 - 3\vec{v}_1\cdot\vec{v}_2 + 12\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - 3(\vec{v}_1\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(4\vec{v}_1\cdot\vec{n} - 5\vec{v}_2\cdot\vec{n}\big)\Big] - \frac{G^2m_1m_2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_2\cdot\vec{n} + 6\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n} - 3\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big] + \frac{G^2m_1m_2}{r^2}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n} - 6\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{n} - 2\vec{v}_2\cdot\vec{n}\big) + 3\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big] + \frac{6G^2m_1m_2}{r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2, \tag{3.30}$$

$$\text{Fig. 3(a8)} = \frac{G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2 - 3v_2^2 - 8\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - 12(\vec{v}_2\cdot\vec{n})^2\big) + 2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - 3(\vec{v}_1\cdot\vec{n})^2\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(4\vec{v}_1\cdot\vec{n} - 5\vec{v}_2\cdot\vec{n}\big)\Big] + \frac{G^2m_1m_2}{r^2}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_2\cdot\vec{n} - 2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n} + \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big] + \frac{G^2m_1m_2}{r^2}\Big[3\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n} - 2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_1\cdot\vec{n} + \dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \tag{3.31}$$

$$\text{Fig. 3(a9)} = \frac{2G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big]$$

$$
-\frac{G^2m_1m_2}{r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big]+\frac{2G^2m_1m_2}{r^2}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2
+\frac{G^2m_1m_2}{r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2, \tag{3.32}
$$

$$
\text{Fig. 3(a10)} =\frac{G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2-\vec{v}_1\cdot\vec{v}_2+4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-2(\vec{v}_1\cdot\vec{n})^2\big)
+\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]+\frac{G^2m_1m_2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\Big]
+\frac{G^2m_1m_2}{r^2}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(3\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n}\big)-2\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big]
-\frac{2G^2m_1m_2}{r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_1, \tag{3.33}
$$

$$
\text{Fig. 3(a11)} =-\frac{G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2-\vec{v}_1\cdot\vec{v}_2+4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-2(\vec{v}_1\cdot\vec{n})^2\big)
+\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]+\frac{G^2m_1m_2}{r^2}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}-2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(3\vec{v}_1\cdot\vec{n}
-2\vec{v}_2\cdot\vec{n}\big)+3\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1+2\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big]-\frac{G^2m_1m_2}{r^2}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1
+\frac{2G^2m_1m_2}{r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_1, \tag{3.34}
$$

$$
\text{Fig. 3(a12)} =-\frac{2G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2-4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]
-\frac{G^2m_1m_2}{r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big]-\frac{G^2m_1m_2}{r^2}\Big[2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}
+\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \tag{3.35}
$$

$$
\text{Fig. 3(a13)} =-\frac{16G^2m_2^2}{r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_1v_2^2, \tag{3.36}
$$

$$
\text{Fig. 3(a14)} =\frac{8G^2m_1m_2}{r^3}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2+\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big], \tag{3.37}
$$

$$
\text{Fig. 3(b1)} =\frac{G^2m_2^2}{2r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_1+\frac{G^2m_2^2}{4r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(3v_1^2+2v_2^2-8(\vec{v}_1\cdot\vec{n})^2-8(\vec{v}_2\cdot\vec{n})^2\big)
+4\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big]+\frac{G^2m_2^2}{2r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}+2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_1\cdot\vec{n}
-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\Big]+\frac{3G^2m_2^2}{4r^2}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1, \tag{3.38}
$$

$$
\text{Fig. 3(b2)} =\frac{G^2m_1m_2}{2r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_1-\frac{G^2m_1m_2}{4r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(2v_1^2-4\vec{v}_1\cdot\vec{v}_2-3v_2^2
+16\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-16(\vec{v}_1\cdot\vec{n})^2\big)+4\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]
-\frac{2G^2m_1m_2}{r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}+4\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_1\cdot\vec{n}-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\Big]
-\frac{6G^2m_1m_2}{r^2}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1, \tag{3.39}
$$

$$
\text{Fig. 3(b3)} =\frac{G^2m_2^2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2-4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]
$$

$$+\frac{G^2m_2^2}{4r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big]+\frac{G^2m_2^2}{4r^2}\Big[2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \tag{3.40}$$

$$\text{Fig. 3(b4)}=-\frac{G^2m_2^2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2-v_2^2-4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-2(\vec{v}_2\cdot\vec{n})^2\big)-\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_1^2+4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-4(\vec{v}_1\cdot\vec{n})^2\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(3\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n}\big)\Big]$$
$$-\frac{G^2m_2^2}{2r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}-\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{a}_1\cdot\vec{n}+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big]$$
$$-\frac{G^2m_2^2}{2r^2}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}-\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}+\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \tag{3.41}$$

$$\text{Fig. 3(b5)}=-\frac{G^2m_1m_2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2-3\vec{v}_1\cdot\vec{v}_2+12\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-7(\vec{v}_1\cdot\vec{n})^2\big)+3\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]+\frac{G^2m_1m_2}{4r^2}\Big[7\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}+\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(3\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1-2\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big]+\frac{G^2m_1m_2}{2r^2}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(5\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n}\big)-\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \tag{3.42}$$

$$\text{Fig. 3(b6)}=\frac{G^2m_1m_2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(2v_1^2-3\vec{v}_1\cdot\vec{v}_2+12\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-5(\vec{v}_1\cdot\vec{n})^2\big)+3\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]+\frac{G^2m_1m_2}{4r^2}\Big[5\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}+\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_1\cdot\vec{n}+2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\Big]+\frac{3G^2m_1m_2}{2r^2}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1, \tag{3.43}$$

$$\text{Fig. 3(b7)}=\frac{G^2m_1m_2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2-4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]+\frac{G^2m_1m_2}{4r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big]+\frac{G^2m_1m_2}{4r^2}\Big[2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \tag{3.44}$$

$$\text{Fig. 3(b8)}=-\frac{8G^2m_2^2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_2^2-4(\vec{v}_2\cdot\vec{n})^2\big)-2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big], \tag{3.45}$$

$$\text{Fig. 3(b9)}=-\frac{6G^2m_1m_2}{r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2-4(\vec{v}_1\cdot\vec{n})^2\big), \tag{3.46}$$

$$\text{Fig. 3(b10)}=\frac{2G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_2^2-2(\vec{v}_2\cdot\vec{n})^2\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big], \tag{3.47}$$

$$\text{Fig. 3(c1)}=-\frac{G^2m_2^2}{2r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_2-\frac{G^2m_2^2}{4r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(7\vec{v}_1\cdot\vec{v}_2-16\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)+6\vec{S}_1\times\vec{n}\cdot\vec{v}_2v_2^2-5\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big]-\frac{G^2m_2^2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big]-\frac{G^2m_2^2}{r^2}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \tag{3.48}$$

$$\text{Fig. 3(c2)}=-\frac{G^2m_1m_2}{2r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_1+\frac{G^2m_1m_2}{4r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(2v_1^2+4\vec{v}_1\cdot\vec{v}_2-3v_2^2$$

$$
\begin{aligned}
& - 16\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} + 16(\vec{v}_1\cdot\vec{n})^2\big) - 4\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big] \\
& + \frac{2G^2m_1m_2}{r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(\vec{v}_1\cdot\vec{n}+\vec{v}_2\cdot\vec{n}\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{a}_1 + \vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big] \\
& + \frac{2G^2m_1m_2}{r^2}\Big[2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{n}+\vec{v}_2\cdot\vec{n}\big) + \dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \qquad (3.49)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 3(c3)} = & -\frac{G^2m_2^2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big] \\
& + \frac{G^2m_2^2}{4r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n} - \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big] - \frac{G^2m_2^2}{2r^2}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2 \\
& - \frac{G^2m_2^2}{4r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2, \qquad (3.50)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 3(c4)} = & -\frac{G^2m_2^2}{2r^3}\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_2^2+2(\vec{v}_2\cdot\vec{n})^2\big) - \frac{G^2m_2^2}{2r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{a}_2\cdot\vec{n} \\
& + 2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2 + 2\vec{S}_1\times\vec{v}_2\cdot\vec{a}_2\Big] \\
& + \frac{G^2m_2^2}{2r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2, \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 3(c5)} = & -\frac{G^2m_1m_2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(2v_2^2+3(\vec{v}_1\cdot\vec{n})^2-8(\vec{v}_2\cdot\vec{n})^2\big) - 4\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big] \\
& + \frac{G^2m_1m_2}{4r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(3\vec{a}_1\cdot\vec{n}+4\vec{a}_2\cdot\vec{n}\big) - \vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_1\cdot\vec{n} + 2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1 \\
& + 2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big] - \frac{3G^2m_1m_2}{2r^2}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1 + \frac{G^2m_1m_2}{2r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_1, \qquad (3.52)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 3(c6)} = & -\frac{G^2m_1m_2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2-2v_2^2-(\vec{v}_1\cdot\vec{n})^2-8(\vec{v}_2\cdot\vec{n})^2\big) \\
& + 4\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}\Big] + \frac{G^2m_1m_2}{4r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{a}_1\cdot\vec{n}-4\vec{a}_2\cdot\vec{n}\big) \\
& - \vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(3\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n}\big) + 2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1 + 2\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2 \\
& - 2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big] - \frac{G^2m_1m_2}{2r^2}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{n}+2\vec{v}_2\cdot\vec{n}\big) + \dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big] \\
& + \frac{G^2m_1m_2}{2r}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_1, \qquad (3.53)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 3(c7)} = & -\frac{G^2m_1m_2}{2r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2-4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\Big] \\
& - \frac{G^2m_1m_2}{4r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n} + \vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big] - \frac{G^2m_1m_2}{4r^2}\Big[2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n} \\
& + \dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \qquad (3.54)
\end{aligned}
$$

$$
\begin{aligned}
\text{Fig. 3(c8)} = & \frac{8G^2m_2^2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{v}_2+v_2^2-4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+4(\vec{v}_2\cdot\vec{n})^2\big) \\
& - \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{n}+2\vec{v}_2\cdot\vec{n}\big)\Big] + \frac{4G^2m_2^2}{r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\vec{v}_2\cdot\vec{n} + \vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\Big] \\
& + \frac{4G^2m_2^2}{r^2}\Big[2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n} + \dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\Big], \qquad (3.55)
\end{aligned}
$$

$$
\text{Fig. 3(c9)} = \frac{6G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2+\vec{v}_1\cdot\vec{v}_2-4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+4(\vec{v}_1\cdot\vec{n})^2\big)
$$

$$- \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \Big] + \frac{4G^2 m_1 m_2}{r^2} \Big[2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \Big]$$

$$+ \frac{8G^2 m_1 m_2}{r^2} \vec{v}_1 \cdot \vec{n} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1, \tag{3.56}$$

$$\text{Fig. 3(c10)} = - \frac{2G^2 m_1 m_2}{r^3} \Big[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \big(\vec{v}_1 \cdot \vec{v}_2 - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \Big]$$

$$- \frac{2G^2 m_1 m_2}{r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \Big] - \frac{2G^2 m_1 m_2}{r^2} \Big[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n}$$

$$+ \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \Big], \tag{3.57}$$

$$\text{Fig. 3(d1)} = \frac{8G^2 m_2^2}{r^3} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \big(\vec{v}_1 \cdot \vec{v}_2 - v_2^2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 4(\vec{v}_2 \cdot \vec{n})^2 \big)$$

$$+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \Big] + \frac{8G^2 m_2^2}{r^2} \vec{v}_2 \cdot \vec{n} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2, \tag{3.58}$$

$$\text{Fig. 3(d2)} = \frac{6G^2 m_1 m_2}{r^3} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \big(v_1^2 - \vec{v}_1 \cdot \vec{v}_2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2 \big)$$

$$+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \Big] + \frac{8G^2 m_1 m_2}{r^2} \vec{v}_1 \cdot \vec{n} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1, \tag{3.59}$$

$$\text{Fig. 3(d3)} = \frac{2G^2 m_1 m_2}{r^3} \vec{v}_2 \cdot \vec{n} \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 - \frac{2G^2 m_1 m_2}{r^2} \vec{v}_2 \cdot \vec{n} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2, \tag{3.60}$$

$$\text{Fig. 3(d4)} = \frac{4G^2 m_2^2}{r^2} \Big[2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \Big], \tag{3.61}$$

$$\text{Fig. 3(d5)} = - \frac{2G^2 m_1 m_2}{r^3} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \big(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2 \big)$$

$$- \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \Big] + \frac{4G^2 m_1 m_2}{r^2} \Big[2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \Big], \tag{3.62}$$

$$\text{Fig. 3(d6)} = \frac{2G^2 m_1 m_2}{r^3} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{v}_2 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \big(v_1^2 - \vec{v}_1 \cdot \vec{v}_2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}$$

$$- 4(\vec{v}_1 \cdot \vec{n})^2 \big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \big(3\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n} \big) \Big] - \frac{2G^2 m_1 m_2}{r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n}$$

$$- \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \Big], \tag{3.63}$$

$$\text{Fig. 3(e1)} = \frac{8G^2 m_2^2}{r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \big(v_2^2 - 4(\vec{v}_2 \cdot \vec{n})^2 \big), \tag{3.64}$$

$$\text{Fig. 3(e2)} = \frac{2G^2 m_1 m_2}{r^3} \Big[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_2 \cdot \vec{n})^2 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{v}_2 \Big], \tag{3.65}$$

$$\text{Fig. 3(e3)} = \frac{2G^2 m_1 m_2}{r^3} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{v}_2 + 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \big(v_1^2 - 6(\vec{v}_1 \cdot \vec{n})^2 \big)$$

$$+ 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \Big], \tag{3.66}$$

$$\text{Fig. 3(e4)} = \frac{2G^2 m_2^2}{r^3} \Big[3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 v_2^2 - 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{v}_2 \Big], \tag{3.67}$$

$$\text{Fig. 3(e5)} = - \frac{2G^2 m_1 m_2}{r^3} \Big[4\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{v}_2 - 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 v_1^2 + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \Big], \tag{3.68}$$

$$\text{Fig. 3(f1)} = - \frac{2G^2 m_2^2}{r^3} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \big(4\vec{v}_1 \cdot \vec{v}_2 + 3v_2^2 - 16\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 16(\vec{v}_2 \cdot \vec{n})^2 \big)$$

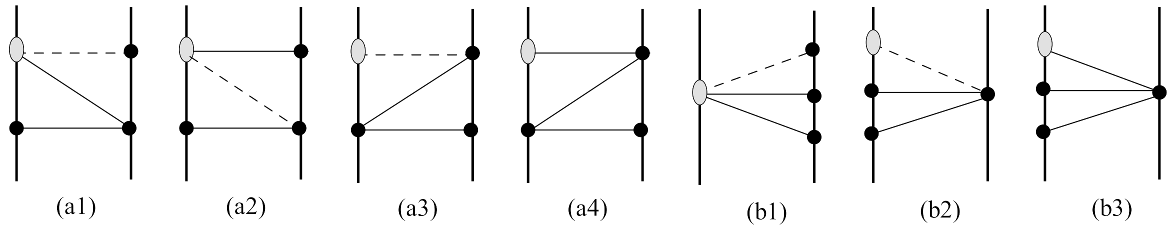


Figure 4. NNLO spin-orbit Feynman diagrams of order G^3 with no loops.

$$+ 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n}\Big] + \frac{8G^2m_2^2}{r^2}\Big[2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\vec{v}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2\Big] - \frac{8G^2m_2^2}{r^2}\vec{v}_2 \cdot \vec{n}\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2, \tag{3.69}$$

$$\text{Fig. 3(f2)} = \frac{2G^2m_1m_2}{r^3}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1\big(3\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\big) - 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\big(v_1^2 + \vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 4(\vec{v}_1 \cdot \vec{n})^2\big) - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\big(3\vec{v}_1 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n}\big)\Big] + \frac{2G^2m_1m_2}{r^2}\Big[3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1\vec{v}_2 \cdot \vec{n} + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\vec{v}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2\Big] + \frac{2G^2m_1m_2}{r^2}\Big[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} - 4\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2\vec{v}_1 \cdot \vec{n} + 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2\Big], \tag{3.70}$$

$$\text{Fig. 3(g1)} = -\frac{2G^2m_2^2}{r^3}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1\big(4\vec{v}_1 \cdot \vec{v}_2 - v_2^2 - 16\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 2(\vec{v}_2 \cdot \vec{n})^2\big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\big(4\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\big)\Big] - \frac{4G^2m_2^2}{r^2}\Big[2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1\vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2\Big] - \frac{4G^2m_2^2}{r^2}\Big[2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} + \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2\Big], \tag{3.71}$$

$$\text{Fig. 3(g2)} = \frac{2G^2m_1m_2}{r^3}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1\big(v_1^2 - 4\vec{v}_1 \cdot \vec{v}_2 + 16\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 2(\vec{v}_1 \cdot \vec{n})^2\big) + 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_1 \cdot \vec{n}\Big] + \frac{2G^2m_1m_2}{r^2}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{a}_1\big(\vec{v}_1 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}\big) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 - 2\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2\Big] + \frac{2G^2m_1m_2}{r^2}\Big[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1\big(\vec{v}_1 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}\big) - 2\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2\Big]. \tag{3.72}$$

3.3 Cubic in G interaction

3.3.1 Three-graviton exchange

For the NNLO spin-orbit interaction we have 7 diagrams at order G^3 with no loops, as shown in figure 4. These three-graviton exchange diagrams are constructed with either one-, two-, or three-graviton spin couplings.

The values of these diagrams are given by

$$\text{Fig. 4(a1)} = -8\frac{G^3m_1m_2^2}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.73}$$

$$\text{Fig. 4(a2)} = -8\frac{G^3m_1m_2^2}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.74}$$

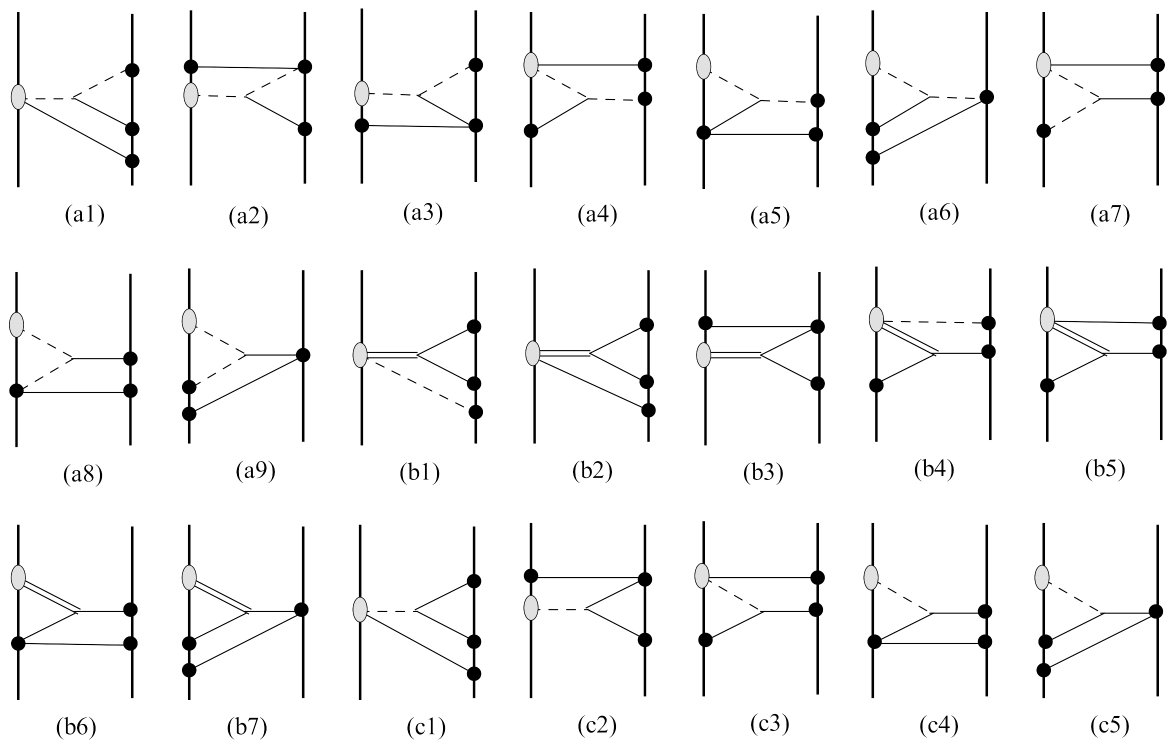


Figure 5. NNLO spin-orbit Feynman diagrams of order G^3 with one loop.

$$\text{Fig. 4(a3)} = -2\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.75}$$

$$\text{Fig. 4(a4)} = 2\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}, \tag{3.76}$$

$$\text{Fig. 4(b1)} = -16\frac{G^3 m_2^3}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.77}$$

$$\text{Fig. 4(b2)} = -\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.78}$$

$$\text{Fig. 4(b3)} = \frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}. \tag{3.79}$$

3.3.2 Cubic self-interaction with two-graviton exchange

For the NNLO spin-orbit interaction we have 21 diagrams at order G^3 with one loop, as shown in figure 5. These diagrams contain both cubic self-interaction and two-graviton worldline couplings.

The values of these diagrams are given by

$$\text{Fig. 5(a1)} = 32\frac{G^3 m_2^3}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.80}$$

$$\text{Fig. 5(a2)} = 8\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.81}$$

$$\text{Fig. 5(a3)} = 8\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.82}$$

$$\text{Fig. 5(a4)} = 8\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}, \tag{3.83}$$

$$\text{Fig. 5(a5)} = 2\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.84}$$

$$\text{Fig. 5(a6)} = 2\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.85}$$

$$\text{Fig. 5(a7)} = -8\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.86}$$

$$\text{Fig. 5(a8)} = -2\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.87}$$

$$\text{Fig. 5(a9)} = -2\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.88}$$

$$\text{Fig. 5(b1)} = -\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.89}$$

$$\text{Fig. 5(b2)} = -\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.90}$$

$$\text{Fig. 5(b3)} = \frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.91}$$

$$\text{Fig. 5(b4)} = 0, \tag{3.92}$$

$$\text{Fig. 5(b5)} = 8\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.93}$$

$$\text{Fig. 5(b6)} = \frac{1}{2}\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.94}$$

$$\text{Fig. 5(b7)} = \frac{1}{2}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.95}$$

$$\text{Fig. 5(c1)} = -2\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.96}$$

$$\text{Fig. 5(c2)} = -\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.97}$$

$$\text{Fig. 5(c3)} = -2\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.98}$$

$$\text{Fig. 5(c4)} = -\frac{1}{2}\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.99}$$

$$\text{Fig. 5(c5)} = -\frac{1}{2}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}. \tag{3.100}$$

Note that the total value of the diagram in figure 5(b4) equals 0, although it does not stand for a short distance contribution.

3.3.3 Two-loop interaction

For the NNLO spin-orbit interaction we have 32 two-loop diagrams at order G^3, as shown in figure 6. These diagrams contain two cubic vertices or one quartic vertex, and even include cubic vertices with time dependence. As explained in [18], they contain two-loop Feynman integrals of three kinds: Factorizable, nested, and irreducible. The factorizable two-loop diagrams do not contribute at the NNLO level, and they yield here purely short distance contributions, of the form $\delta^{(1)}(\vec{r})$, which are contact interaction terms. For other two-loop diagrams calculations should be made, keeping the dimension d general, and the limit $d \to 3$ is only taken in the end.

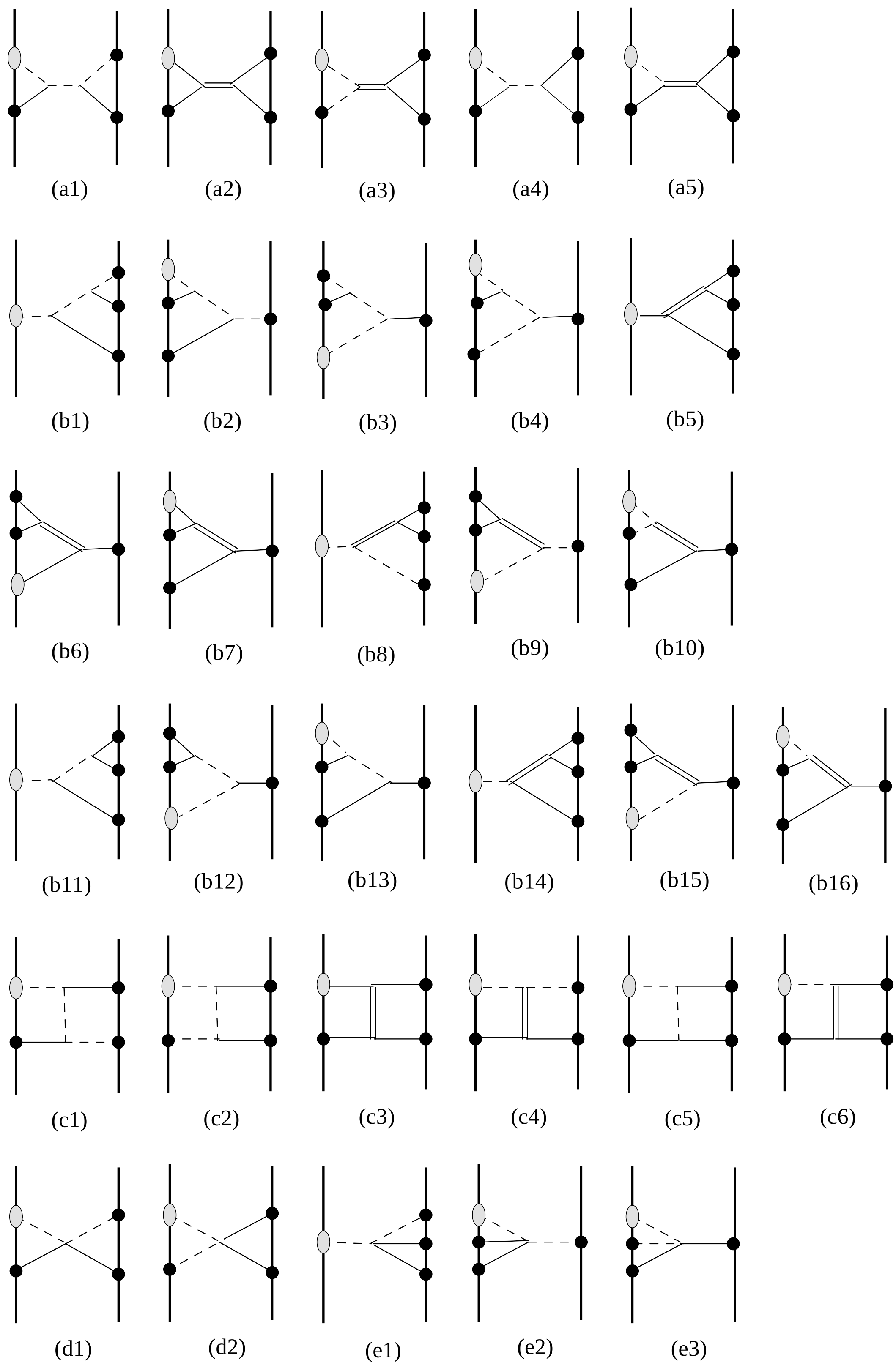


Figure 6. NNLO spin-orbit Feynman diagrams of order G^3 with two loops.

For the irreducible two-loop diagrams, which are the most complicated, irreducible two-loop tensor integrals of order 3 are encountered here. These are reduced using the integration by parts method to a sum of factorizable and nested two-loop integrals, as explained in [18], and see appendix A there. In addition to the irreducible two-loop tensor integrals, which were given in appendix A of [18], eqs. (A11), (A12) there, two further irreducible tensor integrals are required here, and we provide them in appendix A below.

The values of the two-loop diagrams are given in the following:

$$\text{Fig. 6(a1)} = 0, \tag{3.101}$$

$$\text{Fig. 6(a2)} = 0, \tag{3.102}$$

$$\text{Fig. 6(a3)} = 0, \tag{3.103}$$

$$\text{Fig. 6(a4)} = 0, \tag{3.104}$$

$$\text{Fig. 6(a5)} = 0, \tag{3.105}$$

$$\text{Fig. 6(b1)} = -32\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.106}$$

$$\text{Fig. 6(b2)} = -\frac{24}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.107}$$

$$\text{Fig. 6(b3)} = \frac{16}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.108}$$

$$\text{Fig. 6(b4)} = \frac{8}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.109}$$

$$\text{Fig. 6(b5)} = 2\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.110}$$

$$\text{Fig. 6(b6)} = \frac{6}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.111}$$

$$\text{Fig. 6(b7)} = \frac{4}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.112}$$

$$\text{Fig. 6(b8)} = -\frac{2}{5}\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.113}$$

$$\text{Fig. 6(b9)} = -\frac{2}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.114}$$

$$\text{Fig. 6(b10)} = -2\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.115}$$

$$\text{Fig. 6(b11)} = 2\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.116}$$

$$\text{Fig. 6(b12)} = -\frac{1}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.117}$$

$$\text{Fig. 6(b13)} = \frac{1}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.118}$$

$$\text{Fig. 6(b14)} = \frac{2}{5}\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.119}$$

$$\text{Fig. 6(b15)} = -\frac{3}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.120}$$

$$\text{Fig. 6(b16)} = \frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.121}$$

$$\text{Fig. 6(c1)} = -4\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.122}$$

$$\text{Fig. 6(c2)} = 4\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.123}$$

$$\text{Fig. 6(c3)} = 12\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}, \tag{3.124}$$

$$\text{Fig. 6(c4)} = -28\frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.125}$$

$$\text{Fig. 6(c5)} = \frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\left[2\vec{v}_1\times\vec{n}-2\vec{v}_2\times\vec{n}\right], \tag{3.126}$$

$$\text{Fig. 6(c6)} = \frac{G^3 m_1 m_2^2}{r^4}\vec{S}_1\cdot\left[-12\vec{v}_1\times\vec{n}+22\vec{v}_2\times\vec{n}\right], \tag{3.127}$$

$$\text{Fig. 6(d1)} = 0, \tag{3.128}$$

$$\text{Fig. 6(d2)} = 0, \tag{3.129}$$

$$\text{Fig. 6(e1)} = 16\frac{G^3 m_2^3}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.130}$$

$$\text{Fig. 6(e2)} = \frac{16}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}, \tag{3.131}$$

$$\text{Fig. 6(e3)} = -\frac{16}{5}\frac{G^3 m_1^2 m_2}{r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}. \tag{3.132}$$

4 Next-to-next-to-leading order spin-orbit potential and Hamiltonian

Summing up all of the Feynman diagrams from the previous section, we obtain the NNLO spin-orbit interaction potential for a binary system of compact spinning objects as follows:

$$\begin{aligned}
V_{\rm SO}^{\rm NNLO} =& -\frac{5}{16}\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1 v_1^4 - \frac{Gm_2}{4r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(5v_1^2\vec{v}_1\cdot\vec{v}_2 - 3v_1^2v_2^2\\
&+4\vec{v}_1\cdot\vec{v}_2 v_2^2 - 3v_1^4 - 3v_2^4 + 6\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n} - 6\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 + 6\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&+3v_2^2(\vec{v}_1\cdot\vec{n})^2 + 3v_1^2(\vec{v}_2\cdot\vec{n})^2 - 15(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_1^2v_2^2\\
&-2\vec{v}_1\cdot\vec{v}_2v_2^2 - 2(\vec{v}_1\cdot\vec{v}_2)^2 + 3v_2^4 - 6\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2 + 3v_2^2(\vec{v}_1\cdot\vec{n})^2 + 3v_1^2(\vec{v}_2\cdot\vec{n})^2\\
&+15(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{n}v_1^2 - 2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 + 2\vec{v}_1\cdot\vec{n}v_2^2\\
&+2\vec{v}_2\cdot\vec{n}v_2^2 - 6\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)\Big] + \frac{Gm_2}{4r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(4\vec{v}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}\\
&-4\vec{v}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2 + \vec{a}_1\cdot\vec{n}v_2^2 - v_1^2\vec{a}_2\cdot\vec{n} + 4\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2 - 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{a}_2\\
&+3\vec{a}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2 - 3\vec{a}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big) + 2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(v_1^2\vec{v}_2\cdot\vec{n} - \vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+\vec{v}_1\cdot\vec{n}v_2^2 + \vec{v}_2\cdot\vec{n}v_2^2 - 3\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big) - \vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\big(12v_1^2 - 12\vec{v}_1\cdot\vec{v}_2 + 5v_2^2\\
&-4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} + (\vec{v}_2\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(2\vec{v}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2 - \vec{a}_1\cdot\vec{n}v_2^2\\
&+v_1^2\vec{a}_2\cdot\vec{n} - 2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2 + 4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{a}_2 - 3\vec{a}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2 - 3\vec{a}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)\\
&-2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(6\vec{v}_1\cdot\vec{a}_1 - \vec{a}_1\cdot\vec{v}_2 - \vec{v}_1\cdot\vec{a}_2 + 2\vec{v}_2\cdot\vec{a}_2 - \vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)\\
&-\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\big(6v_1^2 - 4\vec{v}_1\cdot\vec{v}_2 + 5v_2^2 - 6\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - (\vec{v}_2\cdot\vec{n})^2\big)\\
&+2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\big(v_1^2\vec{v}_2\cdot\vec{n} + \vec{v}_1\cdot\vec{n}v_2^2 - 3\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\big(v_1^2 - 2v_2^2\\
&+4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - (\vec{v}_1\cdot\vec{n})^2\big) - \vec{S}_1\times\vec{v}_2\cdot\vec{a}_2\big(v_1^2 - (\vec{v}_1\cdot\vec{n})^2\big)\Big]
\end{aligned}$$

$$+\frac{Gm_2}{2r}\Big[\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2\vec{v}_2\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+\vec{v}_1\cdot\vec{n}v_2^2+\vec{v}_2\cdot\vec{n}v_2^2-3\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)$$
$$-\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{n}v_2^2+\vec{v}_2\cdot\vec{n}v_2^2-3\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)-\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\big(3v_1^2+2v_2^2$$
$$-3\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-(\vec{v}_2\cdot\vec{n})^2\big)\Big]-\frac{1}{4}Gm_2\Big[\vec{S}_1\times\vec{n}\cdot\dot{\vec{a}}_1\big(v_2^2-(\vec{v}_2\cdot\vec{n})^2\big)$$
$$+4\vec{S}_1\times\vec{v}_1\cdot\dot{\vec{a}}_1\vec{v}_2\cdot\vec{n}+6\vec{S}_1\times\dot{\vec{a}}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}-\vec{S}_1\times\vec{n}\cdot\dot{\vec{a}}_2\big(v_1^2-(\vec{v}_1\cdot\vec{n})^2\big)$$
$$-2\vec{S}_1\times\vec{v}_1\cdot\dot{\vec{a}}_2\vec{v}_1\cdot\vec{n}+\ddot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(v_2^2-(\vec{v}_2\cdot\vec{n})^2\big)-\ddot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\big(v_2^2-(\vec{v}_2\cdot\vec{n})^2\big)$$
$$+6\ddot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}+\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(5\vec{a}_1\cdot\vec{a}_2+\vec{a}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)$$
$$+2\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(2\vec{v}_1\cdot\vec{a}_2-2\vec{v}_2\cdot\vec{a}_2+\vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1\vec{a}_2\cdot\vec{n}$$
$$-\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(3\vec{a}_1\cdot\vec{a}_2+\vec{a}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)+\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}$$
$$-2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\big(\vec{a}_1\cdot\vec{v}_2+\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\vec{a}_1\cdot\vec{n}$$
$$-2\vec{S}_1\times\vec{a}_1\cdot\vec{a}_2\big(4\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)+\vec{S}_1\times\vec{v}_2\cdot\vec{a}_2\vec{a}_1\cdot\vec{n}+2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(2\vec{v}_1\cdot\vec{a}_2$$
$$-2\vec{v}_2\cdot\vec{a}_2+\vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)+2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_1\big(v_2^2-(\vec{v}_2\cdot\vec{n})^2\big)+4\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}$$
$$-2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{a}_2-2\vec{v}_2\cdot\vec{a}_2+\vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)+2\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}$$
$$+12\dot{\vec{S}}_1\times\vec{a}_1\cdot\vec{v}_2\vec{v}_2\cdot\vec{n}+2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2\big(v_2^2-2\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)$$
$$-4\dot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_2\big(2\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)+2\dot{\vec{S}}_1\times\vec{v}_2\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}\Big]$$
$$+\frac{1}{4}Gm_2r\Big[\vec{S}_1\times\vec{n}\cdot\dot{\vec{a}}_1\vec{a}_2\cdot\vec{n}+\vec{S}_1\times\vec{n}\cdot\dot{\vec{a}}_2\vec{a}_1\cdot\vec{n}+7\vec{S}_1\times\dot{\vec{a}}_1\cdot\vec{a}_2$$
$$+\vec{S}_1\times\vec{a}_1\cdot\dot{\vec{a}}_2+2\dot{\vec{S}}_1\times\vec{n}\cdot\dot{\vec{a}}_2\vec{v}_1\cdot\vec{n}+2\dot{\vec{S}}_1\times\vec{v}_1\cdot\dot{\vec{a}}_2+\ddot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_2\cdot\vec{n}$$
$$-\ddot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}-2\ddot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_2\cdot\vec{n}+7\ddot{\vec{S}}_1\times\vec{v}_1\cdot\vec{a}_2-\ddot{\vec{S}}_1\times\vec{v}_2\cdot\vec{a}_2$$
$$+2\dot{\vec{S}}_1\times\vec{n}\cdot\vec{a}_1\vec{a}_2\cdot\vec{n}+14\dot{\vec{S}}_1\times\vec{a}_1\cdot\vec{a}_2\Big]+\frac{1}{4}Gm_2r^2\ddot{\vec{S}}_1\times\vec{n}\cdot\dot{\vec{a}}_2$$
$$+\frac{G^2m_2^2}{4r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(13v_1^2-41\vec{v}_1\cdot\vec{v}_2+28v_2^2-16\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-8(\vec{v}_1\cdot\vec{n})^2$$
$$-12(\vec{v}_2\cdot\vec{n})^2\big)-2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_1^2+7\vec{v}_1\cdot\vec{v}_2-8v_2^2-56\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+4(\vec{v}_1\cdot\vec{n})^2$$
$$-62(\vec{v}_2\cdot\vec{n})^2\big)+\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(7\vec{v}_1\cdot\vec{n}-38\vec{v}_2\cdot\vec{n}\big)\Big]-\frac{2G^2m_1m_2}{r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(2v_1^2$$
$$+\vec{v}_1\cdot\vec{v}_2-3v_2^2+12\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-16(\vec{v}_1\cdot\vec{n})^2+4(\vec{v}_2\cdot\vec{n})^2\big)-2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{v}_2$$
$$-v_2^2+4\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-4(\vec{v}_1\cdot\vec{n})^2\big)-\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(2\vec{v}_1\cdot\vec{n}-5\vec{v}_2\cdot\vec{n}\big)\Big]$$
$$-\frac{2G^2m_1m_2}{r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(2\vec{a}_1\cdot\vec{n}+\vec{a}_2\cdot\vec{n}\big)+\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(3\vec{v}_1\cdot\vec{n}+2\vec{v}_2\cdot\vec{n}\big)$$
$$-2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\vec{v}_1\cdot\vec{n}+2\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2+2\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2\Big]-\frac{G^2m_2^2}{4r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}$$
$$+4\vec{S}_1\times\vec{n}\cdot\vec{a}_1\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(\vec{a}_1\cdot\vec{n}+14\vec{a}_2\cdot\vec{n}\big)$$
$$-2\vec{S}_1\times\vec{n}\cdot\vec{a}_2\big(\vec{v}_1\cdot\vec{n}-18\vec{v}_2\cdot\vec{n}\big)+14\vec{S}_1\times\vec{v}_1\cdot\vec{a}_1+27\vec{S}_1\times\vec{a}_1\cdot\vec{v}_2+\vec{S}_1\times\vec{v}_1\cdot\vec{a}_2$$
$$+12\vec{S}_1\times\vec{v}_2\cdot\vec{a}_2\Big]-\frac{G^2m_1m_2}{r^2}\Big[6\dot{\vec{S}}_1\times\vec{n}\cdot\vec{v}_1\big(3\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)$$

$$- 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2\big(7\vec{v}_1 \cdot \vec{n} - 5\vec{v}_2 \cdot \vec{n}\big) + 7\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2\Big] - \frac{G^2 m_2^2}{4r^2}\Big[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1\big(3\vec{v}_1 \cdot \vec{n}$$

$$- 4\vec{v}_2 \cdot \vec{n}\big) + 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2\big(\vec{v}_1 \cdot \vec{n} + 15\vec{v}_2 \cdot \vec{n}\big) + 27\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2\Big]$$

$$- \frac{G^2 m_1 m_2}{r}\Big[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 + 6\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2\Big] - \frac{17 G^2 m_2^2}{4r}\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2$$

$$+ \frac{G^3 m_1^2 m_2}{r^4}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2\Big] + \frac{5G^3 m_1 m_2^2}{r^4}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1$$

$$- \vec{S}_1 \times \vec{n} \cdot \vec{v}_2\Big] + \frac{G^3 m_2^3}{r^4}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2\Big] + (1 \leftrightarrow 2). \quad (4.1)$$

This potential still contains higher order time derivatives of the velocity and spin. These can be handled at the level of the EOM through a substitution of lower order EOM. However, it is often more useful to perform the elimination of higher order time derivatives at the level of the potential, and to also transform to a Hamiltonian.

For the reduction of higher order time derivatives we follow the procedure outlined in [24, 33], and its explicit extension for spin variables in [24]. It should be stressed that this procedure is in general not equivalent to a substitution of EOM at the level of the potential, but rather to a redefinition of variables, which removes the higher order time derivatives. Yet, as long as this redefinition contributes only linearly to the sector, its result is identical to an insertion of the lower order EOM. Indeed, this is the case in the NNLO spin-orbit sector considered here. In the point-mass potentials the accelerations start to appear at the 2PN order, so that the redefinitions would contribute quadratically only at the 4PN order. The LO spin-orbit potential at 1.5PN order also contains time derivatives, but the required redefinitions would contribute quadratically only at the quadratic level in the individual spins. The higher order time derivatives can therefore still be removed here through a simple insertion of lower order EOM into the potential.

Next, we can perform a Legendre transformation to obtain a Hamiltonian. For that we need to replace the velocity in terms of canonical momenta, which reads

$$v_1 = \hat{p}_1^i - \frac{1}{2}\hat{p}_1^i \hat{p}_1^2 + \frac{G m_2}{2r}\big[-6\hat{p}_1^i + 7\hat{p}_2^i + n^i \vec{n} \cdot \vec{\hat{p}}_2\big] - \frac{2G}{r^2}\epsilon_{ijk} n^j S_2^k - \frac{3 G m_2}{2 m_1 r^2}\epsilon_{ikj} n^k S_1^j$$

$$+ \frac{G}{2r^2}\big[-5\epsilon_{ikj}\hat{p}_2^k S_2^j \vec{n} \cdot \vec{\hat{p}}_1 + 4\epsilon_{ikj}\hat{p}_2^k S_2^j \vec{n} \cdot \vec{\hat{p}}_2 + 6\epsilon_{ikj} n^k S_2^j \vec{n} \cdot \vec{\hat{p}}_1 \vec{n} \cdot \vec{\hat{p}}_2$$

$$+ 2\epsilon_{ikj} n^k S_2^j \vec{\hat{p}}_1 \cdot \vec{\hat{p}}_2 - 2\hat{p}_2^i \vec{S}_2 \times \vec{n} \cdot \vec{\hat{p}}_1 - 6 n^i \vec{n} \cdot \vec{\hat{p}}_2 \vec{S}_2 \times \vec{n} \cdot \vec{\hat{p}}_1 - 3 n^i \vec{n} \cdot \vec{\hat{p}}_1 \vec{S}_2 \times \vec{n} \cdot \vec{\hat{p}}_2$$

$$+ 6 n^i \vec{n} \cdot \vec{\hat{p}}_2 \vec{S}_2 \times \vec{n} \cdot \vec{\hat{p}}_2 - 5 n^i \vec{S}_2 \times \vec{\hat{p}}_1 \cdot \vec{\hat{p}}_2\big] + \frac{35 G^2 m_1}{4r^3}\epsilon_{ikj} n^k S_2^j + \frac{6 G^2 m_2}{r^3}\epsilon_{ikj} n^k S_2^j$$

$$+ \frac{G m_2}{8 m_1 r^2}\big[16\epsilon_{ijk}\hat{p}_2^j S_1^k \vec{n} \cdot \vec{\hat{p}}_1 + 5\epsilon_{ijk} n^j S_1^k \hat{p}_1^2 - 20\epsilon_{ijk}\hat{p}_2^j S_1^k \vec{n} \cdot \vec{\hat{p}}_2$$

$$+ 24\epsilon_{ijk} n^j S_1^k \vec{n} \cdot \vec{\hat{p}}_1 \vec{n} \cdot \vec{\hat{p}}_2 - 10\hat{p}_1^i \vec{S}_1 \times \vec{n} \cdot \vec{\hat{p}}_1 - 24 n^i \vec{n} \cdot \vec{\hat{p}}_2 \vec{S}_1 \times \vec{n} \cdot \vec{\hat{p}}_1 + 8\hat{p}_2^i \vec{S}_1 \times \vec{n} \cdot \vec{\hat{p}}_2$$

$$+ 24 n^i \vec{n} \cdot \vec{\hat{p}}_2 \vec{S}_1 \times \vec{n} \cdot \vec{\hat{p}}_2 + 16 n^i \vec{S}_1 \times \vec{\hat{p}}_1 \cdot \vec{\hat{p}}_2 - 6\epsilon_{ijk} n^j S_1^k (\vec{n} \cdot \vec{\hat{p}}_2)^2\big] + \frac{7 G^2 m_2}{2r^3}\epsilon_{ijk} n^j S_1^k$$

$$+ \frac{5 G^2 m_2^2}{m_1 r^3}\epsilon_{ijk} n^j S_1^k, \quad (4.2)$$

where we have used the abbreviation $\vec{\hat{p}}_a \equiv \vec{p}_a / m_a$. This results in a compact Hamiltonian:

$$H_{\rm SO}^{\rm NNLO} = \frac{G m_2}{16 r^2}\Big[\vec{S}_1 \times \vec{n} \cdot \vec{\hat{p}}_1\big(4\hat{p}_1^2 \vec{\hat{p}}_1 \cdot \vec{\hat{p}}_2 + \hat{p}_1^2 \hat{p}_2^2 + 10\vec{\hat{p}}_1 \cdot \vec{\hat{p}}_2 \hat{p}_2^2 - 14(\vec{\hat{p}}_1 \cdot \vec{\hat{p}}_2)^2 + 7\hat{p}_1^4 - \hat{p}_2^4$$

$$
\begin{aligned}
&+24\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{n}+12\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+18\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\hat{p}_2^2-15\hat{p}_2^2(\vec{p}_1\cdot\vec{n})^2\\
&-30\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2+6\hat{p}_2^2(\vec{p}_2\cdot\vec{n})^2+75(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2-30\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)\\
&-4\vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(2\hat{p}_1^2\vec{p}_1\cdot\vec{p}_2+\hat{p}_1^2\hat{p}_2^2+2\vec{p}_1\cdot\vec{p}_2\hat{p}_2^2-2(\vec{p}_1\cdot\vec{p}_2)^2+6\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{n}\\
&+6\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\hat{p}_2^2-3\hat{p}_2^2(\vec{p}_1\cdot\vec{n})^2-3\hat{p}_1^2(\vec{p}_2\cdot\vec{n})^2+15(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)\\
&-2\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(6\vec{p}_1\cdot\vec{n}\hat{p}_1^2-12\hat{p}_1^2\vec{p}_2\cdot\vec{n}-2\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+10\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+9\vec{p}_1\cdot\vec{n}\hat{p}_2^2\\
&-14\vec{p}_2\cdot\vec{n}\hat{p}_2^2+21\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2-27\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big)\Big]-\frac{G^2m_1m_2}{8r^3}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(11\hat{p}_1^2\\
&+35\vec{p}_1\cdot\vec{p}_2-44\hat{p}_2^2-144(\vec{p}_1\cdot\vec{n})^2-86(\vec{p}_2\cdot\vec{n})^2\big)-4\vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(11\hat{p}_1^2-14\vec{p}_1\cdot\vec{p}_2\\
&-16\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}+8(\vec{p}_1\cdot\vec{n})^2\big)+3\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(41\vec{p}_1\cdot\vec{n}-28\vec{p}_2\cdot\vec{n}\big)\Big]\\
&+\frac{G^2m_2^2}{16r^3}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(21\hat{p}_1^2+17\vec{p}_1\cdot\vec{p}_2-41\hat{p}_2^2+117\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}+24(\vec{p}_1\cdot\vec{n})^2\\
&-66(\vec{p}_2\cdot\vec{n})^2\big)-\vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(13\hat{p}_1^2+228\vec{p}_1\cdot\vec{p}_2-202\hat{p}_2^2-24\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}+32(\vec{p}_1\cdot\vec{n})^2\\
&-328(\vec{p}_2\cdot\vec{n})^2\big)-2\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(62\vec{p}_1\cdot\vec{n}-17\vec{p}_2\cdot\vec{n}\big)\Big]+\frac{G^3m_1m_2^2}{8r^4}\Big[191\vec{S}_1\times\vec{n}\cdot\vec{p}_1\\
&-312\vec{S}_1\times\vec{n}\cdot\vec{p}_2\Big]+\frac{3G^3m_1^2m_2}{4r^4}\Big[9\vec{S}_1\times\vec{n}\cdot\vec{p}_1-16\vec{S}_1\times\vec{n}\cdot\vec{p}_2\Big]\\
&+\frac{9G^3m_2^3}{4r^4}\Big[5\vec{S}_1\times\vec{n}\cdot\vec{p}_1-8\vec{S}_1\times\vec{n}\cdot\vec{p}_2\Big]+(1\leftrightarrow 2).
\end{aligned}\tag{4.3}
$$

The Poisson brackets are the standard canonical ones, as shown in [17].

4.1 Resolution via canonical transformations

If the EFT Hamiltonian obtained here in eq. (4.3) is physically equivalent to that of [11, 12], then there exists an infinitesimal generator g of a canonical transformation such that

$$
\begin{aligned}
\Delta H &= \{H,g\}=\{H_{\rm N}+H_{\rm 1PN}+H_{\rm LO}^{\rm SO},\ g_{\rm NNLO}^{\rm SO}+g_{\rm NLO}^{\rm SO}+g_{\rm 2PN}\}\\
&=\Delta H_{\rm 2PN}+\Delta H_{\rm 3PN}+\Delta H_{\rm SO}^{\rm NLO}+\Delta H_{\rm SO}^{\rm NNLO},
\end{aligned}\tag{4.4}
$$

where here we have dropped contributions to sectors beyond linear in spin and beyond NNLO, and where

$$\Delta H = H_{\rm EFT}-H_{\rm ADM}.\tag{4.5}$$

Thus, the contribution to the NNLO spin-orbit sector comprises

$$\Delta H_{\rm SO}^{\rm NNLO}=\{H_N,g_{\rm NNLO}^{\rm SO}\}+\{H_{\rm 1PN},g_{\rm NLO}^{\rm SO}\}+\{H_{\rm LO}^{\rm SO},g_{\rm 2PN}\},\tag{4.6}$$

and we also have here contributions to lower orders, given by

$$
\begin{aligned}
\Delta H_{\rm SO}^{\rm NLO}&=\{H_{\rm N},g_{\rm NLO}^{\rm SO}\},\\
\Delta H_{\rm 2PN}&=\{H_{\rm N},g_{\rm 2PN}\},
\end{aligned}\tag{4.7}
$$

so we also require that the canonical transformation is consistent with the equivalence at NLO of the spin-orbit, and 2PN non spinning Hamiltonians.

Similarly to the construction considerations in [24, 28], we find for the infinitesimal generator of PN canonical transformations for the NNLO spin-orbit sector, $g_{\rm NNLO}^{\rm SO}$, the following general form:

$$
\begin{aligned}
g^{\rm SO}_{\rm NNLO} =& \frac{Gm_2}{r}\vec{S}_1\cdot\left[\vec{p}_1\times\vec{p}_2\left(g_1\hat{p}_1^2+g_2\vec{p}_1\cdot\vec{p}_2+g_3\hat{p}_2^2+g_4\left(\vec{p}_1\cdot\vec{n}\right)^2+g_5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\right.\right.\\
&\left.+g_6\left(\vec{p}_2\cdot\vec{n}\right)^2\right)+\vec{p}_1\times\vec{n}\left(g_7\hat{p}_1^2\vec{p}_1\cdot\vec{n}+g_8\hat{p}_1^2\vec{p}_2\cdot\vec{n}+g_9\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{n}\right.\\
&+g_{10}\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{n}+g_{11}\hat{p}_2^2\vec{p}_1\cdot\vec{n}+g_{12}\hat{p}_2^2\vec{p}_2\cdot\vec{n}+g_{13}\vec{p}_1\cdot\vec{n}\left(\vec{p}_2\cdot\vec{n}\right)^2\\
&\left.+g_{14}\left(\vec{p}_1\cdot\vec{n}\right)^2\vec{p}_2\cdot\vec{n}+g_{15}\left(\vec{p}_1\cdot\vec{n}\right)^3+g_{16}\left(\vec{p}_2\cdot\vec{n}\right)^3\right)+\vec{p}_2\times\vec{n}\left(g_{17}\hat{p}_1^2\vec{p}_1\cdot\vec{n}\right.\\
&+g_{18}\hat{p}_1^2\vec{p}_2\cdot\vec{n}+g_{19}\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{n}+g_{20}\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{n}+g_{21}\hat{p}_2^2\vec{p}_1\cdot\vec{n}+g_{22}\hat{p}_2^2\vec{p}_2\cdot\vec{n}\\
&\left.\left.+g_{23}\vec{p}_1\cdot\vec{n}\left(\vec{p}_2\cdot\vec{n}\right)^2+g_{24}\left(\vec{p}_1\cdot\vec{n}\right)^2\vec{p}_2\cdot\vec{n}+g_{25}\left(\vec{p}_1\cdot\vec{n}\right)^3+g_{26}\left(\vec{p}_2\cdot\vec{n}\right)^3\right)\right]\\
&+\frac{G^2m_2}{r^2}\vec{S}_1\cdot\left[\vec{p}_1\times\vec{p}_2\left(g_{27}m_1+g_{28}m_2\right)\right.\\
&+\vec{p}_1\times\vec{n}\left(\vec{p}_1\cdot\vec{n}\left(g_{29}m_1+g_{30}m_2\right)+\vec{p}_2\cdot\vec{n}\left(g_{31}m_1+g_{32}m_2\right)\right)\\
&\left.+\vec{p}_2\times\vec{n}\left(\vec{p}_1\cdot\vec{n}\left(g_{33}m_1+g_{34}m_2\right)+\vec{p}_2\cdot\vec{n}\left(g_{35}m_1+g_{36}m_2\right)\right)\right]. \qquad (4.8)
\end{aligned}
$$

We should also have the generators contributing first to lower orders, as noted in eq. (4.6), so that their coefficients are already set from eq. (4.7). For the NLO spin-orbit sector we have the generator from eq. (7.8) in [24, 28] with its coefficients set to the values

$$
g_1=-\frac{1}{2},\ g_2=0,\ g_3=\frac{1}{2},\ g_4=0,\ g_5=0. \qquad (4.9)
$$

We should also take into account the generator, which contributes first at the 2PN non spinning sector, from eq. (7.10) in [24] with its coefficients set to

$$
g_1=0,\ g_2=-\frac{1}{2},\ g_3=0,\ g_4=0,\ g_5=0,\ g_6=0,\ g_7=-\frac{1}{4}. \qquad (4.10)
$$

Thus, we plug in eq. (4.6) our ansatz for $g^{\rm SO}_{\rm NNLO}$ from eq. (4.8), together with the fixed generators in eqs. (4.9), (4.10), and we compare that to eq. (4.5). Comparing $O(G)$ terms fixes the $O(G)$ coefficients of $g^{\rm SO}_{\rm NNLO}$ to the values

$$
\begin{aligned}
&g_1=\frac{7}{16}, && g_2=-\frac{5}{8}, && g_3=-\frac{1}{4}, && g_4=0, && g_5=-\frac{7}{8}, && g_6=0, && g_7=0,\\
&g_8=\frac{1}{16}, && g_9=0, && g_{10}=-\frac{7}{8}, && g_{11}=\frac{5}{16}, && g_{12}=\frac{1}{4}, && g_{13}=-\frac{15}{16}, && g_{14}=0,\\
&g_{15}=0, && g_{16}=0, && g_{17}=0, && g_{18}=0, && g_{19}=0, && g_{20}=0, && g_{21}=0,\\
&g_{22}=0, && g_{23}=0, && g_{24}=0, && g_{25}=0, && g_{26}=0. && && \qquad (4.11)
\end{aligned}
$$

This eliminates all of the $O(G)$ terms in the difference. Comparing the remaining $O(G^2)$ terms in the difference fixes the $O(G^2)$ coefficients of $g^{\rm SO}_{\rm NNLO}$ to the values

$$
\begin{aligned}
&g_{27}=\frac{9}{16}, && g_{28}=-\frac{1}{2}, && g_{29}=-\frac{39}{8}, && g_{30}=-\frac{3}{4}, && g_{31}=-\frac{31}{8},\\
&g_{32}=-\frac{11}{4}, && g_{33}=2, && g_{34}=-\frac{1}{2}, && g_{35}=0, && g_{36}=-\frac{51}{8}. \qquad (4.12)
\end{aligned}
$$

This eliminates all of the $O(G^2)$ terms, as well as all terms at $O(G^3)$ in the difference. Hence, we have shown that the ADM Hamiltonian and the EFT Potential at NNLO spin-orbit are completely equivalent.

5 Conclusions

In this work we implemented the EFT for gravitating spinning objects in the PN scheme [17] at the NNLO level, which was first treated in [18]. We derived the NNLO spin-orbit interaction potential at the 3.5PN order for rapidly rotating compact objects. Such high PN orders are required for the successful detection of gravitational radiation, as the EOB Hamiltonian, e.g. , requires parameters for the even higher 5PN and 6PN orders in the point-mass case, in order to produce good waveforms. From the NNLO spin-orbit interaction potential, which we obtain here in a Lagrangian form for the first time, we directly derived the corresponding Hamiltonian. We then compared our result to the ADM Hamiltonian result [11], and arrived at a complete agreement between the ADM and EFT results. Therefore, in order to complete the spin dependent conservative sector to 4PN order, it remains to apply the EFT for gravitating spinning objects [17] at NNLO to quadratic level in the spin, taking into account finite size effects was already done in [24].

The spin-orbit sector constitutes the most elaborate spin dependent sector at each order, and accordingly we encountered here a proliferation of the relevant Feynman diagrams, where there are 132 diagrams contributing to this sector, and a significant increase of the computational complexity, e.g. there are 32 two-loop diagrams here. We also recall that as the spin is derivative-coupled, higher-order tensor expressions are required for all integrals involved in the calculations, compared to the non spinning case. However, the computation is made efficient through the use of the NRG fields, which are advantageous also in the spin dependent sectors, together with the various gauge choices included in the EFT for gravitating spinning objects [17]. In addition, we automatized the EFT computations here, and carried out the automated computations in parallel. Hence, it is clear that for higher order corrections automated EFT computations, utilizing the NRG fields, should be implemented, and are most powerful and efficient. It should be stressed that in order to obtain such higher order results, all lower order results are required consistently within one formalism, and so also for that the derivation presented in this work is essential. This work then paves the way for the obtainment of the next-to-NNLO spin-orbit interaction potential at 4.5PN order for rapidly rotating compact objects, once this level of accuracy would be approached.

Acknowledgments

This work has been done within the Labex ILP (reference ANR-10-LABX-63) part of the Idex SUPER, and received financial French state aid managed by the Agence Nationale de la Recherche, as part of the programme Investissements d'Avenir under the reference ANR-11-IDEX-0004-02. ML is grateful for the kind hospitality at the Max Planck Institute for Gravitational Physics (AEI) during the final stages of this work.

A Irreducible two-loop tensor integrals

In the evaluation of the irreducible two-loop diagrams we encounter irreducible two-loop tensor integrals up to order 3. These are reduced using the integration by parts method to a sum of factorizable and nested two-loop integrals, as explained in [18], and see appendix A there. In addition to the irreducible two-loop tensor integrals, which were given in appendix A of [18], the two following reductions are also required here:

$$\int_{\vec{k}_1\vec{k}_2} \frac{k_1^i}{k_1^2\,(p-k_1)^2\,k_2^2\,(p-k_2)^2\,(k_1-k_2)^2} = \frac{p^i}{d-4}\int_{\vec{k}_1\vec{k}_2}\left[\frac{1}{k_1^2\,(p-k_1)^2\,k_2^2\,(p-k_2)^4} - \frac{1}{k_1^2\,(k_1-k_2)^2\,k_2^2\,(p-k_2)^4}\right], \quad \text{(A.1)}$$

$$\int_{\vec{k}_1\vec{k}_2} \frac{k_1^i k_1^j}{k_1^2\,(p-k_1)^2\,k_2^2\,(p-k_2)^2\,(k_1-k_2)^2} = \frac{1}{d-4}\int_{\vec{k}_1\vec{k}_2}\left[\frac{p^ip^j - p^ik_1^j - p^jk_1^i + 2k_1^ik_1^j}{k_1^2\,(p-k_1)^2\,k_2^2\,(p-k_2)^4} - \frac{p^ip^j - p^ik_1^j - p^jk_1^i + 2k_1^ik_1^j}{k_1^2\,(k_1-k_2)^2\,k_2^2\,(p-k_2)^4}\right]. \quad \text{(A.2)}$$

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