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Relations between elliptic multiple zeta values and a special derivation algebra

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Abstract

We investigate relations between elliptic multiple zeta values and describe a method to derive the number of indecomposable elements of given weight and length. Our method is based on representing elliptic multiple zeta values as iterated integrals over Eisenstein series and exploiting the connection with a special derivation algebra. Its commutator relations give rise to constraints on the iterated integrals over Eisenstein series relevant for elliptic multiple zeta values and thereby allow to count the indecomposable representatives. Conversely, the above connection suggests apparently new relations in the derivation algebra. Under `https://tools.aei.mpg.de/emzv` we provide relations for elliptic multiple zeta values over a wide range of weights and lengths.

Contents

1 Introduction

While multiple zeta values (MZVs) have been a very active field of research during the last decades, their elliptic analogues have received more attention only recently. Pioneered by the work of Enriquez [1], Levin [2], Levin and Racinet [3] as well as Brown and Levin [4], many properties of elliptic multiple zeta values (eMZVs) have been identified. While mathematically interesting objects in their own right, eMZVs, the associated elliptic iterated integrals as well as related objects such as multiple elliptic polylogarithms appear in various contexts in quantum field theory and string theory. Well-known examples include the one-loop amplitude in open superstring theory [5] as well as the sunset Feynman integral and its generalizations [6–9]. We would also like to mention a recent application of elliptic functions to the reduction of Feynman integrals using maximal unitarity cuts [10].

Relations between usual MZVs are well understood based on their conjectural structure as a Hopf algebra comodule [11,12]. The number of indecomposable MZVs of given weight and depth is expected to be given by the Broadhurst-Kreimer conjecture [13], which is in line with Zagier's conjecture [14] on the counting of MZVs at fixed weight. Employing numerical calculations in addition, a collection of relations among MZVs has been made available electronically in the MZV datamine [15].

In this article, we investigate relations between eMZVs and classify their *indecomposable* representatives. Apart from the shuffle relations which are immediately implied by the definition as iterated integrals, eMZVs are related by Fay identities. Those identities are the generalization of partial-fraction identities, which appear in the context of usual MZVs. Both shuffle and Fay relations preserve the overall modular weight of the integrand which appears to furnish a natural analogue of the MZVs' transcendental weight. We will describe a systematic way of exploiting the combination of the two types of relations. However, the application of this method to higher weights and lengths suffers from the proliferating combinatorics of the Fay relations.

An alternative and computationally more efficient way of deducing relations between eMZVs consists of employing their Fourier expansion in $q = e^{2\pi i\tau}$, where τ is the modular parameter of the elliptic curve. The q-derivative of eMZVs is known from ref. [1] in terms of Eisenstein series and eMZVs of lower length. Since eMZVs degenerate to MZVs at the cusp $q \to 0$ in a manner described in refs. [1, 16], the supplementing boundary value is available as well. Hence, the differential equation can be integrated to yield the q-expansion of eMZVs recursively to – in principle – arbitrarily high order. Once the q-expansion of an eMZV is available up to a certain power in q, finding relations between eMZVs valid up to this particular power and identifying indecomposable representatives amounts to solving a linear system.

Clearly, the agreement of the respective q-expansions up to a certain power in q is necessary but not sufficient for the validity of eMZV relations. Nevertheless, the method of q-expansions allows to confirm that indeed, Fay and shuffle identities comprise the entity of eMZV relations for a variety of combinations of weights and lengths. Accordingly, this leads us to conjecture that *all* available relations between eMZVs are implied by Fay and shuffle identities. At lengths and weights beyond the reach of our current computer implementation of Fay and shuffle identities, the comparison of q-expansions gives rise to conjectural relations which nicely tie in with the algebraic considerations to be described next.

An obvious idea to surmount the shortcomings of comparing q-expansions is to express eMZVs in terms of Eisenstein series which arise in the q-derivative of eMZVs. Solving the differential equation is recursive in the eMZVs' length and therefore yields iterated integrals over Eisenstein series or *iterated Eisenstein integrals* for short. Contrary to the definition of eMZVs, where the iterated integration is performed over coordinates of the elliptic curve, the integration in their representation via Eisenstein series is performed over the modular parameter of the elliptic curve. Similar iterated integrals, some of them involving more general modular forms, have been studied by Manin in ref. [17]. Those integrals have been revisited by Brown [18] recently, who used them to define *multiple modular values*. A new feature of Brown's approach to iterated integrals of modular forms is that it allows also for an integration along paths which connect two cusps. Among other things, multiple modular values provide a conceptual explanation of the relationship between double zeta values and cusp forms [19].

While the representation of eMZVs as iterated Eisenstein integrals again implies shuffle relations, an analogue of the Fay relations is not known. This led to the first expectation that one can find the number of indecomposable eMZVs by enumerating all shuffle-independent iterated Eisenstein integrals.

While this idea indeed yields the correct number of indecomposable eMZVs of low length and weight, there is another effect appearing at higher weight: in the rewriting of *any* eMZV certain shuffle-independent iterated Eisenstein integrals occur in rigid linear combinations only.

In other words, not all iterated Eisenstein integrals can be expressed in terms of eMZVs. The above rigid linear combinations in turn are implied by relations well known from a special algebra of derivations $\mathfrak{u}$ [20–23]. The patterns we find from investigating various eMZVs exactly match the available data about the derivation algebra in ref. [22, 24]. Consequently, we turn this into a method to infer the number of indecomposable eMZVs at given weight and length. The results of this method agree perfectly with the data obtained from either shuffle and Fay relations. In addition, complete knowledge of relations in the derivation algebra leads to upper bounds on the number of indecomposable eMZVs. Those upper bounds complement the lower bounds from comparing q-expansions.

The algebra of derivations $\mathfrak{u}$ on the one hand side and eMZVs on the other side are linked by a differential equation for the elliptic KZB associator [20, 21] derived by Enriquez [16]. This differential equation implies upper bounds on the number of indecomposable eMZVs. Assuming these upper bounds to be attained, one can extend the knowledge about the derivation algebra $\mathfrak{u}$ substantially: we identify numerous apparently new relations up to and including depth five. There is no conceptual bottleneck in extending the analysis to arbitrary weight and depth.

Practically, the enumeration of shuffle-independent iterated Eisenstein integrals can be performed by mapping them onto words composed from non-commutative letters g. The viability of this mapping ψ relies on the linear independence of iterated Eisenstein integrals, which can be shown. Moreover, the mapping ψ bears similarities to a structure, which appeared already in the context of usual MZVs. Employing a conjectural isomorphism ϕ, MZVs can be rewritten in terms of an alphabet of non-commutative letters f [11,25]. While the number of indecomposable MZVs at given weight is determined by counting *all* shuffle-independent non-commutative words in f, the corresponding problem for eMZVs requires the consideration of the properties of the derivation algebra $\mathfrak{u}$ in addition.

The article is organized as follows: section 2 starts with a small review of eMZVs and sets the stage for combining Fay and shuffle relations with the q-expansion, resulting in an "observational" set of indecomposable eMZVs. Section 3 is devoted to a brief recapitulation of structures present for usual MZVs with particular emphasis on their rewriting in terms of non-commutative words. In section 4 we set up the translation of eMZVs into iterated Eisenstein integrals, investigate their properties and connect the bookkeeping of indecomposable eMZVs with the algebra of derivations $\mathfrak{u}$. In subsection 4.6 we describe a modified version of iterated Eisenstein integrals suitable in particular for the description of eMZVs. Several appendices are complementary to the discussion in the main text. In particular, relations for derivation algebras are collected in appendix C.2.

Relations between eMZVs and among derivations are collected on a web page which can be accessed via `https://tools.aei.mpg.de/emzv`.

2 Relations between elliptic multiple zeta values

After recalling the definition of eMZVs, we are going to explore the implications of Fay and shuffle relations as well as the method of q-expansions. In addition, we will describe how usual MZVs defined via

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \ , \qquad n_r \geq 2 \tag{2.1}$$

arise as constant terms of eMZVs.

2.1 Prerequisites and definitions

In this subsection we will briefly review elliptic iterated integrals define eMZVs. An elaborate introduction from a string theorist's point of view is available in ref. [5]. To get started, let us consider iterated integrals on the punctured elliptic curve $E_\tau^\times$, which is $E_\tau \equiv \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with the origin removed and $\mathrm{Im}\,(\tau) > 0$. We will frequently refer to the modular parameter τ by its exponentiated version

$$q \equiv e^{2\pi i\tau}, \quad \text{such that} \quad 2\pi i \frac{\mathrm{d}}{\mathrm{d}\tau} = -4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q} = -4\pi^2 \frac{\mathrm{d}}{\mathrm{d}\log q}\ . \tag{2.2}$$

Functions A of the modular parameter will be denoted by either $A(\tau)$ or $A(q)$.

Weighting functions. A natural collection of weighting functions for the iterated integration to be defined below is provided by the Eisenstein-Kronecker series $F(z,\alpha,\tau)$ [26, 4]

$$F(z,\alpha,\tau) \equiv \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}\ , \tag{2.3}$$

where θ_1 is the odd Jacobi theta function and the tick denotes a derivative with respect to the first argument. The definition eq. (2.3) immediately yields $F(z,\alpha,\tau) = F(z+1,\alpha,\tau)$, and supplementing an additional, non-holomorphic factor lifts the quasi-periodicity of the Eisenstein-Kronecker series with respect to $z \mapsto z+\tau$ to an honest double-periodicity. The resulting function $\Omega(z,\alpha,\tau)$ on an elliptic curve serves as a generating series for the weighting functions $f^{(n)}(z,\tau)$ in eMZVs:

$$\Omega(z,\alpha,\tau) \equiv \exp\left(2\pi i\alpha \frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}\right) F(z,\alpha,\tau) = \sum_{n=0}^{\infty} f^{(n)}(z,\tau)\alpha^{n-1}\ . \tag{2.4}$$

The functions $f^{(n)}$ are doubly periodic and alternate in their parity,

$$f^{(n)}(z+1,\tau) = f^{(n)}(z+\tau,\tau) = f^{(n)}(z,\tau)\ , \qquad f^{(n)}(-z,\tau) = (-1)^n f^{(n)}(z,\tau)\ . \tag{2.5}$$

Their simplest instances read

$$f^{(0)}(z,\tau) = 1 \qquad f^{(1)}(z,\tau) = \frac{\theta_1'(z,\tau)}{\theta_1(z,\tau)} + 2\pi i \frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}\ , \tag{2.6}$$

and $f^{(1)}$ is in fact the only weighting function with a simple pole on the lattice $\mathbb{Z}+\mathbb{Z}\tau$ including the origin. The remaining $f^{(n)}$ with $n \neq 1$ are non-singular on the entire elliptic curve. As elaborated in [4] and section 3 of [5], the weighting functions $f^{(n)}$ can be expressed in terms of Eisenstein functions and series the latter of which will play a central rôle in the sections below.

Elliptic iterated integrals and eMZVs. Even though the functions $f^{(n)}$ are defined for arbitrary complex arguments z and suitable for integrations along both homology cycles of the elliptic curve, we will restrict our subsequent discussion to real arguments. This is sufficient for studying eMZVs as iterated integrals over the interval $[0,1]$ on the real axis and avoids the necessity for homotopy-invariant completions of the integrands[1]. Hence, any integration variable

[1] A generating series for homotopy-invariant iterated integrals is given in ref. [4], in which the differential forms $f^{(n)}(z)\mathrm{d}z$ are accompanied by $\nu \equiv 2\pi i\, \frac{\mathrm{dIm}\,(z)}{\mathrm{Im}\,(\tau)}$. While any integral based upon a sequence of ν and $\mathrm{d}z$ has a unique homotopy-invariant uplift via admixtures of $f^{(n>0)}(z)\mathrm{d}z$, iterated integrals of $f^{(n)}(z)\mathrm{d}z$ allow for multiple homotopy-invariant completions via ν. A thorough discussion of the issue is provided in ref. [5].

and first argument of $f^{(n)}$ is understood to be real.

Employing the functions $f^{(n)}$, iterated integrals on the elliptic curve $E_\tau^\times$ are defined via

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix} ; z\right) \equiv \int_0^z \mathrm{d}t\, f^{(n_1)}(t-a_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix} ; t\right), \tag{2.7}$$

where the recursion starts with $\Gamma(;z) \equiv 1$. The elliptic iterated integral in eq. (2.7) is said to have *weight* $w = \sum_{i=1}^r n_i$, and the number r of integrations will be referred to as its *length*. Beginning with the above equation, we will usually suppress the second argument τ for the weighting functions $f^{(n)}$ and the elliptic iterated integrals Γ.

Definition (2.7) implies a shuffle relation with respect to the combined letters $A_i \equiv \begin{smallmatrix} n_i \\ a_i \end{smallmatrix}$ describing the weighting functions $f^{(n_i)}(z-a_i)$,

$$\Gamma(A_1, A_2, \dots, A_r; z)\, \Gamma(B_1, B_2, \dots, B_q; z) = \Gamma\left((A_1, A_2, \dots, A_r) \sqcup\!\sqcup (B_1, B_2, \dots, B_q); z\right). \tag{2.8}$$

Another obvious property of elliptic iterated integrals is the reflection identity due to eq. (2.5)

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix} ; z\right) = (-1)^{n_1+n_2+\dots+n_r}\, \Gamma\left(\begin{smallmatrix} n_r & \dots & n_2 & n_1 \\ z-a_r & \dots & z-a_2 & z-a_1 \end{smallmatrix} ; z\right). \tag{2.9}$$

Finally, if all the labels a_i vanish – which, because of the periodicity of f is equivalent to all labels a_i being integer – we will often use the notation

$$\Gamma(n_1, n_2, \dots, n_r; z) \equiv \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ 0 & 0 & \dots & 0 \end{smallmatrix} ; z\right). \tag{2.10}$$

Evaluating the elliptic iterated integrals in eq. (2.10) at $z=1$ gives rise to *elliptic multiple zeta values* or *eMZV*s for short [1]:

$$\begin{aligned} \omega(n_1, n_2, \dots, n_r) &\equiv \int\limits_{0 \le z_i \le z_{i+1} \le 1} f^{(n_1)}(z_1)\mathrm{d}z_1\, f^{(n_2)}(z_2)\mathrm{d}z_2 \dots f^{(n_r)}(z_r)\mathrm{d}z_r \\ &= \Gamma(n_r, \dots, n_2, n_1; 1), \end{aligned} \tag{2.11}$$

where $\ell_\omega = r$ is referred to as the *length* while $w_\omega = \sum_{i=1}^r n_i$ is called the *weight* of an eMZV. The subscript ω refers to the current ω-representation of eMZVs in eq. (2.11), which has a different notion of weight and length compared to the iterated Eisenstein integrals to be defined below in section 4.

Being defined on an elliptic curve, eMZVs depend on its modular parameter τ and furnish a natural genus-one generalization of standard MZVs, which are to be reviewed in section 3.

The shuffle relation eq. (2.8) straightforwardly carries over to eMZVs,

$$\omega(n_1, n_2, \dots, n_r)\, \omega(k_1, k_2, \dots, k_s) = \omega\big((n_1, n_2, \dots, n_r) \sqcup\!\sqcup (k_1, k_2, \dots, k_s)\big), \tag{2.12}$$

whereas the parity property eq. (2.5) of the weighting functions $f^{(n)}$ implies the reflection identity

$$\omega(n_1, n_2, \dots, n_{r-1}, n_r) = (-1)^{n_1+n_2+\dots+n_r}\, \omega(n_r, n_{r-1}, \dots, n_2, n_1). \tag{2.13}$$

Note that this implies the vanishing of odd-weight eMZVs with reversal-symmetric labels:

$$\omega(n_1, n_2, \dots, n_r) = 0 \qquad \text{if } (n_1, n_2, \dots, n_r) = (n_r, \dots, n_2, n_1) \text{ and } \sum_{i=1}^r n_i \text{ odd}. \tag{2.14}$$

Although suppressed in our notation, every eMZV is still a function of the modular parameter τ and inherits a Fourier expansion in q from the restriction of $f^{(n)}$ to real arguments.

$$\omega(n_1,\ldots,n_r;\tau) = \omega_0(n_1,\ldots,n_r) + \sum_{k=1}^{\infty} c_k q^k \ . \tag{2.15}$$

The τ-independent quantity ω_0 in eq. (2.15) is called the *constant term* of ω and will be shown to consist of MZVs and powers of $(2\pi i)^{\pm 1}$ in the next subsection. We will refer to eMZVs for which $c_k = 0$ for all $k \in \mathbb{N}^+$ as *constant*.

Regularization. The function $f^{(1)}(z)$ in eq. (2.6) diverges as $\frac{1}{z}$ and $\frac{1}{z-1}$ for $z \to 0$ and $z \to 1$, respectively. Hence, eMZVs $\omega(n_1,\ldots,n_r)$ with $n_1 = 1$ or $n_r = 1$ are a priori divergent, and require a regularization process similar to shuffle regularization for MZVs [1] (cf. also [27]). A natural choice at genus one is to modify the integration region in eq. (2.7) by a small $\varepsilon > 0$,

$$\int\limits_{\varepsilon \le z_i \le z_{i+1} \le z-\varepsilon} f^{(n_1)}(z_1-a_1)\mathrm{d}z_1\, f^{(n_2)}(z_2-a_2)\mathrm{d}z_2 \ldots f^{(n_r)}(z_r-a_r)\mathrm{d}z_r \ , \tag{2.16}$$

and to expand the integral as a polynomial in $\log(-2\pi i\varepsilon)$. Hereby the branch of the logarithm is chosen such that $\log(-i) = -\frac{\pi i}{2}$. The regularized value of eq. (2.16) is then defined to be the constant term in the ε-expansion. The factor $-2\pi i$ in the expansion parameter $\log(-2\pi i\varepsilon)$ ensures that the limit $\tau \to i\infty$ does not introduce any logarithms, and that eMZVs degenerate to MZVs upon setting $z = 1$ in eq. (2.16). For later reference, we will call eMZVs of the form $\omega(1, n_2, \ldots)$ or $\omega(\ldots, n_{r-1}, 1)$ *divergent*.

For the enumeration of eMZVs, we have employed an infinite alphabet, consisting of the non-negative integers $0, 1, 2, \ldots$ eq. (2.11). There is another way of carrying out this enumeration, which uses a two-letter alphabet instead [4]. The two-letter alphabet descends from a construction of eMZVs via *homotopy-invariant* iterated integrals. Since every eMZV in the infinite alphabet can be rewritten as an eMZV in the two-letter alphabet and vice-versa, one does not lose information by choosing to work with one alphabet or the other.

2.2 Fay and shuffle relations

In this subsection, we analyze relations among eMZVs defined in eq. (2.11) and gather examples of *indecomposable* eMZVs. A set of *indecomposable* eMZVs of weight w_ω and length ℓ_ω is a *minimal* set of eMZVs such that any other eMZV of the same weight and length can be expressed as a linear combination of elements from this set and

- products of eMZVs with strictly positive weights,
- eMZVs of lengths smaller than ℓ_ω or weight lower than w_ω,

where the coefficients comprise MZVs (including rational numbers) and integer powers of $2\pi i$. After exploring the consequences of shuffle and reflection identities eq. (2.12) and eq. (2.13), Fay identities are discussed as a genus-one analogue of the partial-fraction identities among products of $(z-a)^{-1}$ due to the differential forms $\mathrm{dlog}(z-a)$. The weight of eMZVs is preserved under all these identities whereas the length obviously varies in shuffle-relations. In contradistinction to usual MZVs, there is an infinite number of eMZVs for a certain weight, so the counting of indecomposable eMZVs must be performed at fixed length and weight.

Examples of constant eMZVs. The simplest examples of the eMZVs defined in eq. (2.11) are of length one:

$$\omega(n_1) = \begin{cases} -2\,\zeta_{n_1} & : \ n_1 \text{ even} \\ 0 & : \ n_1 \text{ odd} \end{cases} . \tag{2.17}$$

The underlying single integration over the interval $[0,1]$ picks up the constant term in the q-expansion of $f^{(n)}$ (see section 3.3 of ref. [5]) and yields the constants in eq. (2.17) with regularized value $\zeta_0 = -\frac{1}{2}$.

Another distinction between even and odd labels n_i occurs at length $\ell_\omega = 2$. The union of shuffle and reflection identities eq. (2.12) and eq. (2.13) contains more independent relations for even total weight than for odd weight, and the eMZVs are then completely determined by eq. (2.17):

$$\omega(n_1, n_2)\Big|_{n_1+n_2 \text{ even}} = \begin{cases} 2\,\zeta_{n_1}\,\zeta_{n_2} & : \ n_1, n_2 \text{ even} \\ 0 & : \ n_1, n_2 \text{ odd} \end{cases} . \tag{2.18}$$

For eMZVs of odd total weight, on the other hand, shuffle and reflection relations at length two coincide, and $\omega(n_1, n_2)$ are no longer bound to be constant. This correlation between $(-1)^{w_\omega}$ and the length will be turned into a general rule in the next paragraph. In addition, there are also constant eMZVs, which make their appearance only at sufficiently high length. For example, ζ_3 is identified in eq. (2.40) to be an eMZV of weight 3 and length 4. One can show that all constant eMZVs evaluate to products of MZVs and powers of $(2\pi i)^{\pm 1}$, see Proposition 5.3 of ref. [1].

Interesting and boring eMZVs. The lack of a τ-dependence for eMZVs $\omega(n_1, n_2)$ of even weight can be viewed as the analogue of the vanishing of $\omega(n_1)$ for odd weight as observed in eq. (2.17). The general pattern is as follows: whenever weight and length of an eMZV have the same parity (i.e. $(-1)^{w_\omega} = (-1)^{\ell_\omega}$), shuffle and reflection identities eqs. (2.12) and (2.13) allow to determine this eMZV in terms of eMZVs of lower length. Novel indecomposable eMZVs can only occur for opposite parity $(-1)^{w_\omega} = -(-1)^{\ell_\omega}$ such as the odd-weight $\omega(n_1, n_2)$. Accordingly, an eMZV $\omega(n_1, n_2, \ldots, n_r)$ is called *interesting*, if the combination $w_\omega + \ell_\omega$ of weight and length is odd, otherwise we refer to it as *boring*.

Boring eMZVs at length $\ell_\omega = 3$ can arise from four different choices of even and odd labels. For those, the shuffle identities eq. (2.12) allow to reduce them to interesting eMZVs at length two. Explicitly, we have

$$\begin{aligned}
\omega(o_1, o_2, o_3) &= 0 \\
\omega(e_1, e_2, o_3) &= -\,\zeta_{e_1}\,\omega(e_2, o_3) \\
\omega(e_1, o_2, e_3) &= -\,\zeta_{e_1}\,\omega(o_2, e_3) - \zeta_{e_3}\,\omega(e_1, o_2) \\
\omega(o_1, e_2, e_3) &= -\,\zeta_{e_3}\,\omega(o_1, e_2)\,,
\end{aligned} \tag{2.19}$$

where e_i and o_i refer to even and odd labels respectively. Similarly, boring eMZV at length $\ell_\omega = 4$ come in the following (reflection-independent) classes:

$$\begin{aligned}
\omega(o_1, o_2, o_3, o_4) &= 0 \\
\omega(e_1, e_2, e_3, e_4) &= -2\,\zeta_{e_1}\,\zeta_{e_2}\,\zeta_{e_3}\,\zeta_{e_4} - \zeta_{e_4}\,\omega(e_1, e_2, e_3) - \zeta_{e_1}\,\omega(e_4, e_3, e_2) \\
\omega(o_1, o_2, e_3, e_4) &= -\,\zeta_{e_4}\,\omega(o_1, o_2, e_3) \\
\omega(o_1, e_2, o_3, e_4) &= \tfrac{1}{2}\,\omega(o_1, e_2)\,\omega(o_3, e_4) - \zeta_{e_4}\,\omega(o_1, e_2, o_3)
\end{aligned} \tag{2.20}$$

$$\omega(o_1,e_2,e_3,o_4) = \tfrac{1}{2}\,\omega(o_1,e_2)\,\omega(e_3,o_4)$$
$$\omega(e_1,o_2,o_3,e_4) = \tfrac{1}{2}\,\omega(e_1,o_2)\,\omega(o_3,e_4) - \zeta_{e_1}\,\omega(o_2,o_3,e_4) - \zeta_{e_4}\,\omega(e_1,o_2,o_3)\,.$$

Although becoming more involved for higher length, the distinction of cases as well as the decomposition of boring eMZVs can be cast into a nice form, as is explained in appendix A.1. Below, however, we will be concerned with interesting eMZVs mostly. Note that the vanishing of eMZVs with only odd entries is true at all lengths,

$$\omega(o_1,o_2,\ldots,o_r) = 0\ . \tag{2.21}$$

Fay relations among $f^{(n)}$ and elliptic iterated integrals. While reflection and shuffle identities preserve the partition of the modular weight among the integrated $f^{(n_i)}$, so-called Fay relations mix eMZVs involving different values of n_i. They can be traced back to the Fay identity of their generating series eq. (2.4) [4]

$$\begin{aligned}\Omega(z_1,\alpha_1,\tau)\Omega(z_2,\alpha_2,\tau) &= \Omega(z_1,\alpha_1+\alpha_2,\tau)\Omega(z_2-z_1,\alpha_2,\tau)\\ &\quad + \Omega(z_2,\alpha_1+\alpha_2,\tau)\Omega(z_1-z_2,\alpha_1,\tau)\ ,\end{aligned} \tag{2.22}$$

which is valid for any complex z_1,z_2 and follows from the Fay trisecant equation [28]. Relations among $f^{(n)}$ can be read off from eq. (2.22) by isolating monomials in α_1,α_2 [5]

$$\begin{aligned}f^{(n_1)}(t-x)f^{(n_2)}(t) &= -(-1)^{n_1}f^{(n_1+n_2)}(x) + \sum_{j=0}^{n_2}\binom{n_1-1+j}{j}f^{(n_2-j)}(x)f^{(n_1+j)}(t-x)\\ &\quad + \sum_{j=0}^{n_1}\binom{n_2-1+j}{j}(-1)^{n_1+j}f^{(n_1-j)}(x)f^{(n_2+j)}(t)\ .\end{aligned} \tag{2.23}$$

The simplest instance of these Fay relations can be viewed as a genus-one counterpart of partial fraction relations such as $\frac{1}{tx} = \frac{1}{x(t-x)} + \frac{1}{t(x-t)}$:

$$f^{(1)}(t-x)f^{(1)}(t) = f^{(1)}(t-x)f^{(1)}(x) - f^{(1)}(t)f^{(1)}(x) + f^{(2)}(t) + f^{(2)}(x) + f^{(2)}(t-x)\ . \tag{2.24}$$

The Fay relations eq. (2.23) are a very powerful tool for rearranging the elliptic iterated integrals in eq. (2.7). Together with the derivatives of Γ with respect to their argument z and labels a_i [5], they allow for example to recursively remove any appearance of $a_i = z$ in the label of an iterated integral, e.g.

$$\begin{aligned}\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ z & 0 & \ldots & 0\end{smallmatrix};z\right) &= (-1)^r\,\zeta_r\prod_{j=1}^{r}\delta_{n_j,1} - (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_1+n_2 & 0 & n_3 & \ldots & n_r \\ 0 & 0 & 0 & \ldots & 0\end{smallmatrix};z\right)\\ &\quad + \sum_{j=0}^{n_1}(-1)^{n_1+j}\binom{n_2-1+j}{j}\Gamma\left(\begin{smallmatrix} n_1-j & n_2+j & n_3 & \ldots & n_r \\ 0 & 0 & 0 & \ldots & 0\end{smallmatrix};z\right)\\ &\quad + \sum_{j=0}^{n_2}\binom{n_1-1+j}{j}\int_0^z \mathrm{d}t\, f^{(n_2-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1+j & n_3 & \ldots & n_r \\ t & 0 & \ldots & 0\end{smallmatrix};t\right)\ ,\end{aligned} \tag{2.25}$$

see appendix A.2 for a generalization to multiple appearance of $a_i = z$. The zeta value ζ_r in the first line of eq. (2.25) stems from the limit $z \to 0$ of the left hand side for which $f^{(1)}(z)$ can be approximated by $\frac{1}{z}$ [5]. Note that the Kronecker-deltas $\delta_{n_j,1}$ ensure that the notions of weights for MZVs and elliptic iterated integrals are compatible in eq. (2.25).

Fay relations among eMZVs. A rich class of eMZV relations can be inferred from the limit $z \to 1$ of eq. (2.25). On the left hand side, periodicity of $f^{(n)}$ w.r.t. $z \to z+1$ leads to

$$\lim_{z\to 1} \Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ z & 0 & \dots & 0 \end{smallmatrix} ; z \right) = \omega(n_r, \dots, n_2, n_1) \ , \qquad n_1 \neq 1 \text{ or } n_2 \neq 1 \ , \tag{2.26}$$

where cases with $n_1 = n_2 = 1$ require an additional treatment of the poles of the associated $f^{(1)}$ and are therefore excluded. By eq. (2.11), the elliptic iterated integrals on the right hand side reduce to eMZVs under $z \to 1$ once the recursion eq. (2.25) has been applied iteratively to remove any appearance of the argument from the labels. At length two, the resulting eMZV relation is

$$\begin{aligned} \omega(n_2, n_1) &= \zeta_2 \delta_{n_1,1}\delta_{n_2,1} - (-1)^{n_1}\, \omega(0, n_1+n_2) + \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} (-1)^{n_1+j}\, \omega(n_1+j, n_2-j) \\ &+ \sum_{j=0}^{n_1} \binom{n_2-1+j}{j} (-1)^{n_1+j}\, \omega(n_2+j, n_1-j) \ , \qquad n_1 \neq 1 \text{ or } n_2 \neq 1 \ , \end{aligned} \tag{2.27}$$

and length three requires two applications of the recursion in eq. (2.25):

$$\begin{aligned} \omega(n_3, n_2, n_1) &= -\zeta_3 \delta_{n_1,1}\delta_{n_2,1}\delta_{n_3,1} + \zeta_2 \sum_{j=0}^{n_2} \delta_{n_3,1}\delta_{n_1+j,1} \binom{n_1-1+j}{j} \omega(n_2-j) \\ &- (-1)^{n_1}\, \omega(n_3, 0, n_1+n_2) + \sum_{j=0}^{n_1} (-1)^{n_1+j} \binom{n_2-1+j}{j} \omega(n_3, n_2+j, n_1-j) \\ &+ \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} \sum_{k=0}^{n_3} (-1)^{n_1+j+k} \binom{n_1+j-1+k}{k} \omega(n_1+j+k, n_3-k, n_2-j) \\ &+ \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} \sum_{k=0}^{n_1+j} (-1)^{n_1+j+k} \binom{n_3-1+k}{k} \omega(n_3+k, n_1+j-k, n_2-j) \\ &- \sum_{j=0}^{n_2} (-1)^{n_1+j} \binom{n_1-1+j}{j} \omega(0, n_1+n_3+j, n_2-j) \ , \qquad n_1 \neq 1 \text{ or } n_2 \neq 1 \ . \end{aligned} \tag{2.28}$$

It is straightforward to derive higher-length relations (involving any ζ_r with $2 \leq r \leq \ell_\omega$) from further iterations of eq. (2.25) in the limit $z \to 1$. The exclusion of $n_1 = n_2 = 1$ suppresses ζ_2 in eq. (2.27) and ζ_3 in eq. (2.28), and, more generally, the appearance of ζ_r is relegated to eMZV relations of length $r+1$. By the relations derived from $\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_k & n_{k+1} & \dots & n_r \\ z & z & \dots & z & 0 & \dots & 0 \end{smallmatrix} ; z \right)$ in appendix A.2, the same statement applies to generic MZVs, and any MZV will appear in the rewriting of some $\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix} ; z \right)$ with appropriate combinations of $a_j \in \{0, z\}$.

Generating series for Fay relations. In the same way as Fay identities eq. (2.23) among products of $f^{(n)}$ are more compactly encoded in their generating series eq. (2.22), the resulting eMZV relations can be efficiently organized in terms of their generating series

$$\mathcal{I}(\alpha_1, \alpha_2, \dots, \alpha_r) \equiv \sum_{n_1,\dots,n_r=0}^{\infty} \alpha_1^{n_1-1}\alpha_2^{n_2-1} \dots \alpha_r^{n_r-1}\, \omega(n_r, \dots, n_2, n_1) \ . \tag{2.29}$$

In a shorthand where $\alpha_{12\ldots p} \equiv \alpha_1 + \alpha_2 + \ldots + \alpha_p$, the generating series eq. (2.29) allows to rewrite eq. (2.27) and eq. (2.28) as

$$\begin{aligned} \mathcal{I}(\alpha_1,\alpha_2) + \mathcal{I}(-\alpha_1,\alpha_{12}) + \mathcal{I}(\alpha_2,-\alpha_{12}) &= -3\,\zeta_2 \\ \mathcal{I}(\alpha_1,\alpha_2,\alpha_3) + \mathcal{I}(-\alpha_1,\alpha_{12},\alpha_3) + \mathcal{I}(\alpha_2,-\alpha_{12},\alpha_{123}) + \mathcal{I}(\alpha_2,\alpha_3,-\alpha_{123}) &= -\,\zeta_2(\mathcal{I}(\alpha_2)+\mathcal{I}(\alpha_3)) \ , \end{aligned} \tag{2.30}$$

where the right hand sides $\sim \zeta_2$ address the values of n_i excluded from eq. (2.27) and eq. (2.28). Further corollaries of the relation eq. (2.25) among elliptic iterated integrals include

$$\begin{aligned} &\mathcal{I}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) + \mathcal{I}(-\alpha_1,\alpha_{12},\alpha_3,\alpha_4) + \mathcal{I}(\alpha_2,-\alpha_{12},\alpha_{123},\alpha_4) + \mathcal{I}(\alpha_2,\alpha_3,-\alpha_{123},\alpha_{1234}) \\ &\quad + \mathcal{I}(\alpha_2,\alpha_3,\alpha_4,-\alpha_{1234}) = -\,\zeta_2(\mathcal{I}(\alpha_2,\alpha_3)+\mathcal{I}(\alpha_3,\alpha_4)) + \zeta_3(\mathcal{I}(\alpha_2)-\mathcal{I}(\alpha_4)) - \frac{\pi^4}{24} \\ &\mathcal{I}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) + \mathcal{I}(-\alpha_1,\alpha_{12},\alpha_3,\alpha_4,\alpha_5) + \mathcal{I}(\alpha_2,-\alpha_{12},\alpha_{123},\alpha_4,\alpha_5) \\ &\quad + \mathcal{I}(\alpha_2,\alpha_3,-\alpha_{123},\alpha_{1234},\alpha_5) + \mathcal{I}(\alpha_2,\alpha_3,\alpha_4,-\alpha_{1234},\alpha_{12345}) + \mathcal{I}(\alpha_2,\alpha_3,\alpha_4,\alpha_5,-\alpha_{12345}) \\ &\quad = -\,\zeta_2(\mathcal{I}(\alpha_2,\alpha_3,\alpha_4)+\mathcal{I}(\alpha_3,\alpha_4,\alpha_5)) + \zeta_3(\mathcal{I}(\alpha_2,\alpha_3)-\mathcal{I}(\alpha_4,\alpha_5)) - \zeta_4(\mathcal{I}(\alpha_2)+\mathcal{I}(\alpha_5)) \ . \end{aligned} \tag{2.31}$$

The left hand side obviously generalizes to

$$\mathcal{I}(\alpha_1,\alpha_2,\ldots,\alpha_r) + \sum_{j=1}^{r} \mathcal{I}(\alpha_1,\alpha_2,\ldots,\alpha_j,-\alpha_{12\ldots j},\alpha_{12\ldots j+1},\alpha_{j+2},\ldots,\alpha_r) \tag{2.32}$$

Note that the weight is manifestly conserved in eq. (2.30) and eq. (2.31) where a product of MZVs, eMZVs and elliptic iterated integrals is understood to carry the sum of the weights in each factor.

Combining shuffle and Fay relations. Shuffle relations reduce boring eMZVs to interesting eMZVs of lower length, see for instance eqs. (2.18) and (2.19). At first glance, this appears to attribute more significance to Fay relations among interesting eMZVs, e.g. eq. (2.27) at odd n_1+n_2 and eq. (2.28) at even $n_1+n_2+n_3$. The former yields length-two relations such as

$$\omega(0,5) = \omega(2,3) \ , \qquad \omega(3,4) = -2\,\omega(0,7) + \omega(2,5) \ , \tag{2.33}$$

which by themselves leave $1 + \lfloor \frac{1}{3}(n_1+n_2) \rfloor$ eMZVs at length $\ell_\omega = 2$ and weight $w_\omega = n_1 + n_2$ independent [29]. However, Fay relations eq. (2.28) among boring eMZVs at length three turn out to contain additional information about interesting $\omega(n_1,n_2)$. For example, writing eq. (2.28) with $(n_1,n_2,n_3) = (1,0,2)$,

$$\omega(0,3,0) - \omega(1,2,0) - \omega(2,0,1) + \omega(3,0,0) = 0 \ , \tag{2.34}$$

followed by a shuffle-reduction of the boring eMZVs via eq. (2.19) yields the length-two relation

$$\omega(1,2) = 2\,\zeta_2\,\omega(0,1) - \omega(0,3) \ . \tag{2.35}$$

This relation would be inaccessible from Fay relations at length two and identifies $\omega(0,3)$ to be the unique indecomposable $\omega(n_1,n_2)$ at weight three, which is short of the above $1+\lfloor \frac{1}{3}(n_1+n_2) \rfloor$ counting. Hence – when combined with shuffle-relations – Fay relations among boring eMZVs at length $\ell_\omega + 1$ provide more information than their counterparts among interesting eMZVs at length ℓ_ω. The need for Fay relations at length $\ell_\omega + 1$ to classify indecomposable eMZVs at length ℓ_ω is reminiscent of double-shuffle relations among MZVs. For example, the relation

$$\zeta_{5,7} = \frac{14}{9}\,\zeta_{3,9} + \frac{28}{5}\,\zeta_5\,\zeta_7 - \frac{121285}{12438}\,\zeta_{12} \tag{2.36}$$

is inaccessible from double-shuffle relations of depth two and requires higher-depth input [19].

Indecomposable eMZVs. By applying the shuffle-reduction eq. (2.19) to higher-weight instances of the length-three Fay relation eq. (2.28), any length-two eMZV can be expressed in terms of products of ζ_{2k} and $\omega(0, 2n-1)$:

$$\begin{aligned} \omega(n_1,n_2)\Big|_{n_1+n_2 \text{ odd}} &= (-1)^{n_1}\,\omega(0, n_1+n_2) + 2\delta_{n_1,1}\,\zeta_{n_2}\,\omega(0,1) - 2\delta_{n_2,1}\,\zeta_{n_1}\,\omega(0,1) \\ &+ 2 \sum_{p=1}^{\lceil \frac{1}{2}(n_2-3)\rceil} \binom{n_1+n_2-2p-2}{n_1-1} \zeta_{n_1+n_2-2p-1}\,\omega(0,2p+1) \\ &- 2 \sum_{p=1}^{\lceil \frac{1}{2}(n_1-3)\rceil} \binom{n_1+n_2-2p-2}{n_2-1} \zeta_{n_1+n_2-2p-1}\,\omega(0,2p+1)\,, \end{aligned} \tag{2.37}$$

which implies that no eMZVs at length two other than $\omega(0, 2n-1)$ are indecomposable. This relation can be straightforwardly proven using the techniques of subsection 2.3.

Accordingly, the richest source of relations between interesting eMZVs at length three are the length-four Fay relations at even weight together with the shuffle reduction eq. (2.20) of the boring eMZVs therein. The indecomposable eMZVs can be chosen to include $\omega(0,0,2n)$ by analogy with eq. (2.37), and additional indecomposable eMZVs such as $\omega(0,3,5)$ occur at weight $w_\omega \geq 8$, e.g.

$$\begin{aligned} \omega(1,1,2) &= \frac{13}{12}\,\zeta_4 - \zeta_2\,\omega(0,1)^2 + \omega(0,1)\,\omega(0,3) + 3\,\zeta_2\,\omega(0,0,2) - \frac{1}{2}\,\omega(0,0,4) \\ \omega(0,6,2) &= -\frac{21}{2}\,\zeta_8 + 2\,\omega(0,3)\,\omega(0,5) - 14\,\zeta_6\,\omega(0,0,2) - 6\,\zeta_4\,\omega(0,0,4) - \frac{9}{2}\,\omega(0,0,8) - \frac{2}{5}\,\omega(0,3,5)\,. \end{aligned} \tag{2.38}$$

Similarly, the set of indecomposable length-three eMZVs at weights ten and twelve can be chosen as $\{\omega(0,0,10), \omega(0,3,7)\}$ and $\{\omega(0,0,12), \omega(0,3,9)\}$, respectively. The weight-twelve relation

$$\begin{aligned} \omega(0,5,7) = &- 140\,\zeta_{10}\,\omega(0,0,2) - 14\,\zeta_8\,\omega(0,0,4) + \frac{28}{3}\,\omega(0,5)\,\omega(0,7) \\ &- \frac{119}{6}\,\omega(0,0,12) + \frac{14}{9}\,\omega(0,3,9) - \frac{550396}{6219}\,\zeta_{12} \end{aligned} \tag{2.39}$$

will play an essential rôle later on.

While even single MZVs are special cases of length-one eMZVs by eq. (2.17), odd MZVs do not show up in any relation for an eMZV of $\ell_\omega \leq 3$. When applying the above procedure to higher lengths, ζ_3 is identified to be an eMZV by length-four relations such as

$$\omega(0,1,2,0) = \frac{1}{4}\,\omega(0,3) - \frac{5}{2}\omega(0,0,0,3) - \frac{\zeta_3}{4}\,. \tag{2.40}$$

The appearance of ζ_3 in eMZV relations at length $\ell_\omega = 4, 5$ is governed by the right hand side of eq. (2.31), and similar relations are expected to hold for any odd MZV by eq. (2.25) and eq. (A.3). Further support stems from the description of the eMZVs' constant terms through the Drinfeld associator [30–32] in eq. (2.47) below.

Usual MZVs show up in many relations between eMZVs such as eq. (2.40). While crucial for matching the constant term for the eMZVs in question, we will not count them as indecomposable eMZVs. Instead, they will arise as suitably chosen boundary conditions for a differential equation to be elaborated upon below.

Table 1 shows a possible (non-canonical) choice of indecomposable eMZVs for weights up to 14 and length up to and including five. The need for higher-length Fay relations increases the computational complexity in the classification of indecomposable eMZVs using the above procedure. Hence, comparing the τ-dependence will enter as an additional method in the next subsection to extend the results in the table to higher lengths and weights. Still, shuffle, reflection and Fay relations were assembled completely at $\ell_\omega = 2$, at $\ell_\omega = 3$ with $w_\omega \leq 14$, at $\ell_\omega = 4$ with $w_\omega \leq 9$ as well as at $\ell_\omega = 5$ with $w_\omega \leq 6$, and additional eMZV relations at those weights and lengths have been ruled out on the basis of their q-expansion. Continuing the search for

w_ω \ ℓ_ω	2	3	4
1	$\omega(0,1)$		$\omega(0,0,1,0)$
3	$\omega(0,3)$		$\omega(0,0,0,3)$
5	$\omega(0,5)$		$\omega(0,0,0,5)$
			$\omega(0,0,2,3)$
7	$\omega(0,7)$		$\omega(0,0,0,7)$
			$\omega(0,0,2,5)$
			$\omega(0,0,4,3)$
9	$\omega(0,9)$		$\omega(0,0,0,9)$
			$\omega(0,0,2,7)$
			$\omega(0,0,4,5)$
			$\omega(0,1,3,5)$
11	$\omega(0,11)$		$\omega(0,0,0,11)$
			$\omega(0,0,2,9)$
			$\omega(0,0,4,7)$
			$\omega(0,1,3,7)$
			$\omega(0,3,3,5)$
13	$\omega(0,13)$		$\omega(0,0,0,13)$
			$\omega(0,0,2,11)$
			$\omega(0,0,4,9)$
			$\omega(0,1,3,9)$
			$\omega(0,1,5,7)$
			$\omega(0,3,3,7)$
			$\omega(0,3,5,5)$

w_ω \ ℓ_ω	2	3	4	5
2		$\omega(0,0,2)$		$\omega(0,0,0,0,2)$
4		$\omega(0,0,4)$		$\omega(0,0,0,0,4)$
				$\omega(0,0,0,1,3)$
6		$\omega(0,0,6)$		$\omega(0,0,0,0,6)$
				$\omega(0,0,0,1,5)$
				$\omega(0,0,0,2,4)$
				$\omega(0,0,2,2,2)$
8		$\omega(0,0,8)$		$\omega(0,0,0,0,8)$
		$\omega(0,3,5)$		$\omega(0,0,0,1,7)$
				$\omega(0,0,0,2,6)$
				$\omega(0,0,1,2,5)$
				$\omega(0,0,2,2,4)$
10		$\omega(0,0,10)$		$\omega(0,0,0,0,10)$
		$\omega(0,3,7)$		$\omega(0,0,0,1,9)$
				and 7 more
12		$\omega(0,0,12)$		$\omega(0,0,0,0,12)$
		$\omega(0,3,9)$		$\omega(0,0,0,1,11)$
				$\omega(0,0,0,2,12)$
				and 11 more
14		$\omega(0,0,14)$		$\omega(0,0,0,0,14)$
		$\omega(0,3,11)$		$\omega(0,0,0,1,13)$
		$\omega(0,5,9)$		$\omega(0,0,0,2,12)$
				and many more

Table 1: A possible choice of indecomposable eMZVs up to weight 14 and length 5. A table containing the elements missing here is available at `https://tools.aei.mpg.de/emzv`.

indecomposable eMZVs as described in previous and subsequent subsections leads to table 2, in which the number of indecomposable eMZVs for a certain length and weight are noted. Basis rules for rewriting each eMZV in terms of those indecomposable elements can be obtained in digital form from the web page `https://tools.aei.mpg.de/emzv` and are available up to and including weights $30, 18, 12, 10$ for length $3, 4, 5, 6$ respectively.

ℓ_ω \ w_ω	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		2		3		3		3	
4	1		1		2		3		4		5		7		8		10		x
5		1		2		4		6		9		13		x		x		x	
6	1		2		4		8		13		x		x		x		x		x
7		1		2		x		x		x		x		x		x		x	

Table 2: Number $N(\ell_\omega, w_\omega)$ of indecomposable eMZVs at length ℓ_ω and weight w_ω.

2.3 Constant term and q-expansion

The system of relations discussed in the previous section did not require any information on the eMZVs' functional dependence on the modular parameter τ. In this section, we introduce a method to determine the Fourier expansion in $q = e^{2\pi i\tau}$. This will not only lead to crosschecks for the above eMZV-relations but also yields a computationally more efficient approach to classifying indecomposable eMZVs at higher length and weight. The method to be described below, however, does not allow to prove relations, because it relies on comparing a finite number of terms in the q-expansion of eMZVs, cf. eq. (2.15). Nevertheless, in order to remain on the safe side, we considered the expansion for powers of q up to 160.

Constant term. The constant term of an eMZV can be determined explicitly using results of ref. [1]. By construction, the elliptic KZB associator $A(\tau)$ is the generating series of eMZVs,

$$e^{\pi i[y,x]}A(\tau) \equiv \sum_{r\geq 0}(-1)^r \sum_{n_1,n_2,\ldots,n_r\geq 0} \omega(n_1,n_2,\ldots,n_r)\mathrm{ad}_x^{n_r}(y)\ldots\mathrm{ad}_x^{n_2}(y)\mathrm{ad}_x^{n_1}(y) \ , \tag{2.41}$$

and captures the monodromy of the elliptic KZB equation [20, 21] along the path $[0,1]$. The prefactor $e^{\pi i[y,x]}$ is adjusted to the regularization scheme in eq. (2.16). The variables x and y generate a complete, free algebra $\mathbb{C}\langle\langle x,y\rangle\rangle$ of formal power series with complex coefficients, whose multiplication is the concatenation product, and the convention for the adjoint action is

$$\mathrm{ad}_x(y) \equiv [x,y] \ , \qquad \mathrm{ad}_x^n(y) = \underbrace{[x,\ldots[x,[x,}_{n \text{ times}}y]]\ldots] \ . \tag{2.42}$$

Note that the appearance of eMZVs in eq. (2.41) along with non-commutative words in x and y allows for an alternative enumeration scheme using a two-letter alphabet discussed in subsection 2.1.

Enriquez proved that $A(\tau)$ admits the asymptotic expansion as $\tau \to i\infty$ [1]

$$A(\tau) = \Phi(\tilde{y},t)\, e^{2\pi i\tilde{y}}\, \Phi(\tilde{y},t)^{-1} + \mathcal{O}(e^{2\pi i\tau}) \ , \tag{2.43}$$

where $\mathcal{O}(e^{2\pi i\tau})$ refers to the non-constant terms in eq. (2.15) exclusively. In the above equation, the genus-one alphabet consisting of x,y is translated into a genus-zero alphabet involving

$$t \equiv [y,x] \ , \qquad \tilde{y} \equiv \frac{\mathrm{ad}_x}{e^{2\pi i \mathrm{ad}_x}-1}(y) \ , \tag{2.44}$$

and Φ denotes the Drinfeld associator [30–32]

$$\Phi(a,b) \equiv \sum_{W\in\langle a,b\rangle} \zeta^{ш}(W)\cdot W \ . \tag{2.45}$$

The sum over $W \in \langle a,b\rangle$ includes all non-commutative words in letters a and b, and $\zeta^{ш}(W)$ denote shuffle-regularized MZVs [33] which are uniquely determined from eq. (3.1), the shuffle product eq. (3.2) and the definition $\zeta^{ш}(a) = \zeta^{ш}(b) = 0$ for words of length one. Consequently, the first few terms of $\Phi(a,b)$ are given by

$$\Phi(a,b) = 1 - \zeta_2[a,b] - \zeta_3[a+b,[a,b]] + \ldots \ . \tag{2.46}$$

From eqs. (2.41) and (2.43), the generating series of constant terms $\omega_0(n_1,...,n_r)$ of eMZVs is immediately obtained as

$$\sum_{r\geq 0}(-1)^r \sum_{n_1,\ldots,n_r\geq 0} \omega_0(n_1,\ldots,n_r)\mathrm{ad}_x^{n_r}(y)\ldots\mathrm{ad}_x^{n_1}(y) = e^{\pi i[y,x]}\,\Phi(\tilde{y},t)\,e^{2\pi i\tilde{y}}\,\Phi(\tilde{y},t)^{-1} \ . \tag{2.47}$$

In order to transfer information from the right hand side of eq. (2.47) to the constant terms of eMZVs on the left hand side, it remains to expand words in the alphabet $\{\tilde{y},t\}$ in eq. (2.44) as formal series of words in the alphabet $\{\mathrm{ad}_x^n(y)\,|\,n\geq 0\}$ and then to compare the coefficients of both sides.

Perhaps surprisingly, the case where all $n_i \neq 1$ is very simple to treat. In that case, only the middle term $e^{2\pi i\tilde{y}}$ from eq. (2.47) yields a non-trivial contribution, and therefore we have

$$\omega_0(n_1,n_2,\ldots,n_r)\Big|_{n_i\neq 1} = \begin{cases} 0 & \text{if at least one } n_i \text{ is odd, and all } n_i\neq 1 \\ \dfrac{1}{r!}\displaystyle\prod_{i=1}^{r}(-2\,\zeta_{n_i}) & \text{if all } n_i \text{ are even} \end{cases} \ . \tag{2.48}$$

In particular, given that $f^{(0)} \equiv 1$, one finds

$$\omega(\underbrace{0,0,\ldots,0}_{n \text{ times}}) = \frac{1}{n!} \ . \tag{2.49}$$

On the other hand, in presence of $n_i = 1$ at some places, a general formula for the constant term is very cumbersome. Simple instances include

$$\begin{gathered} \omega_0(1,0) = -\frac{i\pi}{2}\ , \qquad \omega_0(1,0,0) = -\frac{i\pi}{4}\ , \qquad \omega_0(1,0,0,0) = -\frac{i\pi}{12} - \frac{\zeta_3}{24\,\zeta_2} \\ \omega_0(0,1,1,0,0) = \frac{\zeta_2}{15}\ , \qquad \omega_0(1,0,1,1,0,0) = -\frac{i\pi\,\zeta_2}{30} - \frac{\zeta_3}{8} - \frac{17\,\zeta_5}{96\,\zeta_2}\ , \end{gathered} \tag{2.50}$$

and replacing $n_i = 0$ by even values $n_i = 2k$ amounts to multiplication with $-2\,\zeta_{2k}$.

***q*-expansion.** The q-dependent terms in the expansion can be determined using the known form of the τ-derivative of eMZVs. In Théorème 3.3 of ref. [1], the derivative of a generating functional for eMZVs is presented, which translates as follows into derivatives of individual eMZVs in our conventions:

$$2\pi i\frac{\mathrm{d}}{\mathrm{d}\tau}\,\omega(n_1,\ldots,n_r) = -4\pi^2 q\frac{\mathrm{d}}{\mathrm{d}q}\,\omega(n_1,\ldots,n_r)$$

$$
\begin{aligned}
&= n_1\, \mathrm{G}_{n_1+1}\, \omega(n_2,\dots,n_r) - n_r\, \mathrm{G}_{n_r+1}\, \omega(n_1,\dots,n_{r-1}) \\
&+ \sum_{i=2}^{r} \Big\{ (-1)^{n_i} (n_{i-1}+n_i)\, \mathrm{G}_{n_{i-1}+n_i+1}\, \omega(n_1,\dots,n_{i-2},0,n_{i+1},\dots,n_r) \\
&\quad - \sum_{k=0}^{n_{i-1}+1} (n_{i-1}-k) \binom{n_i+k-1}{k} \mathrm{G}_{n_{i-1}-k+1}\, \omega(n_1,\dots,n_{i-2},k+n_i,n_{i+1},\dots,n_r) \\
&\quad + \sum_{k=0}^{n_i+1} (n_i-k) \binom{n_{i-1}+k-1}{k} \mathrm{G}_{n_i-k+1}\, \omega(n_1,\dots,n_{i-2},k+n_{i-1},n_{i+1},\dots,n_r) \Big\}\,.
\end{aligned} \tag{2.51}
$$

The *Eisenstein series* $\mathrm{G}_k \equiv \mathrm{G}_k(\tau)$ on the right hand side are defined by[2]

$$
\begin{aligned}
\mathrm{G}_0(\tau) &\equiv -1 \\
\mathrm{G}_k(\tau) &\equiv \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^k} \qquad k \text{ even},\ k>0\,,
\end{aligned} \tag{2.52}
$$

and clearly vanish for odd values of k. Positive even values of k admit a series expansion in the modular parameter:

$$
\mathrm{G}_k(\tau) = 2\,\zeta_k + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} \qquad k \text{ even},\ k>0\,. \tag{2.53}
$$

Using the above formulæ and the known expansion of the Eisenstein series G_k in eq. (2.53), one can recursively obtain the explicit q-expansion for any eMZV: The length of eMZVs on the right hand side of eq. (2.51) is decreased by one compared to the left hand side, and the recursion terminates with the constant eMZVs at length one given by eq. (2.17).

In addition, one finds from eq. (2.51) that only divergent eMZVs with $n_1 = 1$ or $n_r = 1$ lead to the non-modular G_2, see the discussion around eq. (2.16). In all other situations which lead to the non-modular G_2 in the last three lines, the respective terms cancel out. Not surprisingly, the interesting and boring character of eMZVs is preserved by eq. (2.51): the decreased length on the right hand side is compensated by an increased weight.

Also, note that the differential equation eq. (2.51) contains no MZV terms. In fact, the only way through which MZVs enter the stage of eMZVs is by means of the constant term eq. (2.43) of the elliptic associator. As mentioned earlier, this constant term can be thought of as a boundary-value prescription for the differential equation eq. (2.51), thereby determining eMZVs uniquely.

eMZV relations from the q-expansion. Now that q-expansions are available for any eMZV, their relations can be alternatively found and ruled out by comparing the expansions order by order in q. In practice, one writes down an ansatz comprised from interesting eMZVs and products thereof with uniform weight and an upper bound on the length of interest, each term supplemented with fudge coefficients. Afterwards, one calculates the q-expansions of all constituents up to a certain order. This allows to fix the coefficients and check their validity up to

[2]The two cases $k = 1, 2$ require the Eisenstein summation prescription

$$
\sum_{m,n\in\mathbb{Z}} a_{m,n} \equiv \lim_{N\to\infty} \lim_{M\to\infty} \sum_{n=-N}^{N} \sum_{m=-M}^{M} a_{m,n}\ .
$$

– in principle – arbitrary order. Our computer implementation of this approach was far more efficient compared to the analysis of reflection, shuffle and Fay identities and lead to substantial parts of the data shown in tables 1 and 2. However, since the comparison of q-expansions has to be cut off at some chosen power of q, a proof for relations using the method is impossible by construction. Thus, table 2 merely provides a lower bound for the number of indecomposable eMZVs for a given weight and length which turns out to be saturated for all cases considered in section 4, where corresponding upper bounds are derived.

3 Multiple zeta values and the ϕ-map

In this section we gather information on the structure of MZVs, which are to be compared with those found for eMZVs in section 4 below. While represented as nested sums in eq. (2.1) in section 2, they can alternatively be defined as iterated integrals

$$\begin{aligned} \zeta_{n_1,n_2,\ldots,n_r} &= \int\limits_{0\leq z_i\leq z_{i+1}\leq 1} \omega_1 \underbrace{\omega_0\ldots\omega_0}_{n_1-1} \omega_1 \underbrace{\omega_0\ldots\omega_0}_{n_2-1} \ldots \omega_1 \underbrace{\omega_0\ldots\omega_0}_{n_r-1} \\ &= \zeta(1\underbrace{0\ldots0}_{n_1-1}1\underbrace{0\ldots0}_{n_2-1}\ldots1\underbrace{0\ldots0}_{n_r-1}) \end{aligned} \tag{3.1}$$

over the differential forms $\omega_0 \equiv \frac{\mathrm{d}z}{z}$ and $\omega_1 \equiv \frac{\mathrm{d}z}{1-z}$ with all z_i on the real line. The MZV $\zeta_{n_1,\ldots,n_r}$ is said to have *weight* $w = \sum_{i=1}^r n_i$ and *depth* r. Written in terms of words W composed from the letters 0 and 1, which correspond to the differential forms ω_0 and ω_1 in eq. (3.1), respectively, ζ's satisfy the shuffle product:

$$\zeta(W_1)\,\zeta(W_2) = \zeta(W_1 ⧢ W_2)\,. \tag{3.2}$$

There is also a second product structure on MZVs, the stuffle product. Its simplest instance reads

$$\zeta_m\;\zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}\,. \tag{3.3}$$

It follows from either eq. (3.2) or eq. (3.3) that the $\mathbb{Q}$-span $\mathcal{Z}$ of all multiple zeta values is a subalgebra of $\mathbb{R}$. It is conjectured that $\mathcal{Z}$ is graded by the weight of the MZVs

$$\mathcal{Z} = \bigoplus_{w=0}^{\infty} \mathcal{Z}_w\,, \tag{3.4}$$

where the dimensions d_w of $\mathcal{Z}_w$ have been conjectured to be $d_w = d_{w-2} + d_{w-3}$ where $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$ [14]. A possible choice of basis elements for each weight w is given in table 3, for higher weights consult ref. [15].

Single ζ-functions of even weight are rather different from their odd-weight counterparts: all single zeta values of even weight $2n$ can be expressed as rational multiples of π^{2n}, which renders them transcendental numbers immediately. For odd single zeta values, however, there is no analogous property: there are no known relations relating two single zeta values of distinct odd weight, and in fact no such relations are expected. Also, none of the odd ζ-values has been proven to be transcendental so far: the only known facts are the irrationality of ζ_3 as well as the existence of an infinite number of odd irrational ζ's [34, 35].

w	2	3	4	5	6	7	8	9	10	11	12
$\mathcal{Z}_w$	ζ_2	ζ_3	ζ_2^2	ζ_5 $\zeta_2\,\zeta_3$	ζ_3^2 ζ_2^3	ζ_7 $\zeta_2\,\zeta_5$ $\zeta_2^2\,\zeta_3$	$\zeta_{3,5}$ $\zeta_3\,\zeta_5$ $\zeta_2\,\zeta_3^2$ ζ_2^4	ζ_9 ζ_3^3 $\zeta_2\,\zeta_7$ $\zeta_2^2\,\zeta_5$ $\zeta_2^3\,\zeta_3$	$\zeta_{3,7}$ $\zeta_3\,\zeta_7$ ζ_5^2 $\zeta_2\,\zeta_{3,5}$ $\zeta_2\,\zeta_3\,\zeta_5$ $\zeta_2^2\,\zeta_3^2$ ζ_2^5	$\zeta_{3,3,5}$ $\zeta_{3,5}\,\zeta_3$ ζ_{11} $\zeta_3^2\,\zeta_5$ $\zeta_2\,\zeta_3^3$ $\zeta_2\,\zeta_9$ $\zeta_2^2\,\zeta_7$ $\zeta_2^3\,\zeta_5$ $\zeta_2^4\,\zeta_3$	$\zeta_{1,1,4,6}$ $\zeta_{3,9}$ $\zeta_3\,\zeta_9$ $\zeta_5\,\zeta_7$ ζ_3^4 $\zeta_2\,\zeta_{3,7}$ $\zeta_2^2\,\zeta_{3,5}$ $\zeta_2\,\zeta_5^2$ $\zeta_2\,\zeta_3\,\zeta_7$ $\zeta_2^2\,\zeta_3\,\zeta_5$ $\zeta_2^3\,\zeta_3^2$ ζ_2^6
d_w	1	1	1	2	2	3	4	5	7	9	12

Table 3: A possible choice for the basis elements of $\mathcal{Z}_w$ for $2 \leq w \leq 12$.

3.1 Hopf algebra structure of MZVs

The basis elements in table 3 have been chosen by convenience preferring short and simple ζ's. Instead, it would be preferable to find a language in which the basis elements take a canonical form. This language does indeed exist: it is the graded Hopf algebra comodule $\mathcal{U}$, which is composed from words

$$f_{2i_1+1} \dots f_{2i_r+1}\, f_2^k, \quad \text{with} \quad r, k \geq 0 \quad \text{and} \quad i_1, \dots, i_r \geq 1 \tag{3.5}$$

of weight $w = 2(i_1 + \ldots + i_r) + r + 2k$. While the non-commutative letters f_{2i+1} constitute a commutative Hopf algebra, the Hopf algebra comodule is obtained upon adding powers of f_2, which commute with all f_{2i+1} [12]. Writing down all non-commutative words of the form in eq. (3.5), one indeed finds the dimension of $\mathcal{U}_w$ to match the dimension d_w of $\mathcal{Z}_w$.

How are the MZVs related to the non-commutative words in eq. (3.5)? The construction of a map from the $\mathbb{Q}$-algebra $\mathcal{Z}$ in eq. (3.4) to $\mathcal{U}$ is slightly involved and relies in particular on knowing, or rather excluding, algebraic relations between MZVs, which in turn is related to apparently very hard problems of transcendence theory.

In order to circumvent this issue, one lifts MZVs ζ to so-called motivic MZVs $\zeta^{\mathfrak{m}}$, which have a more elaborate definition [36,12,37], but which still satisfy the same shuffle and stuffle product formulæ as the MZVs eq. (3.2), eq. (3.3). Moreover, passing from MZVs to motivic MZVs has the advantage that many of the desirable, but currently unproven facts about MZVs are in fact proven for motivic MZVs. In particular, the commutative algebra $\mathcal{H}$ of motivic multiple zeta values is by definition graded for the weight, and carries a well-defined motivic coaction, first written down by Goncharov [36] and further studied by Brown [12,11,37].

With the availability of $\mathcal{H}$ the only remaining piece is the construction of an isomorphism ϕ of graded algebra comodules

$$\phi : \mathcal{H} \to \mathcal{U}\,. \tag{3.6}$$

The map ϕ, which assigns to each motivic MZV a linear combination of the non-commutative words defined in eq. (3.5), is thoroughly described and explored in ref. [11]. The map ϕ can, however, be determined from algebraic constraints only up to a normalization. This normalization can be fixed by demanding single ζ's to be directly translated into one-letter words

$$\phi(\zeta_w^{\mathfrak{m}}) = f_w\,, \qquad f_{2k} \equiv \frac{\zeta_{2k}}{(\zeta_2)^k}\, f_2^k\,. \tag{3.7}$$

As pointed out in ref. [11], the map ϕ preserves all relations between the motivic MZVs, for example (cf. eq. (3.3)):

$$\phi(\zeta^{\mathfrak{m}}_m \zeta^{\mathfrak{m}}_n) = \phi(\zeta^{\mathfrak{m}}_{m,n}) + \phi(\zeta^{\mathfrak{m}}_{n,m}) + \phi(\zeta^{\mathfrak{m}}_{m+n}) \,. \tag{3.8}$$

In addition one finds for example:

$$\begin{aligned} \phi(\zeta^{\mathfrak{m}}_{3,9}) &= -6\, f_5 f_7 - 15\, f_7 f_5 - 27\, f_9 f_3 \\ \phi(\zeta^{\mathfrak{m}}_{3,3,5}) &= -5\, f_5 f_3 f_3 + \frac{4}{7}\, f_5 f_2^3 - \frac{6}{5}\, f_7 f_2^2 - 45\, f_9 f_2 \,. \end{aligned} \tag{3.9}$$

See ref. [25] for higher-weight examples of the ϕ-map and its applications in the context of the low-energy expansion of superstring tree-level amplitudes.

4 Indecomposable eMZVs, Eisenstein series and the derivation algebra

As described in section 2, indecomposable eMZVs at a certain weight and length can be in principle inferred from considering reflection, shuffle and Fay relations. For higher weights and lengths, however, it is favorable to employ a computer implementation based on comparing q-expansions of eMZVs which in turn can be obtained recursively from eq. (2.51). In this section we are going to provide an algorithm which does not only deliver the appropriate indecomposable elements as listed in table 1 but as well explains their number at a given length and weight.

As described in the previous section, the appropriate mathematical idea for standard motivic MZVs is to map them to the non-commutative words composed from letters f_w in eq. (3.5) using the map ϕ. For the elliptic case we will construct a isomorphism ψ relating the ω-representation of eMZVs to non-commutative words composed from letters g_w, which in turn arise as labels of iterated Eisenstein integrals γ to be defined below.

4.1 Iterated Eisenstein integrals

Given that the q-expansion of eMZVs can be iteratively generated from the Eisenstein series G_k employing eq. (2.51), we will now describe eMZVs based on combinations of G_k. Instead of representing eMZVs as elliptic iterated integrals as in section 2, we will write them as iterated integrals over Eisenstein series G_k where the iterated integration is now performed over the modular parameter τ (or equivalently q).

Iterated integrals over Eisenstein series arise as a subclass of iterated integrals of modular forms, which have been studied in refs. [17,18]. In this section, we will briefly review some of the key definitions in order to embed the subsequent presentation of eMZVs into a broader context.

Iterated integrals of modular forms or *iterated Shimura integrals* [17, 18] are defined via

$$\int\limits_{i\infty>\tau_1>\tau_2>\cdots>\tau} \mathrm{d}\tau_1 \, (X_1 - \tau_1 Y_1)^{k_1-2} \mathcal{F}_{k_1}(\tau_1) \, \mathrm{d}\tau_2 \, (X_2 - \tau_2 Y_2)^{k_2-2} \mathcal{F}_{k_2}(\tau_2) \cdots \\ \cdots \mathrm{d}\tau_n \, (X_n - \tau_n Y_n)^{k_n-2} \mathcal{F}_{k_n}(\tau_n) \,, \tag{4.1}$$

where $\mathcal{F}_k(\tau)$ is a modular form of weight k and the modular group acts on commutative variables X_i and Y_i as to render eq. (4.1) modular invariant. The divergences in these integrals caused by the constant terms in the q-expansion of the modular forms can be regularized in a

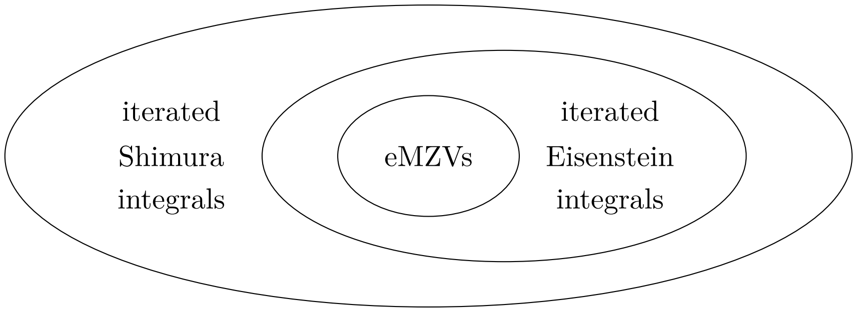

Figure 1: Relation between different type of iterated integrals discussed.

manner described in ref. [18]. The key idea of this regularization procedure is to separate the constant part from the remaining q-series for each $\mathcal{F}_{k_j}(q)$ and to associate a different integration prescription to it. The mathematical justification of this procedure is furnished by the theory of *tangential base points* [38]. In the present case, one regularizes the integral with respect to the tangential base point $\vec{1}_\infty$ [18].

In the context of eMZVs in eq. (2.11), we encounter special cases of the iterated Shimura integrals defined above, evaluated at $X_i = 1$ and $Y_i = 0$. Furthermore, the τ-derivative of eMZVs in eq. (2.51) involves no modular forms $\mathcal{F}_k$ other than Eisenstein series G_k. This motivates to study the following *iterated Eisenstein integrals* as building blocks for eMZVs,

$$\begin{aligned}\gamma(k_1, k_2, \ldots, k_n; q) &\equiv \frac{1}{4\pi^2} \int_{0 \le q' \le q} \mathrm{dlog}\, q' \;\; \gamma(k_1, \ldots, k_{n-1}; q')\, \mathrm{G}_{k_n}(q') \\ &= \frac{1}{(4\pi^2)^n} \int_{0 \le q_i < q_{i+1} \le q} \mathrm{dlog}\, q_1 \;\; \mathrm{G}_{k_1}(q_1) \;\; \mathrm{dlog}\, q_2 \;\; \mathrm{G}_{k_2}(q_2) \;\; \ldots \;\; \mathrm{dlog}\, q_n\, \mathrm{G}_{k_n}(q_n) \;, \end{aligned} \tag{4.2}$$

where the number n of integrations will be referred to as the *length* ℓ_γ, and the *weight* is given by $w_\gamma = \sum_{i=1}^n k_i$. The definition in eq. (4.2) as an iterated integral immediately implies

$$\frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \gamma(k_1, k_2, \ldots, k_n; q) = \frac{\mathrm{G}_{k_n}(q)}{4\pi^2}\, \gamma(k_1, k_2, \ldots, k_{n-1}; q) \tag{4.3}$$

$$\gamma(n_1, n_2, \ldots, n_r; q)\, \gamma(k_1, k_2, \ldots, k_s; q) = \gamma\big((n_1, n_2, \ldots, n_r) \sqcup\!\sqcup\, (k_1, k_2, \ldots, k_s); q\big) \;, \tag{4.4}$$

where the dependence on q will be suppressed in most cases: $\gamma(\ldots) \equiv \gamma(\ldots; q)$. The integrals in eq. (4.2) generally diverge due to the constant term in $\mathrm{G}_{k_1} = 2\,\zeta_{k_1} + \mathcal{O}(q)$ and can be regularized using the procedure discussed around eq. (4.7) while preserving eq. (4.3) and eq. (4.4).

As will be explained in detail below, eMZVs can be expressed in terms of particular linear combinations of iterated Eisenstein integrals in eq. (4.2) such that all possible divergences cancel. An alternative description of eMZVs which manifests the absence of divergences and admits convenient formulæ for their q-expansion will be given in subsection 4.6. The convergent linear combinations of eq. (4.2) occurring in eMZVs will turn out to be governed by a special derivation algebra $\mathfrak{u}$. The situation is summarized in figure 1: eMZVs are a special case of iterated Eisenstein integrals eq. (4.2) which in turn span a subspace of iterated Shimura integrals eq. (4.1).

Regularization. Even though eMZVs can be assembled from convergent iterated integrals over modular parameters – see subsection 4.6 – we shall sketch a regularization procedure for the iterated Eisenstein integrals in eq. (4.2) to render individual terms in the subsequent description of eMZVs well-defined. Let us consider the simplest case, namely that of an iterated integral of length one:

$$\gamma(k) = \frac{1}{4\pi^2} \int_0^q \mathrm{G}_k(q')\, \mathrm{dlog}\, q'. \tag{4.5}$$

The term $(\mathrm{G}_k(q') - 2\,\zeta_k)\ \mathrm{dlog}\, q'$ is straightforward to integrate from 0 to q, since it has no poles on the integration domain $0 \leq q' \leq q$. On the other hand, integration of the term $2\,\zeta_k\, \mathrm{dlog}\, q'$ in isolation requires regularization, due to the presence of a simple pole at $q' = 0$. The regularization scheme employed in this case, however, is entirely analogous to the regularization scheme for multiple polylogarithms, MZVs or eMZVs: One introduces a small parameter $\varepsilon > 0$, then expands the integral

$$2\,\zeta_k \int_\varepsilon^q \mathrm{dlog}\, q' = 2\,\zeta_k (\log q - \log \epsilon) \tag{4.6}$$

as a polynomial in $\log(\varepsilon)$ and finally takes the constant term in this expansion. Using this procedure in the length one case, one obtains from eq. (4.5)

$$\gamma(k) = \frac{1}{4\pi^2} \int_0^q \mathrm{G}_k(q')\, \mathrm{dlog}\, q' = \frac{1}{4\pi^2} \left(2\,\zeta_k \log q + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{m,n=1}^{\infty} \frac{m^{k-2}}{n} q^{mn} \right). \tag{4.7}$$

The regularization procedure for a general iterated Eisenstein integral $\gamma(k_1, k_2, \ldots, k_n)$ as in 4.2 is deduced from the length one case, using the shuffle product formula. Full details can be found in [18].

4.2 eMZVs as iterated Eisenstein integrals

As a first example on how to express eMZVs in terms of iterated Eisenstein integrals, let us consider eq. (2.51) for two simple types of eMZVs (recalling eq. (2.2) and $\mathrm{G}_0 \equiv -1$):

$$2\pi i \frac{\mathrm{d}}{\mathrm{d}\tau}\, \omega(0, n) = -4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q}\, \omega(0, n) = -2n\, \zeta_{n+1}\, \mathrm{G}_0 - n\, \mathrm{G}_{n+1}, \quad n \text{ odd} \tag{4.8a}$$

$$2\pi i \frac{\mathrm{d}}{\mathrm{d}\tau}\, \omega(0, 0, n) = -4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q}\, \omega(0, 0, n) = n\, \omega(0, n+1)\, \mathrm{G}_0\ , \qquad n \text{ even}\ . \tag{4.8b}$$

Integration over $\mathrm{dlog}\, q$ relates the eMZVs on the left hand side to iterated Eisenstein integrals of the form eq. (4.2), and the absence of constant terms within τ-derivatives guarantees that the integral converges. This insight will actually be the key ingredient to the simplified representation of eMZVs described in subsection 4.6 below. The rewriting in eqs. (4.8a) and (4.8b) can be generalized for all eMZVs: using the differential equation (2.51) one can represent their derivative as a sum over Eisenstein series G_{2k},

$$\frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \omega(n_1, n_2, \ldots, n_r) = \frac{1}{4\pi^2} \sum_{k=0}^{\infty} \xi_{2k}(n_1, n_2, \ldots, n_r)\, \mathrm{G}_{2k}\ , \tag{4.9}$$

where the coefficients $\xi_{2k}(n_1, \ldots, n_r)$ are linear combinations of eMZVs of weight $n_1 + \ldots + n_r + 1 - 2k$ and length $r - 1$. An example of this decomposition is spelled out below eq. (4.11).

For the eMZVs appearing in the coefficients $\xi_{2k}(n_1, \ldots, n_r)$ of eq. (4.9), the procedure can be repeated to successively reduce the length. Finally, any eMZV can be rewritten in terms of the *iterated Eisenstein integrals* in eq. (4.2). Since the τ-derivative on the right hand side of eq. (4.9) cannot have a constant term in q, its integral over $\mathrm{d}\log q$ is convergent and the first entries of the resulting iterated Eisenstein integrals for any eMZV are interlocked as $\gamma(k, \ldots) + 2\zeta_k\, \gamma(0, \ldots)$.

Examples. Let us return to the examples eq. (4.8a) and eq. (4.8b). The differential equation eq. (4.3) immediately implies

$$\omega(0,n) = n\big(\,\gamma(n+1) + 2\,\zeta_{n+1}\,\gamma(0)\big), \qquad n \text{ odd} \tag{4.10a}$$

$$\omega(0,0,n) = -\frac{1}{3}\,\zeta_n - n(n+1)\big(\,\gamma(n+2,0) + 2\,\zeta_{n+2}\,\gamma(0,0)\big), \quad n \text{ even}\,, \tag{4.10b}$$

where $-\frac{1}{3}\,\zeta_n$ arises as an integration constant w.r.t. $\log q$. Even though all the above iterated Eisenstein integrals $\gamma(n+1), \gamma(n+2,0), \gamma(0)$ and $\gamma(0,0)$ individually require regularization – see the discussion around eq. (4.7) – any divergence cancels in the linear combinations of schematic form $\gamma(k,\ldots) + 2\zeta_k\,\gamma(0,\ldots)$ in eqs. (4.10a) and (4.10b).

The conversion of eMZVs into γ's amounts to recursively applying the differential equation eq. (2.51) and casting it into the form eq. (4.9). At each step, an instance of G is separated until one has reached eMZVs of the form in eqs. (4.10a) and (4.10b) exclusively. After converting those into γ's, one reverts the direction and successively integrates using eq. (4.2), supplementing integration constants from eq. (2.47).

Let us demonstrate the conversion into iterated Eisenstein integrals γ for $\omega(0,3,5)$. Employing eq. (2.51), one finds

$$4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\, q}\,\omega(0,3,5) = -15\,\mathrm{G}_4\,\omega(0,5) + 42\,\omega(0,9) + 3\,\omega(4,5)\,, \tag{4.11}$$

i.e. we have $\xi_4(0,3,5) = -15\,\omega(0,5)$ and $\xi_0(0,3,5) = -42\,\omega(0,9) - 3\,\omega(4,5)$ in the notation of eq. (4.9). While $\omega(0,5)$ and $\omega(0,9)$ can be readily converted into γ's using eqs. (4.10a) and (4.10b), we will have to take another derivative[3] for $\omega(4,5)$:

$$\begin{aligned} 4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\, q}\,\omega(4,5) &= 9\,\mathrm{G}_{10}\,\omega(0) - 15\,\mathrm{G}_4\,\omega(6) + 42\,\omega(10) \\ &= 9\,\mathrm{G}_{10} + 30\,\zeta_6\,\mathrm{G}_4 + 84\,\zeta_{10}\,\mathrm{G}_0\ . \end{aligned} \tag{4.12}$$

Performing the integration eq. (4.2) then leads to

$$\omega(4,5) = 9\,\gamma(10) + 30\,\zeta_6\,\gamma(4) + 84\,\zeta_{10}\,\gamma(0)\,, \tag{4.13}$$

which – after plugged into eq. (4.11) – yields

$$4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\, q}\,\omega(0,3,5) = -75\,\mathrm{G}_4(\gamma(6) + 2\,\zeta_6\,\gamma(0)) + 405\,\gamma(10) + 90\,\zeta_6\,\gamma(4) + 1008\,\zeta_{10}\,\gamma(0)\,. \tag{4.14}$$

After a last integration of the type in eq. (4.2) one finally obtains

$$\omega(0,3,5) = -405\,\gamma(10,0) - 75\,\gamma(6,4) - \zeta_6(150\,\gamma(0,4) + 90\,\gamma(4,0)) - 1008\,\zeta_{10}\,\gamma(0,0)\,, \tag{4.15}$$

which casts the first indecomposable length-three eMZV beyond eq. (4.10b) into the language of iterated Eisenstein integrals and fits into the pattern $\gamma(k,\ldots) + 2\zeta_k\,\gamma(0,\ldots)$ for the first entries. Further examples of expressing eMZVs as iterated Eisenstein integrals are listed in appendix B.2.

Conversion of weight and length. Length and weight are different between the representation of eMZVs in terms of iterated Eisenstein integrals γ and the ω-representation. Denoting

[3] Alternatively, one could use eq. (2.37), but for illustrational purposes we will perform the recursion explicitly.

length and weight for γ and ω by $(\ell_\gamma,\, w_\gamma)$ and $(\ell_\omega,\, w_\omega)$, respectively, one finds straightforwardly

$$\ell_\gamma = \ell_\omega - 1 \qquad \text{and} \qquad w_\gamma = \ell_\omega - 1 + w_\omega = \ell_\gamma + w_\omega \tag{4.16}$$

such that

$$\gamma(k_1, k_2, \ldots, k_n) \leftrightarrow \text{eMZV in } \omega\text{-rep. with } \ell_\omega = n+1 \text{ and } w_\omega = -n + \sum_{j=1}^{n} k_j \tag{4.17a}$$

$$\omega(n_1, n_2, \ldots, n_r) \leftrightarrow \text{Eisenstein integral with } \ell_\gamma = r-1 \text{ and } w_\gamma = r - 1 + \sum_{j=1}^{r} n_j \ . \tag{4.17b}$$

Those formulæ, however, are valid for the *maximal component* only: as illustrated e.g. in eq. (4.10b), the presentation of eMZVs in terms of iterated Eisenstein integrals involves different lengths ℓ_γ and weights w_γ. Correspondingly, the maximal component is comprised from all terms in an eMZVs γ-representation, which are of length ℓ_γ and weight w_ω. Below, we will exclude γ's, which can be represented as shuffle products, from the maximal component. Iterated Eisenstein integrals of length $\ell_\gamma - 2, \ell_\gamma - 4, \ldots$ as well as any terms in which weight is carried by MZVs do not belong to the maximal component.

The examples in eq. (4.10b) and eq. (4.15) give rise to maximal components

$$\omega(0,0,n) = -n(n+1)\,\gamma(n+2,0) + \text{non-maximal terms} \tag{4.18}$$

$$\omega(0,3,5) = -405\,\gamma(10,0) - 75\,\gamma(6,4) + \text{non-maximal terms} \ , \tag{4.19}$$

which are defined up to shuffle products of lower-length iterated Eisenstein integrals.

Considering eq. (4.17a), one can create γ's corresponding to ω-representations of negative weight. Since weighting functions $f^{(m)}$ are not defined for negative weight, $\gamma(k_1, k_2, \ldots, k_n)$ with $\sum_{j=1}^{n} k_j < n$ are clearly incompatible with the definition of eMZVs in eq. (2.11). However, the connection with the derivation algebra $\mathfrak{u}$ in subsection 4.3 below will assign a meaning to those γ's in the context of relations between eMZVs at length $\ell_\omega \geq 6$.

Counting of indecomposable eMZVs. What are the advantages of translating eMZVs into iterated Eisenstein integrals? We would like to derive the set of indecomposable eMZVs with given length and weight from purely combinatorial considerations, similar to writing down all non-commutative words of letters f for standard MZVs (cf. eq. (3.5)). In particular, each indecomposable eMZV in table 1 should be related to a particular combination of shuffle-independent γ's. Correspondingly, the counting of indecomposable γ's of appropriate weight and length should be related to the numbers in table 2.

In order to assess the viability of iterated Eisenstein integrals γ for this purpose, it is worthwhile to recall the following observations:

(a) By construction, constant terms are absent in the differential eq. (2.51) for eMZVs. This interlocks the first entries of iterated Eisenstein integrals representing eMZVs in rigid combinations of $\gamma(k, \ldots) + 2\zeta_k\,\gamma(0, \ldots)$. Hence, it is sufficient for counting purposes to focus on $\gamma(k_1, k_2, \ldots, k_r)$ with $k_1 \neq 0$.

(b) The choice of indecomposable eMZVs in table 1 contains no further divergent representative besides $\omega(0,1) = \gamma(2) + 2\,\zeta_2\,\gamma(0)$. For any weight and length considered, divergences in eMZVs are captured by products with $\gamma(2)$ instead of shuffle-irreducible integrals of higher length such as $\gamma(2,4)$. We will assume the continuation of this pattern and confine the choice of labels for all other Eisenstein integrals γ at length $\ell_\gamma \geq 2$ to the set $\{0,4,6,\ldots\}$.

(c) The shuffle relations eq. (4.4) allow to reduce various linear combinations of iterated Eisenstein integrals to lower length, e.g.

$$\gamma(4,4) = \frac{1}{2}\,\gamma(4)^2 \quad \text{and} \quad \gamma(6)\,\gamma(4) = \gamma(4,6) + \gamma(6,4)\,, \tag{4.20}$$

and the bookkeeping of indecomposable eMZVs boils down to classifying shuffle-independent Eisenstein integrals γ. At length $\ell_\gamma = 2$ and weight $w_\gamma = 10$, possible indecomposable elements read $\gamma(10,0)$ and $\gamma(6,4)$, because $\gamma(4,6)$ can be obtained using shuffling of γ's of lower length. Similarly, $\ell_\gamma = 2$ and $w_\gamma = 12$ leaves no indecomposable eMZVs beyond $\gamma(12,0)$ and $\gamma(8,4)$.

Let us compare the survey of available Eisenstein integrals with the indecomposable eMZVs in table 1. Eisenstein integrals of length one immediately match with the maximal component of indecomposable eMZVs $\omega(0,2n+1)$ of length two using eq. (4.10a), so the first non-trivial tests occur at length $\ell_\omega = 3$, i.e. $\ell_\gamma = 2$.

Via eq. (4.10b) one finds indeed $\gamma(4,0), \gamma(6,0)$ and $\gamma(8,0)$ to represent the maximal component of $\omega(0,0,2), \omega(0,0,4)$ and $\omega(0,0,6)$, respectively. For $w_\gamma = 10$, which corresponds to $w_\omega = 8$, one can write down two distinct indecomposable elements: $\gamma(10,0)$ and $\gamma(6,4)$. This nicely ties in with the appearance of a second indecomposable eMZV at $\ell_\omega = 3$ and $w_\omega = 8$, see eqs. (4.18) and (4.19).

Similarly, the aforementioned indecomposable eMZVs $\gamma(12,0)$ and $\gamma(8,4)$ at weight $w_\gamma = 12$ are in concordance with the $w_\omega = 10$ entry of table 1,

$$\begin{aligned} \omega(0,0,10) &= -\frac{\zeta_{10}}{3} - 110\,\gamma(12,0) - 220\,\zeta_{12}\,\gamma(0,0) \\ \omega(0,3,7) &= -294\,\gamma(8,4) - 1848\,\gamma(12,0) + \text{non-maximal terms}\,. \end{aligned} \tag{4.21}$$

The appearance of the indecomposable eMZVs $\omega(0,3,5)$ and $\omega(0,3,7)$ beyond $\omega(0,0,2n)$ matches the existence of shuffle-independent Eisenstein integrals $\gamma(6,4)$ and $\gamma(8,4)$ in addition to $\gamma(10,0)$ and $\gamma(12,0)$.

Surprises from weight-twelve eMZVs and beyond. The literal application of the above reasoning to iterated Eisenstein integrals of weight $w_\gamma = 14$ suggests indecomposable eMZVs

$$\omega(0,0,12) \text{ from } \gamma(14,0), \quad \omega(0,3,9) \text{ from } \gamma(10,4) \quad \text{and} \quad \omega(0,5,7) \text{ from } \gamma(8,6)\,. \tag{4.22}$$

This, however, clashes with the findings noted in table 1: at $\ell_\omega = 3$ and $w_\omega = 12$ we find only *two* indecomposable eMZVs $\omega(0,0,12)$ and $\omega(0,3,9)$, whereas the above counting of appropriate interated Eisenstein integrals would suggest *three* indecomposable eMZVs. In particular, $\omega(0,5,7)$ can be expressed in terms of the two indecomposable eMZVs as written in eq. (2.39).

In order to explain the discrepancy between indecomposable eMZVs and shuffle-independent iterated Eisenstein integrals, let us inspect the first instance at $w_\gamma = 14, \ell_\gamma = 2$, which corresponds to $w_\omega = 12, \ell_\omega = 3$. The natural candidates for indecomposable eMZVs besides $\omega(0,0,12)$ have the following γ-representations,

$$\omega(0,3,9) = -\,315\,\gamma(8,6) - 729\,\gamma(10,4) - 5616\,\gamma(14,0) + \text{non-maximal terms} \tag{4.23}$$

$$\omega(0,5,7) = -\,490\,\gamma(8,6) - 1134\,\gamma(10,4) - 5642\,\gamma(14,0) + \text{non-maximal terms}\,,$$

and the relation eq. (2.39) for $\omega(0,5,7)$ leaves only $\omega(0,0,12)$ and $\omega(0,3,9)$ indecomposable. In general, there seem to be non-obvious restrictions to the Eisenstein integrals γ appearing in eMZVs, beyond the observations (a), (b) and (c). In table 4, we have noted the deviations from the expected pattern at lengths $\ell_\omega \leq 5$. Interestingly, the Eisenstein integrals $\gamma(8,6)$ and $\gamma(10,4)$

ℓ_ω \ w_ω	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		3_{2}		3		4_{3}		4_{3}	
4	1		1		2		3		4		6_{5}		8_{7}		10_{8}		13_{10}		16_{12}
5		1		2		4		6		10_{9}		14_{13}		21_{17}		28_{23}		39_{30}	

Table 4: Number of indecomposable eMZVs at length ℓ according to the counting of $\gamma(k_1, k_2, \ldots, k_n)$ suggested by the above observations (a), (b) and (c). The black numbers denote the number of shuffle-independent γ's with $k_i = 0, 4, 6, \ldots$ and $k_1 \neq 0$ while the red numbers indicate a deviating number of indecomposable eMZVs found from reflection-, shuffle- and Fay relations or the q-expansion.

enter eq. (4.23) and thus any other eMZV of the same weight and length in the combination

$$35\,\gamma(8,6) + 81\,\gamma(10,4) \tag{4.24}$$

exclusively. The above quantity is the first in a series of links to the derivation algebra $\mathfrak{u}$ introduced and discussed in the next subsection.

4.3 A relation to the derivation algebra $\mathfrak{u}$

The explanation of the deviating numbers for indecomposable eMZVs compared to shuffle-independent Eisenstein integrals in the last subsection can be provided starting from the following differential equation for the elliptic associator $A(q)$ defined in eq. (2.41) [1]:

$$\frac{\mathrm{d}}{\mathrm{dlog}\,q}(e^{i\pi[y,x]}A(q)) = \frac{1}{4\pi^2}\left(\sum_{n=0}^{\infty}(2n-1)\,\mathrm{G}_{2n}(q)\epsilon_{2n}\right)(e^{i\pi[y,x]}A(q))\,. \tag{4.25}$$

The Eisenstein series G_{2n} in eq. (4.25) are accompanied by derivations ϵ_{2n} which act on the non-commutative variables x and y in the expansion of $A(q)$ via

$$\epsilon_{2n}(x) = (\mathrm{ad}_x)^{2n}(y)\;, \qquad n \geq 0 \tag{4.26a}$$

$$\epsilon_{2n}(y) = [y, (\mathrm{ad}_x)^{2n-1}(y)] + \sum_{1\leq j<n}(-1)^j[(\mathrm{ad}_x)^j(y), (\mathrm{ad}_x)^{2n-1-j}(y)]\;, \quad n > 0 \tag{4.26b}$$

$$\epsilon_0(y) = 0\,. \tag{4.26c}$$

They generate a Lie subalgebra $\mathfrak{u}$ of the algebra of all derivations on the free Lie algebra generated by x, y [16, 20, 21]. The relations originating from eq. (4.26) have been studied extensively in ref. [22]. Beyond

$$[\epsilon_{2n}, \epsilon_2] = 0\;, \tag{4.27}$$

there are several non-obvious relations such as

$$
\begin{aligned}
0 &= [\epsilon_{10}, \epsilon_4] - 3\,[\epsilon_8, \epsilon_6]\,, && (4.28a)\\
0 &= 2\,[\epsilon_{14}, \epsilon_4] - 7\,[\epsilon_{12}, \epsilon_6] + 11\,[\epsilon_{10}, \epsilon_8]\,, && (4.28b)\\
0 &= 80\,[\epsilon_{12}, [\epsilon_4, \epsilon_0]] + 16\,[\epsilon_4, [\epsilon_{12}, \epsilon_0]] - 250\,[\epsilon_{10}, [\epsilon_6, \epsilon_0]]\\
&\quad - 125\,[\epsilon_6, [\epsilon_{10}, \epsilon_0]] + 280\,[\epsilon_8, [\epsilon_8, \epsilon_0]] - 462\,[\epsilon_4, [\epsilon_4, \epsilon_8]] - 1725\,[\epsilon_6, [\epsilon_6, \epsilon_4]]\,. && (4.28c)
\end{aligned}
$$

The rôle of ϵ_2 as a central element in eq. (4.27) is reminiscent of the above observation (b): any appearance of the non-modular G_2 can be captured by powers of $\gamma(2)$. Moreover, a peculiar linear combination of $\gamma(8,6)$ and $\gamma(10,4)$ has been observed in eq. (4.24) to appear in all eMZVs at $\ell_\omega = 3$ and $w_\omega = 12$. Upon identifying labels in γ with those of derivations ϵ_{2n} as suggested by eq. (4.25), one could attribute the selection rule on $\gamma(8,6)$ and $\gamma(10,4)$ to eq. (4.28a).

This connection will be made more precise in the subsequent. For this purpose, iterated Eisenstein integrals will be rewritten in a non-commutative language very similar to the one discussed for usual MZVs in section 3. In particular we are led to an eMZV analogue of the ϕ-map, which provided the key to a canonical representation of MZVs.

Eisenstein integrals as non-commutative words. As a first step to make the connection between eMZVs and the algebra of derivations manifest, let us translate iterated Eisenstein integrals into words composed from non-commutative generators $g_0, g_2, g_4, \ldots$,

$$\psi\big[\gamma(k_1, k_2, \ldots, k_n)\big] \equiv \frac{g_{k_n} g_{k_{n-1}} \cdots g_{k_2} g_{k_1}}{\prod_{j=1}^n (k_j - 1)}\ . \tag{4.29}$$

The normalization $g_k/(k-1)$ of the non-commutative alphabet is suggested by the combinations $(k-1)\,\mathrm{G}_k$ in eq. (4.25) and the factors of $n_i\,\mathrm{G}_{n_i+1}$ in eq. (4.10a).

The non-commutative letters g_k are naturally endowed with a shuffle product. The ψ-map defined by eq. (4.29) then satisfies

$$\psi\big[\gamma(n_1, n_2, \ldots, n_r)\gamma(k_1, k_2, \ldots, k_s)\big] = \psi\big[\gamma(n_1, n_2, \ldots, n_r)\big] ⧢ \psi\big[\gamma(k_1, k_2, \ldots, k_s)\big]\ . \tag{4.30}$$

The linear combination of $\gamma(8,6)$ and $\gamma(10,4)$ appearing in the eMZVs with $w_\omega = 12$ and $\ell_\omega = 3$ are mapped to

$$\psi\big[35\,\gamma(8,6) + 81\,\gamma(10,4)\big] = g_6 g_8 + 3 g_4 g_{10}\,. \tag{4.31}$$

Hence, the image of any $w_\omega = 12, \ell_\omega = 3$ eMZV under eq. (4.29) is annihilated by the differential operator

$$[\partial_{10}, \partial_4] - 3[\partial_8, \partial_6]\,. \tag{4.32}$$

once differentiation of a non-commutative word in g_i is defined to act on the leftmost letter

$$\partial_j g_{k_1} \ldots g_{k_n} = \delta_{j,k_1} g_{k_2} \ldots g_{k_n}\,. \tag{4.33}$$

This differentiation rule satisfies a Leibniz property w.r.t. the shuffle product eq. (4.30) and appeared already in the context of the representation of motivic MZVs in terms of non-commutative letters f_i [11], see the discussion in section 3. Note furthermore that the recursive construction of the eMZVs' ψ-image via eq. (4.9) with coefficients $\xi_{2k}(n_1, \ldots, n_r)$ determined by the differential equation (2.51) is similar to the recursive evaluation of the ϕ-map [11]: The coefficients ξ_{2k+1}

of $\phi(\zeta^{\mathfrak{m}}) = \sum_{3\leq 2k+1\leq w} f_{2k+1}\xi_{2k+1}$ for some motivic MZV of weight w are determined by the component of weight $(2k+1)\otimes(w-2k-1)$ in the coaction. Hence, the τ-derivative in the form eq. (4.9) plays a rôle similar to the coaction of motivic MZVs.

Non-commutative differentiation and the derivation algebra $\mathfrak{u}$. The similarity between eqs. (4.28a) and (4.32) suggests to identify derivations ϵ_{2m} with derivatives with respect to the non-commutative letters ∂_{2m}. Indeed, we will verify in three steps that the derivations ϵ_{2m} encode the action of ∂_{2m} on the ψ-image of the KZB associator eq. (2.41) and therefore on the ψ-image of any eMZVs:

(i) integrate the differential equation eq. (4.25) of the KZB associator,

$$e^{i\pi[y,x]}(A(q)-A(0)) = e^{i\pi[y,x]}\frac{1}{4\pi^2}\sum_{n=0}^{\infty}\int_0^q \mathrm{dlog}\, q' \;\; \mathrm{G}_{2n}(q')\epsilon_{2n}A(q') \; . \tag{4.34}$$

(ii) apply the ψ-map defined in eq. (4.29):

$$\psi[A(q)-A(0)] = \sum_{n=0}^{\infty}\epsilon_{2n}g_{2n}\psi[A(q)] \; , \tag{4.35}$$

using the fact that integration against $\frac{(2n-1)}{4\pi^2}\,\mathrm{G}_{2n}$ amounts to left-concatenation with g_{2n}

(iii) act with ∂_{2m} such that the sum over n collapses by eq. (4.33),

$$\partial_{2m}\psi[A(q)-A(0)] = \partial_{2m}\psi[A(q)] = \partial_{2m}\sum_{n=0}^{\infty}\epsilon_{2n}g_{2n}\psi[A(q)] = \epsilon_{2m}\psi[A(q)] \; . \tag{4.36}$$

where we used that the derivative ∂_{2m} annihilates the boundary term $A(0)$. This happens because $A(0)$ is a trivial iterated Eisenstein integral and thus translates into a word in the letters g of length 0.

This is the reason, why any relation among the derivations ϵ_i defines a differential operator via $\epsilon_i \to \partial_i$ which annihilates the ψ image of any eMZV. Explicitly:

$$\forall\; E\in\mathfrak{u} \text{ such that } E(x)=E(y)=0 \quad \Rightarrow \quad E\big|_{\epsilon_{2m}\to\partial_{2m}}\psi[\omega(n_1,\ldots,n_r)] = 0 \; . \tag{4.37}$$

Thus, any relation in $\mathfrak{u}$ obstructs the appearance of one linear combination of iterated Eisenstein integrals eq. (4.2) among eMZVs and reduces the counting of indecomposable representatives at lengths and weights governed by the conversion rules eq. (4.17a).

4.4 Systematics of relations in the derivation algebra

Naturally, we have been testing the relation between eMZVs and the derivation algebra $\mathfrak{u}$ established in the previous subsection by comparing q-expansions: using this method, we have identified indecomposable eMZVs up to weights $w_\omega = 30, 18, 8, 5, 3$ for $\ell_\omega = 3, 4, 5, 6, 7$. For all those situations, the relations noted in ref. [22] were capable of explaining the difference between the number of indecomposable eMZVs determined from the methods of section 2 and the naïve counting of shuffle-independent $\gamma(k_1, k_2, \ldots, k_n)$ subject to $k_1 \neq 0$ and $k_i \neq 2$, cf. the three observations around eq. (4.20).

While our computational power for identifying indecomposable eMZVs runs out at the limits noted above, we can still extract information about the derivation algebra $\mathfrak{u}$ from eMZVs of higher weight and length: instead of explicitly determining the indecomposable elements, we can count the number of shuffle-independent iterated Eisenstein integrals and find the number of relations originating from the derivation algebra.

Counting relations from the algebra of derivations $\mathfrak{u}$ for a given weight and depth works as follows: we start with an ansatz for a relation E of the form

$$0 \stackrel{!}{=} \sum_{\{n_1,n_2,\ldots,n_r\}} \alpha_{n_1,n_2,\ldots,n_r}[[\ldots[[\partial_{n_1},\partial_{n_2}],\partial_{n_3}],\ldots],\partial_{n_r}] \tag{4.38}$$

with rational fudge coefficients $\alpha_{n_1,n_2,\ldots,n_r}$ and $\{n_1,n_2,\ldots,n_r\}$ composed of $n_i = 0,4,6,\ldots$ of appropriate weight and length. The number r of partial derivatives in the nested commutators of eq. (4.38) (or the number of ϵ_n in the dual derivations respectively) is referred to as *depth*. Of course, the summation in eq. (4.38) is restricted to nested commutators which are independent under Jacobi identities.

Considering eq. (4.37), the above ansatz for E should annihilate all ψ-images of eMZVs of the length and weight considered. Using a sufficiently large set of eMZVs, one can easily fix all fudge coefficients in the ansatz and thus extract relations.

Using this method, we find perfect agreement of eq. (4.38) as an operator equation acting on eMZVs with the relations in the derivation algebra available in refs. [22, 39]. In the following paragraphs we will review their classification and extend the explicit results to higher commutator-depths.

Special rôle of ϵ_2. As already observed above, none of the indecomposable eMZVs besides $\omega(0,1)$ does contain an Eisenstein integral involving G_2. This reflects the rôle of ϵ_2 as a central element, as noted in eq. (4.27). Hence, it is sufficient to study commutator relations without ϵ_2.

Irreducible versus reducible relations. Any relation in the derivation algebra $\mathfrak{u} \ni E = 0$ of the form eq. (4.38) yields an infinity of higher-depth corollaries by repeated adjoint action of ϵ_n:

$$E = 0 \quad \Rightarrow \quad \mathrm{ad}_{n_1,n_2,\ldots,n_k}(E) \equiv [\epsilon_{n_1},[\epsilon_{n_2},[\ldots,[\epsilon_{n_k},E]\ldots]]] = 0\ . \tag{4.39}$$

Any instance of eq. (4.39) with $k > 0$ and E denoting a vanishing combination of ϵ_n-commutators is called a *reducible relation*, whereas relations that cannot be cast into the form $\mathrm{ad}_{n_1,n_2,\ldots,n_k}(E) = 0$ are referred to as *irreducible*. For instance, the simplest non-obvious relation eq. (4.28a) is irreducible and gives rise to reducible relations such as

$$[\epsilon_n,[\epsilon_{10},\epsilon_4] - 3[\epsilon_8,\epsilon_6]] = 0 \tag{4.40}$$

and generalizations to higher depth. They affect the bookkeeping of irreducible eMZVs starting from $w_\gamma = 14$ and $\ell_\gamma = 3$, which corresponds to $w_\omega = 11$ and $\ell_\omega = 4$.

A correspondence between cusp forms of weight w and irreducible relations at depth d and weight $w+2(d-1)$ has been discussed in ref. [22]. In the same way as the number of cusp forms at modular weight w is given by

$$\chi_w \equiv \begin{cases} \lfloor \frac{w}{12} \rfloor - 1 & : \ w = 2 \bmod 12 \\ \lfloor \frac{w}{12} \rfloor & : \ \text{other even values of } w \end{cases}\ , \tag{4.41}$$

we expect $\chi_{w-2(d-1)}$ irreducible relations at weight w and depth d relevant to eMZVs of non-negative weight w_ω (see eq. (4.16) for its relation to the weight of the iterated Eisenstein integral). In table 5, this conjectural counting is exemplified up to $w_\gamma = 30$ with a notation $r^d_{w_\gamma}$ for such irreducible relations. Relations of depth two can be cast into a closed formula [24]

w_γ \ ℓ_γ	2	3	4	5	6	7	8	9	10
12	0	0	0	0	0	0	0	0	0
14	r^2_{14}	0	0	0	0	0	0	0	0
16	0	r^3_{16}	0	0	0	0	0	0	0
18	r^2_{18}	0	r^4_{18}	0	0	0	0	0	0
20	r^2_{20}	r^3_{20}	0	r^5_{20}	0	0	0	0	0
22	r^2_{22}	r^3_{22}	r^4_{22}	0	r^6_{22}	0	0	0	0
24	r^2_{24}	r^3_{24}	r^4_{24}	r^5_{24}	0	r^7_{24}	0	0	0
26	$2 \times r^2_{26}$	r^3_{26}	r^4_{26}	r^5_{26}	r^6_{26}	0	r^8_{26}	0	0
28	r^2_{28}	$2 \times r^3_{28}$	r^4_{28}	r^5_{28}	r^6_{28}	r^7_{28}	0	r^9_{28}	0
30	$2 \times r^2_{30}$	r^3_{30}	$2 \times r^4_{30}$	r^5_{30}	r^6_{30}	r^7_{30}	r^8_{30}	0	r^{10}_{30}

Table 5: Irreducible relations r^ℓ_w. Up to weight 30 there are no more than two relations at a particular weight and length, which will, however, change proceeding to higher weight and length. An actual list of the first irreducible relations is available in appendix C.1.

$$0 = \sum_{i=1}^{2n+2p-1} \frac{[\epsilon_{2p+2n-i+1}, \epsilon_{i+1}]}{(2p+2n-i-1)!} \left\{ \frac{(2n-1)! B_{i-2p+1}}{(i-2p+1)!} + \frac{(2p-1)! B_{i-2n+1}}{(i-2n+1)!} \right\} , \tag{4.42}$$

where $p, n \geq 1$ denote arbitrary integers and B_n are Bernoulli numbers. Each term of eq. (4.42) carries weight $2(p+n+1)$, e.g. the weight-14 relation eq. (4.28a) follows from any partition of $p+n=6$, and the weight 18 relation eq. (4.28b) from any partition of $p+n=8$.

Irreducible relations at higher depth can be obtained in electronic form from the website `https://tools.aei.mpg.de/emzv`, whereas relations of depth three at $w = 16, 20$ and depth four at $w = 18, 22$ are provided in ref. [22]. New relations beyond those in said reference are conjectural and obtained from the differential operators eq. (4.38) annihilating all eMZVs. This approach to relations in the derivation algebra appears computationally more efficient to us than evaluating the action of elements of the derivation algebra on generators x and y of the free Lie algebra via eq. (4.26).

Vanishing nested commutators. Starting from $w_\gamma = 8$ and $\ell_\gamma = 5$, we find that the ψ-image of any eMZV with appropriate weight and length is annihilated by operators of the form

$$[[[[\partial_4, \partial_0], \partial_0], \partial_0], \partial_{2m}] \, . \tag{4.43}$$

The reason becomes clear by considering $\gamma(4,0,0,0)$, one of the corresponding Eisenstein integrals. By eq. (4.16), related eMZVs are bound to have $\ell_\omega = 5$ and $w_\omega = 0$, but the only known eMZV with these properties is $\omega(0,0,0,0,0) = 1/120$ which cannot equal the non-constant

$\gamma(4,0,0,0)$. Hence, the latter does not occur among eMZVs and signals the irreducible relation

$$[[[\epsilon_4,\epsilon_0],\epsilon_0],\epsilon_0] = 0 \ , \tag{4.44}$$

which in turn implies that $[[[\partial_4,\partial_0],\partial_0],\partial_0]$ annihilates the elliptic associator by eq. (4.36). The relation eq. (4.44) can be understood from the organization of $\mathfrak{u}$ in terms of representations of SL_2: considering ϵ_{2m} as the lowest-weight state in a $(2m-1)$-dimensional module, the highest-weight vector $\mathrm{ad}_0^{2m-2}\epsilon_{2m}$ is annihilated by further adjoint action of ϵ_0.

Further irreducible relations of this type include

$$\mathrm{ad}_0^{p-1}\epsilon_p = \underbrace{[\epsilon_0,\ldots[\epsilon_0,[\epsilon_0,}_{p-1 \text{ times}}\epsilon_p]]\ldots] = 0 \ , \qquad p=4,6,8,\ldots \ , \tag{4.45}$$

corresponding to the Eisenstein integral $\gamma(p,0^{p-1})$ with would-be eMZV partners of vanishing w_ω. Different partitions of the weight in eq. (4.45) lead to further relations such as

$$[[[[[[[\epsilon_4,\epsilon_0],\epsilon_4],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0] = 0 \ , \qquad [[[[[[[[[\epsilon_4,\epsilon_0],\epsilon_6],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0],\epsilon_0] = 0 \ . \tag{4.46}$$

Since all their permutations via $\epsilon_4 \leftrightarrow \epsilon_0$ or $\epsilon_6 \leftrightarrow \epsilon_0$ can be identified as a reducible relation descending from eq. (4.45), we expect no further irreducible relations at $d = w_\gamma = 8$ or 10 besides eq. (4.46).

Additional generators of the Lie algebra Consider the free Lie algebra $\mathfrak{k} = \mathbb{L}(z_3,z_5,z_7,\ldots)$ generated by one element in every odd degree strictly greater than one. As mentioned on page 6 of ref. [22], every generator z_j of $\mathfrak{k}$ defines a derivation $\widetilde{z}_j$ of depth j and weight $2j$ of the free Lie algebra on two generators x,y, and satisfies $[\widetilde{z}_{2k+1},\epsilon_{2m}] \in \mathfrak{u}$ for every $\epsilon_{2m}\in\mathfrak{u}$. More precisely, the elements ϵ_0,ϵ_2 are annihilated by the elements $\widetilde{z}_j$

$$0 = [\widetilde{z}_{2k+1},\epsilon_0] = [\widetilde{z}_{2k+1},\epsilon_2] \ , \qquad k=1,2,3,\ldots \ , \tag{4.47}$$

and their commutators with $\epsilon_4,\epsilon_6,\ldots$ can be constructed using the techniques of [22], e.g.

$$[\widetilde{z}_3,\epsilon_4] = -\frac{1}{14}[[\epsilon_4,\epsilon_0],[\epsilon_6,\epsilon_0]] + \frac{1}{42}[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_6]]] - \frac{1}{7}[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_4]]] \, . \tag{4.48}$$

They give rise to further reducible relations, starting from length five at weights $20,24,26,\ldots$ by the commutator of $\widetilde{z}_3$ with the depth-two relations in eq. (4.28) or eq. (4.42).

4.5 Counting relations between nested commutators

Example. In order to demonstrate the virtue of the derivation algebra as a counting formalism for indecomposable eMZVs, let us consider $w_\gamma = 20, \ell_\gamma = 5$ as a specific example, which corresponds to $w_\omega = 15, \ell_\omega = 6$. This is the first situation, where all four types of relations described in the previous section have to be taken into account in order to arrive at what we believe is the correct counting of eMZVs.

The naïve enumeration of shuffle-independent γ's with $k_1 \neq 0$ and $k_i \neq 2$ leads to 55 distinct elements. Each relation of depth 5 and weight 20 in the derivation algebra will lower this number according to eq. (4.37).

Let us first consider reducible relations. Starting from table 5, one can construct the following reducible relations by adjoint action of ϵ_n (recalling the notation $r^d_{w_\gamma}$ for irreducible relations of depth d and weight w_γ as well as $\mathrm{ad}_{n_1,n_2,\ldots,n_k} r_i^j \equiv [\epsilon_{n_1}, [\epsilon_{n_2}, [\ldots, [\epsilon_{n_k}, r^d_{w_\gamma}] \ldots]]])$:

$$\begin{array}{rcl} \mathrm{ad}_{6,0,0}\, r^2_{14} & \leftrightarrow & 3 \text{ permutations }, \\ \mathrm{ad}_{4,0}\, r^3_{16} & \leftrightarrow & 2 \text{ permutations }, \end{array} \qquad \begin{array}{rcl} \mathrm{ad}_{0,0,0}\, r^2_{20} & \leftrightarrow & 1 \text{ permutation} \\ \mathrm{ad}_{0,0}\, r^3_{20} & \leftrightarrow & 1 \text{ permutation} \end{array} . \tag{4.49}$$

In addition, there is one relation each descending from the vanishing nested commutator eq. (4.44) and the additional Lie algebra generator $\widetilde{z}_3$,

$$[[[[\epsilon_4, \epsilon_0], \epsilon_0], \epsilon_0], \epsilon_{12}] = 0 \quad \text{and} \quad [\widetilde{z}_3, r^2_{14}] \,, \tag{4.50}$$

which makes a total of 9 reducible relations.

Indeed, starting with an ansatz of the form eq. (4.38), we find ten distinct relations: while eqs. (4.49) and (4.50) are confirmed, our method explicitly delivers the new irreducible relation r^5_{20} expected from table 5. To our knowledge this is the first appearance of an explicit relation at depth 5 in $\mathfrak{u}$, which is written out in appendix C.5. Correspondingly, we find the number of indecomposable eMZVs at $(\ell_\gamma, w_\gamma) = (5, 20)$ (or $(\ell_\omega, w_\omega) = (6, 15)$) to be 45.

General. In order to repeat the counting procedure from the above example for a variety of weights and lengths, the following tables give an overview of the required ingredients: The numbers of shuffle-independent iterated Eisenstein integrals compatible with observations (a) and (b) in subsection 4.2 are gathered in table 6 and have to be compared with the counting of relations in $\mathfrak{u}$ seen in table 7. Once the offset between (w_γ, ℓ_γ) and (w_ω, ℓ_ω) in eq. (4.17a) is

ℓ_γ \ w_γ	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	1	1	1	2	2	3	3	4	4	5	5	6	6	7
3	0	0	1	1	2	3	4	6	8	10	13	16	19	23	27	31
4	0	0	1	1	2	4	6	10	14	21	28	39	50	66	82	104
5	0	0	1	1	3	5	9	15	24	37	55	80	113	156	211	280
6	0	0	1	1	3	6	11	21	35	59	93	146	217	322	459	649
7	0	0	1	1	4	7	15	28	51	89	150	245	389	602	910	1347

Table 6: Shuffle-independent $\gamma(k_1, \ldots, k_n)$ subject to $k_1 \neq 0$ and $k_i \neq 2$ at various weights w_γ and lengths ℓ_γ.

d \ w_γ	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
2	0	0	0	0	0	0	1	0	1	1	1	1	2	1	2	2	2	2	3	2
3	0	0	0	0	0	0	1	1	2	3	4	5	7	8	10	12	14	16	19	21
4	0	1	0	0	0	0	1	1	4	5	9	13	19	?	?	?	?	?	?	?
5	0	1	0	1	1	1	2	2	6	10	?	?	?	?	?	?	?	?	?	?
6	0	1	1	2	2	3	5	6	11	?	?	?	?	?	?	?	?	?	?	?

Table 7: Relations in the derivation algebra at various weights w_γ and depths d, excluding the central element ϵ_2.

taken into account, one arrives at the numbers of indecomposable eMZVs in the ω-representation noted in table 8.

ℓ_ω \ w_ω	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	1		1		1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		2		3		3		3		4		4	
4	1		1		2		3		4		5		7		8		10		12		14		16
5		1		2		4		6		9		13		17		23		30		37		47	
6	1		2		4		8		13		22		31		45		?		?		?		?
7		1		4		8		16		29		48		?		?		?		?		?	

Table 8: Numbers $N(\ell_\omega, w_\omega)$ of indecomposable eMZVs in their ω-representation. This is an extended version of table 2, where the black results are obtained by explicitly determining q-expansions while results printed in blue originate from testing relations between nested commutators as described around eq. (4.38).

From the above data, one readily arrives at all-weight statements on the number of indecomposable eMZVs of length $\ell_\omega \leq 4$:

- At length $\ell_\omega = 2$, there is obviously one indecomposable eMZV at each odd weight w_ω.
- At length $\ell_\omega = 3$, the number of indecomposable eMZVs at even weight w_ω is $\lceil \frac{1}{6} w_\omega \rceil$. This follows from comparing the number $\lceil \frac{w_\omega}{4} \rceil - 1$ of admissible $\gamma(k_1, k_2)$ $(k_1 > k_2,\ k_i \neq 2)$ at weight $w_\omega > 4$ with the counting of depth-two relations in $\mathfrak{u}$ governed by eq. (4.41).
- At length $\ell_\omega = 4$, the number of indecomposable eMZVs at odd weight w_ω is conjectured to be $\lfloor \frac{1}{2} + \frac{1}{48}(w_\omega + 5)^2 \rfloor$. This conjecture stems from extrapolating [40] the data available at $w_\omega \leq 37$. The extrapolation will remain valid, if the counting of irreducible r_w^3 keeps on following the cusp forms.

Starting from the next length, $\ell_\omega = 5$ or $\ell_\gamma = 4$, an effect well-known from the algebra of MZVs kicks in: because the lowest non-trivial relation from the derivation algebra $\mathfrak{u}$ exists at weight 14 depth 2, there is the possibility to obtain the "relation of a relation" $\mathrm{ad}_{r_{14}^2}(r_{14}^2) = 0$ at weight 28, depth 4. This effect, which appears in iterated form for higher depth, as well as the action of the generators of the free Lie algebra $\mathfrak{k}$ described in subsection 4.4 render the counting at higher depth difficult. Correspondingly, a closed formula, e.g. a generating series for the number of indecomposable eMZVs at given length and weight is still lacking and some of the entries in table 7 are left undetermined.

4.6 A simpler representation of the eMZV subspace

From the discussion in the previous subsections it became clear that eMZVs can be nicely represented in terms of iterated Eisenstein integrals eq. (4.2). While those integrals have to be regularized individually as pointed out in the context of eq. (4.7), the representation of eMZVs cannot not involve any divergences upon integrating their τ-derivative eq. (2.51). In this section we would like to manifest this property and define a modified version of iterated Eisenstein integrals γ_0, which are individually convergent by construction. By using the γ_0-language, one will trade some of the connections to periods and motives [18] inherent in the γ-language for compactness of representation. A further advantage of the γ_0-language to be introduced is a better accessibility of the q-expansions of eMZVs.

Modified iterated Eisenstein integrals. Already in subsection 4.2 it was remarked that the τ derivative of eMZVs determined by the differential equation (2.51) cannot contain any constant terms. Therefore, it is an obvious idea to subtract the constants from the non-trivial Eisenstein series before defining their iterated integrals:

$$\begin{aligned} \mathrm{G}^0_0 &\equiv -1 \\ \mathrm{G}^0_k &\equiv \mathrm{G}_k - 2\,\zeta_k = \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} \,, \qquad k \text{ even, } k \neq 0\,. \end{aligned} \tag{4.51}$$

Using this definition, one can rewrite eqs. (4.8a) and (4.8b) as

$$\frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(0,n) = \frac{n}{4\pi^2}\,\mathrm{G}^0_{n+1}, \qquad n \text{ odd} \tag{4.52a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(0,0,n) = \frac{n}{4\pi^2}\,\omega(0,n+1)\,\mathrm{G}^0_0, \qquad n \text{ even}\,, \tag{4.52b}$$

and the differential equation (2.51) for generic eMZVs can be easily cast into the form

$$\frac{\mathrm{d}}{\mathrm{d}\log q}\,\omega(n_1,n_2,\ldots,n_r) = \frac{1}{4\pi^2}\sum_{k=0}^{\infty} \rho_{2k}(n_1,n_2,\ldots,n_r)\,\mathrm{G}^0_{2k}\,. \tag{4.53}$$

In complete analogy to eq. (4.9), the coefficients $\rho_{2k}(n_1,\ldots,n_r)$ are linear combinations of eMZVs with weight $n_1+\ldots+n_r+1-2k$ and length $r-1$, the only difference being that Eisenstein series in eq. (2.51) are now expanded via $\mathrm{G}_k = \mathrm{G}^0_k - 2\,\zeta_k\,\mathrm{G}^0_0$ whenever $k \neq 0$.

From the form eq. (4.53) of the differential eq. (2.51), it is straightforward to introduce modified iterated Eisenstein integrals γ_0 via

$$\begin{aligned} \gamma_0(k_1,k_2,\ldots,k_n;q) &\equiv \frac{1}{4\pi^2}\int_{0\leq q'\leq q} \mathrm{d}\log q'\ \gamma_0(k_1,\ldots,k_{n-1};q')\,\mathrm{G}^0_{k_n}(q')\,, \qquad k_1 \neq 0 \\ &= \frac{1}{(4\pi^2)^n}\int_{0\leq q_i<q_{i+1}\leq q} \mathrm{d}\log q_1\ \mathrm{G}^0_{k_1}(q_1)\ \mathrm{d}\log q_2\ \mathrm{G}^0_{k_2}(q_2)\ \ldots\ \mathrm{d}\log q_n\,\mathrm{G}^0_{k_n}(q_n)\,, \end{aligned} \tag{4.54}$$

whose definition as an iterated integral implies

$$\frac{\mathrm{d}}{\mathrm{d}\log q}\,\gamma_0(k_1,k_2,\ldots,k_n;q) = \frac{\mathrm{G}^0_{k_n}(q)}{4\pi^2}\,\gamma_0(k_1,k_2,\ldots,k_{n-1};q) \tag{4.55}$$

$$\gamma_0(n_1,n_2,\ldots,n_r;q)\,\gamma_0(k_1,k_2,\ldots,k_s;q) = \gamma_0\big((n_1,n_2,\ldots,n_r)\sqcup\!\sqcup\,(k_1,k_2,\ldots,k_s);q\big)\,. \tag{4.56}$$

The notion of weight and length are not altered w.r.t. the definition for γ. There are several advantages of employing this modified class of iterated Eisenstein integrals γ_0 for the description of eMZVs:

- Logarithmic divergences for $q \to 0$ as present in eq. (4.2) do not occur after setting $k_1 \neq 0$. Modified iterated Eisenstein integrals γ_0 are perfectly well-defined objects which do not require regularization.
- The number of terms necessary to express eMZVs as combinations of iterated Eisenstein integrals γ_0 is significantly lower than for γ.
- The absence of constant terms in the expansion of $\mathrm{G}^0_{k_1}$ propagates to any convergent

iterated Eisenstein integral,

$$\gamma_0(k_1, k_2, \ldots, k_n; 0) = 0 \ . \tag{4.57}$$

Note that we will again suppress the dependence on q in most cases: $\gamma_0(\ldots) \equiv \gamma_0(\ldots; q)$.

Examples. Let us return to the examples eq. (4.52a) and eq. (4.52b). The differential eq. (4.55) immediately implies

$$\omega(0,n) = n\,\gamma_0(n+1), \qquad n \text{ odd} \tag{4.58a}$$

$$\omega(0,0,n) = -\frac{1}{3}\,\zeta_n - n(n+1)\,\gamma_0(n+2,0), \quad n \text{ even}\,, \tag{4.58b}$$

where $-\frac{1}{3}\,\zeta_n$ arises as an integration constant w.r.t. $\log q$. Indeed, these expressions are convergent by definition and significantly shorter than their counterparts in eqs. (4.10a) and (4.10b).

For illustrational purposes let us also revisit the example $\omega(0,3,5)$. Its derivative

$$4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \omega(0,3,5) = 30\,\zeta_6\,\omega(0,3) - 15(\mathrm{G}_4^0 - 2\,\zeta_4\,\mathrm{G}_0^0)\,\omega(0,5) + 45\,\omega(0,9) \tag{4.59}$$

amounts to $\rho_4(0,3,5) = -15\,\omega(0,5)$ and $\rho_0(0,3,5) = 30\,\zeta_4\,\omega(0,5) - 45\,\omega(0,9) - 30\,\zeta_6\,\omega(0,3)$ in the notation of eq. (4.9) and can be translated to modified Eisenstein integrals via eq. (4.58a):

$$4\pi^2 \frac{\mathrm{d}}{\mathrm{dlog}\, q}\, \omega(0,3,5) = 90\,\zeta_6\,\gamma_0(4) - 75\,(\mathrm{G}_4^0 - 2\,\zeta_4\,\mathrm{G}_0^0)\,\gamma_0(6) + 405\,\gamma_0(10) \tag{4.60}$$

Integration using eq. (4.54) yields the following alternative representation to eq. (4.15),

$$\omega(0,3,5) = -90\,\zeta_6\,\gamma_0(4,0) + 150\,\zeta_4\,\gamma_0(6,0) - 75\,\gamma_0(6,4) - 405\,\gamma_0(10,0)\,. \tag{4.61}$$

Further examples of eMZVs expressed in the language of modified iterated Eisenstein integrals can be found in appendix B.

***q*-expansion.** In contrast to the γ-language used in the last section, there is no caveat on regularization when performing the integrals over q_j in the definition eq. (4.54) of γ_0. The q-expansion stems from the expression for Eisenstein series in eq. (4.51) and can be cast into a closed form (with 0^n denoting a sequence of n entries $0,0,\ldots,0$):

$$\begin{aligned}\gamma_0(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_r, 0^{p_r-1}) &= \prod_{j=1}^{r} \Big(-\frac{2(2\pi i)^{k_j - 2p_j}}{(k_j-1)!} \Big) \\ &\times \sum_{m_i,n_i=1}^{\infty} \frac{m_1^{k_1-1} m_2^{k_2-1} \ldots m_r^{k_r-1} q^{m_1 n_1 + m_2 n_2 + \ldots + m_r n_r}}{(m_1 n_1)^{p_1} (m_1 n_1 + m_2 n_2)^{p_2} \ldots (m_1 n_1 + m_2 n_2 + \ldots + m_r n_r)^{p_r}}\,.\end{aligned} \tag{4.62}$$

An even more compact representation can be achieved using the divisor sum

$$\sigma_k(n) \equiv \sum_{d|n} d^k\,, \tag{4.63}$$

which allows to rewrite eq. (4.62) as

$$\gamma_0(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_r, 0^{p_r-1}) = \prod_{j=1}^{r} \Big(-\frac{2(2\pi i)^{k_j - 2p_j}}{(k_j-1)!} \Big) \tag{4.64}$$

$$\times \sum_{0<n_1<n_2<\cdots<n_r} \frac{\sigma_{k_1-1}(n_1)\sigma_{k_2-1}(n_2-n_1)\ldots\sigma_{k_r-1}(n_r-n_{r-1})q^{n_r}}{n_1^{p_1}n_2^{p_2}\ldots n_r^{p_r}} .$$

The above expression bears some resemblance to the sum representation eq. (2.1) of MZVs. One could wonder if rearrangements of the sums could yield a genus-one analogue of stuffle relations. However, both the appearance of the divisor sums and the q-dependence prevent such manipulations. In fact, we did not observe a single relation among iterated Eisenstein integrals γ_0 beyond the shuffle relations eq. (4.4) up to weights $44, 31, 22, 19$ for length $2, 3, 4, 5$ respectively.

Given the above γ_0-representation of the simplest eMZVs, we arrive at two closed forms for q-expansions

$$\omega(0,k) = \delta_{k,1}\frac{i\pi}{2} + \frac{2(-1)^{(k+1)/2}(2\pi)^{k-1}}{(k-1)!}\sum_{m,n=1}^{\infty}\frac{m^{k-1}}{n}q^{mn} , \qquad k \text{ odd} \tag{4.65}$$

$$\omega(0,0,k) = -\frac{1}{3}\zeta_k + \frac{2(-1)^{(k+2)/2}(2\pi)^{k-2}}{(k-1)!}\sum_{m,n=1}^{\infty}\frac{m^{k-1}}{n^2}q^{mn} , \qquad k \text{ even} , \tag{4.66}$$

while further expressions for interesting $\omega(0,0,\ldots,0,k)$ at higher length are given in appendix B.1.

Connection with the derivation algebra. A manifestly convergent description of eMZVs in terms of modified iterated Eisenstein integrals γ_0 comes with a price at the end of the day: the constant terms which have been omitted in the definition (4.54) have to be restored in order to establish a connection with the derivation algebra. In particular, the translation of modified iterated Eisenstein integrals into the language of non-commutative words built from letters g_{2k} described in subsection 4.3 involves various shifts $\sim \zeta_{k_n}\, g_0$,

$$\begin{aligned}\psi[\gamma_0(k_1,0^{p_1},k_2,0^{p_2},\ldots,k_n,0^{p_n})] &= (-1)^{p_1+p_2+\ldots+p_n} \\ \times\ (g_0)^{p_n}\Big(\frac{g_{k_n}}{k_n-1} - 2\,\zeta_{k_n}\,g_0\Big)\cdots(g_0)^{p_2}&\Big(\frac{g_{k_2}}{k_2-1} - 2\,\zeta_{k_2}\,g_0\Big)(g_0)^{p_1}\Big(\frac{g_{k_1}}{k_1-1} - 2\,\zeta_{k_1}\,g_0\Big)\end{aligned} \tag{4.67}$$

where $(g_0)^n$ refers to n adjacent letters g_0. Furthermore, the concatenation of words is understood to act linearly, e.g. $g_2(\zeta_4\, g_0 + g_4)g_8 = \zeta_4\, g_2g_0g_8 + g_2g_4g_8$. Nevertheless, the counting of indecomposable eMZVs remains unmodified when projecting to the maximal component of their γ-representation, see the discussion below eq. (4.17a).

5 Conclusions

In this work we have been studying the systematics of relations between eMZVs. Our results support the conjecture that the entity of relations can be traced back to reflection, shuffle and Fay identities.

The numbers $N(\ell_\omega, w_\omega)$ of indecomposable eMZVs at any weight and length can be explained once their connection to a special derivation algebra is taken into account: Any eMZV can be expressed in terms of iterated integrals over Eisenstein series whose appearance in turn is governed by the derivation algebra.

Our results for the numbers $N(\ell_\omega, w_\omega)$ of indecomposable eMZVs for various weights w_ω and lengths ℓ_ω are listed in table 8. In addition, there are all-weight formulæ available for $\ell_\omega \leq 4$

and odd values of $w_\omega + \ell_\omega$,

$$N(2, w_\omega) = 1 \ , \qquad N(3, w_\omega) = \left\lceil \frac{1}{6} w_\omega \right\rceil \ , \qquad N(4, w_\omega) = \left\lfloor \frac{1}{2} + \frac{1}{48}(w_\omega + 5)^2 \right\rfloor \ , \tag{5.1}$$

where the expression for $N(4, w_\omega)$ is conjectural. Because of the diversity of constraints originating from the derivation algebra as described in section 4, a closed formula for all weights and lengths is challenging to find and not yet available.

Explicit q-expansions for eMZVs are accessible using a slightly modified version of iterated Eisenstein integrals described in subsection 4.6. The resulting closed expression can be found in eq. (4.64).

The improved understanding of eMZVs raises a variety of follow-up questions, starting with a connection of the underlying elliptic iterated integrals with recent results on Feynman integrals [6–9]. In particular, the techniques which led to the q-expansions of eMZVs furnish a convenient starting point to connect with the functions ELi introduced in ref. [7] and generalized in ref. [9].

The appearance of eMZVs in one-loop scattering amplitudes of the open superstring [5] suggested a systematic study of indecomposable eMZVs. The results of the current article should pave the way towards a compact form of string corrections at higher orders in α' and might even lead to a glimpse of an all-order pattern. The existence of such a description is not unlikely: for open-string tree-level amplitudes a recursive formula based on the Drinfeld associator. It was found by extending an initial observation in ref. [41] into a recursive computation of the complete α'-expansions in ref. [42]. Similarly, the α'-expansion at one-loop might be accessible by using the elliptic associators discussed in ref. [16].

The α'-expansion of the closed-string four-point amplitude at genus one has been investigated in refs. [43–45], see [46, 47] for generalizations to five external states. The functions appearing in those amplitudes include non-holomorphic Eisenstein series and a variety of their generalizations which have been analyzed in ref. [45]. It would be interesting to establish a connection between these non-holomorphic functions and modular-invariant combinations of eMZVs and their counterpart originating from the other homology cycle.

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Appendix

A eMZV relations

A.1 Decomposition of boring eMZVs

By eq. (2.17), all the above examples of shuffle-reductions of boring eMZVs can be identified as special cases of the following general identity

$$\omega(B)\Big|_{\text{boring}} = \sum_{k=1}^{\infty} C_{2k} \sum_{\substack{B=A_1A_2\ldots A_{2k} \\ \omega(A_i)\ \text{interesting}}} \omega(A_1)\,\omega(A_2)\ldots\omega(A_{2k})\ , \tag{A.1}$$

which remains conjectural beyond length six and the rational coefficients $C_2=\frac{1}{2},\ C_4=-\frac{1}{8},\ldots$ are defined by the generating series

$$\sqrt{1+x} = 1+\sum_{k=1}^{\infty} C_{2k}x^k = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+\ldots\ . \tag{A.2}$$

The arguments $B\equiv\{n_1,n_2,\ldots,n_r\}$ of the boring eMZV on the left-hand side are deconcatenated[4] into smaller tuples $A_j=\{a_1^j,a_2^j,\ldots,a_{m_j}^j\}$ such that all eMZVs $\omega(A_j)$ are interesting. Only even numbers of interesting $\omega(A_j)$ are compatible with the boring nature of $\omega(B)$, and the concatenation A_jA_{j+1} in eq. (A.1) is defined to yield $\{a_1^j,\ldots,a_{m_j}^j,a_1^{j+1},\ldots,a_{m_{j+1}}^j\}$.

Note that the first appearance of $C_4=-\frac{1}{8}$ can be seen from the second case $\omega(n_1,n_2,n_3,n_4)$ in eq. (2.20). The vanishing of eMZVs with all entries odd (cf. eq. (2.21)) follows from the absence of deconcatenations into tuples A_j with $\omega(A_j)$ interesting.

A.2 More general Fay identities

The relation eq. (2.25) among elliptic iterated integrals yields various Fay identities in the limit $z\to 1$ and generalizes as follows to multiple appearances of the argument among the labels:

$$\begin{aligned}
\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_k & n_{k+1} & \ldots & n_r \\ z & z & \ldots & z & 0 & \ldots & 0\end{smallmatrix};z\right) &= (-1)^k\,\zeta(\underbrace{0\ldots0}_{r-k}\underbrace{1\ldots1}_{k})\prod_{j=1}^{r}\delta_{n_j,1} \\
&- (-1)^{n_k}\int_0^z \mathrm{d}t\, f^{(n_k+n_{k+1})}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{k-1} & 0 & n_{k+2} & \ldots & n_r \\ t & \ldots & t & 0 & 0 & \ldots & 0\end{smallmatrix};t\right) \\
&+ \sum_{j=0}^{n_{k+1}}\binom{n_k-1+j}{j}\int_0^z \mathrm{d}t\, f^{(n_{k+1}-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{k-1} & n_k+j & n_{k+2} & \ldots & n_r \\ t & \ldots & t & t & 0 & \ldots & 0\end{smallmatrix};t\right) \\
&+ \sum_{j=0}^{n_k}\binom{n_{k+1}-1+j}{j}(-1)^{n_k+j}\int_0^z \mathrm{d}t\, f^{(n_k-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_{k-1} & n_{k+1}+j & n_{k+2} & \ldots & n_r \\ t & \ldots & t & 0 & 0 & \ldots & 0\end{smallmatrix};t\right)\ .
\end{aligned} \tag{A.3}$$

The MZV in the first line stems from $\lim_{z\to0} G(\underbrace{z,\ldots,z}_{k},\underbrace{0,\ldots,0}_{r-k};z)$ see eq. (3.1). The first novel eMZV relations follow from the limit $z\to1$ of eq. (A.3) at $k=2$ and $r=4,5$:

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & n_3 & n_4 \\ z & z & 0 & 0\end{smallmatrix};z\right) = -\frac{1}{4}\,\zeta_4\,\delta_{n_1,1}\delta_{n_2,1}\delta_{n_3,1}\delta_{n_4,1} - (-1)^{n_2}\int_0^z \mathrm{d}t\, f^{(n_2+n_3)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & 0 & n_4 \\ t & 0 & 0\end{smallmatrix};t\right)$$

[4] For example, the $k=1$ part of eq. (A.1) encompasses those deconcatenations $B=A_1A_2$ into $A_1=\{n_1,n_2,\ldots,n_j\}$ and $A_2=\{n_{j+1},\ldots,n_r\}$ where $\omega(n_1,n_2,\ldots,n_j)$ and $\omega(n_{j+1},\ldots,n_r)$ are interesting eMZVs.

$$+\sum_{j=0}^{n_3}\binom{n_2-1+j}{j}\int_0^z \mathrm{d}t\, f^{(n_3-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & n_2+j & n_4 \\ t & t & 0\end{smallmatrix};t\right) \tag{A.4}$$
$$+\sum_{j=0}^{n_2}\binom{n_3-1+j}{j}(-1)^{n_2+j}\int_0^z \mathrm{d}t\, f^{(n_2-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & n_3+j & n_4 \\ t & 0 & 0\end{smallmatrix};t\right)$$

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & n_3 & n_4 & n_5 \\ z & z & 0 & 0 & 0\end{smallmatrix};z\right)=(2\,\zeta_5-\zeta_2\,\zeta_3)\Big(\prod_{j=1}^{5}\delta_{n_j,1}\Big)-(-1)^{n_2}\int_0^z \mathrm{d}t\, f^{(n_2+n_3)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & 0 & n_4 & n_5 \\ t & 0 & 0 & 0\end{smallmatrix};t\right)$$
$$+\sum_{j=0}^{n_3}\binom{n_2-1+j}{j}\int_0^z \mathrm{d}t\, f^{(n_3-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & n_2+j & n_4 & n_5 \\ t & t & 0 & 0\end{smallmatrix};t\right) \tag{A.5}$$
$$+\sum_{j=0}^{n_2}\binom{n_3-1+j}{j}(-1)^{n_2+j}\int_0^z \mathrm{d}t\, f^{(n_2-j)}(t)\,\Gamma\left(\begin{smallmatrix} n_1 & n_3+j & n_4 & n_5 \\ t & 0 & 0 & 0\end{smallmatrix};t\right)$$

In particular, note that the MZV product $\zeta_2\,\zeta_3$ is absent in eq. (2.25) at $r=5$. Also, note that the divergent nature of $f^{(1)}$ causes extra complications in the limit $\zeta\to 1$ of eq. (A.4) if $n_i=1$ for $i=1,2,3,4$ and eq. (A.5) if $n_2=n_3=n_4=1$ and one of $n_1=1$ or $n_5=1$.

B Iterated Eisenstein integrals versus eMZVs: examples

In this appendix, we supplement further examples for the conversion of eMZVs into modified iterated Eisenstein integrals as defined in eq. (4.54).

B.1 Conversion of $\omega(0,0,\ldots,0,n)$

For eMZVs with only one non-zero entry, a closed formula can be given for their conversion into iterated Eisenstein integrals. At length $\ell_\omega=4$ and $\ell_\omega=5$, eqs. (4.58a) and (4.58b) can be generalized to

$$\omega(0,0,0,n)=\frac{n}{3!}\,\gamma_0(n+1)+n(n+1)(n+2)\,\gamma_0(n+3,0,0) \tag{B.1}$$
$$\omega(0,0,0,0,n)=-\frac{2\,\zeta_n}{5!}-\frac{n}{3!}(n+1)\,\gamma_0(n+2,0)-n(n+1)(n+2)(n+3)\,\gamma_0(n+4,0,0,0)\ , \tag{B.2}$$

where n is chosen to be odd in eq. (B.1) and even in eq. (B.2). At arbitrary length ℓ, we have

$$\omega(\underbrace{0,0,\ldots,0}_{\ell-1},n)=\begin{cases}\displaystyle\sum_{\substack{i=1,3,5,\\ \ldots,\ell-1}}\frac{\gamma_0(n+i,0^{i-1})}{(\ell-i)!}\prod_{j=0}^{i-1}(n+j) & :\ \ell\text{ even, } n\text{ odd}\\ \displaystyle -\frac{2\,\zeta_n}{\ell!}-\sum_{\substack{i=2,4,6,\\ \ldots,\ell-1}}\frac{\gamma_0(n+i,0^{i-1})}{(\ell-i)!}\prod_{j=0}^{i-1}(n+j) & :\ \ell\text{ odd, } n\text{ even}\end{cases}\ , \tag{B.3}$$

whose q-expansion can be inferred from the special case

$$\gamma_0(k,0^{p-1})=-\frac{2(2\pi i)^{k-2p}}{(k-1)!}\sum_{n=1}^{\infty}\frac{\sigma_{k-1}(n)q^n}{n^p} \tag{B.4}$$

of eq. (4.64), see eq. (4.63) for the definition of the divisor sum $\sigma_k(n)$.

B.2 Conversion of indecomposable eMZVs at $\ell_\omega \geq 3$

Among the indecomposable eMZVs beyond $\omega(0,\ldots,0,n)$, the simplest case $\omega(0,3,5)$ is converted to (modified) iterated Eisenstein integrals in eqs. (4.15) and (4.61). Beyond that, we find for example

$$\begin{aligned}
\omega(0,3,7) &= -1848\,\gamma(12,0) - 294\,\gamma(8,4) + \text{nmt} \\
&= -1848\,\gamma_0(12,0) - 294\,\gamma_0(8,4) - 75\big(\gamma_0(6)\big)^2 + 588\,\zeta_4\,\gamma_0(8,0) - 504\,\zeta_8\,\gamma_0(4,0) \\
\omega(0,3,9) &= -5616\,\gamma(14,0) - 729\,\gamma(10,4) - 315\,\gamma(8,6) + \text{nmt} \\
&= -5616\,\gamma_0(14,0) - 729\,\gamma_0(10,4) - 315\,\gamma_0(8,6) - 210\,\gamma_0(6)\,\gamma_0(8) \\
&\quad + 1458\,\zeta_4\,\gamma_0(10,0) + 630\,\zeta_6\,\gamma_0(8,0) - 630\,\zeta_6\,\gamma_0(6,0) - 1350\,\zeta_{10}\,\gamma_0(4,0) \\
\omega(0,3,11) &= -13695\,\gamma(16,0) - 1452\,\gamma(12,4) - 990\,\gamma(10,6) + \text{nmt} \\
&= -13695\,\gamma_0(16,0) - 1452\,\gamma_0(12,4) - 735/2\big(\gamma_0(8)\big)^2 - 990\,\gamma_0(10,6) - 270\,\gamma_0(6)\,\gamma_0(10) \\
&\quad + 2904\,\zeta_4\,\gamma_0(12,0) + 1980\,\zeta_6\,\gamma_0(10,0) - 1980\,\zeta_{10}\,\gamma_0(6,0) - 2772\,\zeta_{12}\,\gamma_0(4,0) \\
\omega(0,5,9) &= -30105\,\gamma(16,0) - 5445\,\gamma(12,4) - 3105\,\gamma(10,6) + \text{nmt} \\
&= -30105\,\gamma_0(16,0) - 5445\,\gamma_0(12,4) - 3105\,\gamma_0(10,6) - 735/2\big(\gamma_0(8)\big)^2 \\
&\quad + 10890\,\zeta_4\,\gamma_0(12,0) + 6210\,\zeta_6\,\gamma_0(10,0) - 5850\,\zeta_{10}\,\gamma_0(6,0) - 8910\,\zeta_{12}\,\gamma_0(4,0)
\end{aligned} \tag{B.5}$$

at length three, and

$$\begin{aligned}
\omega(0,0,2,3) &= 252\,\gamma(8,0,0) - 18\,\gamma(4,4,0) + \frac{5}{6}\,\gamma(6) + \text{nmt} \\
&= 252\,\gamma_0(8,0,0) - 18\,\gamma_0(4,4,0) + \frac{5}{6}\,\gamma_0(6) - 72\,\zeta_4\,\gamma_0(4,0,0) \\
\omega(0,0,2,5) &= 2826\,\gamma(10,0,0) + 150\,\gamma(6,4,0) + 180\,\gamma(6,0,4) + \frac{7}{6}\,\gamma(8) + \text{nmt} \\
&= 2826\,\gamma_0(10,0,0) + 150\,\gamma_0(6,4,0) + 180\,\gamma_0(6,0,4) + \frac{7}{6}\,\gamma_0(8) \\
&\quad - 660\,\zeta_4\,\gamma_0(6,0,0) + 180\,\zeta_6\,\gamma_0(4,0,0) \\
\omega(0,0,4,3) &= -2340\,\gamma(10,0,0) - 300\,\gamma(6,4,0) - 120\,\gamma(6,0,4) + \frac{7}{6}\,\gamma(8) + \text{nmt} \\
&= -2340\,\gamma_0(10,0,0) - 300\,\gamma_0(6,4,0) - 120\,\gamma_0(6,0,4) - 60\,\gamma_0(4)\,\gamma_0(6,0) + \frac{7}{6}\,\gamma_0(8) \\
&\quad + 480\,\zeta_4\,\gamma_0(6,0,0) - 1080\,\zeta_6\,\gamma_0(4,0,0) - 3\,\zeta_4\,\gamma_0(4)
\end{aligned} \tag{B.6}$$

at length four, where "nmt" refers to non-maximal terms as explained after eq. (4.17a). The q-expansion of the constituents is given by eq. (4.64).

C Examples for relations in the derivation algebra $\mathfrak{u}$

C.1 Known relations

Irreducible relations $r^{\text{depth}}_{\text{weight}}$ are listed in table 5. For depth two, all relations can be obtained from eq. (4.42). At depth three, we can confirm the relations listed in eq. (4.28c) as well as [22]:

$$\begin{aligned}
r^3_{20}: \quad 0 &= 1050[\epsilon_0,[\epsilon_6,\epsilon_{14}]] - 6580[\epsilon_0,[\epsilon_8,\epsilon_{12}]] + 4320[\epsilon_4,[\epsilon_0,\epsilon_{16}]] - 10970[\epsilon_4,[\epsilon_4,\epsilon_{12}]] \\
&\quad + 166675[\epsilon_4,[\epsilon_6,\epsilon_{10}]] - 17150[\epsilon_6,[\epsilon_0,\epsilon_{14}]] - 500675[\epsilon_6,[\epsilon_6,\epsilon_8]] + 30184[\epsilon_8,[\epsilon_0,\epsilon_{12}]]
\end{aligned}$$

$$+80388[\epsilon_8,[\epsilon_4,\epsilon_8]]-17325[\epsilon_{10},[\epsilon_0,\epsilon_{10}]] \tag{C.1}$$

$$\begin{aligned} r_{22}^3: \quad 0 =\ & 40[\epsilon_0,[\epsilon_6,\epsilon_{16}]]-280[\epsilon_0,[\epsilon_8,\epsilon_{14}]]+910[\epsilon_0,[\epsilon_{10},\epsilon_{12}]]-360[\epsilon_4,[\epsilon_0,\epsilon_{18}]] \\ & -11535[\epsilon_4,[\epsilon_6,\epsilon_{12}]]+6069[\epsilon_4,[\epsilon_8,\epsilon_{10}]]+1320[\epsilon_6,[\epsilon_0,\epsilon_{16}]]+15140[\epsilon_6,[\epsilon_4,\epsilon_{12}]] \\ & -7150[\epsilon_6,[\epsilon_6,\epsilon_{10}]]-1820[\epsilon_8,[\epsilon_0,\epsilon_{14}]]-12922[\epsilon_8,[\epsilon_6,\epsilon_8]]+858[\epsilon_{10},[\epsilon_0,\epsilon_{12}]] \end{aligned} \tag{C.2}$$

$$\begin{aligned} r_{18}^4: \quad 0 =\ & [\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_{12}]]]-\frac{215}{74}[\epsilon_0,[\epsilon_0,[\epsilon_8,\epsilon_{10}]]]-\frac{2323}{518}[\epsilon_0,[\epsilon_4,[\epsilon_6,\epsilon_8]]]+\frac{218}{37}[\epsilon_0,[\epsilon_6,[\epsilon_4,\epsilon_8]]] \\ & +\frac{60}{407}[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]]+\frac{285561[\epsilon_4,[\epsilon_0,[\epsilon_6,\epsilon_8]]]}{5698}+\frac{8599[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_{10}]]]}{1628} \\ & +\frac{53855}{444}[\epsilon_4,[\epsilon_4,[\epsilon_4,\epsilon_6]]]-\frac{691}{333}[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{12}]]]-\frac{19853}{518}[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_8]]] \\ & +\frac{17275}{333}[\epsilon_6,[\epsilon_6,[\epsilon_0,\epsilon_6]]]+\frac{3455}{518}[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]+\frac{49565}{518}[\epsilon_8,[\epsilon_0,[\epsilon_4,\epsilon_6]]] \\ & -\frac{691}{74}[\epsilon_{10},[\epsilon_0,[\epsilon_0,\epsilon_8]]]+\frac{691}{111}[\epsilon_{12},[\epsilon_0,[\epsilon_0,\epsilon_6]]]-\frac{60}{37}[\epsilon_{14},[\epsilon_0,[\epsilon_0,\epsilon_4]]]-\frac{87595[\epsilon_6,[\epsilon_4,[\epsilon_0,\epsilon_8]]]}{1554} \end{aligned} \tag{C.3}$$

$$\begin{aligned} r_{22}^4: \quad 0 =\ & [\epsilon_0,[\epsilon_0,[\epsilon_8,\epsilon_{14}]]]+\frac{192903}{230}[\epsilon_0,[\epsilon_4,[\epsilon_6,\epsilon_{12}]]]-\frac{861492}{805}[\epsilon_0,[\epsilon_6,[\epsilon_4,\epsilon_{12}]]] \\ & +\frac{134488}{161}[\epsilon_0,[\epsilon_6,[\epsilon_6,\epsilon_{10}]]]+\frac{6588}{805}[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{18}]]]+\frac{269217}{805}[\epsilon_4,[\epsilon_0,[\epsilon_6,\epsilon_{12}]]] \\ & -\frac{39418}{115}[\epsilon_4,[\epsilon_0,[\epsilon_8,\epsilon_{10}]]]-\frac{13253}{115}[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_{14}]]]-\frac{18221}{115}[\epsilon_4,[\epsilon_4,[\epsilon_6,\epsilon_8]]] \\ & +\frac{33109}{322}[\epsilon_6,[\epsilon_0,[\epsilon_6,\epsilon_{10}]]]+\frac{25095129[\epsilon_6,[\epsilon_4,[\epsilon_0,\epsilon_{12}]]]}{37375}+\frac{11266827[\epsilon_6,[\epsilon_4,[\epsilon_4,\epsilon_8]]]}{5750} \\ & -\frac{786557}{644}[\epsilon_6,[\epsilon_6,[\epsilon_0,\epsilon_{10}]]]+\frac{80233[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]]}{1265}+\frac{21742068[\epsilon_8,[\epsilon_0,[\epsilon_6,\epsilon_8]]]}{6325} \\ & -\frac{112835}{253}[\epsilon_8,[\epsilon_4,[\epsilon_0,\epsilon_{10}]]]+\frac{403764}{115}[\epsilon_8,[\epsilon_4,[\epsilon_4,\epsilon_6]]]+\frac{644938}{575}[\epsilon_8,[\epsilon_6,[\epsilon_0,\epsilon_8]]] \\ & -\frac{103859}{115}[\epsilon_{10},[\epsilon_0,[\epsilon_4,\epsilon_8]]]+\frac{301851[\epsilon_{12},[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]}{8050}+\frac{734133}{805}[\epsilon_{12},[\epsilon_0,[\epsilon_4,\epsilon_6]]] \\ & -\frac{1913}{115}[\epsilon_{14},[\epsilon_0,[\epsilon_0,\epsilon_8]]]+\frac{672}{115}[\epsilon_{16},[\epsilon_0,[\epsilon_0,\epsilon_6]]]-\frac{972}{805}[\epsilon_{18},[\epsilon_0,[\epsilon_0,\epsilon_4]]] \\ & -\frac{1015637[\epsilon_0,[\epsilon_4,[\epsilon_8,\epsilon_{10}]]]}{1150}-\frac{27458211[\epsilon_6,[\epsilon_6,[\epsilon_4,\epsilon_6]]]}{3220}-\frac{23679[\epsilon_0,[\epsilon_0,[\epsilon_{10},\epsilon_{12}]]]}{8050} \\ & -\frac{493889[\epsilon_{10},[\epsilon_0,[\epsilon_0,\epsilon_{12}]]]}{8050}-\frac{372888[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{16}]]]}{10465}-\frac{23054063[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_{12}]]]}{52325} \end{aligned} \tag{C.4}$$

C.2 New relations

At depth 5 we explicitly isolated the irreducible relation r_{20}^5, which is apparently new:

$$\begin{aligned} r_{22}^5: \quad 0 =\ & 2206388620800\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_{16}]]]]-8366188740000\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_{14}]]]] \\ & +12305858292000\,[\epsilon_0,[\epsilon_0,[\epsilon_0,[\epsilon_8,\epsilon_{12}]]]]-1834700544000\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{16}]]]] \\ & +35687825530800\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_4,\epsilon_{12}]]]]-109425220173750\,[\epsilon_0,[\epsilon_4,[\epsilon_0,[\epsilon_6,\epsilon_{10}]]]] \\ & -39970750599360\,[\epsilon_0,[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_{12}]]]]-380488416808500\,[\epsilon_0,[\epsilon_4,[\epsilon_4,[\epsilon_4,\epsilon_8]]]] \end{aligned}$$

$$
\begin{aligned}
&+ 13171256280000\,[\epsilon_0,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]]] + 220479512028750\,[\epsilon_0,[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_{10}]]]]\\
&- 498847136287500\,[\epsilon_0,[\epsilon_6,[\epsilon_0,[\epsilon_6,\epsilon_8]]]] + 220479512028750\,[\epsilon_0,[\epsilon_6,[\epsilon_4,[\epsilon_0,\epsilon_{10}]]]]\\
&- 458212979593200\,[\epsilon_0,[\epsilon_6,[\epsilon_4,[\epsilon_4,\epsilon_6]]]] + 17540335312500\,[\epsilon_0,[\epsilon_6,[\epsilon_6,[\epsilon_0,\epsilon_8]]]]\\
&- 34407225652800\,[\epsilon_0,[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_{12}]]]] - 97419791414400\,[\epsilon_0,[\epsilon_8,[\epsilon_0,[\epsilon_4,\epsilon_8]]]]\\
&- 197536749664800\,[\epsilon_0,[\epsilon_8,[\epsilon_4,[\epsilon_0,\epsilon_8]]]] + 22970739577500\,[\epsilon_0,[\epsilon_{10},[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]]\\
&+ 161385266688750\,[\epsilon_0,[\epsilon_{10},[\epsilon_0,[\epsilon_4,\epsilon_6]]]] + 611566848000\,[\epsilon_4,[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_{16}]]]]\\
&- 58836403790864\,[\epsilon_4,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_{12}]]]] + 134572047805000\,[\epsilon_4,[\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_{10}]]]]\\
&+ 965866444426884\,[\epsilon_4,[\epsilon_0,[\epsilon_4,[\epsilon_4,\epsilon_8]]]] + 92810063342256\,[\epsilon_4,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{12}]]]]\\
&- 204658497503460\,[\epsilon_4,[\epsilon_4,[\epsilon_0,[\epsilon_4,\epsilon_8]]]] + 541534390897500\,[\epsilon_4,[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_8]]]]\\
&- 215755493216250\,[\epsilon_4,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]] - 1490371718737200\,[\epsilon_4,[\epsilon_6,[\epsilon_0,[\epsilon_4,\epsilon_6]]]]\\
&+ 1032598095322950\,[\epsilon_4,[\epsilon_6,[\epsilon_4,[\epsilon_0,\epsilon_6]]]] + 298655975581600\,[\epsilon_4,[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_8]]]]\\
&- 220479512028750\,[\epsilon_4,[\epsilon_{10},[\epsilon_0,[\epsilon_0,\epsilon_6]]]] + 54837332264496\,[\epsilon_4,[\epsilon_{12},[\epsilon_0,[\epsilon_0,\epsilon_4]]]]\\
&- 6941740260000\,[\epsilon_6,[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_{14}]]]] - 220479512028750\,[\epsilon_6,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_{10}]]]]\\
&+ 231883232831250\,[\epsilon_6,[\epsilon_0,[\epsilon_0,[\epsilon_6,\epsilon_8]]]] + 519528504682200\,[\epsilon_6,[\epsilon_0,[\epsilon_4,[\epsilon_4,\epsilon_6]]]]\\
&- 220479512028750\,[\epsilon_6,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]] - 2120947122294000\,[\epsilon_6,[\epsilon_4,[\epsilon_0,[\epsilon_4,\epsilon_6]]]]\\
&- 1538522546497950\,[\epsilon_6,[\epsilon_4,[\epsilon_4,[\epsilon_0,\epsilon_6]]]] + 249423568143750\,[\epsilon_6,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_8]]]]\\
&- 266963903456250\,[\epsilon_6,[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_6]]]] + 23162632092600\,[\epsilon_8,[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_{12}]]]]\\
&+ 184988881773150\,[\epsilon_8,[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_8]]]] + 310347440367510\,[\epsilon_8,[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_8]]]]\\
&- 183822714075000\,[\epsilon_8,[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_6]]]] + 171943360038450\,[\epsilon_8,[\epsilon_8,[\epsilon_0,[\epsilon_0,\epsilon_4]]]]\\
&- 22551859687500\,[\epsilon_{10},[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_{10}]]]] - 240755752121625\,[\epsilon_{10},[\epsilon_0,[\epsilon_0,[\epsilon_4,\epsilon_6]]]]\\
&- 104628710038125\,[\epsilon_{10},[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_6]]]] - 14987648446875\,[\epsilon_{10},[\epsilon_6,[\epsilon_0,[\epsilon_0,\epsilon_4]]]]\\
&+ 11918038532400\,[\epsilon_{12},[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_8]]]] + 46293152724000\,[\epsilon_{12},[\epsilon_4,[\epsilon_0,[\epsilon_0,\epsilon_4]]]]\\
&- 8366188740000\,[\epsilon_{14},[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_6]]]]\,. \qquad (C.5)
\end{aligned}
$$

The complete set of all irreducible relations known to us is available from

`https://tools.aei.mpg.de/emzv`

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