Annales Henri Poincaré

Convergent Null Data Expansions at Space-Like Infinity of Stationary Vacuum Solutions

Andrés E. Aceña

Abstract. We present a characterization of the asymptotics of all asymptotically flat, stationary solutions with non-vanishing ADM mass to Einstein's vacuum field equations. This characterization is given in terms of two sequences of symmetric trace free tensors (we call them the 'null data'), which determine a formal expansion of the solution, and which are in a one to one correspondence to Hansen's multipoles. We obtain necessary and sufficient growth estimates on the null data to define an absolutely convergent series in a neighborhood of spatial infinity. This provides a complete characterization of all asymptotically flat, stationary vacuum solutions to the field equations with non-vanishing ADM mass.

1. Introduction

A stationary vacuum spacetime is given by a triplet $(\tilde{M}, \tilde{g}_{\mu\nu}, \xi^\mu)$, where $\tilde{M}$ is a four-dimensional manifold, $\tilde{g}_{\mu\nu}$ is a Lorentzian metric with signature $(+ - - -)$ that satisfy Einstein's vacuum equations $Ric[\tilde{g}] = 0$, and ξ^μ is a time-like Killing vector field with complete orbits. The metric can be written locally as

$$\tilde{g} = V(dt + \gamma_a d\tilde{x}^a)^2 + V^{-1}\tilde{h}_{ab}d\tilde{x}^a d\tilde{x}^b, \quad a, b = 1, 2, 3, \tag{1}$$

where V, γ_a and $\tilde{h}_{ab}$ depend only on the spatial coordinates $\tilde{x}^a$. As shown by Geroch [10] the description of this spacetime can be done in terms of fields defined on an abstract three-dimensional manifold $\tilde{N}$ which is obtained as the quotient space of $\tilde{M}$ with respect to the trajectories of ξ^μ. The fields V, γ_a, $\tilde{h}_{ab}$ on $\tilde{M}$ can be considered as pull-backs under the projection map of fields on $\tilde{N}$. The latter will be denoted by the same symbols. In the following we shall work in terms of the fields V, γ_a, $\tilde{h}_{ab}$ on $\tilde{N}$, where $\tilde{h}_{ab}$ is a negative definite metric on $\tilde{N}$.

The vacuum Einstein's field equations in $\tilde{M}$ imply that on $\tilde{N}$ the quantity

$$\omega_a = -V^2\tilde{\epsilon}_{abc}\tilde{D}^b\gamma^c$$

is curl-free, i.e.

$$\tilde{D}_{[a}\omega_{b]} = 0\,,$$

where $\tilde{D}$ is the covariant derivative with respect to $\tilde{h}_{ab}$ and $\tilde{\epsilon}_{abc} = \tilde{\epsilon}_{[abc]}$, $\tilde{\epsilon}_{123} = |\det \tilde{h}_{ab}|^{\frac{1}{2}}$. We are interested in the asymptotics of the spacetime at spatial infinity, so it will be assumed that $\tilde{N}$ is diffeomorphic to the complement of a closed ball $\bar{B}_R(0)$ in $\mathbb{R}^3$. Thus $\tilde{N}$ is simply connected and there exists a scalar field ω such that

$$\tilde{D}_a\omega = \omega_a\,.$$

Instead of working with V and ω it is convenient to use the combinations

$$\tilde{\phi}_M = \frac{V^2 + \omega^2 - 1}{4V}\,,$$

$$\tilde{\phi}_S = \frac{\omega}{2V}\,,$$

introduced by Hansen [11]. In this setting Einstein's vacuum field equations are then equivalent to

$$\Delta_{\tilde{h}}\tilde{\phi}_A = 2R[\tilde{h}]\tilde{\phi}_A\,, \quad A = M, S\,, \tag{2}$$

$$R_{ab}[\tilde{h}] = 2\big[(\tilde{D}_a\tilde{\phi}_M)(\tilde{D}_b\tilde{\phi}_M) + (\tilde{D}_a\tilde{\phi}_S)(\tilde{D}_b\tilde{\phi}_S) - (\tilde{D}_a\tilde{\phi}_K)(\tilde{D}_b\tilde{\phi}_K)\big]\,, \tag{3}$$

where $\tilde{\phi}_K = \left(\frac{1}{4} + \tilde{\phi}_M^2 + \tilde{\phi}_S^2\right)^{\frac{1}{2}}$. Equations (2), (3) will be referred to as the *stationary vacuum field equations*. We are looking for solutions of (2) and (3). The spacetime $(\tilde{M}, \tilde{g}_{\mu\nu}, \xi^\mu)$ can be reconstructed from $(\tilde{N}, \tilde{h}_{ab}, \tilde{\phi}_M, \tilde{\phi}_S)$ (cf. [7] for a detailed discussion).

The asymptotic flatness condition is usually stated by assuming $(\tilde{N}, \tilde{h}_{ab})$ to admit a smooth conformal extension in the following way: there exist a smooth Riemannian manifold (N, h_{ab}) and a function $\Omega \in C^2(N) \cap C^\infty(\tilde{N})$ such that $N = \tilde{N} \cup \{i\}$, where i is a single point,

$$\Omega > 0 \quad \text{on} \quad \tilde{N}\,,$$

$$h_{ab} = \Omega^2\tilde{h}_{ab} \quad \text{on} \quad \tilde{N}\,,$$

$$\Omega|_i = 0\,, \quad D_a\Omega|_i = 0\,, \quad D_aD_b\Omega|_i = -2h_{ab}|_i\,, \tag{4}$$

where D is the covariant derivative operator defined by h. This makes N diffeomorphic to an open ball in $\mathbb{R}^3$, with center at the point i, which represents space-like infinity. From now on we assume $\tilde{N}$ to be asymptotically flat in the stated sense.

Considering $\tilde{N}$ to be diffeomorphic to the complement of a closed ball $\bar{B}_R(0)$ in $\mathbb{R}^3$ is natural in the present context. It corresponds to the idea of an isolated system, where the material sources are confined to a bounded region outside of which is vacuum. Lichnerowicz [14] has shown that if $\tilde{N}$ is diffeomorphic to $\mathbb{R}^3$ then $\tilde{N}$ is flat.

Reula [15] has shown existence and uniqueness of asymptotically flat solutions to (2), (3), in terms of a boundary value problem, when data are prescribed on the sphere $\partial\tilde{N}$. In order to be able to control the precise asymptotic behavior of the

spacetime, however, it would be convenient to have a complete description of the asymptotically flat stationary vacuum solutions in terms of asymptotic quantities.

Candidates for this task are Hansen's multipoles [11]. With the previous assumptions Hansen proposes a definition of multipoles, which extends Geroch's definition of multipoles for asymptotically flat static spacetimes [9] to the stationary case. He defines the conformal potentials

$$\phi_A = \Omega^{-\frac{1}{2}} \tilde{\phi}_A\,, \quad A = M, S,\tag{5}$$

and two sequences of tensor fields near i through

$$P^A = \phi_A\,, \quad P_a^A = D_a P^A\,, \quad P_{a_2 a_1}^A = \mathcal{C}\left(D_{a_2} P_{a_1}^A - \frac{1}{2} P^A R_{a_2 a_1}\right),\tag{6}$$

$$P_{a_{s+1}\ldots a_1}^A = \mathcal{C}\left[D_{a_{s+1}} P_{a_s \ldots a_1}^A - \frac{1}{2} s(2s-1) P_{a_{s+1}\ldots a_3}^A R_{a_2 a_1}\right], \quad A = M, S,\tag{7}$$

where R_{ab} is the Ricci tensor of h_{ab} and $\mathcal{C}$ is the projector onto the symmetric trace free part of the respective tensor fields. The multipole moments are then defined as the tensors

$$\nu^A = P^A(i)\,, \quad \nu_{a_p \ldots a_1}^A = P_{a_p \ldots a_1}^A(i)\,, \quad A = M, S, \quad p = 1, 2, 3, \ldots\tag{8}$$

Keeping aside the monopoles, ν^A, we will denote the two sequences of remaining multipoles by

$$\mathcal{D}_{mp}^A = \{\nu_{a_1}^A, \nu_{a_2 a_1}^A, \nu_{a_3 a_2 a_1}^A, \ldots\}\,, \quad A = M, S\,.$$

The multipole moments are proposed as a way to characterize solutions of (2), (3). So a natural question is to what extent do the multipoles determine the metric h and the potentials ϕ_M, ϕ_S. For this to be the case the metric and the potentials should be real analytic even at i in suitable coordinates and conformal rescaling. Beig and Simon [3] and Kundu [13] have shown that the metric and the potentials do extend in a suitable gauge as real analytic fields to i if it is assumed that

$$(\nu^M)^2 + (\nu^S)^2 \neq 0\,.$$

As explained in [16] (cf. also [4]), in order for a solution of (2), (3) to lead to an asymptotically flat spacetime $\tilde{M}$ it is necessary that $\nu^S = 0$. So, we assume from now on that

$$\nu^M \neq 0\,, \quad \nu^S = 0\,.\tag{9}$$

In [3] and [13] it is also shown that for given multipoles there is a unique formal expansion of a 'formal solution' to the stationary field equations, but it is not touched upon the convergence of the expansion.

Bäckdahl and Herberthson [2] have found, assuming a given asymptotically flat solution of the stationary field equations, necessary bounds on the multipoles.

The question that remains open is under which conditions a pair of sequences, taken as the multipoles, do indeed determine a convergent expansion of a stationary solution. This question has been studied for the axisymmetric case by Bäckdahl [1]. In the static case there is only one sequence of multipoles. Friedrich [8] has used as data a sequence of trace-free symmetric tensors, referred to as null data, which

are different but related to the multipoles. He has shown that imposing certain types of estimates on the null data is necessary and sufficient for the existence of asymptotically flat static spacetimes. However, so far the existence question has never been answered for the general stationary case.

Using a conformal factor that is specified later on, we define the following two sequences of trace-free symmetric tensors at infinity

$$\mathcal{D}_n^\phi = \left\{ \mathcal{C}(D_{a_1}\phi)(i), \mathcal{C}(D_{a_2}D_{a_1}\phi)(i), \mathcal{C}(D_{a_3}D_{a_2}D_{a_1}\phi)(i), \dots \right\},$$
$$\mathcal{D}_n^S = \left\{ S_{a_2a_1}(i), \mathcal{C}(D_{a_3}S_{a_2a_1})(i), \mathcal{C}(D_{a_4}D_{a_3}S_{a_2a_1})(i), \dots \right\}, \tag{10}$$

where $\phi = \phi_S$ and S_{ab} is the trace free part of the Ricci tensor of h. These two sequences are referred to again as the *null data*.

The purpose of this work is to derive, under the assumption (9), necessary and sufficient conditions for the null data $\mathcal{D}_n^\phi$, $\mathcal{D}_n^S$, to determine apart from gauge conditions (unique) real analytic solutions of (2) and (3) and thus to provide a complete characterization of all possible asymptotically flat solutions to the stationary vacuum field equations. This generalizes the work by Friedrich [8] from the static to the stationary case in a way discussed later on.

For the same reasons that justify $\tilde{N}$ to be considered diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$, we shall treat the case in which N may comprise a small neighborhood of the point i, without worrying about the behavior of the solution in the large (note that in terms of $\tilde{h}$ a neighborhood of i covers an infinite domain extending to space-like infinity).

For our analysis it is convenient to remove the conformal gauge freedom and use, following Beig and Simon [3],

$$\Omega = \frac{1}{2}m^{-2}\left[\left(1 + 4\tilde{\phi}_M^2 + 4\tilde{\phi}_S^2\right)^{\frac{1}{2}} - 1\right]. \tag{11}$$

With this conformal factor they derive fall-off conditions and then show that under some assumptions the rescaled metric can be extended in suitable coordinates on a suitable neighborhood of space-like infinity as a metric which is real analytic at i. The potentials ϕ_M and ϕ_S are then also real analytic at i, so that the multipoles are well defined. Using this gauge, and taking into account that the angular momentum monopole vanish, we get

$$\nu^M = m, \quad \nu_a^M = 0.$$

We express now the tensors in $\mathcal{D}_n^\phi$, $\mathcal{D}_n^S$ in terms of an h-orthonormal frame $c_{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, at i. Denoting by $D_{\mathbf{a}}$ the covariant derivative in the direction of $c_{\mathbf{a}}$,

$$\mathcal{D}_n^{\phi*} = \left\{ \mathcal{C}(D_{\mathbf{a}_1}\phi)(i), \mathcal{C}(D_{\mathbf{a}_2}D_{\mathbf{a}_1}\phi)(i), \mathcal{C}(D_{\mathbf{a}_3}D_{\mathbf{a}_2}D_{\mathbf{a}_1}\phi)(i), \dots \right\}, \tag{12}$$
$$\mathcal{D}_n^{S*} = \left\{ S_{\mathbf{a}_2\mathbf{a}_1}(i), \mathcal{C}(D_{\mathbf{a}_3}S_{\mathbf{a}_2\mathbf{a}_1})(i), \mathcal{C}(D_{\mathbf{a}_4}D_{\mathbf{a}_3}S_{\mathbf{a}_2\mathbf{a}_1})(i), \dots \right\}. \tag{13}$$

These tensors, which are defined uniquely up to rigid rotations in $\mathbb{R}^3$, will be referred to as the *null data of h in the frame $c_{\mathbf{a}}$*.

If the metric h and the potential ϕ are real analytic near i, then there exist constants $M, r > 0$ such that the components of these tensors satisfy the estimates

$$|\mathcal{C}(D_{\mathbf{a}_p} \ldots D_{\mathbf{a}_1} \phi)(i)| \leq \frac{Mp!}{r^p}, \quad \mathbf{a}_p, \ldots, \mathbf{a}_1 = 1, 2, 3, \quad p = 0, 1, 2, \ldots,$$

$$|\mathcal{C}(D_{\mathbf{a}_p} \ldots D_{\mathbf{a}_1} S_{\mathbf{bc}})(i)| \leq \frac{Mp!}{r^p}, \quad \mathbf{a}_p, \ldots, \mathbf{a}_1, \mathbf{b}, \mathbf{c} = 1, 2, 3, \quad p = 0, 1, 2, \ldots. \quad (14)$$

Although these estimates are similar to the Cauchy estimates, known to hold for the derivatives of analytic functions, they are not the same. The difference being that here the estimates are on the symmetric trace free part of the derivatives instead of being directly on the derivatives. These estimates are derived from Cauchy estimates in Section 3. Remarkably, the statement that these estimates are not only necessary but also sufficient to have an analytic solution of the stationary field equations is also true. This constitutes our main result, given in the following theorem.

Theorem 1.1. *Suppose $m \neq 0$ and*

$$\hat{\mathcal{D}}_n^\phi = \{\psi_{\mathbf{a}_1}, \psi_{\mathbf{a}_2 \mathbf{a}_1}, \psi_{\mathbf{a}_3 \mathbf{a}_2 \mathbf{a}_1}, \ldots\}, \quad (15)$$

$$\hat{\mathcal{D}}_n^S = \{\Psi_{\mathbf{a}_2 \mathbf{a}_1}, \Psi_{\mathbf{a}_3 \mathbf{a}_2 \mathbf{a}_1}, \Psi_{\mathbf{a}_4 \mathbf{a}_3 \mathbf{a}_2 \mathbf{a}_1}, \ldots\}, \quad (16)$$

are two infinite sequences of symmetric, trace free tensors given in an orthonormal frame at the origin of a 3-dimensional Euclidean space. If there exist constants $M, r > 0$ such that the components of these tensors satisfy the estimates

$$|\psi_{\mathbf{a}_p \ldots \mathbf{a}_1}| \leq \frac{Mp!}{r^p}, \quad \mathbf{a}_p, \ldots, \mathbf{a}_1 = 1, 2, 3, \quad p = 1, 2, \ldots,$$

$$|\Psi_{\mathbf{a}_p \ldots \mathbf{a}_1 \mathbf{bc}}| \leq \frac{Mp!}{r^p}, \quad \mathbf{a}_p, \ldots, \mathbf{a}_1, \mathbf{b}, \mathbf{c} = 1, 2, 3, \quad p = 0, 1, 2, \ldots,$$

then there exists an analytic, asymptotically flat, stationary vacuum solution $(\tilde{h}, \tilde{\phi}_M, \tilde{\phi}_S)$ with mass monopole m and zero angular momentum monopole, unique up to isometries, so that the null data implied by $h = \frac{1}{4} m^{-4} [(1 + 4\tilde{\phi}_M^2 + 4\tilde{\phi}_S^2)^{\frac{1}{2}} - 1]^2 \tilde{h}$ and $\phi_S = 2^{\frac{1}{2}} m [(1 + 4\tilde{\phi}_M^2 + 4\tilde{\phi}_S^2)^{\frac{1}{2}} - 1]^{-\frac{1}{2}} \tilde{\phi}_S$ in a suitable frame $c_{\mathbf{a}}$ as described above satisfy

$$\mathcal{C}(D_{\mathbf{a}_q} \ldots D_{\mathbf{a}_1} \phi_S)(i) = \psi_{\mathbf{a}_q \ldots \mathbf{a}_1}, \quad \mathbf{a}_q, \ldots, \mathbf{a}_1 = 1, 2, 3, \quad q = 1, 2, \ldots,$$

$$\mathcal{C}(D_{\mathbf{a}_q} \ldots D_{\mathbf{a}_3} S_{\mathbf{a}_2 \mathbf{a}_1})(i) = \Psi_{\mathbf{a}_q \ldots \mathbf{a}_1}, \quad \mathbf{a}_q, \ldots, \mathbf{a}_1 = 1, 2, 3, \quad q = 2, 3, \ldots.$$

Two sequences of data of the form (15), (16), not necessarily satisfying any estimates, will be referred to as *abstract null data*.

The type of estimates imposed here on the abstract null data does not depend on the orthonormal frame in which they are given. Since these estimates are necessary as well as sufficient, all possible asymptotically flat solutions of the stationary vacuum field equations are characterized by the null data.

Corvino and Schoen [6] and Chruściel and Delay [5] have proven that it is possible to deform given general asymptotically flat vacuum data in an annulus in order to glue that data to stationary vacuum data in the asymptotic region. In

relation with those works, as they need a family of asymptotically flat stationary solutions to perform the gluing procedure, our result gives a complete survey of the possible stationary asymptotics that can be attained beyond the known exact solutions.

As both, the multipoles and the null data, determine the metric and the potentials, there is a bijective map between them. The sets $\mathcal{D}_n^\phi$, $\mathcal{D}_n^S$ and $\mathcal{D}_{mp}^M$, $\mathcal{D}_{mp}^S$ thus contain the same information. We prefer to work with the null data because the expressions are linear in ϕ and S_{ab}.

This work contains the static case as a special case. Starting from (1) the static case can be attained by making $\gamma_a = 0$, which gives $\omega = 0$, $\tilde{\phi}_S = 0$ and $\phi_S = 0$. This implies that all tensors in $\mathcal{D}_n^\phi$ are zero. Conversely, if all tensors in $\mathcal{D}_n^\phi$ are zero then all tensors in $\mathcal{D}_{mp}^S$ are zero and by Xanthopoulos' work [17] the spacetime is static. So we are left with $\mathcal{D}_n^S$ as the free data in the static case.

Friedrich [8] has given the same result for the static case using a different conformal metric. Let us assume for now that we are in the static case, then Friedrich uses a metric $\breve{h}$, which is conformally related to our metric h by

$$\breve{h} = \breve{\Omega}^2 h \,, \tag{17}$$

where

$$\breve{\Omega} = \frac{4\left[(1 + m^2\Omega)^{\frac{1}{2}} + m\Omega^{\frac{1}{2}}\right]}{\left[(1 + m^2\Omega)^{\frac{1}{2}} + m\Omega^{\frac{1}{2}} + 1\right]^2} \,.$$

Using $\breve{h}$ he defines a sequence of symmetric trace-free tensors $\breve{\mathcal{D}}_n$ in the same way as we defined $\mathcal{D}_n^S$ in (10). He shows that imposing estimates of the type (14) on the tensors in $\breve{\mathcal{D}}_n$ is necessary and sufficient for the existence of an asymptotically flat static vacuum solution of the Einstein's equations. To see that this result is equivalent to our result in the static case, we have to show that having estimates of the type (14) on the tensors in $\mathcal{D}_n^S$ imply estimates of the same type on the tensors in $\breve{\mathcal{D}}_n$ and vice versa. This is done through Theorem 1.1 and relation (17). If the tensors in $\mathcal{D}_n^S$ satisfy estimates of the type (14) then there exist h and Ω analytic, and then $\breve{h}$ given by (17) is also analytic, thus the tensors in $\breve{\mathcal{D}}_n$ satisfy estimates of the type (14), the converse is shown in the same way using Friedrich's result. Hence this work generalizes the work by Friedrich [8] from the static to the stationary case. The procedure that we use in the present work follows similar steps and several of the technics in [8] will be used. For completeness we include them.

Theorem 1.1 will be proven in terms of the conformal metric h. Thus we shall express in Section 2 the stationary vacuum field equations as 'conformal stationary vacuum field equations'. In Section 3 we show, by going to the space-spinor formalism, that the abstract null data indeed determine the expansion coefficients of a certain type of formal expansions of solutions to the conformal stationary vacuum field equations uniquely. Showing convergence in this way appears difficult,

however. For this reason the underlying geometry of the problem is used in Section 4 to cast the problem in a certain setting, where it becomes a characteristic initial value problem. The setting is necessarily singular, as the set where data is prescribed contains a vertex, but the convergence problem can now be handled. In Section 5 it is shown how to determine a formal solution to a subset of the conformal field equations from a given set of abstract null data. Then, in Section 6, the convergence of the series so obtained is shown. In Section 7 it is shown that the obtained solution satisfy the full set of conformal field equations. Finally, in Section 8, the convergence result is translated into a gauge which is regular near i, allowing us to prove Theorem 1.1.

2. The stationary field equations in the conformal setting

The existence problem will be analyzed completely in terms of the conformally rescaled metric h, so we need to express the stationary field equations in terms of the conformal fields. By a constant conformal rescaling it can always be achieved that $m = 1$. For simplicity we use this scale from now on.

If we directly transform the fields in (2) and (3) we arrive at a system of equations that is singular at i. To overcome this problem we follow the work of Beig and Simon [3]. Using (11) as the conformal factor, which together with (5) imply

$$\Omega = \phi_M^2 + \phi_S^2 - 1\,,\tag{18}$$

and standard formulae for conformal transformations, they rewrite the stationary field equations, arriving at the following equivalent system of equations:

$$\Delta\phi_A = -\frac{1}{2}\left[R - \frac{5}{2}D_a\Omega D^a\Omega + 10(1+\Omega)\pi_a{}^a\right]\phi_A\,,\quad \text{A=M,S}\,,$$

$$D_aD_b\Omega = -\Omega R_{ab} - \frac{1}{3}h_{ab}R + \left(\Omega + \frac{2}{3}\right)h_{ab}D_c\Omega D^c\Omega$$
$$-4\left(\Omega + \frac{2}{3}\right)(\Omega+1)h_{ab}\pi_c{}^c - \frac{1}{2}(\Omega-1)D_a\Omega D_b\Omega + 2\Omega^2\pi_{ab}\,,$$

$$D_aR = 7D^b\Omega D_aD_b\Omega + 3R_{ab}D^b\Omega + 4(3\Omega - 2)\pi_b{}^bD_a\Omega$$
$$-\frac{3}{2}D_b\Omega D^b\Omega D_a\Omega - 6\Omega\pi_{ab}D^b\Omega - 2(7\Omega+4)D_a\pi_b{}^b\,,$$

$$D_{[c}R_{b]a} = 2(3\Omega - 1)\pi_d{}^d h_{a[b}D_{c]}\Omega - h_{a[b}D_{c]}\Omega D_d\Omega D^d\Omega$$
$$-2(\Omega-1)h_{a[b}\pi_{c]d}D^d\Omega - 2(2\Omega+1)h_{a[b}D_{c]}\pi_d{}^d$$
$$+2h_{a[b}D_{c]}D_d\Omega D^d\Omega + \frac{1}{2}D_{[c}\Omega D_{b]}D_a\Omega - (\Omega-4)\pi_{a[b}D_{c]}\Omega$$
$$+2\Omega D_{[c}\pi_{b]a} + \frac{1}{2}R_{a[b}D_{c]}\Omega + h_{a[b}R_{c]d}D^d\Omega\,,$$

where

$$\pi_{ab} \equiv D_a\phi_M D_b\phi_M + D_a\phi_S D_b\phi_S \tag{19}$$

has been introduced just as a useful notation.

These equations are regular even at i. They form a quasi-linear, overdetermined system of PDE's which implies, by applying formal derivatives to some of the equations, elliptic equations for all unknowns in a suitable gauge. Considering the fall-off conditions on the fields, Beig and Simon [3] deduced a certain smoothness of the conformal fields at i. Invoking a general theorem of Morrey on elliptic systems of this type they concluded that the solutions are in fact real analytic at i. Later Kennefick and O'Murchadha [12] showed that the fall-off conditions are reasonable, as they are implied by the spacetime being asymptotically flat. To avoid introducing additional constraints by taking derivatives, we shall deal with the system as it is.

For our purposes it is convenient to make some changes to this system. We separate the Ricci tensor into its trace free part and the Ricci scalar,

$$R_{ab} = S_{ab} + \frac{1}{3}h_{ab}R. \tag{20}$$

We also get rid of π_{ab} by using (19) in the other equations. From (18) we see that Ω, ϕ_M and ϕ_S are not independent, we use this equation to get rid of ϕ_M in the other equations. With these changes and the change of notation $\phi_S \to \phi$ the system of equations takes the form

$$\Delta\phi = -\phi\left\{\frac{1}{2}R + \frac{5}{1+\Omega-\phi^2}\left[\frac{1}{4}\phi^2 D^a\Omega D_a\Omega \right.\right. \tag{21}$$

$$\left.\left. -(1+\Omega)\phi D^a\Omega D_a\phi + (1+\Omega)^2 D^a\phi D_a\phi\right]\right\},$$

$$D_aD_b\Omega = -\Omega S_{ab} - \frac{1}{3}(1+\Omega)h_{ab}R \tag{22}$$

$$+ \frac{1}{1+\Omega-\phi^2}\left\{\frac{1}{2}\left[1+(-1+\Omega)\phi^2\right]D_a\Omega D_b\Omega\right.$$

$$-\frac{1}{3}(2+3\Omega)\phi^2 h_{ab}D^c\Omega D_c\Omega - 2\Omega^2\phi D_{(a}\Omega D_{b)}\phi$$

$$+\frac{4}{3}(1+\Omega)(2+3\Omega)\phi h_{ab}D^c\Omega D_c\phi + 2\Omega^2(1+\Omega)D_a\phi D_b\phi$$

$$\left.-\frac{4}{3}(1+\Omega)^2(2+3\Omega)h_{ab}D^c\phi D_c\phi\right\},$$

$$D_aR = \frac{1}{1+\Omega-\phi^2}\left\{2(4+7\Omega)\phi D^b\Omega D_b D_a\phi - 4(1+\Omega)(4+7\Omega)D^b\phi D_b D_a\phi\right. \tag{23}$$

$$+\left[3+(-3+7\Omega)\phi^2\right]D^b\Omega S_{ba} - 2\Omega(4+7\Omega)\phi D^b\phi S_{ba}$$

$$\left.+\frac{1}{3}(4+7\Omega)\phi^2 RD_a\Omega - \frac{2}{3}(1+\Omega)(4+7\Omega)\phi RD_a\phi\right\}$$

$$+ \frac{1}{3(1+\Omega-\phi^2)^2}\left\{\frac{1}{2}\phi^2\left[-12+(40+21\Omega)\phi^2\right]D^b\Omega D_b\Omega D_a\Omega\right.$$

$$-2\phi\left[-18(1+\Omega)+(46+61\Omega+21\Omega^2)\phi^2\right]D^b\Omega D_b\phi D_a\Omega$$

$$+2(1+\Omega)\left[-24(1+\Omega)+(52+61\Omega+21\Omega^2)\phi^2\right]D^b\phi D_b\phi D_a\Omega$$

$$-\phi\left[12(1+\Omega)+(16+61\Omega+21\Omega^2)\phi^2\right]D^b\Omega D_b\Omega D_a\phi$$

$$+4(1+\Omega)\left[6(1+\Omega)+(22+61\Omega+21\Omega^2)\phi^2\right]D^b\Omega D_b\phi D_a\phi$$

$$\left.-4(1+\Omega)^2(28+61\Omega+21\Omega^2)\phi D^b\phi D_b\phi D_a\phi\right\},$$

$$D_{[c}S_{b]a} = \frac{1}{1+\Omega-\phi^2}\left\{\Omega\phi D_a D_{[b}\phi D_{c]}\Omega - 2\Omega(1+\Omega)D_a D_{[b}\phi D_{c]}\phi\right. \tag{24}$$

$$-\frac{2}{3}(1+\Omega)\phi h_{a[b}D_{c]}D_d\phi D^d\Omega + \frac{4}{3}(1+\Omega)^2 h_{a[b}D_{c]}D_d\phi D^d\phi$$

$$+\frac{1}{2}\left[1+(-1+\Omega)\phi^2\right]S_{a[b}D_{c]}\Omega - \Omega^2\phi S_{a[b}D_{c]}\phi$$

$$-\frac{1}{3}\Omega\phi^2 h_{a[b}S_{c]d}D^d\Omega + \frac{2}{3}\Omega(1+\Omega)\phi h_{a[b}S_{c]d}D^d\phi$$

$$+\frac{1}{18}(-2+\Omega)\phi^2 R h_{a[b}D_{c]}\Omega - \frac{1}{9}(-2+\Omega)(1+\Omega)\phi R h_{a[b}D_{c]}\phi$$

$$\left.+2\phi D_a\Omega D_{[b}\Omega D_{c]}\phi - 4(1+\Omega)D_a\phi D_{[b}\Omega D_{c]}\phi\right\}$$

$$+\frac{1}{9(1+\Omega-\phi^2)^2}\left\{\frac{1}{2}\phi^2\left[3+2(-5+3\Omega)\phi^2\right]h_{a[b}D_{c]}\Omega D^d\Omega D_d\Omega\right.$$

$$-\phi\left[6(1+\Omega)+(-13-4\Omega+6\Omega^2)\phi^2\right]h_{a[b}D_{c]}\phi D^d\Omega D_d\Omega$$

$$-2(-7-4\Omega+6\Omega^2)\phi^3 h_{a[b}D_{c]}\Omega D^d\Omega D_d\phi$$

$$+4(1+\Omega)^2\left[3+2(-5+3\Omega)\phi^2\right]h_{a[b}D_{c]}\phi D^d\Omega D_d\phi$$

$$+2(1+\Omega)\left[-3(1+\Omega)+(-4-4\Omega+6\Omega^2)\phi^2\right]h_{a[b}D_{c]}\Omega D^d\phi D_d\phi$$

$$\left.-4(1+\Omega)^2(-7-4\Omega+6\Omega^2)\phi h_{a[b}D_{c]}\phi D^d\phi D_d\phi\right\}.$$

Besides (21), (22), (23), (24) we need an equation for the metric or for the frame field and the connection coefficients. This equation is just (20),

$$R_{ab}[h] = S_{ab} + \frac{1}{3}h_{ab}R, \tag{25}$$

where the expression on the left hand side is understood as the Ricci operator acting on the metric h.

The system of equations (25), (21), (22), (23), (24), together with conditions (4) and the condition

$$R|_i = -(6+8D^a\phi D_a\phi)|_i, \tag{26}$$

implied by (4), will be referred to as the *conformal stationary vacuum field equations* for the unknown fields

$$h_{ab}, \phi, \Omega, R, S_{ab}. \tag{27}$$

3. The exact sets of equations argument

To see that it is possible to construct solutions to the conformal stationary vacuum field equations from the null data we study expansions of the conformal fields (27) in normal coordinates.

We assume from now on N to be small enough to coincide with a convex h-normal neighborhood of i. Let $c_{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, be an h-orthonormal frame field on N which is parallelly transported along the h-geodesics through i and let x^a denote normal coordinates centered at i so that $c^b{}_{\mathbf{a}} \equiv \langle dx^b, c_{\mathbf{a}} \rangle = \delta^b{}_{\mathbf{a}}$ at i. We refer to such a frame as *normal frame centered at i*. Its dual frame will be denoted by $\chi^{\mathbf{c}} = \chi^{\mathbf{c}}{}_b dx^b$. In the following all tensor fields, except the frame field $c_{\mathbf{a}}$ and the coframe field $\chi^{\mathbf{c}}$, will be expressed in terms of this frame field, so that the metric is given by $h_{\mathbf{ab}} \equiv h(c_{\mathbf{a}}, c_{\mathbf{b}}) = -\delta_{\mathbf{ab}}$. With $D_{\mathbf{a}} \equiv D_{c_{\mathbf{a}}}$ denoting the covariant derivative in the $c_{\mathbf{a}}$ direction, the connection coefficients with respect to $c_{\mathbf{a}}$ are defined by $D_{\mathbf{a}} c_{\mathbf{c}} = \Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}} c_{\mathbf{b}}$.

An analytic tensor field $T_{\mathbf{a}_1 \dots \mathbf{a}_k}$ on N has in the normal coordinates x^a a *normal expansion* at i, which can be written

$$T_{\mathbf{a}_1 \dots \mathbf{a}_k}(x) = \sum_{p \geq 0} \frac{1}{p!} x^{c_p} \dots x^{c_1} D_{\mathbf{c}_p} \dots D_{\mathbf{c}_1} T_{\mathbf{a}_1 \dots \mathbf{a}_k}(i), \tag{28}$$

where we assume from now on that the summation convention does not distinguish between bold face and other indices.

Since $h_{\mathbf{ab}} = -\delta_{\mathbf{ab}}$, it remains to be seen how to obtain normal expansions for

$$\phi, \Omega, R, S_{\mathbf{ab}}, \tag{29}$$

using the field equations and the null data. The algebra necessary for doing this simplifies considerably in the space-spinor formalism. To do the transition we introduce the constant van der Waerden symbols $\alpha^{AB}{}_a$, $\alpha^a{}_{AB}$, $a = 1, 2, 3$, $A, B = 0, 1$, which are symmetric in AB and whose components, if read as matrices, are

$$\alpha^{AB}{}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha^{AB}{}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \quad \alpha^{AB}{}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\alpha^1{}_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha^2{}_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \alpha^3{}_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The relation between tensors *given in the frame $c_{\mathbf{a}}$* and space-spinors is made by

$$T^{\mathbf{a}_1 \dots \mathbf{a}_p}{}_{\mathbf{b}_1 \dots \mathbf{b}_q} \to T^{A_1 B_1 \dots A_p B_p}{}_{C_1 D_1 \dots C_q D_q} \equiv T^{\mathbf{a}_1 \dots \mathbf{a}_p}{}_{\mathbf{b}_1 \dots \mathbf{b}_q} \alpha^{A_1 B_1}{}_{a_1} \dots \alpha^{b_q}{}_{C_q D_q}.$$

With the summation rule also applying to capital indices we get

$$\delta^b{}_a = \alpha^b{}_{AB}\alpha^{AB}{}_a\,, \quad -\delta_{ab}\alpha^a{}_{AB}\alpha^b{}_{CD} = -\epsilon_{A(C}\epsilon_{D)B} \equiv h_{ABCD}\,,$$
$$a,b = 1,2,3\,, \quad A,B,C,D = 0,1\,,$$

where the constant ϵ-spinor satisfies $\epsilon_{AB} = -\epsilon_{BA}$, $\epsilon_{01} = 1$. It is used to move indices according to the rules $\iota_B = \iota^A\epsilon_{AB}$, $\iota^A = \epsilon^{AB}\iota_B$, so that $\epsilon_A{}^B$ corresponds to the Kronecker delta.

As the spinors are in general complex, we need a way to sort out those that arise from real tensors. For this we define

$$\tau^{AA'} = \epsilon_0{}^A\epsilon_0{}^{A'} + \epsilon_1{}^A\epsilon_1{}^{A'}\,.$$

Primed indices take values 0, 1 and the summation rule also applies to them. A bar denotes complex conjugation and indices acquire a prime under complex conjugation, an exception being $\epsilon_{A'B'}$, the complex conjugate of ϵ_{AB}. We define

$$\xi^+_{A...H} = \tau_A{}^{A'}..\tau_H{}^{H'}\bar{\xi}_{A'...H'}\,.$$

Then a space spinor field $T_{A_1B_1...A_pB_p} = T_{(A_1B_1)...(A_pB_p)}$ arises from a real tensor field $T_{\mathbf{a}_1...\mathbf{a}_p}$ if and only if

$$T_{A_1B_1...A_pB_p} = (-1)^p T^+_{A_1B_1...A_pB_p}\,. \tag{30}$$

Any spinor field $T_{A...H}$ admits a decomposition into products of totally symmetric spinor fields and epsilon spinors which can be written schematically in the form

$$T_{A...H} = T_{(A...H)} + \sum \epsilon's \times \text{symmetrized contractions of } T\,. \tag{31}$$

It will be important that if $T_{A_1B_1...A_pB_p}$ arises from $T_{\mathbf{a}_1...\mathbf{a}_p}$ then

$$T_{(A_1B_1...A_pB_p)} = \mathcal{C}(T_{\mathbf{a}_1...\mathbf{a}_p})\alpha^{a_1}{}_{A_1B_1}\cdots\alpha^{a_p}{}_{A_pB_p}\,.$$

To discuss vector analysis in terms of spinors, a complex frame field and its dual 1-form field are defined by

$$c_{AB} = \alpha^a{}_{AB}c_{\mathbf{a}}\,, \quad \chi^{AB} = \alpha^{AB}{}_a\chi^{\mathbf{a}}\,,$$

so that $h(c_{AB}, c_{CD}) = h_{ABCD}$. From this one sees that c_{00} and c_{11} are null vectors orthogonal to c_{01}. The derivative of a function f in the direction of c_{AB} is denoted by $c_{AB}(f) = f_{,a}c^a{}_{AB}$ and the spinor connection coefficients are defined by

$$\Gamma_{AB}{}^C{}_D = \frac{1}{2}\Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}}\alpha^a{}_{AB}\alpha^{CH}{}_b\alpha^c{}_{DH}\,, \quad \text{so that} \quad \Gamma_{ABCD} = \Gamma_{(AB)(CD)}\,.$$

The covariant derivative of a spinor field ι^A is then given by

$$D_{AB}\iota^C = c_{AB}(\iota^C) + \Gamma_{AB}{}^C{}_D\iota^D\,.$$

If it is required to satisfy the Leibniz rule with respect to tensor products, then covariant derivatives in the $c_{\mathbf{a}}$-frame formalism translate under contractions with

the van der Waerden symbols into spinor covariant derivatives and vice versa. We also have

$$(D_{CD}D_{EF} - D_{EF}D_{CD})\iota^A = r^A{}_{BCDEF}\iota^B, \tag{32}$$

$$r_{ABCDEF} = \frac{1}{2}\left[\left(S_{ABCE} - \frac{1}{6}Rh_{ABCE}\right)\epsilon_{DF} \tag{33}\right.$$
$$\left. + \left(S_{ABDF} - \frac{1}{6}Rh_{ABDF}\right)\epsilon_{CE}\right],$$

where R is the Ricci scalar of h and $S_{ABCD} = S_{\mathbf{ab}}\alpha^a_{AB}\alpha^b_{CD} = S_{(ABCD)}$ represents the trace free part of the Ricci tensor of h.

Equations (21), (24) take in the space-spinor formalism the form

$$D^P{}_B D_{AP}\phi = -\frac{1}{4}\epsilon_{AB}\phi\left(R + \frac{10}{1+\Omega-\phi^2}\left[\frac{1}{4}\phi^2 D_{PQ}\Omega D^{PQ}\Omega \tag{34}\right.\right.$$
$$\left.\left. -(1+\Omega)\phi D^{PQ}\Omega D_{PQ}\phi + (1+\Omega)^2 D_{PQ}\phi D^{PQ}\phi\right]\right),$$

$$D^P{}_A S_{BCDP} \tag{35}$$
$$= \frac{1}{1+\Omega-\phi^2}\left\{\Omega\phi D_A{}^P\Omega D_{(BC}D_{D)P}\phi - 2\Omega(1+\Omega)D_A{}^P\phi D_{(BC}D_{D)P}\phi\right.$$
$$+ (1+\Omega)\phi D_{PQ}\Omega D_{(BC}D^{PQ}\phi\epsilon_{D)A} - 2(1+\Omega)^2 D_{PQ}\phi D_{(BC}D^{PQ}\phi\epsilon_{D)A}$$
$$+ \frac{1}{2}\left[1 + (-1+\Omega)\phi^2\right]D_A{}^P\Omega S_{PBCD} - \Omega^2\phi D_A{}^P\phi S_{PBCD}$$
$$+ \frac{1}{2}\Omega\phi^2 D^{PQ}\Omega S_{PQ(BC}\epsilon_{D)A} - \Omega(1+\Omega)\phi D^{PQ}\phi S_{PQ(BC}\epsilon_{D)A}$$
$$+ \frac{1}{6}(1+\Omega)\phi^2 R D_{(BC}\Omega\epsilon_{D)A} - \frac{1}{3}(1+\Omega)^2\phi R D_{(BC}\phi\epsilon_{D)A}$$
$$+ 2\phi\left(D_A{}^P\phi D_{P(B}\Omega D_{CD)}\Omega - D_A{}^P\Omega D_{P(B}\Omega D_{CD)}\phi\right)$$
$$\left. + 4(1+\Omega)\left(D_A{}^P\Omega D_{P(B}\phi D_{CD)}\phi - D_A{}^P\phi D_{P(B}\phi D_{CD)}\Omega\right)\right\}$$
$$+ \frac{1}{(1+\Omega-\phi^2)^2}\left\{\frac{1}{24}\phi^2\left[-6 + (20+3\Omega)\phi^2\right]D^{PQ}\Omega D_{PQ}\Omega D_{(BC}\Omega\epsilon_{D)A}\right.$$
$$- \frac{1}{6}(14 + 23\Omega + 3\Omega^2)\phi^3 D^{PQ}\Omega D_{PQ}\phi D_{(BC}\Omega\epsilon_{D)A}$$
$$+ \frac{1}{6}(1+\Omega)\left[6(1+\Omega) + (8 + 23\Omega + 3\Omega^2)\phi^2\right]D^{PQ}\phi D_{PQ}\phi D_{(BC}\Omega\epsilon_{D)A}$$
$$- \frac{1}{12}\phi\left[-12(1+\Omega) + (26 + 23\Omega + 3\Omega^2)\phi^2\right]D^{PQ}\Omega D_{PQ}\Omega D_{(BC}\phi\epsilon_{D)A}$$
$$+ \frac{1}{3}(1+\Omega)^2\left[-6 + (20+3\Omega)\phi^2\right]D^{PQ}\Omega D_{PQ}\phi D_{(BC}\phi\epsilon_{D)A}$$
$$\left. - \frac{1}{3}(1+\Omega)^2\left(14 + 23\Omega + 3\Omega^2\right)\phi D^{PQ}\phi D_{PQ}\phi D_{(BC}\phi\epsilon_{D)A}\right\}.$$

Equations (22), (23) are translated into the space-spinor formalism by making the index replacements $a \to AB$, $b \to CD$, $c \to EF$.

We use equations (34), (35), the spinor version of equations (22), (23) and the theory of 'exact sets of fields' to prove the following result.

Lemma 3.1. *Let there be two given sequences*

$$\hat{\mathcal{D}}_n^\phi = \{\psi_{A_1 B_1}, \psi_{A_2 B_2 A_1 B_1}, \psi_{A_3 B_3 A_2 B_2 A_1 B_1}, \ldots\},$$

$$\hat{\mathcal{D}}_n^S = \{\Psi_{A_2 B_2 A_1 B_1}, \Psi_{A_3 B_3 A_2 B_2 A_1 B_1}, \Psi_{A_4 B_4 A_3 B_3 A_2 B_2 A_1 B_1}, \ldots\},$$

of totally symmetric spinors satisfying the reality condition (30). *Assume that there exists a solution* h, ϕ, Ω, R, S_{ABCD} *to the conformal stationary field equations* (25), (21), (22), (23), (24) *satisfying* (4), (26) *so that the spinors given by* $\hat{\mathcal{D}}_n^\phi, \hat{\mathcal{D}}_n^S$ *coincide with the null data* $\mathcal{D}_n^{\phi*}, \mathcal{D}_n^{S*}$ *given by* (12), (13) *of the metric* h *in terms of an* h-*orthonormal normal frame centered at* i, *i.e.,*

$$\psi_{A_p B_p \ldots A_1 B_1} = D_{(A_p B_p} \ldots D_{A_1 B_1)} \phi(i), \quad p \geq 1, \tag{36}$$

$$\Psi_{A_p B_p \ldots A_1 B_1} = D_{(A_p B_p} \ldots D_{A_3 B_3} S_{A_2 B_2 A_1 B_1)}(i), \quad p \geq 2. \tag{37}$$

Then the coefficients of the normal expansions (28) *of the fields* (29), *i.e.*

$$D_{A_p B_p} \ldots D_{A_1 B_1} \phi(i), \qquad D_{A_p B_p} \ldots D_{A_1 B_1} \Omega(i),$$

$$D_{A_p B_p} \ldots D_{A_1 B_1} R(i), \qquad D_{A_p B_p} \ldots D_{A_1 B_1} S_{ABCD}(i), \quad p \geq 0,$$

are uniquely determined by the data $\hat{\mathcal{D}}_n^\phi, \hat{\mathcal{D}}_n^S$ *and satisfy the reality conditions.*

Proof. It holds $\phi(i) = 0$, $D_{AB}\phi(i) = \psi_{AB}$ and $S_{ABCD}(i) = \Psi_{ABCD}$ by assumption and the expansion coefficients for Ω and R of lowest order are given by (4) and (26). Assume the expansion coefficients of ϕ and Ω up to order p and the expansion coefficients of R and S_{ABCD} up to order $p-1$ are known.

To discuss the induction step we start with $D_{A_{p+1} B_{p+1}} \ldots D_{A_1 B_1} \phi(i)$ and its decomposition in the form (31). By assumption, the totally symmetric part of it is given by $\psi_{A_{p+1} B_{p+1} \ldots A_1 B_1}$. The other terms in the decomposition contain contractions. Let us consider A_i contracted with A_j. We can commute the operators $D_{A_i B_i}$ and $D_{A_j B_j}$ with other covariant derivatives, generating by (32) and (33) only terms of lower order, until we have

$$D_{A_{p+1} B_{p+1}} \ldots D_{A_{i+1} B_{i+1}} D_{A_{i-1} B_{i-1}} \ldots D_{A_{j+1} B_{j+1}} D_{A_{j-1} B_{j-1}}$$
$$\ldots D_{A_1 B_1} D^P{}_{B_i} D_{P B_j} \phi(i).$$

Equation (34) then shows how to express the resulting term by quantities of lower order that are already known.

For $D_{A_{p+1} B_{p+1}} \ldots D_{A_1 B_1} \Omega(i)$ and $D_{A_p B_p} \ldots D_{A_1 B_1} R(i)$ we just use the spinor versions of (22) and (23) to express them by quantities of lower order.

Finally, dealing with $D_{A_p B_p} \ldots D_{A_1 B_1} S_{CDEF}(i)$ is quite similar to $D_{A_{p+1} B_{p+1}} \ldots D_{A_1 B_1} \phi(i)$. The symmetric term is known by the data. If a contraction is performed between a derivative index and one of C, D, E, F then (35) is

used after interchanging derivatives. If the contraction is between two derivatives, the general identities

$$D_{H(A}D^H{}_{B)}S_{CDEF} = -2S_{H(CDE}S_{F)}{}^H{}_{AB} + \frac{1}{3}RS_{H(CDE}h_{F)}{}^H{}_{AB}\,,$$

$$D_{AB}D^{AB}S_{CDEF} = -2D_F{}^G D_G{}^H S_{CDEH} + 3S_{GH(CD}S_{E)F}{}^{GH} + \frac{1}{2}RS_{CDEF}\,,$$

implied by (32), (33), together with (35) show that the corresponding term can be expressed in terms of quantities of lower order. The induction step is completed.

That the expansion coefficients satisfy the reality condition is a consequence of the formalism and the fact that they are satisfied by the data.   □

In order to show the convergence of the formal series determined in the previous lemma we need to impose estimates on the free coefficients given by $\hat{\mathcal{D}}_n^\phi, \hat{\mathcal{D}}_n^S$. For this we have the following result.

Lemma 3.2. *A necessary condition for the formal series determined in Lemma* 3.1 *to be absolutely convergent near the origin is that the data given by* $\hat{\mathcal{D}}_n^\phi$, $\hat{\mathcal{D}}_n^S$ *satisfy estimates of the type*

$$|\psi_{A_p B_p \ldots A_1 B_1}| \le \frac{p!M}{r^p}\,, \quad p = 1, 2, 3, \ldots\,, \tag{38}$$

$$|\Psi_{A_p B_p \ldots A_1 B_1 CDEF}| \le \frac{p!M}{r^p}\,, \quad p = 0, 1, 2, \ldots\,, \tag{39}$$

with some constants $M, r > 0$.

We skip the proof of this lemma because it uses the same argument as the proof of Lemma 3.2 in [8].

Lemma 3.1 shows that the null data determines a formal solution to the stationary field equations. As shown by Beig and Simon [3], the multipole moments do the same. Thus there is a bijective map Θ from the null data to the multipoles sequences, $\Theta : \{\mathcal{D}_n^\phi, \mathcal{D}_n^S\} \to \{\mathcal{D}_{mp}^M, \mathcal{D}_{mp}^S\}$. Instead of using this argument, we can try to gain more information on the relation starting from (6), (7). It is convenient to work in space-spinor form, that means that we are using the h-orthonormal frame and normal coordinates previously defined. We get the following result.

Lemma 3.3. *The spinor fields* $P_{A_p B_p \ldots A_1 B_1}^M$, $P_{A_p B_p \ldots A_1 B_1}^S$, *near* i, *given by* (6), (7), *are of the form*

$$P_{A_p B_p \ldots A_1 B_1}^M \tag{40}$$

$$= -\frac{1}{2}\left(1 + \Omega - \phi^2\right)^{-\frac{1}{2}}\left(1 + 2\Omega - \phi^2\right) D_{(A_p B_p} \ldots D_{A_3 B_3} S_{A_2 B_2 A_1 B_1)}$$

$$- \left(1 + \Omega - \phi^2\right)^{-\frac{1}{2}} \phi D_{(A_p B_p} \ldots D_{A_1 B_1)}\phi$$

$$+ \frac{1}{2}\left(1 + \Omega - \phi^2\right)^{-\frac{3}{2}} \phi \left(p - 2\Omega^2\right) D_{(A_p B_p}\Omega D_{A_{p-1} B_{p-1}} \ldots D_{A_1 B_1)}\phi$$

$$- \left(1 + \Omega - \phi^2\right)^{-\frac{3}{2}} (1 + \Omega)\left(p - 2\Omega^2\right) D_{(A_p B_p} \phi D_{A_{p-1} B_{p-1}} \dots D_{A_1 B_1)} \phi$$
$$+ F^M_{A_p B_p \dots A_1 B_1}, \quad p \geq 3,$$

$$P^S_{A_p B_p \dots A_1 B_1} = D_{(A_p B_p} \dots D_{A_1 B_1)} \phi + F^S_{A_p B_p \dots A_1 B_1}, \quad p \geq 2, \tag{41}$$

with symmetric spinor-valued functions $F^M_{A_p B_p \dots A_1 B_1}$ and $F^S_{A_p B_p \dots A_1 B_1}$. The function $F^M_{A_p B_p \dots A_1 B_1}$, $p \geq 3$, is at each point a real linear combination of symmetrized tensor products of

$$D_{(A_{q-1} B_{q-1}} \dots D_{A_1 B_1)} \phi, \quad D_{(A_q B_q} \dots D_{A_3 B_3} S_{A_2 B_2 A_1 B_1)}, \quad D_{AB} \Omega, \quad 2 \leq q \leq p-1,$$

with coefficients that depend on Ω and ϕ. The function $F^S_{A_p B_p \dots A_1 B_1}$, $p \geq 2$, is a real linear combination of symmetrized tensor products of

$$D_{(A_{q-2} B_{q-2}} \dots D_{A_1 B_1)} \phi, \quad D_{(A_q B_q} \dots D_{A_3 B_3} S_{A_2 B_2 A_1 B_1)}, \quad 2 \leq q \leq p.$$

Proof. From (18) we get

$$\phi_M = \left(1 + \Omega - \phi^2\right)^{\frac{1}{2}},$$

and by direct calculations from (6), (7) we see that (40) is valid for $p = 3$ and that (41) is valid for $p = 2$, with the stated properties for $F^M_{A_3 B_3 A_2 B_2 A_1 B_1}$ and $F^S_{A_2 B_2 A_1 B_1}$. Assuming that the lemma is true for $p \leq k$, inserting (40) and (41) into the recursion relation (7), and using the symmetrized spinor version of (22), we see that the lemma is true for $p = k + 1$. $\square$

Using (6), (7), (8) and the identification (36), (37) we get for the lower order multipoles

$$\nu^M_{A_1 B_1} = 0, \quad \nu^M_{A_2 B_2 A_1 B_1} = -\frac{1}{2} \Psi_{A_2 B_2 A_1 B_1} - \psi_{(A_2 B_2} \psi_{A_1 B_1)}, \tag{42}$$

$$\nu^M_{A_3 B_3 A_2 B_2 A_1 B_1} = -\frac{1}{2} \Psi_{A_3 B_3 A_2 B_2 A_1 B_1} - 3\psi_{(A_3 B_3} \psi_{A_2 B_2 A_1 B_1)}, \tag{43}$$

$$\nu^S_{A_1 B_1} = \psi_{A_1 B_1}, \quad \nu^S_{A_2 B_2 A_1 B_1} = \psi_{A_2 B_2 A_1 B_1}. \tag{44}$$

Also restricting (40) and (41) to i and with the identification (36), (37) we get

$$\nu^M_{A_p B_p \dots A_1 B_1} = -\frac{1}{2} \Psi_{A_p B_p \dots A_1 B_1} - p\psi_{(A_p B_p} \psi_{A_{p-1} B_{p-1} \dots A_1 B_1)} \tag{45}$$
$$+ f^M_{A_p B_p \dots A_1 B_1}, \quad p \geq 3,$$

$$\nu^S_{A_p B_p \dots A_1 B_1} = \psi_{A_p B_p \dots A_1 B_1} + f^S_{A_p B_p \dots A_1 B_1}, \quad p \geq 2, \tag{46}$$

where $f^M_{A_p B_p \dots A_1 B_1}$, $p \geq 3$, is a real linear combination of symmetrized tensor products of

$$\psi_{A_{q-1} B_{q-1} \dots A_1 B_1}, \quad \Psi_{A_q B_q \dots A_1 B_1}, \quad 2 \leq q \leq p-1,$$

and $f^S_{A_p B_p \dots A_1 B_1}$, $p \geq 2$, is a real linear combination of symmetrized tensor products of

$$\psi_{A_{q-2} B_{q-2} \dots A_1 B_1}, \quad \Psi_{A_q B_q \dots A_1 B_1}, \quad 2 \leq q \leq p.$$

Equations (42), (43), (44), (45) and (46) give a nonlinear map Θ, that can be read as a map

$$\Theta : \{\hat{\mathcal{D}}_n^\phi, \hat{\mathcal{D}}_n^S\} \to \{\hat{\mathcal{D}}_{mp}^M, \hat{\mathcal{D}}_{mp}^S\}$$

of the set of *abstract null data* into the set of *abstract multipoles* (i.e., sequences of symmetric spinors not necessarily derived from a metric). It is now possible to show that the map can be inverted.

Corollary 3.4. *The map Θ that maps sequences of abstract null data $\{\hat{\mathcal{D}}_n^\phi, \hat{\mathcal{D}}_n^S\}$ onto sequences of abstract multipoles $\{\hat{\mathcal{D}}_{mp}^M, \hat{\mathcal{D}}_{mp}^S\}$ is bijective.*

Proof. From (43), (44) we see that $f_{A_3 B_3 A_2 B_2 A_1 B_1}^M = 0$, $f_{A_2 B_2 A_1 B_1}^S = 0$, with this and the stated properties for $f_{A_p B_p \dots A_1 B_1}^M$ and $f_{A_p B_p \dots A_1 B_1}^S$ an inverse for Θ can be constructed inverting the relations (45) and (46) recursively. $\square$

Hence, for a given metric h, the sequences of multipoles and the sequences of null data in a given standard frame carry the same information on h. As said, we prefer to work with the null data because they are linear in ϕ and S_{ABCD}.

4. The characteristic initial value problem

After showing that the null data determine the solution, one would have to show that the estimates (38), (39) imply Cauchy estimates for the expansion coefficients

$$|D_{A_p B_p} \dots D_{A_1 B_1} T(i)| \le \frac{p! M}{r^p} \,, \quad A_p, B_p, \dots, A_1, B_1 = 0, 1 \,, \quad p = 0, 1, 2, \dots \,,$$

where T is any of $\phi, \Omega, R, S_{ABCD}$. This would ensure the convergence of the normal expansion at i. The induction procedure used so far for calculating the expansion coefficients from the null data generates additional non-linear terms each time one interchanges a derivative or uses the conformal field equations. Thus, it does not seem suited for deriving estimates. Instead, we use the intrinsic geometric nature of the problem and the data to formulate the problem as a boundary value problem to which Cauchy-Kowalevskaya type arguments apply.

As the fields $h, \phi, \Omega, R, S_{ABCD}$ are real analytic in the normal coordinates x^a and a standard frame c_{AB} centered at i, they can be extended near i by analyticity into the complex domain and considered as holomorphic fields on a complex analytic manifold N_c. Choosing N_c to be a sufficiently small neighborhood of i, we can assume the extended coordinates, again denoted by x^a, to define a holomorphic coordinate system on N_c which identifies N_c with an open neighborhood of the origin in $\mathbb{C}^3$. The original manifold N is then a real, 3-dimensional, real analytic submanifold of the real, 6-dimensional, real analytic manifold underlying N_c. Under the analytic extension the main differential geometric concepts and formulas remain valid. The coordinates x^a and the extended frame, again denoted by c_{AB}, satisfy the same defining equations and the extended fields, denoted again by $h, \phi, \Omega, R, S_{ABCD}$, satisfy the conformal stationary vacuum field equations as before.

The analytic function $\Gamma = \delta_{ab} x^a x^b$ on N extends to a holomorphic function on N_c. On N it vanishes only at i, but the set

$$\mathcal{N}_i = \left\{ p \in N_c | \Gamma(p) = 0 \right\},$$

is an irreducible analytical set such that $\mathcal{N}_i \backslash \{i\}$ is a 2-dimensional complex submanifold of N_c. It is the cone swept out by the complex null geodesics through i and we will refer to it as the *null cone at i*.

Now let $u \to x^a(u)$ be a null geodesic through i such that $x^a(0) = 0$. Its tangent vector is then of the form $\dot{x}^{AB} = \iota^A \iota^B$ with a spinor field $\iota^A = \iota^A(u)$ satisfying $D_{\dot{x}} \iota^A = 0$ along the geodesic. Then

$$\phi(u) = \phi\big(x(u)\big), \tag{47}$$

$$S_0(u) = \dot{x}^a \dot{x}^b S_{ab}\big(x(u)\big) = \iota^A \iota^B \iota^C \iota^D S_{ABCD}\big(x(u)\big), \tag{48}$$

are analytic functions of u with Taylor expansion

$$\phi(u) = \sum_{p=0}^{\infty} \frac{1}{p!} u^p \frac{d^p \phi}{du^p}(0), \quad S_0(u) = \sum_{p=0}^{\infty} \frac{1}{p!} u^p \frac{d^p S_0}{du^p}(0),$$

where

$$\frac{d^p \phi}{du^p}(0) = \dot{x}^{a_p} \ldots \dot{x}^{a_1} D_{a_p} \ldots D_{a_1} \phi(0) = \iota^{A_p} \iota^{B_p} \ldots \iota^{A_1} \iota^{B_1} D_{(A_p B_p} \ldots D_{A_1 B_1)} \phi(i),$$

$$\frac{d^p S_0}{du^p}(0) = \iota^{A_p} \iota^{B_p} \ldots \iota^{A_1} \iota^{B_1} \iota^C \iota^D \iota^E \iota^F D_{(A_p B_p} \ldots D_{A_1 B_1} S_{CDEF)}(i).$$

This shows that knowing these expansion coefficients for initial null vectors $\iota^A \iota^B$ covering an open subset of the null directions at i is equivalent to knowing the null data $\hat{\mathcal{D}}_n^\phi, \hat{\mathcal{D}}_n^S$ of the metric h.

Our problem can thus be formulated as the boundary value problem for the conformal stationary vacuum equations with data given by the functions (47), (48) on $\mathcal{N}_i$, where the $\iota^A \iota^B$ are parallelly propagated null vectors tangent to $\mathcal{N}_i$.

$\mathcal{N}_i$ is not a smooth hypersurface but an analytic set with a vertex at the point i, and we need a setting in which the mechanism of calculating the expansion coefficients allows us to derive estimates on the coefficients from the conditions imposed on the data. That is done in the following subsections.

4.1. The geometric gauge

We need to choose a gauge suitably adapted to the singular set $\mathcal{N}_i$. The coordinates and the frame field will then necessarily be singular and the frame will no longer define a smooth lift to the bundle of frames but a subset which becomes tangent to the fibres over some points.

We will use the principal bundle of normalized spin frames $SU(N) \xrightarrow{\pi} N$ with structure group $SU(2)$, which is the group of complex 2×2 matrices $(s^A{}_B)_{A,B=0,1}$ satisfying

$$\epsilon_{AB} s^A{}_C s^B{}_D = \epsilon_{CD}, \quad \tau_{AB'} s^A{}_C \bar{s}^{B'}{}_{D'} = \tau_{CD'}. \tag{49}$$

The $2:1$ covering homomorphism of $SU(2)$ onto $SO(3,\mathbb{R})$ is performed via

$$SU(2) \ni s^A{}_B \to s^a{}_b = \alpha^a{}_{AB} s^A{}_C s^B{}_D \alpha^{CD}{}_b \in SO(3,\mathbb{R})\,.$$

Under holomorphic extension the map above extends to a $2:1$ covering homomorphism of the group $SL(2,\mathbb{C})$ onto the group $SO(3,\mathbb{C})$, where $SL(2,\mathbb{C})$ denotes the group of complex 2×2 matrices satisfying only the first of conditions (49).

A point $\delta \in SU(N)$ is given by a pair of spinors $\delta = (\delta_0^A, \delta_1^A)$ at a given point of N which satisfies

$$\epsilon(\delta_A, \delta_B) = \epsilon_{AB}\,, \quad \epsilon(\delta_A, \delta_{B'}^+) = \tau_{AB'}\,, \tag{50}$$

and the action of the structure group is given for $s \in SU(2)$ by

$$\delta \to \delta \cdot s \quad \text{where} \quad (\delta \cdot s)_A = s^B{}_A \delta_B\,.$$

The projection π maps a frame δ into its base point in N. The bundle of spin frames is mapped by a $2:1$ bundle morphism $SU(N) \xrightarrow{p} SO(N)$ onto the bundle $SO(N) \xrightarrow{\pi'} N$ of oriented, orthonormal frames on N so that $\pi' \circ p = \pi$. For any spin frame δ we can identify by (50) the matrix $(\delta_B^A)_{A,B=0,1}$ with an element of the group $SU(2)$. With this reading the map p will be assumed to be realized by

$$SU(N) \ni \delta \to p(\delta)_{AB} = \delta_A^E \delta_B^F c_{EF} \in SO(N)\,,$$

where c_{AB} denotes the normal frame field on N introduced before. We refer to $p(\delta)$ as the frame associated with the spin frame δ.

Under holomorphic extension the bundle $SU(N) \xrightarrow{\pi} N$ is extended to the principal bundle $SL(N_c) \xrightarrow{\pi} N_c$ of spin frames $\delta = (\delta_0^A, \delta_1^A)$ at given points of N_c which satisfy only the first of conditions (50). Its structure group is $SL(2,\mathbb{C})$. The bundle $SU(N) \xrightarrow{\pi} N$ is embedded into $SL(N_c) \xrightarrow{\pi} N_c$ as a real analytic subbundle. The bundle morphism p extends to a $2:1$ bundle morphism, again denoted by p, of $SL(N_c) \xrightarrow{\pi} N_c$ onto the bundle $SO(N_c) \xrightarrow{\pi'} N_c$ of oriented, normalized frames of N_c with structure group $SO(3,\mathbb{C})$. We shall make use of several structures on $SM(N_c)$.

With each $\alpha \in sl(2,\mathbb{C})$, i.e., $\alpha = (\alpha^A{}_B)$ with $\alpha_{AB} = \alpha_{BA}$, is associated a *vertical vector field* Z_α tangent to the fibres, which is given at $\delta \in SL(N_c)$ by $Z_\alpha(\delta) = \frac{d}{dv}(\delta \cdot \exp(v\alpha))|_{v=0}$, where $v \in \mathbb{C}$ and $\exp$ denotes the exponential map $sl(2,\mathbb{C}) \to SL(2,\mathbb{C})$.

The $\mathbb{C}^3$-valued *soldering form* $\sigma^{AB} = \sigma^{(AB)}$ maps a tangent vector $X \in T_\delta SL(N_c)$ onto the components of its projection $T_\delta(\pi)X \in T_{\pi(\delta)}N_c$ in the frame $p(\delta)$ associated with δ so that $T_\delta(\pi)X = \langle \sigma^{AB}, X \rangle p(\delta)_{AB}$. It follows that $\langle \sigma^{AB}, Z_\alpha \rangle = 0$ for any vertical vector field Z_α.

The $sl(2,\mathbb{C})$-valued *connection form* $\omega^A{}_B$ on $SL(N_c)$ transforms with the adjoint transformation under the action of $SL(2,\mathbb{C})$ and maps any vertical vector field Z_α onto its generator so that $\langle \omega^A{}_B, Z_\alpha \rangle = \alpha^A{}_B$.

With $x^{AB} = x^{(AB)} \in \mathbb{C}^3$ is associated the *horizontal vector field* H_x on $SL(N_c)$ which is horizontal in the sense that $\langle \omega^A{}_B, H_x \rangle = 0$ and which satisfies

$\langle \sigma^{AB}, H_x \rangle = x^{AB}$. Denoting by $H_{AB}, A, B = 0, 1$, the horizontal vector fields satisfying $\langle \sigma^{AB}, H_{CD} \rangle = h^{AB}{}_{CD}$, it follows that $H_x = x^{AB} H_{AB}$. An integral curve of a horizontal vector field projects onto an h-geodesic and represents a spin frame field which is parallelly transported along this geodesic.

A holomorphic spinor field ψ on N_c is represented on $SL(N_c)$ by a holomorphic spinor-valued function $\psi_{A_1 \ldots A_j}(\delta)$ on $SL(N_c)$, given by the components of ψ in the frame δ. We shall use the notation $\psi_k = \psi_{(A_1 \ldots A_j)_k}, k = 0, .., j$, where $(\ldots \ldots)_k$ denotes the operation 'symmetrize and set k indices equal to 1 the rest equal to 0'. These functions completely specify ψ if ψ is symmetric. They are then referred to as the *essential components of ψ*.

4.2. The submanifold $\hat{N}$ of $SL(N_c)$

Using the available geometrical structure we construct a three-dimensional submanifold $\hat{N}$ of $SL(N_c)$ in such a way that it induces coordinates in N_c. By the construction procedure the induced coordinates are suitable adapted to the set $\mathcal{N}_i$.

We start by choosing a spin frame δ^* such that $\pi(\delta^*) = i$ and $p(\delta^*)_{AB} = c_{AB}$. The curve

$$\mathbb{C} \ni v \to \delta(v) = \delta^* \cdot s(v) \in SL(N_c) \,,$$

$$s(v) = exp(v\alpha) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in sl(2, \mathbb{C}) \,, \qquad (51)$$

defines a vertical, 1-dimensional, holomorphic submanifold I of $SL(N_c)$ on which v defines a coordinate. The associated family of frames $e_{AB}(v)$ at i is given by $e_{AB}(v) = s^C{}_A(v) s^D{}_B(v) c_{CD}$, and explicitly by

$$e_{00} = c_{00} + 2v c_{01} + v^2 c_{11} \,, \quad e_{01}(v) = c_{01} + v c_{11} \,, \quad e_{11}(v) = c_{11} \,.$$

We perform the following construction in a neighborhood of I. If it is chosen small enough all the following statements will be correct.

The set I is moved with the flow of H_{11} to obtain a holomorphic 2-manifold U_0 of $SL(N_c)$. We denote by w the parameter on the integral curves of H_{11} that vanishes on I, and we extend v to U_0 by assuming it to be constant on the integral curves of H_{11}. All these integral curves are mapped by π onto the null geodesic $\gamma(w)$ with affine parameter w and tangent vector $\gamma'(0) = c_{11}$ at $\gamma(0) = i$. The parameter v specifies which frame fields are parallelly propagated along γ.

U_0 is now moved with the flow of H_{00} to obtain a holomorphic 3-submanifold $\hat{N}$ of $SL(N_c)$. We denote by u the parameter on the integral curves of H_{00} that vanishes on U_0 and we extend v and w to $\hat{N}$ by assuming them to be constant along the integral curves of H_{00}. The functions $z^1 = u, z^2 = v, z^3 = w$ define holomorphic coordinates on $\hat{N}$. We denote again π the restriction of the projection to $\hat{N}$.

The projections of the integral curves of H_{00} with a fixed value of w sweep out, together with γ, the null cone $\mathcal{N}_{\gamma(w)}$ near $\gamma(w)$, which is generated by the null geodesics through the point $\gamma(w)$. On the null geodesics u is an affine parameter which vanishes at $\gamma(w)$ while v parametrizes the different generators. The set $W_0 = \{w = 0\}$ projects onto $\mathcal{N}_i \backslash \gamma$ and will define the initial data set for our

problem. The map π induces a biholomorphic diffeomorphism of $\hat{N}' \equiv \hat{N} \backslash U_0$ onto $\pi(\hat{N}')$. The singularity of the gauge at points of U_0 consists in π dropping rank on U_0, where $\partial_v = Z_\alpha$. The null curve $\gamma(w)$ will be referred to as the *the singular generator of $\mathcal{N}_i$ in the gauge determined by the spin frame δ^* resp. the corresponding frame c_{AB} at i.*

The soldering an the connection form pull back to holomorphic 1-forms on $\hat{N}$, which will be denoted again by σ^{AB} and $\omega^A{}_B$. If the pull back of the curvature form $\Omega^A{}_B = \frac{1}{2} r^A{}_{BCDEF} \sigma^{CD} \wedge \sigma^{EF}$ to $\hat{N}$ is denoted again by $\Omega^A{}_B$, then the soldering and the connection form satisfy the structural equations

$$d\sigma^{AB} = -\omega^A{}_C \wedge \sigma^{CB} - \omega^B{}_C \wedge \sigma^{AC}, \quad d\omega^A{}_B = -\omega^A{}_C \wedge \omega^C{}_B + \Omega^A{}_B.$$

Using the way in which $\hat{N}$ is constructed, and in terms of the coordinates z^a, we get $\sigma^{AB} = \sigma^{AB}{}_a dz^a$ on $\hat{N}'$, where

$$(\sigma^{AB}{}_a) = \begin{pmatrix} 1 & \sigma^{00}{}_2 & \sigma^{00}{}_3 \\ 0 & \sigma^{01}{}_2 & \sigma^{01}{}_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{O}(u^3) & \mathcal{O}(u^2) \\ 0 & u + \mathcal{O}(u^3) & \mathcal{O}(u^2) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{as} \quad u \to 0.$$

On $\hat{N}'$ there exist unique, holomorphic vector fields e_{AB} which satisfy

$$\langle \sigma^{AB}, e_{CD} \rangle = h^{AB}{}_{CD}.$$

If one writes $e_{AB} = e^a{}_{AB} \partial_{z^a}$, then

$$(e^a{}_{AB}) = \begin{pmatrix} 1 & e^1{}_{01} & e^1{}_{11} \\ 0 & e^2{}_{01} & e^2{}_{11} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{O}(u^2) & \mathcal{O}(u^2) \\ 0 & \frac{1}{2u} + \mathcal{O}(u) & \mathcal{O}(u) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{as} \quad u \to 0.$$

We shall write

$$e^a{}_{AB} = e^{*a}{}_{AB} + \hat{e}^a{}_{AB},$$

with singular part

$$e^{*a}{}_{AB} = \delta^a_1 \epsilon_A{}^0 \epsilon_B{}^0 + \delta^a_2 \frac{1}{u} \epsilon_{(A}{}^0 \epsilon_{B)}{}^1 + \delta^a_3 \epsilon_A{}^1 \epsilon_B{}^1,$$

and holomorphic functions $\hat{e}^a{}_{AB}$ on $\hat{N}$ which satisfy

$$\hat{e}^a_{AB} = \mathcal{O}(u) \quad \text{as} \quad u \to 0. \tag{52}$$

We define the connection coefficients on $\hat{N}'$ by $\omega^A{}_B = \Gamma_{CD}{}^A{}_B \sigma^{CD}$ with $\Gamma_{CDAB} \equiv \langle \omega_{AB}, e_{CD} \rangle$, so that $\Gamma_{ABCD} = \Gamma_{(AB)(CD)}$, and from the definition of the frame

$$\Gamma_{00AB} = 0 \quad \text{on} \quad \hat{N}, \quad \Gamma_{11AB} = 0 \quad \text{on} \quad U_0,$$

and it follows that

$$\Gamma_{ABCD} = \Gamma^*_{ABCD} + \hat{\Gamma}_{ABCD},$$

with singular part

$$\Gamma^*_{ABCD} = -\frac{1}{u} \epsilon_{(A}{}^0 \epsilon_{B)}{}^1 \epsilon_C{}^0 \epsilon_D{}^0,$$

and holomorphic functions $\hat{\Gamma}_{ABCD}$ on $\hat{N}$ which satisfy

$$\hat{\Gamma}_{ABCD} = \mathcal{O}(u) \quad \text{as} \quad u \to 0. \tag{53}$$

4.3. Tensoriality and expansion type

As the induced map π of $\hat{N}$ into N_c is singular on U_0, not every holomorphic function of the z^a can arise as a pull-back to $\hat{N}$ of a holomorphic function on N_c. The latter must have a special type of expansion in terms of the z^a which reflects the particular relation between the 'angular' coordinate v and the 'radial' coordinate u. We take from [8] the following definition and lemma.

Definition 4.1. A holomorphic function f on $\hat{N}$ is said to be of v-finite expansion type k_f, with k_f an integer, if it has in terms of the coordinates u, v, and w a Taylor expansion at the origin of the form

$$f = \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{2m+k_f} f_{m,n,p} u^m v^n w^p \,,$$

where it is assumed that $f_{m,n,p} = 0$ if $2m + k_f < 0$.

Lemma 4.1. Let $\phi_{A_1 \ldots A_j}$ be a holomorphic, symmetric, spinor-valued function on $SL(N_c)$. Then the restrictions of its essential components $\phi_k = \phi_{(A_1 \ldots A_j)_k}$, $0 \leq k \leq j$, to $\hat{N}$ satisfy

$$\partial_v \phi_k = (j - k)\phi_{k+1} \,, \quad k = 0, \ldots, j \,, \quad on \quad U_0 \,,$$

(where we set $\phi_{j+1} = 0$) and ϕ_k is of expansion type $j - k$.

4.4. The null data on W_0

As we have seen, prescribing the null data is equivalent to knowing ϕ and S_0 in the null cone. Now we need to know how this fit into our particular gauge. For this we derive an expansion of the restriction of ϕ and S_0 to the hypersurface W_0.

Consider the normal frame c_{AB} on N_c near i which agrees at i with the frame associated with δ^* and denote the null data of h in this frame by

$$\mathcal{D}_n^{\phi *} = \left\{ D^*_{(A_p B_p} \ldots D^*_{A_1 B_1)} \phi(i), \ p = 1, 2, 3, \ldots \right\} \,,$$
$$\mathcal{D}_n^{S *} = \left\{ D^*_{(A_p B_p} \ldots D^*_{A_1 B_1} S^*_{ABCD)}(i), \ p = 0, 1, 2, 3, \ldots \right\} \,.$$

Choose now a fixed value of v and consider $s(v)$ as in (51), then the vector $H_{00}(\delta^* \cdot s)$ projects onto the null vector $e_{00} = s^A{}_0 s^B{}_0 c_{AB}$ at i and is tangent to a null geodesic $\eta = \eta(u, v)$ on $\mathcal{N}_i$ with affine parameter u, $u = 0$ at i. The integral curve of H_{00} through $\delta^* \cdot s$ projects onto this null geodesic. Using the explicit expression for $s = s(v)$ follows that

$$\phi(u, v) = \phi|_{\eta(u,v)} = \sum_{m=0}^{\infty} \frac{1}{m!} u^m D_{00}^m \phi|_{\eta(0,v)}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} u^m s^{A_m}{}_0 s^{B_m}{}_0 \ldots s^{A_1}{}_0 s^{B_1}{}_0 D^*_{(A_m B_m} \ldots D^*_{A_1 B_1)} \phi(i)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{2m} \psi_{m,n} u^m v^n \,, \tag{54}$$

with

$$\psi_{m,n} = \frac{1}{m!}\binom{2m}{n}D^*_{(A_mB_m}\ldots D^*_{A_1B_1)_n}\phi(i)\,,\quad 0 \le n \le 2m\,.$$

In the same way

$$S_0(u,v) = S_{0000}|_{\eta(u,v)} \tag{55}$$

$$= s^A{}_0 s^B{}_0 s^C{}_0 s^D{}_0 S^*_{ABCD}|_{\eta(u,v)} = \sum_{m=0}^{\infty}\sum_{n=0}^{2m+4}\Psi_{m,n}u^m v^n\,,$$

with

$$\Psi_{m,n} = \frac{1}{m!}\binom{2m+4}{n}D^*_{(A_mB_m}\ldots D^*_{A_1B_1}S^*_{ABCD)_n}(i)\,,\quad 0 \le n \le 2m\,.$$

This shows how to determine $\phi(u,v), S_0(u,v)$ from the null data $\mathcal{D}_n^{\phi*}, \mathcal{D}_n^{S*}$ and vice versa.

5. The conformal stationary vacuum field equations on $\hat{N}$

Now we can use the frame calculus in its standard form. Given the fields Ω, ϕ, R and S_{ABCD}, and using the frame e_{AB} and the connection coefficients Γ_{ABCD} on $\hat{N}$, we set

$$r_{ABCDEF} \equiv e_{CD}(\Gamma_{EFAB}) - e_{EF}(\Gamma_{CDAB}) + \Gamma_{EF}{}^K{}_C\Gamma_{DKAB}$$

$$+ \Gamma_{EF}{}^K{}_D\Gamma_{CKAB} - \Gamma_{CD}{}^K{}_E\Gamma_{KFAB} - \Gamma_{CD}{}^K{}_F\Gamma_{EKAB}$$

$$+ \Gamma_{EF}{}^K{}_B\Gamma_{CDAK} - \Gamma_{CD}{}^K{}_B\Gamma_{EFAK} - t_{CD}{}^{GH}{}_{EF}\Gamma_{GHAB}\,,$$

and we define there the quantities $t_{AB}{}^{EF}{}_{CD}$, R_{ABCDEF}, A_{AB}, Σ_{AB}, Φ_{AB}, Π_{AB}, Σ_{ABCD} and H_{ABCD} by

$$t_{AB}{}^{EF}{}_{CD}e^a{}_{EF} \equiv 2\Gamma_{AB}{}^E{}_{(C}e^a{}_{D)E} - 2\Gamma_{CD}{}^E{}_{(A}e^a{}_{B)E}$$

$$- e^a{}_{CD,b}e^b{}_{AB} + e^a{}_{AB,b}e^b{}_{CD}\,,$$

$$R_{ABCDEF} \equiv r_{ABCDEF} - \frac{1}{2}\left[\left(S_{ABCE} - \frac{1}{6}Rh_{ABCE}\right)\epsilon_{DF}\right.$$

$$\left. + \left(S_{ABDF} - \frac{1}{6}Rh_{ABDF}\right)\epsilon_{CE}\right]\,,$$

$$A_{AB} \equiv D_{AB}\phi - \phi_{AB}\,,$$

$$\Sigma_{AB} \equiv D_{AB}\Omega - \Omega_{AB}\,,$$

$$\Phi_{AB} \equiv D^P{}_B\phi_{AP} + \frac{1}{4}\epsilon_{AB}\phi\left(R + \frac{10}{1+\Omega-\phi^2}\left[\frac{1}{4}\phi^2\Omega_{PQ}\Omega^{PQ}\right.\right.$$

$$\left.\left. - (1+\Omega)\phi\Omega^{PQ}\phi_{PQ} + (1+\Omega)^2\phi_{PQ}\phi^{PQ}\right]\right)\,,$$

$$\Pi_{AB} \equiv D_{AB}R - \frac{1}{1+\Omega-\phi^2}\Big\{ 2(4+7\Omega)\phi\Omega^{PQ}D_{AB}\phi_{PQ}$$

$$- 4(1+\Omega)(4+7\Omega)\phi^{PQ}D_{AB}\phi_{PQ}$$

$$+ \big[3+(-3+7\Omega)\phi^2\big]\Omega^{PQ}S_{PQAB} - 2\Omega(4+7\Omega)\phi\phi^{PQ}S_{PQAB}$$

$$+ \frac{1}{3}(4+7\Omega)\phi^2 R\Omega_{AB} - \frac{2}{3}(1+\Omega)(4+7\Omega)\phi R\phi_{AB}\Big\}$$

$$- \frac{1}{3(1+\Omega-\phi^2)^2}\Big\{ \frac{1}{2}\phi^2\big[-12+(40+21\Omega)\phi^2\big]\Omega^{PQ}\Omega_{PQ}\Omega_{AB}$$

$$- 2\phi\big[-18(1+\Omega)+(46+61\Omega+21\Omega^2)\phi^2\big]\Omega^{PQ}\phi_{PQ}\Omega_{AB}$$

$$+ 2(1+\Omega)\big[-24(1+\Omega)+(52+61\Omega+21\Omega^2)\phi^2\big]\phi^{PQ}\phi_{PQ}\Omega_{AB}$$

$$- \phi\big[12(1+\Omega)+(16+61\Omega+21\Omega^2)\phi^2\big]\Omega^{PQ}\Omega_{PQ}\phi_{AB}$$

$$+ 4(1+\Omega)\big[6(1+\Omega)+(22+61\Omega+21\Omega^2)\phi^2\big]\Omega^{PQ}\phi_{PQ}\phi_{AB}$$

$$- 4(1+\Omega)^2(7+3\Omega)(4+7\Omega)\phi\phi^{PQ}\phi_{PQ}\phi_{AB}\Big\},$$

$$\Sigma_{ABCD} \equiv D_{AB}\Omega_{CD} + \Omega S_{ABCD} + \frac{1}{3}(1+\Omega)Rh_{ABCD}$$

$$- \frac{1}{1+\Omega-\phi^2}\Big\{ \frac{1}{2}\big[1+(-1+\Omega)\phi^2\big]\Omega_{AB}\Omega_{CD}$$

$$- \Omega^2\phi(\Omega_{AB}\phi_{CD}+\Omega_{CD}\phi_{AB}) + 2\Omega^2(1+\Omega)\phi_{AB}\phi_{CD}$$

$$- \frac{4}{3}(2+3\Omega)\Big[\frac{1}{4}\phi^2\Omega_{PQ}\Omega^{PQ} - (1+\Omega)\phi\Omega_{PQ}\phi^{PQ}$$

$$+ (1+\Omega)^2\phi_{PQ}\phi^{PQ}\Big]h_{ABCD}\Big\},$$

$$H_{ABCD} \equiv D^P{}_A S_{BCDP}$$

$$- \frac{1}{1+\Omega-\phi^2}\Big\{ \Omega\phi\Omega_A{}^P D_{(BC}\phi_{D)P} - 2\Omega(1+\Omega)\phi_A{}^P D_{(BC}\phi_{D)P}$$

$$+ (1+\Omega)\phi\Omega_{PQ}D_{(BC}\phi^{PQ}\epsilon_{D)A} - 2(1+\Omega)^2\phi_{PQ}D_{(BC}\phi^{PQ}\epsilon_{D)A}$$

$$+ \frac{1}{2}\big[1+(-1+\Omega)\phi^2\big]\Omega_A{}^P S_{PBCD} - \Omega^2\phi\phi_A{}^P S_{PBCD}$$

$$+ \frac{1}{2}\Omega\phi^2\Omega^{PQ}S_{PQ(BC}\epsilon_{D)A} - \Omega(1+\Omega)\phi\phi^{PQ}S_{PQ(BC}\epsilon_{D)A}$$

$$+ \frac{1}{6}(1+\Omega)\phi^2 R\Omega_{(BC}\epsilon_{D)A} - \frac{1}{3}(1+\Omega)^2\phi R\phi_{(BC}\epsilon_{D)A}$$

$$+ 2\phi\big(\phi_A{}^P\Omega_{P(B}\Omega_{CD)} - \Omega_A{}^P\Omega_{P(B}\phi_{CD)}\big)$$

$$+ 4(1+\Omega)\big(\Omega_A{}^P\phi_{P(B}\phi_{CD)} - \phi_A{}^P\phi_{P(B}\Omega_{CD)}\big)\Big\}$$

$$- \frac{1}{3(1 + \Omega - \phi^2)^2} \left\{ \frac{1}{8} \phi^2 \left[-6 + (20 + 3\Omega)\phi^2 \right] \Omega^{PQ} \Omega_{PQ} \Omega_{(BC}\epsilon_{D)A} \right.$$

$$- \frac{1}{2}(14 + 23\Omega + 3\Omega^2)\phi^3 \Omega^{PQ} \phi_{PQ} \Omega_{(BC}\epsilon_{D)A}$$

$$+ \frac{1}{2}(1 + \Omega)\left[6(1 + \Omega) + (8 + 23\Omega + 3\Omega^2)\phi^2 \right] \phi^{PQ} \phi_{PQ} \Omega_{(BC}\epsilon_{D)A}$$

$$- \frac{1}{4}\phi \left[-12(1 + \Omega) + (26 + 23\Omega + 3\Omega^2)\phi^2 \right] \Omega^{PQ} \Omega_{PQ} \phi_{(BC}\epsilon_{D)A}$$

$$+ (1 + \Omega)^2 \left[-6 + (20 + 3\Omega)\phi^2 \right] \Omega^{PQ} \phi_{PQ} \phi_{(BC}\epsilon_{D)A}$$

$$\left. - (1 + \Omega)^2 (7 + \Omega)(2 + 3\Omega)\phi\phi^{PQ} \phi_{PQ} \phi_{(BC}\epsilon_{D)A} \right\} .$$

The tensor fields on the left hand side have been introduced as labels for the equations and for discussing in an ordered manner the interdependencies of the equations. In terms of these tensor fields, the conformal stationary vacuum equations read

$$t_{AB}{}^{EF}{}_{CD}e^a{}_{EF} = 0, \quad R_{ABCDEF} = 0, \quad A_{AB} = 0, \quad \Sigma_{AB} = 0,$$
$$\Phi_{AB} = 0, \quad \Pi_{AB} = 0, \quad \Sigma_{ABCD} = 0, \quad H_{ABCD} = 0.$$

The first equation is Cartan's first structural equation with the requirement that the metric connection be torsion free. The second equation is Cartan's second structural equation, requiring the Ricci tensor to coincide with the appropriate combination of the trace free tensor S_{ab} and the scalar R. The third and fourth equations define the symmetric spinors ϕ_{AB} and Ω_{AB} respectively. The rest of the equations have already been considered.

We want to calculate, using our particular gauge, a formal expansion of the conformal fields using the initial data in the form $\phi(u, v)$, $S_0(u, v)$. As the system of conformal stationary vacuum field equations is an overdetermined system, we have to choose a subsystem of it. In the rest of this section we choose a particular subsystem, writing the chosen equations in our gauge, and at the end we see how a formal expansion is determined by these equations and the initial data.

5.1. The $A_{00} = 0$ equation

The first equation that needs particular attention is the equation $A_{00} = 0$. In our gauge it reads

$$\partial_u \phi = \phi_{00}.$$

This equation is used in the following to calculate ϕ_{00} each time we know ϕ as a function of u. In particular, as ϕ will be prescribed on W_0 as part of the initial data, this equation allows us to calculate ϕ_{00} there immediately.

5.2. The '∂_u-equations'

We now present what we will refer to as the '∂_u-equations'. These equations are chosen because they have the following features. They are a system of PDE's for the set of functions $\hat{e}^a{}_{A1}$, $\hat{\Gamma}_{A1CD}$, Ω, Ω_{AB}, ϕ_{A1}, R, S_1, S_2, S_3 and S_4, which comprise

all the unknowns with the exceptions of the free data ϕ, S_0 and the derived function ϕ_{00}. They are all interior equations on the hypersurfaces $\{w = w_0\}$ in the sense that only derivatives in the directions of u and v are involved, in particular, if we consider the hypersurface $\{w = 0\}$, they are all inner equations in $\mathcal{N}_i$. Also they split into a hierarchy that will be presented in the following section.

The ∂_u-equations:

Equations $t_{AB}{}^{EF}{}_{00}e^a{}_{EF} = 0$:

$$\partial_u \hat{e}^1{}_{01} + \frac{1}{u}\hat{e}^1{}_{01} = -2\hat{\Gamma}_{0101} + 2\hat{\Gamma}_{0100}\hat{e}^1{}_{01} \, ,$$

$$\partial_u \hat{e}^2{}_{01} + \frac{1}{u}\hat{e}^2{}_{01} = \frac{1}{u}\hat{\Gamma}_{0100} + 2\hat{\Gamma}_{0100}\hat{e}^2{}_{01} \, ,$$

$$\partial_u \hat{e}^1{}_{11} = -2\hat{\Gamma}_{1101} + 2\hat{\Gamma}_{1100}\hat{e}^1{}_{01} \, ,$$

$$\partial_u \hat{e}^2{}_{11} = \frac{1}{u}\hat{\Gamma}_{1100} + 2\hat{\Gamma}_{1100}\hat{e}^2{}_{01} \, .$$

Equations $R_{AB00EF} = 0$:

$$\partial_u \hat{\Gamma}_{0100} + \frac{2}{u}\hat{\Gamma}_{0100} - 2\hat{\Gamma}^2_{0100} = \frac{1}{2}S_0 \, ,$$

$$\partial_u \hat{\Gamma}_{0101} + \frac{1}{u}\hat{\Gamma}_{0101} - 2\hat{\Gamma}_{0100}\hat{\Gamma}_{0101} = \frac{1}{2}S_1 \, ,$$

$$\partial_u \hat{\Gamma}_{0111} + \frac{1}{u}\hat{\Gamma}_{0111} - 2\hat{\Gamma}_{0100}\hat{\Gamma}_{0111} = \frac{1}{2}S_2 - \frac{1}{12}R \, ,$$

$$\partial_u \hat{\Gamma}_{1100} + \frac{1}{u}\hat{\Gamma}_{1100} - 2\hat{\Gamma}_{0100}\hat{\Gamma}_{1100} = S_1 \, ,$$

$$\partial_u \hat{\Gamma}_{1101} - 2\hat{\Gamma}_{1100}\hat{\Gamma}_{0101} = S_2 + \frac{1}{12}R \, ,$$

$$\partial_u \hat{\Gamma}_{1111} - 2\hat{\Gamma}_{1100}\hat{\Gamma}_{0111} = S_3 \, .$$

Equation $\Sigma_{00} = 0$:

$$\partial_u \Omega = \Omega_{00} \, .$$

Equations $\Phi_{A0} = 0$:

$$\partial_u \phi_{01} = \frac{1}{2u}(\partial_v \phi_{00} - 2\phi_{01}) + \hat{e}^1{}_{01}\partial_u \phi_{00} + \hat{e}^2{}_{01}\partial_u \phi_{00} - 2\hat{\Gamma}_{0101}\phi_{00} + 2\hat{\Gamma}_{0100}\phi_{01} \, ,$$

$$\partial_u \phi_{11} - \frac{1}{2u}(\partial_v \phi_{01} - \phi_{11}) - \hat{e}^1{}_{01}\partial_u \phi_{01} - \hat{e}^2{}_{01}\partial_v \phi_{01} = -\hat{\Gamma}_{0111}\phi_{00} + \hat{\Gamma}_{0100}\phi_{11}$$
$$- \frac{1}{4}\phi \left\{ R + \frac{10}{1 + \Omega - \phi^2}\left[\frac{1}{2}\phi^2 \Omega_{00}\Omega_{11} - \frac{1}{2}\left[\phi\Omega_{01} - 2(1 + \Omega)\phi_{01} \right]^2 \right. \right.$$
$$\left. \left. - (1 + \Omega)\phi \left(\Omega_{00}\phi_{11} + \Omega_{11}\phi_{00} \right) + 2(1 + \Omega)^2 \phi_{00}\phi_{11} \right] \right\} \, .$$

Equations $\Sigma_{00CD} = 0$:

$$\partial_u \Omega_{00} = -\Omega S_0 + \frac{1}{1 + \Omega - \phi^2} \left\{ \frac{1}{2} \left[1 + (-1 + \Omega)\phi^2 \right] \Omega_{00}^2 \right.$$
$$\left. - 2\Omega^2 \phi \Omega_{00} \phi_{00} + 2\Omega^2 (1 + \Omega) \phi_{00}^2 \right\},$$

$$\partial_u \Omega_{01} = -\Omega S_1 + \frac{1}{1 + \Omega - \phi^2} \left\{ \frac{1}{2} \left[1 + (-1 + \Omega)\phi^2 \right] \Omega_{00} \Omega_{01} \right.$$
$$\left. - \Omega^2 \phi \left(\Omega_{00} \phi_{01} + \Omega_{01} \phi_{00} \right) + 2\Omega^2 (1 + \Omega) \phi_{00} \phi_{01} \right\},$$

$$\partial_u \Omega_{11} = -\Omega S_2 - \frac{1}{3}(1 + \Omega)R$$
$$+ \frac{1}{3(1 + \Omega - \phi^2)} \left\{ \frac{1}{2} \left[3 - (11 + 9\Omega)\phi^2 \right] \Omega_{00} \Omega_{11} \right.$$
$$+ 2(2 + 3\Omega) \left[\phi \Omega_{01} - 2(1 + \Omega)\phi_{01} \right]^2$$
$$\left. + (8 + 20\Omega + 9\Omega^2) \left[\phi(\Omega_{00}\phi_{11} + \Omega_{11}\phi_{00}) - 2(1 + \Omega)\phi_{00}\phi_{11} \right] \right\}.$$

Equation $\Pi_{00} = 0$:

$$\partial_u R - \frac{1}{1 + \Omega - \phi^2} \left\{ 2(4 + 7\Omega)\phi(\Omega_{11}\partial_u\phi_{00} - 2\Omega_{01}\partial_u\phi_{01} + \Omega_{00}\partial_u\phi_{11}) \right.$$
$$\left. - 4(1 + \Omega)(4 + 7\Omega)(\phi_{11}\partial_u\phi_{00} - 2\phi_{01}\partial_u\phi_{01} + \phi_{00}\partial_u\phi_{11}) \right\}$$
$$= \frac{1}{1 + \Omega - \phi^2} \left\{ \left[3 + (-3 + 7\Omega)\phi^2 \right] (\Omega_{11}S_0 - 2\Omega_{01}S_1 + \Omega_{00}S_2) \right.$$
$$- 2\Omega(4 + 7\Omega)\phi(\phi_{11}S_0 - 2\phi_{01}S_1 + \phi_{00}S_2)$$
$$\left. + \frac{1}{3}(4 + 7\Omega)\phi \left[\phi\Omega_{00} - 2(1 + \Omega)\phi_{00} \right] R \right\}$$
$$+ \frac{1}{3(1 + \Omega - \phi^2)^2} \left\{ \phi^2 \left[-12 + (40 + 21\Omega)\phi^2 \right] (\Omega_{00}\Omega_{11} - \Omega_{01}^2)\Omega_{00} \right.$$
$$- 2\phi \left[-18(1 + \Omega) + (46 + 61\Omega + 21\Omega^2)\phi^2 \right] (\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\Omega_{00}$$
$$+ 4(1 + \Omega) \left[-24(1 + \Omega) + (52 + 61\Omega + 21\Omega^2)\phi^2 \right] (\phi_{00}\phi_{11} - \phi_{01}^2)\Omega_{00}$$
$$- 2\phi \left[12(1 + \Omega) + (16 + 61\Omega + 21\Omega^2)\phi^2 \right] (\Omega_{00}\Omega_{11} - \Omega_{01}^2)\phi_{00}$$
$$+ 4(1 + \Omega) \left[6(1 + \Omega) + (22 + 61\Omega + 21\Omega^2)\phi^2 \right] (\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\phi_{00}$$
$$\left. - 8(1 + \Omega)^2(7 + 3\Omega)(4 + 7\Omega)\phi(\phi_{00}\phi_{11} - \phi_{01}^2)\phi_{00} \right\}.$$

Equations $H_{0(ABC)_k} = 0$, k=0,1,2,3 :

$$\partial_u S_1 - \frac{1}{2u}\left(\partial_v S_0 - 4S_1\right) - \hat{e}^1{}_{01}\partial_u S_0 - \hat{e}^2{}_{01}\partial_v S_0$$

$$+ \frac{1}{1+\Omega-\phi^2}\left\{\Omega\left[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\right]\partial_u\phi_{00} - \Omega\left[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\right]\partial_u\phi_{01}\right\}$$

$$= -4\hat{\Gamma}_{0101}S_0 + 4\hat{\Gamma}_{0100}S_1$$

$$- \frac{1}{1+\Omega-\phi^2}\left\{\frac{1}{2}\left[1 + (-1+\Omega)\phi^2\right](\Omega_{01}S_0 - \Omega_{00}S_1) - \Omega^2\phi(\phi_{01}S_0 - \phi_{00}S_1)\right.$$

$$\left. + 2\left[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\right](\Omega_{00}\phi_{01} - \Omega_{01}\phi_{00})\right\},$$

$$\partial_u S_2 - \frac{1}{2u}(\partial_v S_1 - 3S_2) - \hat{e}^1{}_{01}\partial_u S_1 - \hat{e}^2{}_{01}\partial_v S_1$$

$$+ \frac{1}{3(1+\Omega-\phi^2)}\left\{-(1+\Omega)\left[\phi\Omega_{11} - 2(1+\Omega)\phi_{11}\right]\partial_u\phi_{00}\right.$$

$$+ (2+5\Omega)\left[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\right]\partial_u\phi_{01} - (1+2\Omega)\left[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\right]\partial_u\phi_{11}$$

$$\left. - 2\Omega\left[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\right]\left[\frac{1}{2u}(\partial_v\phi_{01} - \phi_{11}) + \hat{e}^1{}_{01}\partial_u\phi_{01} + \hat{e}^2{}_{01}\partial_v\phi_{01}\right]\right\}$$

$$= -\hat{\Gamma}_{0111}S_0 - 2\hat{\Gamma}_{0101}S_1 + 3\hat{\Gamma}_{0100}S_2$$

$$- \frac{1}{1+\Omega-\phi^2}\left\{\frac{2}{3}\Omega\left[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\right]\left[\hat{\Gamma}_{0111}\phi_{00} - \hat{\Gamma}_{0100}\phi_{11}\right]\right.$$

$$+ \frac{1}{2}\left[1 + (-1+\Omega)\phi^2\right](\Omega_{01}S_1 - \Omega_{00}S_2) - \Omega^2\phi(\phi_{01}S_1 - \phi_{00}S_2)$$

$$- \frac{1}{6}\Omega\phi^2(\Omega_{11}S_0 - 2\Omega_{01}S_1 + \Omega_{00}S_2) + \frac{1}{3}\Omega(1+\Omega)\phi(\phi_{11}S_0 - 2\phi_{01}S_1 + \phi_{00}S_2)$$

$$- \frac{1}{18}(1+\Omega)\phi\left[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\right]R$$

$$\left. + 2\left[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\right](\Omega_{00}\phi_{01} - \Omega_{01}\phi_{00})\right\}$$

$$+ \frac{1}{9(1+\Omega-\phi^2)^2}\left\{\frac{1}{4}\phi^2\left[-6 + (20 + 3\Omega)\phi^2\right](\Omega_{00}\Omega_{11} - \Omega_{01}^2)\Omega_{00}\right.$$

$$- \frac{1}{2}(14 + 23\Omega + 3\Omega^2)\phi^3(\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\Omega_{00}$$

$$+ (1+\Omega)\left[6(1+\Omega) + (8 + 23\Omega + 3\Omega^2)\phi^2\right](\phi_{00}\phi_{11} - \phi_{01}^2)\Omega_{00}$$

$$- \frac{1}{2}\phi\left[-12(1+\Omega) + (26 + 23\Omega + 3\Omega^2)\phi^2\right](\Omega_{00}\Omega_{11} - \Omega_{01}^2)\phi_{00}$$

$$+ (1+\Omega)^2\left[-6 + (20 + 3\Omega)\phi^2\right](\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\phi_{00}$$

$$\left. - 2(1+\Omega)^2(7+\Omega)(2+3\Omega)\phi(\phi_{00}\phi_{11} - \phi_{01}^2)\phi_{00}\right\},$$

$$\partial_u S_3 - \frac{1}{2u}(\partial_v S_2 - 2S_3) - \hat{e}^1{}_{01}\partial_u S_2 - \hat{e}^2{}_{01}\partial_v S_2 + \frac{1}{3(1+\Omega-\phi^2)}$$

$$\left\{ -2(1+\Omega)\big[\phi\Omega_{11} - 2(1+\Omega)\phi_{11}\big]\partial_u\phi_{01} + \Omega\big[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\big]\partial_u\phi_{11} \right.$$

$$+ 2(2+3\Omega)\big[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\big]\left[\frac{1}{2u}(\partial_v\phi_{01} - \phi_{11}) + \hat{e}^1{}_{01}\partial_u\phi_{01} + \hat{e}^2{}_{01}\partial_v\phi_{01}\right]$$

$$\left. - (2+5\Omega)\big[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\big]\left[\frac{1}{2u}\partial_v\phi_{11} + \hat{e}^1{}_{01}\partial_u\phi_{11} + \hat{e}^2{}_{01}\partial_v\phi_{11}\right] \right\}$$

$$= -2\hat{\Gamma}_{0111}S_1 + 2\hat{\Gamma}_{0100}S_3$$

$$- \frac{1}{1+\Omega-\phi^2}\left\{ -\frac{2}{3}(2+3\Omega)\big[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\big]\big[\hat{\Gamma}_{0111}\phi_{00} - \hat{\Gamma}_{0100}\phi_{11}\big] \right.$$

$$+ \frac{2}{3}(2+5\Omega)\big[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\big]\big[\hat{\Gamma}_{0111}\phi_{01} - \hat{\Gamma}_{0101}\phi_{11}\big]$$

$$+ \frac{1}{2}\big[1 + (-1+\Omega)\phi^2\big](\Omega_{01}S_2 - \Omega_{00}S_3) - \Omega^2\phi(\phi_{01}S_2 - \phi_{00}S_3)$$

$$- \frac{1}{3}\Omega\phi^2(\Omega_{11}S_1 - 2\Omega_{01}S_2 + \Omega_{00}S_3) + \frac{2}{3}\Omega(1+\Omega)\phi(\phi_{11}S_1 - 2\phi_{01}S_2 + \phi_{00}S_3)$$

$$- \frac{1}{9}(1+\Omega)\phi\big[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\big]R$$

$$\left. + 2\big[\phi\Omega_{11} - 2(1+\Omega)\phi_{11}\big](\Omega_{00}\phi_{01} - \Omega_{01}\phi_{00}) \right\}$$

$$+ \frac{2}{9(1+\Omega-\phi^2)^2}\left\{ \frac{1}{4}\phi^2\big[-6 + (20+3\Omega)\phi^2\big](\Omega_{00}\Omega_{11} - \Omega_{01}^2)\Omega_{01} \right.$$

$$- \frac{1}{2}(14 + 23\Omega + 3\Omega^2)\phi^3(\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\Omega_{01}$$

$$+ (1+\Omega)\big[6(1+\Omega) + (8 + 23\Omega + 3\Omega^2)\phi^2\big](\phi_{00}\phi_{11} - \phi_{01}^2)\Omega_{01}$$

$$- \frac{1}{2}\phi\big[-12(1+\Omega) + (26 + 23\Omega + 3\Omega^2)\phi^2\big](\Omega_{00}\Omega_{11} - \Omega_{01}^2)\phi_{01}$$

$$+ (1+\Omega)^2\big[-6 + (20+3\Omega)\phi^2\big](\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\phi_{01}$$

$$\left. - 2(1+\Omega)^2(7+\Omega)(2+3\Omega)\phi(\phi_{00}\phi_{11} - \phi_{01}^2)\phi_{01} \right\},$$

$$\partial_u S_4 - \frac{1}{2u}(\partial_v S_3 - S_4) - \hat{e}^1{}_{01}\partial_u S_3 - \hat{e}^2{}_{01}\partial_v S_3$$

$$+ \frac{1}{1+\Omega-\phi^2}\left\{ -(1+\Omega)\big[\phi\Omega_{11} - 2(1+\Omega)\phi_{11}\big]\partial_u\phi_{11} \right.$$

$$+ (2+3\Omega)\big[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\big]\left[\frac{1}{2u}\partial_v\phi_{11} + \hat{e}^1{}_{01}\partial_u\phi_{11} + \hat{e}^2{}_{01}\partial_v\phi_{11}\right]$$

$$\left. - (1+2\Omega)\big[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\big]\left(\hat{e}^1{}_{11}\partial_u\phi_{11} + \hat{e}^2{}_{11}\partial_v\phi_{11} + \partial_w\phi_{11}\right) \right\}$$

$$= -3\hat{\Gamma}_{0111}S_2 + 2\hat{\Gamma}_{0101}S_3 + \hat{\Gamma}_{0100}S_4$$

$$+ \frac{1}{1+\Omega-\phi^2}\Bigg\{ 2(2+3\Omega)\big[\phi\Omega_{01} - 2(1+\Omega)\phi_{01}\big]\big[\hat{\Gamma}_{0111}\phi_{01} - \hat{\Gamma}_{0101}\phi_{11}\big]$$

$$- 2(1+2\Omega)\big[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\big]\big[\hat{\Gamma}_{1111}\phi_{01} - \hat{\Gamma}_{1101}\phi_{11}\big]$$

$$- \frac{1}{2}\big[1 + (-1+\Omega)\phi^2\big](\Omega_{01}S_3 - \Omega_{00}S_4) + \Omega^2\phi(\phi_{01}S_3 - \phi_{00}S_4)$$

$$+ \frac{1}{2}\Omega\phi^2(\Omega_{11}S_2 - 2\Omega_{01}S_3 + \Omega_{00}S_4) - \Omega(1+\Omega)\phi(\phi_{11}S_2 - 2\phi_{01}S_3 + \phi_{00}S_4)$$

$$+ \frac{1}{6}(1+\Omega)\phi\big[\phi\Omega_{11} - 2(1+\Omega)\phi_{11}\big]R$$

$$- 2\phi\big[\Omega_{11}(\Omega_{01}\phi_{01} - \Omega_{11}\phi_{00}) + \phi_{11}(\Omega_{00}\Omega_{11} - \Omega_{01}^2)\big]$$

$$+ 4(1+\Omega)\big[\phi_{11}(\Omega_{01}\phi_{01} - \Omega_{00}\phi_{11}) + \Omega_{11}(\phi_{00}\phi_{11} - \phi_{01}^2)\big]\Bigg\}$$

$$+ \frac{1}{3(1+\Omega-\phi^2)^2}\Bigg\{ \frac{1}{4}\phi^2\big[-6 + (20+3\Omega)\phi^2\big](\Omega_{00}\Omega_{11} - \Omega_{01}^2)\Omega_{11}$$

$$- \frac{1}{2}(14 + 23\Omega + 3\Omega^2)\phi^3(\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\Omega_{11}$$

$$+ (1+\Omega)\big[6(1+\Omega) + (8+23\Omega+3\Omega^2)\phi^2\big](\phi_{00}\phi_{11} - \phi_{01}^2)\Omega_{11}$$

$$- \frac{1}{2}\phi\big[-12(1+\Omega) + (26+23\Omega+3\Omega^2)\phi^2\big](\Omega_{00}\Omega_{11} - \Omega_{01}^2)\phi_{11}$$

$$+ (1+\Omega)^2\big[-6 + (20+3\Omega)\phi^2\big](\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\phi_{11}$$

$$- 2(1+\Omega)^2(7+\Omega)(2+3\Omega)\phi(\phi_{00}\phi_{11} - \phi_{01}^2)\phi_{11}\Bigg\}$$

5.3. The ∂_u-equations hierarchy

The system of ∂_u-equations splits into two groups, referred to as G1 and G2. Each of these groups splits into a hierarchy, which is defined as follows:

G1.1: $R_{000001} = 0$,
G1.2: $t_{01}{}^{EF}{}_{00}e^2{}_{EF} = 0$,
G1.3: $t_{01}{}^{EF}{}_{00}e^1{}_{EF} = 0$, $R_{010001} = 0$, $\Sigma_{00} = 0$, $\Sigma_{0000} = 0$, $\Sigma_{0001} = 0$, $\Phi_{00} = 0$,
   $H_{0000} = 0$,
G1.4: $R_{110001} = 0$, $\Sigma_{0011} = 0$, $\Phi_{10} = 0$, $\Pi_{00} = 0$, $H_{0001} = 0$,
G1.5: $R_{000011} = 0$,
G1.6: $R_{010011} = 0$,
G1.7: $t_{11}{}^{EF}{}_{00}e^1{}_{EF} = 0$,
G1.8: $t_{11}{}^{EF}{}_{00}e^2{}_{EF} = 0$,

G2.1: $H_{0011} = 0$,
G2.2: $R_{110011} = 0$,
G2.3: $H_{0111} = 0$.

For dealing with the unknowns we separate them into three groups, x_1, x_2 and x_3. The unknowns involved in G1 are collected in x_1, that is $x_1 = (\hat{e}^1\,_{01}, \hat{e}^2\,_{01}, \hat{e}^1\,_{11}, \hat{e}^2\,_{11}, \hat{\Gamma}_{0100}, \hat{\Gamma}_{0101}, \hat{\Gamma}_{0111}, \hat{\Gamma}_{1100}, \hat{\Gamma}_{1101}, \Omega, \Omega_{00}, \Omega_{01}, \Omega_{11}, \phi_{01}, \phi_{11}, R, S_1, S_2)$. The set x_2 consist of the unknowns of x_1 plus ϕ, S_0 and ϕ_{00}. The unknowns in G2 are collected in x_3, that is $x_3 = (\hat{\Gamma}_{1111}, S_3, S_4)$. So all the unknowns are included in the union of x_2 and x_3.

The defining property of the hierarchy is the following feature. If ϕ and S_0 are prescribed on $\{w = w_0\}$ then G1.1 reduces to an ODE. Once we have its solution, G1.2 reduces to an ODE. Given its solution, G1.3 reduces to a system of ODE's, with coefficients that are calculated by operations interior to $\{w = w_0\}$ from the previously known or calculated functions. This procedure continues till G1.8. So, given ϕ and S_0 on $\{w = w_0\}$ and the appropriate initial data on $U_0 \cap \{w = w_0\}$, the set x_1 can be determined on $\{w = w_0\}$ by solving a sequence of ODE's in the independent variable u.

The process to be followed with G2 is very similar, with the exception that to solve G2.3 it is necessary to know also $\partial_w \phi_{11}$ on $\{w = w_0\}$, this problem can be overcome solving G1 recursively and then analyzing G2.

5.4. The '∂_w-equations'

Our initial data, ϕ and S_0, is prescribed on W_0, and to determine their evolution off W_0 we need the equation $A_{11} = 0$, which reads

$$\partial_w \phi + \hat{e}^1\,_{11} \partial_u \phi + \hat{e}^2\,_{11} \partial_v \phi = \phi_{11}\,,$$

and the equation $H_{1(ABC)_0} + H_{0(ABC)_1} = 0$, which is given by

$$\partial_w S_0 - \partial_u S_2 + \hat{e}^1\,_{11} \partial_u S_0 + \hat{e}^2\,_{11} \partial_v S_0$$

$$- \frac{1}{3(1 + \Omega - \phi^2)} \left\{ (2 + 5\Omega) \left[\phi \Omega_{11} - 2(1 + \Omega)\phi_{11} \right] \partial_u \phi_{00} \right.$$

$$- 4(1 + \Omega) \left[\phi \Omega_{01} - 2(1 + \Omega)\phi_{01} \right] \partial_u \phi_{01} + (2 + \Omega) \left[\phi \Omega_{00} - 2(1 + \Omega)\phi_{00} \right] \partial_u \phi_{11}$$

$$\left. - 2\Omega \left[\phi \Omega_{00} - 2(1 + \Omega)\phi_{00} \right] \left[\frac{1}{2u} (\partial_v \phi_{01} - \phi_{11}) + \hat{e}^1\,_{01} \partial_u \phi_{01} + \hat{e}^2\,_{01} \partial_v \phi_{01} \right] \right\}$$

$$= 4\hat{\Gamma}_{1101} S_0 - 4\hat{\Gamma}_{1100} S_1$$

$$+ \frac{1}{1 + \Omega - \phi^2} \left\{ \frac{2}{3}\Omega \left[\phi \Omega_{00} - 2(1 + \Omega)\phi_{00} \right] \left[\hat{\Gamma}_{0111}\phi_{00} - \hat{\Gamma}_{0100}\phi_{11} \right] \right.$$

$$+ \frac{1}{2} \left[1 + (-1 + \Omega)\phi^2 \right] (\Omega_{11} S_0 - \Omega_{00} S_2) - \Omega^2 \phi (\phi_{11} S_0 - \phi_{00} S_2)$$

$$+ \frac{1}{3}\Omega\phi^2 (\Omega_{11} S_0 - 2\Omega_{01} S_1 + \Omega_{00} S_2) - \frac{2}{3}\Omega(1 + \Omega)\phi(\phi_{11} S_0 - 2\phi_{01} S_1 + \phi_{00} S_2)$$

$$+ \frac{1}{9}(1 + \Omega)\phi \left[\phi \Omega_{00} - 2(1 + \Omega)\phi_{00} \right] R$$

$$\left. + 2 \left[\phi \Omega_{00} - 2(1 + \Omega)\phi_{00} \right] (\Omega_{00}\phi_{11} - \Omega_{11}\phi_{00}) \right\}$$

$$+ \frac{2}{9(1+\Omega-\phi^2)^2} \left\{ \frac{1}{4}\phi^2 \left[-6 + (20+3\Omega)\phi^2 \right] (\Omega_{00}\Omega_{11} - \Omega_{01}^2)\Omega_{00} \right.$$

$$- \frac{1}{2}(14 + 23\Omega + 3\Omega^2)\phi^3(\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\Omega_{00}$$

$$+ (1+\Omega)\left[6(1+\Omega) + (8 + 23\Omega + 3\Omega^2)\phi^2\right](\phi_{00}\phi_{11} - \phi_{01}^2)\Omega_{00}$$

$$- \frac{1}{2}\phi\left[-12(1+\Omega) + (26 + 23\Omega + 3\Omega^2)\phi^2 \right](\Omega_{00}\Omega_{11} - \Omega_{01}^2)\phi_{00}$$

$$+ (1+\Omega)^2 \left[-6 + (20 + 3\Omega)\phi^2 \right] (\Omega_{00}\phi_{11} - 2\Omega_{01}\phi_{01} + \Omega_{11}\phi_{00})\phi_{00}$$

$$\left. - 2(1+\Omega)^2 (7+\Omega)(2+3\Omega)\phi(\phi_{00}\phi_{11} - \phi_{01}^2)\phi_{00} \right\}.$$

These two equations will be referred to as the '∂_w-equations'.

5.5. The initial conditions for the ∂_u-equations

The initial conditions for the ∂_u-equations follow from our gauge conditions (52), (53) which imply

$$\hat{e}^a{}_{A1}|_I = 0, \quad a = 1, 2, \quad A = 0, 1,$$

$$\hat{\Gamma}_{A1CD}|_I = 0, \quad A, C, D = 0, 1.$$

From (4) we get

$$\Omega|_I = 0,$$

$$\Omega_{AB}|_I = 0, \quad A, B = 0, 1,$$

$$R|_I = -6 - 8\partial_u\phi|_I\partial_u\partial_v^2\phi|_I + 4\left(\partial_u\partial_v\phi|_I\right)^2,$$

and from the required spinorial behavior in order to have analytic solutions, as discussed in Section 4.3,

$$\phi_{A1}|_I = \frac{1}{2}\partial_u\partial_v^{1+A}\phi|_I, \quad A = 0, 1, \tag{56}$$

$$S_k|_I = \frac{(4-k)!}{4!}\partial_v^k S_0|_I, \quad k = 1, 2, 3, 4,$$

where $A_{00} = 0$ has been used.

5.6. Calculating the formal expansion

As the system of equations is overdetermined, we have chosen a subsystem in order to calculate a formal expansion of the solution. It will be shown later on that the expansion obtained using this subsystem lead to a formal solution of the full system of equations.

We prescribe ϕ and S_0 on W_0 as our datum and the initial conditions on I for the ∂_u-equations are given in Section 5.5. Following what has been said in Section 5.3 we successively integrate the subsystems on G1 to determine all components of x_1 on W_0.

We give now an inductive argument involving G1 and the ∂_w-equations to show that $\partial_w^k x_2|_{W_0}$ can be determined for all k.

From our initial data and what has been said we know already $\partial_w^k x_2|_{W_0}$ for $k = 0$. As inductive hypothesis we assume as known

$$\partial_w^p x_2|_{W_0}\,, \quad 0 \le p \le k-1\,, \quad k \ge 1\,.$$

Applying formally ∂_w^{k-1} to the ∂_w-equations, and restricting them to W_0, we find $\partial_w^k \phi|_{W_0}$ and $\partial_w^k S_0|_{W_0}$ in terms of known functions. We apply formally ∂_w^k to G1. This is a system of PDE's where the unknowns are $\partial_w^k x_1$. Keeping the hierarchy and considering the functions that we already know on W_0, it again becomes a sequence of ODE's, which can be integrated on W_0 given the appropriate initial conditions on I.

The initial conditions for the frame coefficients and the connection coefficients are obtained from the gauge requirements (52), (53) which imply

$$\partial_w^k \hat{e}^a{}_{A1}|_I = 0\,, \quad a = 1,2\,, \quad A = 0,1\,,$$

$$\partial_w^k \hat{\Gamma}_{A1CD}|_I = 0\,, \quad A,C,D = 0,1\,.$$

From the spinorial behavior as discussed in Section 4.3 we obtain the following set of initial conditions.

$$\partial_w^k \phi_{01}|_I = \frac{1}{2}\partial_u \partial_v \partial_w^k \phi|_I\,,$$

$$\partial_w^k \phi_{11}|_I = \frac{1}{2}\partial_u \partial_v^2 \partial_w^k \phi|_I\,,$$

$$\partial_w^k S_1|_I = \frac{1}{4}\partial_v \partial_w^k S_0|_I\,,$$

$$\partial_w^k S_2|_I = \frac{1}{12}\partial_v^2 \partial_w^k S_0|_I\,.$$

By restricting the equations $\Sigma_{11} = 0$, $\Sigma_{11CD} = 0$ and $\Pi_{11} = 0$ to U_0 and using that $\Omega|_I = 0$, $\Omega_{AB}|_I = 0$ we get

$$\partial_w^k \Omega|_I = 0\,,$$

$$\partial_w^k \Omega_{A1}|_I = 0\,, \quad A = 0,1\,,$$

$$\partial_w \Omega_{00}|_{U_0} = \left[-\frac{1}{3}R + \frac{8}{3(1-\phi^2)}(\phi\Omega_{00}\phi_{11} - 2\phi_{00}\phi_{11} + 2\phi_{01}^2)\right]\bigg|_{U_0}\,, \tag{57}$$

$$\partial_w R|_{U_0} = \left[3\Omega_{00}S_4 + \frac{8}{1-\phi^2}\left\{(\phi\Omega_{00} - 2\phi_{00})\partial_w \phi_{11}\right.\right. \tag{58}$$

$$\left. + 4\phi_{01}\partial_w \phi_{01} - 2\phi_{11}\partial_w \phi_{00} - \frac{1}{3}\phi R\phi_{11}\right\}$$

$$\left. + \frac{8}{3(1-\phi^2)^2}\phi_{11}\left\{(3 + 11\phi^2)\Omega_{00}\phi_{11} - 28\phi(\phi_{00}\phi_{11} - \phi_{01}^2)\right\}\right]\bigg|_{U_0}\,.$$

Applying ∂_w^{k-1} to (57), (58) and evaluating them at I by using the known functions from the inductive hypothesis and the previously stated initial conditions we get $\partial_w^k \Omega_{00}|_I$ and $\partial_w^k R|_I$.

Now we have all the needed initial conditions, thus we know

$$\partial_w^k x_2|_{W_0}$$

and the induction step is completed.

The procedure with G2 is quite similar. Once we know $\partial_w^k x_2|_{W_0}$ for all k, G2.1 reduces to an ODE, which can be integrated on W_0 given the corresponding initial condition. Once we know the solution of G2.1, G2.2 also reduces to an ODE, and finally also G2.3 reduces to an ODE. The initial conditions for G2 are given in Section 5.5.

The inductive step is very similar to the inductive step for x_2. We assume

$$\partial_w^p x_3|_{W_0}, \quad 0 \le p \le k-1, \quad k \ge 1,$$

to be known. We apply formally ∂_w^k to the equations in G2. If we stick to the hierarchy this system again reduces in the prescribed order to a system of ODE's for $\partial_w^k x_3$, which can be integrated given the corresponding initial conditions. Those are

$$\partial_w^k \hat{\Gamma}_{1111}|_I = 0,$$

$$\partial_w^k S_3|_I = \frac{1}{24} \partial_v^3 \partial_w^k S_0|_I,$$

$$\partial_w^k S_4|_I = \frac{1}{24} \partial_v^4 \partial_w^k S_0|_I,$$

obtained from (53) and Section 4.3.

Now we know

$$\partial_w^k x_3|_{W_0}$$

and the induction step is complete.

If we now call X any of the quantities included in x_2 and x_3, that is, X comprises all the unknown quantities that we are solving for, the procedure just stated shows that we know $\partial_w^k X|_{W_0}$ for all k. Expanding these functions around $i = \{u = 0, v = 0, w = 0\}$ gives

$$\partial_u^m \partial_v^n \partial_w^p X|_i \quad \forall\, m, n, p,$$

and the procedure gives a unique sequence of expansion coefficients for all the functions in X.

Lemma 5.1. *The procedure described above determines at the point $O = (u = 0, v = 0, w = 0)$ from the data ϕ, S_0, given on W_0 according to (54), (55), a unique sequence of expansion coefficients*

$$\partial_u^m \partial_v^n \partial_w^p f(O), \quad m, n, p = 0, 1, 2, \dots,$$

where f stands for any of the functions $\hat{e}^a{}_{AB}, \hat{\Gamma}_{ABCD}, \phi, \phi_{AB}, \Omega, \Omega_{AB}, R, S_k$.

If the corresponding Taylor series are absolutely convergent in some neighborhood P of O, they define a solution to the equation $A_{00} = 0$, to the ∂_u-equations and to the ∂_w-equations on P which satisfies on $P \cap U_0$ equations (56) and $\Sigma_{11} = 0$, $\Sigma_{11CD} = 0$, $\Pi_{11} = 0$.

By Lemma 4.1 we know that all spinor-valued functions should have a specific v-finite expansion type. The following lemma, whose proof is quite similar to the proof in [8], will be important for the convergence proof.

Lemma 5.2. *If the data ϕ, S_0 are given on W_0 as in (54), (55), the formal expansions of the fields obtained in Lemma 5.1 correspond to ones of functions of v-finite expansion types given by*

$$k_{\hat{e}^1{}_{AB}} = -A - B\,, \quad k_{\hat{e}^2{}_{AB}} = 3 - A - B\,, \quad AB = 01, 11\,,$$
$$k_{\hat{\Gamma}_{01AB}} = 2 - A - B\,, \quad k_{\hat{\Gamma}_{11AB}} = 1 - A - B\,, \quad A, B = 0, 1\,,$$
$$k_\phi = 0\,, \quad k_{\phi_{AB}} = 2 - A - B\,, \quad A, B = 0, 1\,,$$
$$k_\Omega = 0\,, \quad k_{\Omega_{AB}} = 2 - A - B\,, \quad A, B = 0, 1\,,$$
$$k_R = 0\,,$$
$$k_{S_j} = 4 - j\,, \quad j = 0, 1, 2, 3, 4\,.$$

6. Convergence of the formal expansion

In the previous section we have seen how to calculate a formal expansion for $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ, ϕ_{AB}, Ω, Ω_{AB}, R, S_k given $\phi|_{W_0}$ and $S_0|_{W_0}$, or, what is the same, given the null data. From Lemma 3.2 we know which are the necessary conditions on the null data in order to have analytic solutions of the conformal field equations. In this section we show that those conditions, (38) and (39), are also sufficient for the formal expansion determined in the previous section to be absolutely convergent.

So we start considering the abstract null data as given by two sequences

$$\hat{\mathcal{D}}_n^\phi = \{\psi_{A_1 B_1}, \psi_{A_2 B_2 A_1 B_1}, \psi_{A_3 B_3 A_2 B_2 A_1 B_1}, \ldots\}\,,$$
$$\hat{\mathcal{D}}_n^S = \{\Psi_{A_2 B_2 A_1 B_1}, \Psi_{A_3 B_3 A_2 B_2 A_1 B_1}, \Psi_{A_4 B_4 A_2 A_3 B_3 A_2 B_2 A_1 B_1}, \ldots\}\,,$$

of totally symmetric spinors satisfying the reality condition (30) and we construct $\phi|_{W_0}$ and $S_0|_{W_0}$, by setting in the expansions (54), (55)

$$D^*_{(A_m B_m} \ldots D^*_{A_1 B_1)}\phi(i) = \psi_{A_m B_m \ldots A_1 B_1}\,, \quad m \geq 1\,,$$
$$D^*_{(A_m B_m} \ldots D^*_{A_1 B_1} S^*_{ABCD)}(i) = \Psi_{A_m B_m \ldots A_1 B_1 ABCD}\,, \quad m \geq 0\,.$$

Observing Lemma 3.2, one finds as a necessary condition for the functions ϕ, S_0 on W_0 to determine an analytic solution to the conformal static vacuum field equations that its non-vanishing Taylor coefficients at the point O satisfy estimates of the form

$$|\partial_u^m \partial_v^n \phi(0)| \leq \binom{2m}{n} m! n! \frac{M}{r^m}\,, \quad m \geq 0\,, \quad 0 \leq n \leq 2m\,, \tag{59}$$

$$|\partial_u^m \partial_v^n S_0(0)| \leq \binom{2m+4}{n} m! n! \frac{M}{r^m}\,, \quad m \geq 0\,, \quad 0 \leq n \leq 2m + 4\,. \tag{60}$$

This conditions are also sufficient for $\phi(u, v)$ and $S_0(u, v)$ to be holomorphic functions on W_0. So the null data gives rise to two analytic functions, ϕ and S_0, on W_0.

From $A_{00} = 0$ we have $\phi_{00} = \partial_u \phi$, so having $\phi|_{W_0}$ we have $\phi_{00}|_{W_0}$, which is also an analytic function on W_0.

Following Lemma 6.1 in [8], we can derive from (59), (60), slightly different type of estimates for $\phi(u, v)$, $S_0(u, v)$, which are more convenient in our case.

Lemma 6.1. *Let e be the Euler number. For given ρ_ϕ, ρ_{S_0}, both in $\mathbb{R}$, such that $0 < \rho_\phi < e^2$, $0 < \rho_{S_0} < e^2$, there exist positive constants $\tilde{c}_\phi$, r_ϕ, $\tilde{c}_{S_0}$, r_{S_0}, so that (59), (60), imply estimates of the form*

$$|\partial_u^m \partial_v^n \phi| \leq \tilde{c}_\phi \frac{r_\phi^{m-1} m! \rho_\phi^n n!}{(m+1)^2 (n+1)^2}, \quad m \geq 0, \quad 0 \leq n \leq 2m, \tag{61}$$

$$|\partial_u^m \partial_v^n S_0| \leq \tilde{c}_{S_0} \frac{r_{S_0}^m m! \rho_{S_0}^n n!}{(m+1)^2 (n+1)^2}, \quad m \geq 0, \quad 0 \leq n \leq 2m+4. \tag{62}$$

We can present our estimates.

Lemma 6.2. *Assume $\phi = \phi(u, v)$, $S_0 = S_0(u, v)$ are holomorphic functions defined on some open neighborhood U of $O = \{u = 0, v = 0, w = 0\}$ in $W_0 = \{w = 0\}$ which have expansions of the form*

$$\phi(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{2m} \psi_{m,n} u^m v^n,$$

$$S_0(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4} \Psi_{m,n} u^m v^n,$$

so that its Taylor coefficients at the point O satisfy estimates of the type (61), (62) with some positive constants $\tilde{c}_\phi$, r_ϕ, $\tilde{c}_{S_0}$, r_{S_0}, and $\rho_\phi < \frac{1}{3}$, $\rho_{S_0} < \frac{1}{3}$. Then there exist positive constants

$$r, \rho, c_{\hat{e}^a{}_{AB}}, c_{\hat{\Gamma}_{ABCD}}, c_\phi, c_{\phi_{AB}}, c_\Omega, c_{\Omega_{AB}}, c_R, c_{S_k}$$

so that the expansion coefficients determined from ϕ and S_0 in Lemma 5.1 satisfy for $m, n, p = 0, 1, 2, \ldots$

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c_f \frac{r^{m+p+q_f} (m+p)! \rho^n n!}{(m+1)^2 (n+1)^2 (p+1)^2}, \tag{63}$$

where f stands for any of the functions

$$\hat{e}^a{}_{AB}, \hat{\Gamma}_{ABCD}, \phi, \phi_{AB}, \Omega, \Omega_{AB}, R, S_k,$$

and

$$q_{\hat{e}^a{}_{AB}} = q_{\hat{\Gamma}_{ABCD}} = q_\phi = q_\Omega = q_{\Omega_{AB}} = -1, \quad q_{\phi_{AB}} = q_R = q_{S_k} = 0.$$

Remark. Taking into account the v-finite expansion types of the functions f (Lemma 5.2), we can replace the right hand sides in the estimates above by zero if n is large enough relative to m. This will not be pointed out at each step and for convenience the estimates will be written as above.

We take the following four lemmas from [8]. The first states the necessary part of the estimates, and the other three are needed in order to manipulate the estimates in the proof of Lemma 6.2.

Lemma 6.3. *If f is holomorphic near O, there exist positive constants c, r_0, ρ_0 such that*

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c \frac{r^{m+p}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}, \quad m,n,p = 0,1,2,\ldots$$

for any $r \geq r_0$, $\rho \geq \rho_0$. If in addition $f(0,v,0) = 0$, the constants can be chosen such that

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c \frac{r^{m+p-1}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}, \quad m,n,p = 0,1,2,\ldots$$

for any $r \geq r_0$, $\rho \geq \rho_0$.

Lemma 6.4. *For any non-negative integer n there is a positive constant C, $C > 1$, independent of n so that*

$$\sum_{k=0}^{n} \frac{1}{(k+1)^2(n-k+1)^2} \leq C \frac{1}{(n+1)^2}.$$

In the following C will always denote the constant above.

Lemma 6.5. *For any integers m, n, k, j, with $0 \leq k \leq m$, and $0 \leq j \leq n$ resp. $0 \leq j \leq n-1$ holds*

$$\binom{m}{k}\binom{n}{j} \leq \binom{m+n}{k+j} \quad resp. \quad \binom{m}{k}\binom{n-1}{j} \leq \binom{m+n}{k+j}.$$

Lemma 6.6. *Let m, n, p be non-negative integers and f_i, $i = 1,\ldots,N$, be smooth complex valued functions of u, v, w on some neighborhood U of O whose derivatives satisfy on U (resp. at a given point $p \in U$) estimates of the form*

$$|\partial_u^j \partial_v^k \partial_w^l f_i| \leq c_i \frac{r^{j+l+q_i}(j+l)!\rho^k k!}{(j+1)^2(k+1)^2(l+1)^2}$$

for $0 \leq j \leq m$, $0 \leq k \leq n$, $0 \leq l \leq p$, with some positive constants c_i, r, ρ and some fixed integers q_i (independent of j, k, l). Then one has on U (resp. at p) the estimates

$$|\partial_u^m \partial_v^n \partial_w^p (f_1 \cdot \cdots \cdot f_N)| \leq C^{3(N-1)} c_1 \cdot \cdots \cdot c_N \frac{r^{m+p+q_1+\cdots q_N}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}. \quad (64)$$

Remark. This lemma remains true if m, n, p are replaced in (64) by integers m', n', p' with $0 \leq m' \leq m$, $0 \leq n' \leq n$, $0 \leq p' \leq p$.

The factor $C^{3(N-1)}$ in (64) can be replaced by $C^{(3-s)(N-1)}$ if s of the integers m, n, p vanish.

Proof of Lemma 6.2. The proof is by induction, following the procedure which led to Lemma 5.1. A general outline is as follows. We start leaving the choice of the constants r, ρ, c_f, open. We use the induction hypothesis and the equations that lead to Lemma 5.1 to derive estimates for the derivatives of the next order. These estimates are of the form

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c_f \frac{r^{m+p+q_f}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2} A_f \,, \tag{65}$$

with certain constants A_f which depend on m, n, p and the constants c_f, r and ρ. Sometimes superscripts will indicate to which order of differentiability particular constants A_f refer. In the way we will have to make assumptions on r to proceed with the induction step. We shall collect these conditions and the constants A_f, or estimates for them, and at the end it will be shown that the constants c_f, r and ρ can be adjusted so that all conditions are satisfied and $A_f \leq 1$. This will complete the induction proof.

In order not to write long formulas that do not add to the understanding of the procedure, we state here some properties that are used to simplify the estimates:

- As a corollary of Lemma 6.6 we have:
  If

  $$|\partial_u^j \partial_v^k \partial_w^l g| \leq c_g \frac{r^{j+l-1}(j+l)!\rho^k k!}{(j+1)^2(k+1)^2(l+1)^2}$$

  for $0 \leq j \leq m$, $0 \leq k \leq n$, $0 \leq l \leq p$, where g is ϕ or Ω, and if $r > \frac{C^3}{2}[c_\Omega + (c_\Omega^2 + 4c_\phi^2)^{\frac{1}{2}}]$, then

  $$\left|\partial_u^m \partial_v^n \partial_w^p \left(\frac{1}{1+\Omega-\phi^2}\right)\right| \leq \frac{1}{C^3} \frac{1}{1 - \frac{C^3}{r}\left(c_\Omega + \frac{C^3}{r}c_\phi^2\right)} \frac{r^{m+p}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}\,.$$

- If $r \geq C^3[c_\Omega + (c_\Omega^2 + 2c_\phi^2)^{\frac{1}{2}}]$ then

  $$\frac{1}{1 - \frac{C^3}{r}\left(c_\Omega + \frac{C^3}{r}c_\phi^2\right)} \leq 2\,. \tag{66}$$

- After calculating the estimates and using (66) we find that all the A's satisfy inequalities of the form

  $$A \leq \alpha + \sum_{i=1}^{9} \frac{\alpha_i}{r^i}\,,$$

  where α, α_i are constants that do not depend on r. If $\alpha_i = 0$ then we have to show that we can make $\alpha \leq 1$. If the α_i's are not zero we can take a constant a, $0 < a < 1$, and require that $\alpha \leq a$ and then choose r large enough such that $\sum_{i=1}^{8} \frac{\alpha_i}{r^i} \leq 1 - a$. In the estimates that follows, we shall not write the explicit expressions for the α_i's, as they do not play any role if we are able to make r big enough at the end of the procedure.

From now on we consider that a function in a modulus sign is evaluated at the origin O.

From the analyticity of $\phi_{00}(u, v)$ we also get that, for given $\rho_{\phi_{00}} \in \mathbb{R}$, $0 < \rho_{\phi_{00}} < \frac{1}{3}$, there exist positive constants $c_{\phi_{00}}$, $r_{\phi_{00}}$, such that

$$|\partial_u^m \partial_v^n \phi_{00}| \leq c_{\phi_{00}} \frac{r_{\phi_{00}}^m m! \rho_{\phi_{00}}^n n!}{(m+1)^2 (n+1)^2}, \quad m \geq 0, \quad 0 \leq n \leq 2m+2.$$

As $\phi(0, v) = 0$ the inequalities (61), (62) are maintained if we change the constants for bigger constants. We choose

$$c_\phi = max\{\tilde{c}_\phi, c_{\phi_{00}}\}, \tag{67}$$

$$c_{S_0} = max\left\{\tilde{c}_{S_0}, \frac{64}{3} C^3 c_{\phi_{00}}^2\right\}. \tag{68}$$

Also we require the constants r, ρ to satisfy

$$r \geq max\{r_\phi, r_{S_0}, r_{\phi_{00}}\},$$
$$\rho \geq max\{\rho_\phi, \rho_{S_0}, \rho_{\phi_{00}}\}, \tag{69}$$

but we leave the choice of the precise value open. So we have

$$|\partial_u^m \partial_v^n \partial_w^0 \phi| \leq c_\phi \frac{r^{m-1} m! \rho^n n!}{(m+1)^2 (n+1)^2}, \quad m \geq 0, \quad 0 \leq n \leq 2m,$$

$$|\partial_u^m \partial_v^n \partial_w^0 S_0| \leq c_{S_0} \frac{r^m m! \rho^n n!}{(m+1)^2 (n+1)^2}, \quad m \geq 0, \quad 0 \leq n \leq 2m+4,$$

$$|\partial_u^m \partial_v^n \partial_w^0 \phi_{00}| \leq c_{\phi_{00}} \frac{r^m m! \rho^n n!}{(m+1)^2 (n+1)^2}, \quad m \geq 0, \quad 0 \leq n \leq 2m+2.$$

From the frame properties $\hat{e}^a{}_{AB}|_{U_0} = 0$, $\hat{\Gamma}_{ABCD}|_{U_0} = 0$ follows

$$|\partial_u^0 \partial_v^n \partial_w^p \hat{\Gamma}_{ABCD}| = 0, \quad |\partial_u^0 \partial_v^n \partial_w^p \hat{e}^a{}_{AB}| = 0.$$

The conditions on the conformal factor, $\Omega|_I = 0$, $\Omega_{AB}|_I = 0$, give

$$|\partial_u^0 \partial_v^n \partial_w^0 \Omega| = 0, \quad |\partial_u^0 \partial_v^n \partial_w^0 \Omega_{AB}| = 0.$$

Using Lemma 4.1 we get the relations:

$$\phi_{A1}|_{U_0} = \frac{1}{2} \partial_v^{1+A} \phi_{00}|_{U_0}, \quad A = 0, 1, \tag{70}$$

$$S_k|_{U_0} = \frac{(4-k)!}{4!} \partial_v^k S_0|_{U_0}, \quad k = 1, 2, 3, 4, \tag{71}$$

which imply

$$|\partial_u^0 \partial_v^n \partial_w^0 \phi_{A1}| \leq \left\{ \begin{array}{ll} \frac{1}{2} c_{\phi 00} \frac{\rho^{n+1+A}(n+1+A)!}{(n+2+A)^2}, & n \leq 1-A \\ 0, & n > 1-A \end{array} \right\}$$

$$= c_{\phi A1} \frac{\rho^n n!}{(n+1)^2} A_{\phi A1}^{m=0,p=0},$$

$$A_{\phi A1}^{m=0,p=0} = \frac{1}{2} \frac{c_{\phi 00}}{c_{\phi A1}} \rho^{1+A} h_{A,n} \leq \frac{1}{2} \frac{c_{\phi 00}}{c_{\phi A1}} \rho^{1+A}, \tag{72}$$

$$h_{A,n} = \left\{ \begin{array}{ll} \frac{(n+1+A)!}{n!} \frac{(n+1)^2}{(n+2+A)^2}, & 0 \leq n \leq 1-A \\ 0, & n > 1-A \end{array} \right\} \leq 1,$$

and similarly

$$|\partial_u^0 \partial_v^n \partial_w^0 S_k| \leq c_{S_k} \frac{\rho^n n!}{(n+1)^2} A_{S_k}^{m=0,p=0},$$

$$A_{S_k}^{m=0,p=0} \leq \frac{c_{S_0}}{c_{S_k}} \rho^k. \tag{73}$$

Taking into account that R is a scalar and the initial condition $R|_i = -6 - 16 \left(\phi_{00}\phi_{11} - \phi_{01}^2 \right)|_i$, we get

$$|\partial_u^0 \partial_v^n \partial_w^0 R| \leq \left\{ \begin{array}{ll} 6 + \frac{73}{36} \rho^2 c_{\phi 00}^2, & n = 0 \\ 0, & n > 0 \end{array} \right\}$$

$$= c_R \frac{\rho^n n!}{(n+1)^2} A_R^{m=0,p=0},$$

$$A_R^{m=0,p=0} \leq \frac{1}{c_R} \left(6 + \frac{73}{36} \rho^2 c_{\phi 00}^2 \right).$$

We have obtained so far the estimates for $m = 0$, $p = 0$ and general n. Now we should consider the equations in G1 to get in an inductive form estimates for the quantities in x_1, that means, estimates for $|\partial_u^m \partial_v^n \partial_w^0 x_1|$, considering as known estimates of this type for $|\partial_u^l \partial_v^n \partial_w^0 x_1|$ with $0 \leq l < m$. And once we have this estimates we should do the same procedure with G2 to get estimates for $|\partial_u^m \partial_v^n \partial_w^0 x_3|$. These estimates, i.e. estimates for $p = 0$, can be obtained from the estimates for general p that appears later replacing C^3 by C^2 and p by 0. The estimates for general p are also more restrictive, so we do not enumerate the estimates for $p = 0$ here.

We continue with the induction procedure by considering that the estimates are satisfied for $|\partial_u^m \partial_v^n \partial_w^l X|$ for $0 \leq l < p$, and try to determine conditions for performing the induction step.

We start by formally applying $\partial_u^m \partial_v^n \partial_w^{p-1}$ to the equation $A_{11} = 0$ and taking the modulus at the origin. We get

$$|\partial_u^m \partial_v^n \partial_w^p \phi| \leq |\partial_u^m \partial_v^n \partial_w^{p-1} \phi_{11}| + |\partial_u^m \partial_v^n \partial_w^{p-1} (\hat{e}^1{}_{11} \partial_u \phi)| + |\partial_u^m \partial_v^n \partial_w^{p-1} (\hat{e}^2{}_{11} \partial_v \phi)|.$$

To estimate the terms in the r.h.s. of this inequality we have, using the induction hypothesis,

$$|\partial_u^m \partial_v^n \partial_w^{p-1} \phi_{11}| \leq c_{\phi_{11}} \frac{r^{m+p-1}(m+p-1)!\rho^n n!}{(m+1)^2(n+1)^2 p^2},$$

$$|\partial_u^m \partial_v^n \partial_w^{p-1}(\hat{e}^1\,_{11}\partial_u \phi)|$$

$$\leq \sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p-1} \binom{m}{j}\binom{n}{k}\binom{p-1}{l} |\partial_u^j \partial_v^k \partial_w^l \hat{e}^1\,_{11}||\partial_u^{m-j+1}\partial_v^{n-k}\partial_w^{p-l-1}\phi|$$

$$\leq \sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p-1} \frac{\binom{m}{j}\binom{p-1}{l}}{\binom{m+p}{j+l}}$$

$$\times \frac{c_{\hat{e}^1\,_{11}} c_\phi r^{m+p-2}(m+p)!\rho^n n!}{(j+1)^2(k+1)^2(l+1)^2(m-j+2)^2(n-k+1)^2(p-l)^2}$$

$$\leq C^3 c_{\hat{e}^1\,_{11}} c_\phi \frac{r^{m+p-2}(m+p)!\rho^n n!}{(m+2)^2(n+1)^2 p^2},$$

and similarly

$$|\partial_u^m \partial_v^n \partial_w^{p-1}(\hat{e}^2\,_{11}\partial_v \phi)| \leq C^3 c_{\hat{e}^2\,_{11}} c_\phi \frac{r^{m+p-3}(m+p-1)!\rho^n(n+1)!}{(m+1)^2(n+2)^2 p^2}.$$

Using these inequalities and writing $|\partial_u^m \partial_v^n \partial_w^p \phi|$ in the form (65), we obtain

$$A_\phi^{p\geq 1} = \frac{1}{c_\phi}\left[\frac{(p+1)^2}{p^2(m+p)} c_{\phi_{11}} + \frac{(p+1)^2(m+1)^2}{p^2(m+2)^2} \frac{C^3}{r} c_{\hat{e}^1\,_{11}} c_\phi \right.$$
$$\left. \frac{(p+1)^2(n+1)^3}{p^2(m+p)(n+2)^2} \frac{C^3}{r^2} \rho c_{\hat{e}^2\,_{11}} c_\phi \right].$$

Taking into account the v-finite expansion types of the terms involved, we see that $A_\phi^{p\geq 1} = 0$ if $n > 2m$, and thus

$$A_\phi^{p\geq 1} \leq \frac{4}{c_\phi}\left(c_{\phi_{11}} + \frac{C^3}{r} c_{\hat{e}^1\,_{11}} c_\phi + 2\frac{C^3}{r^2}\rho c_{\hat{e}^2\,_{11}} c_\phi \right) = 4\frac{c_{\phi_{11}}}{c_\phi} + \sum_{i=1}^{9} \frac{(\alpha_\phi^{p\geq 1})_i}{r^i}.$$

The procedure with the rest of the equations is similar to the one presented for the equation $A_{11} = 0$, the only difference being that if an equation is singular with u^{-1} terms we have first to multiply it by u, formally apply $\partial_u^{m+1}\partial_v^n \partial_w^{p-1}$, and then estimate the modulus. Therefore we shall not repeat the details that led from the equations to the estimates, as we shall not state the v-finite expansion type at each step. What we will state is which equation is used for deriving that particular estimate.

Applying formally $\partial_u^m \partial_v^n \partial_w^{p-1}$ to the equation $D_{11}\phi_{00} = D_{00}\phi_{11}$, which follows from $A_{00} = 0$ and $A_{11} = 0$, we obtain

$$A_{\phi_{00}}^{p\geq 1} \leq 4\frac{c_{\phi_{11}}}{c_{\phi_{00}}} + \sum_{i=1}^{9} \frac{(\alpha_{\phi_{00}}^{p\geq 1})_i}{r^i}.$$

Multiplying $H_{1(ABC)_0} + H_{0(ABC)_1} = 0$ by u and formally applying $\partial_u^{m+1}\partial_v^n\partial_w^{p-1}$ we get

$$A_{S_0}^{p\geq 1} \leq \frac{4}{c_{S_0}}\left[c_{S_2} + \frac{16}{3}C^3(c_{\phi_{00}}c_{\phi_{11}} + c_{\phi_{01}}^2)\right] + \sum_{i=1}^{9}\frac{(\alpha_{S_0}^{p\geq 1})_i}{r^i}\,.$$

In the same way as we used (70), (71) to obtain (72), (73) we get

$$A_{\phi_{A1}}^{m=0,p\geq 1} \leq \frac{1}{2}\frac{c_{\phi_{00}}}{c_{\phi_{A1}}}\rho^{1+A}\,,$$

$$A_{S_k}^{m=0,p\geq 1} \leq \frac{c_{S_0}}{c_{S_k}}\rho^k\,.$$

Restricting $\Sigma_{11} = 0$ and $\Sigma_{11CD} = 0$ to U_0 we find that on U_0

$$\Omega = 0\,,\quad \Omega_{01} = 0\,,\quad \Omega_{11} = 0\,,$$

$$\partial_w\Omega_{00} = -\frac{1}{3}R + \frac{8}{3(1-\phi^2)}(\phi\Omega_{00}\phi_{11} - 2\phi_{00}\phi_{11} + 2\phi_{01}^2)\,.$$

Taking formal derivatives of these equations we get

$$A_\Omega^{m=0} = 0\,,\quad A_{\Omega_{01}}^{m=0} = 0\,,\quad A_{\Omega_{11}}^{m=0} = 0\,,$$

and

$$A_{\Omega_{00}}^{m=0,p\geq 1} \leq \frac{4}{3}\frac{1}{c_{\Omega_{00}}}\left[c_R + 32C^2(c_{\phi_{00}}c_{\phi_{11}} + c_{\phi_{01}}^2)\right] + \sum_{i=1}^{9}\frac{(\alpha_{\Omega_{00}}^{m=0,p\geq 1})_i}{r^i}\,.$$

Restricting $\Pi_{11} = 0$ to U_0 gives

$$\partial_w R = 3\Omega_{00}S_4 + \frac{8}{1-\phi^2}\left[-2\phi_{11}\partial_w\phi_{00} + 4\phi_{01}\partial_w\phi_{01} + (\phi\Omega_{00} - 2\phi_{00})\partial_w\phi_{11}\right.$$

$$\left.-\frac{1}{3}\phi R\phi_{11}\right] + \frac{8}{3(1-\phi^2)^2}\phi_{11}\left[(3+11\phi^2)\Omega_{00}\phi_{11} - 28\phi(\phi_{00}\phi_{11} - \phi_{01}^2)\right]\,,$$

so that

$$A_R^{m=0,p\geq 1} \leq \frac{64C^2}{c_R}(c_{\phi_{00}}c_{\phi_{11}} + c_{\phi_{01}}^2) + \sum_{i=1}^{9}\frac{(\alpha_R^{m=0,p\geq 1})_i}{r^i}\,.$$

We complete the calculation of the A's by using the ∂_u-equations. We have to calculate the estimates in the order given by the hierarchy presented in Section 5.3 but for simplicity we present the estimates in the order the ∂_u-equations were stated in Section 5.2.

$t_{AB}{}^{EF}{}_{00}e^a{}_{EF} = 0$:

$$A^{m \geq 1}_{\hat{e}^1{}_{01}} \leq \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{e}^1{}_{01}})_i}{r^i},$$

$$A^{m \geq 1}_{\hat{e}^2{}_{01}} \leq \frac{1}{2} \frac{c_{\hat{\Gamma}_{0100}}}{c_{\hat{e}^2{}_{01}}} + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{e}^2{}_{01}})_i}{r^i},$$

$$A^{m \geq 1}_{\hat{e}^1{}_{11}} \leq \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{e}^1{}_{11}})_i}{r^i},$$

$$A^{m \geq 1}_{\hat{e}^2{}_{11}} \leq \frac{c_{\hat{\Gamma}_{1100}}}{c_{\hat{e}^2{}_{11}}} + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{e}^2{}_{11}})_i}{r^i}.$$

$R_{AB00EF} = 0$:

$$A^{m \geq 1}_{\hat{\Gamma}_{0100}} \leq \frac{2}{3} \frac{c_{S_0}}{c_{\hat{\Gamma}_{0100}}} + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{\Gamma}_{0100}})_i}{r^i},$$

$$A^{m \geq 1}_{\hat{\Gamma}_{0101}} \leq \frac{c_{S_1}}{c_{\hat{\Gamma}_{0101}}} + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{\Gamma}_{0101}})_i}{r^i},$$

$$A^{m \geq 1}_{\hat{\Gamma}_{0111}} \leq \frac{c_{S_2}}{c_{\hat{\Gamma}_{0111}}} + \frac{c_R}{6c_{\hat{\Gamma}_{0111}}} + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{\Gamma}_{0111}})_i}{r^i},$$

$$A^{m \geq 1}_{\hat{\Gamma}_{1100}} \leq 2\frac{c_{S_1}}{c_{\hat{\Gamma}_{1100}}} + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{\Gamma}_{1100}})_i}{r^i},$$

$$A^{m \geq 1}_{\hat{\Gamma}_{1101}} \leq \frac{4}{c_{\hat{\Gamma}_{1101}}} \left(c_{S_2} + \frac{1}{12}c_R \right) + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{\Gamma}_{1101}})_i}{r^i},$$

$$A^{m \geq 1}_{\hat{\Gamma}_{1111}} \leq \frac{4c_{S_3}}{c_{\hat{\Gamma}_{1111}}} + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\hat{\Gamma}_{1111}})_i}{r^i}.$$

$\Sigma_{00} = 0$:

$$A^{m \geq 1}_{\Omega} \leq \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\Omega})_i}{r^i}.$$

$\Phi_{00} = 0$:

$$A^{m \geq 1}_{\phi_{01}} \leq \frac{c_{\phi_{00}}}{c_{\phi_{01}}}\rho + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\phi_{01}})_i}{r^i}.$$

$\Phi_{10} = 0$:

$$A^{m \geq 1}_{\phi_{11}} \leq \frac{c_{\phi_{01}}}{c_{\phi_{11}}}\rho + \sum_{i=1}^{9} \frac{(\alpha^{m \geq 1}_{\phi_{11}})_i}{r^i}.$$

$\Sigma_{00CD} = 0$:

$$A_{\Omega_{00}}^{m\geq 1} \leq \sum_{i=1}^{9} \frac{(\alpha_{\Omega_{00}}^{m\geq 1})_i}{r^i},$$

$$A_{\Omega_{01}}^{m\geq 1} \leq \sum_{i=1}^{9} \frac{(\alpha_{\Omega_{01}}^{m\geq 1})_i}{r^i},$$

$$A_{\Omega_{11}}^{m\geq 1} \leq \frac{4}{3c_{\Omega_{11}}}\left[c_R + 32C^3(c_{\phi_{00}}c_{\phi_{11}} + c_{\phi_{01}}^2)\right] + \sum_{i=1}^{9} \frac{(\alpha_{\Omega_{11}}^{m\geq 1})_i}{r^i}.$$

$\Pi_{00} = 0$:

$$A_R^{m\geq 1} \leq \frac{64C^3}{c_R}(c_{\phi_{00}}c_{\phi_{11}} + c_{\phi_{01}}^2) + \sum_{i=1}^{9} \frac{(\alpha_R^{m\geq 1})_i}{r^i}.$$

$H_{0(ABC)_k} = 0$:

$$A_{S_1}^{m\geq 1} \leq \frac{c_{S_0}}{c_{S_1}}\rho + \sum_{i=1}^{9} \frac{(\alpha_{S_1}^{m\geq 1})_i}{r^i},$$

$$A_{S_2}^{m\geq 1} \leq \frac{1}{c_{S_2}}\left[\rho c_{S_1} + \frac{8}{3}C^3(c_{\phi_{00}}c_{\phi_{11}} + c_{\phi_{01}}^2)\right] + \sum_{i=1}^{9} \frac{(\alpha_{S_2}^{m\geq 1})_i}{r^i},$$

$$A_{S_3}^{m\geq 1} \leq \frac{1}{c_{S_3}}\left\{\rho c_{S_2} + \frac{8}{3}C^3\left[\rho(c_{\phi_{00}}c_{\phi_{11}} + 2c_{\phi_{01}}^2) + \frac{3}{2}c_{\phi_{01}}c_{\phi_{11}}\right]\right\} + \sum_{i=1}^{9} \frac{(\alpha_{S_3}^{m\geq 1})_i}{r^i},$$

$$A_{S_4}^{m\geq 1} \leq \frac{1}{c_{S_4}}\left[\rho c_{S_3} + 4C^3 c_{\phi_{11}}\left(2\rho c_{\phi_{01}} + \frac{8}{3}c_{\phi_{00}} + c_{\phi_{11}}\right)\right] + \sum_{i=1}^{9} \frac{(\alpha_{S_4}^{m\geq 1})_i}{r^i}.$$

We now have to show that all the constants can be chosen in a way that makes all the A's less or equal than 1. So, introducing a constant a, $0 < a < 1$, the following inequalities need to be satisfied:

$$\frac{1}{2}\frac{c_{\phi_{00}}}{c_{\phi_{01}}}\rho \leq 1, \qquad (74)$$

$$\frac{1}{2}\frac{c_{\phi_{00}}}{c_{\phi_{11}}}\rho^2 \leq 1, \qquad (75)$$

$$\frac{c_{S_0}}{c_{S_1}}\rho \leq 1, \quad \frac{c_{S_0}}{c_{S_2}}\rho^2 \leq 1, \quad \frac{c_{S_0}}{c_{S_3}}\rho^3 \leq 1, \quad \frac{c_{S_0}}{c_{S_4}}\rho^4 \leq 1, \qquad (76)$$

$$\frac{1}{c_R}\left(6 + \frac{73}{36}\rho^2 c_{\phi_{00}}^2\right) \leq 1, \qquad (77)$$

$$4\frac{c_{\phi_{11}}}{c_\phi} \leq a, \quad 4\frac{c_{\phi_{11}}}{c_{\phi_{00}}} \leq a, \quad \frac{4}{c_{S_0}}\left[c_{S_2} + \frac{16}{3}C^3(c_{\phi_{00}}c_{\phi_{11}} + c_{\phi_{01}}^2)\right] \leq a, \qquad (78)$$

$$\frac{4}{3}\frac{1}{c_{\Omega_{00}}}\left[c_R + 32C^2(c_{\phi_{00}}c_{\phi_{11}} + c_{\phi_{01}}^2)\right] \leq a, \qquad (79)$$

$$\frac{1}{c_R} 64 C^2 (c_{\phi_{00}} c_{\phi_{11}} + c_{\phi_{01}}^2) \le a, \qquad (80)$$

$$\frac{1}{2} \frac{c_{\hat{\Gamma}_{0100}}}{c_{\hat{e}^2{}_{01}}} \le a, \quad \frac{c_{\hat{\Gamma}_{1100}}}{c_{\hat{e}^2{}_{11}}} \le a, \quad \frac{2}{3} \frac{c_{S_0}}{c_{\hat{\Gamma}_{0100}}} \le a, \quad \frac{c_{S_1}}{c_{\hat{\Gamma}_{0101}}} \le a, \qquad (81)$$

$$\frac{1}{c_{\hat{\Gamma}_{0111}}} \left(c_{S_2} + \frac{1}{6} c_R \right) \le a, \quad 2 \frac{c_{S_1}}{c_{\hat{\Gamma}_{1100}}} \le a, \qquad (82)$$

$$\frac{4}{c_{\hat{\Gamma}_{1101}}} \left(c_{S_2} + \frac{1}{12} c_R \right) \le a, \quad 4 \frac{c_{S_3}}{c_{\hat{\Gamma}_{1111}}} \le a, \qquad (83)$$

$$\frac{c_{\phi_{00}}}{c_{\phi_{01}}} \rho \le a, \qquad (84)$$

$$\frac{c_{\phi_{01}}}{c_{\phi_{11}}} \rho \le a, \qquad (85)$$

$$\frac{4}{3} \frac{1}{c_{\Omega_{11}}} \left[c_R + 32 C^3 (c_{\phi_{00}} c_{\phi_{11}} + c_{\phi_{01}}^2) \right] \le a, \qquad (86)$$

$$\frac{1}{c_R} 64 C^3 (c_{\phi_{00}} c_{\phi_{11}} + c_{\phi_{01}}^2) \le a, \qquad (87)$$

$$\frac{c_{S_0}}{c_{S_1}} \rho \le a, \quad \frac{1}{c_{S_2}} \left[c_{S_1} \rho + \frac{8}{3} C^3 (c_{\phi_{00}} c_{\phi_{11}} + c_{\phi_{01}}^2) \right] \le a, \qquad (88)$$

$$\frac{1}{c_{S_3}} \left[c_{S_2} \rho + \frac{4}{3} C^3 (2 \rho c_{\phi_{00}} c_{\phi_{11}} + 4 \rho c_{\phi_{01}}^2 + 3 c_{\phi_{01}} c_{\phi_{11}}) \right] \le a, \qquad (89)$$

$$\frac{1}{c_{S_4}} \left[c_{S_3} \rho + \frac{4}{3} C^3 c_{\phi_{11}} (8 c_{\phi_{00}} + 6 \rho c_{\phi_{01}} + 3 c_{\phi_{11}}) \right] \le a. \qquad (90)$$

Now we have to show that we can choose the constants such that these inequalities will be satisfied.

We start by setting

$$c_{\phi_{01}} \equiv \frac{\rho}{a} c_{\phi_{00}},$$

with which we satisfy (74) and (84). Next we set

$$c_{\phi_{11}} \equiv \frac{\rho^2}{a^2} c_{\phi_{00}},$$

so that (75) and (85) are satisfied.

We continue by setting

$$c_{S_1} \equiv \frac{\rho}{a} c_{S_0}, \quad c_{S_2} \equiv \frac{\rho^2}{a^2} \left(c_{S_0} + \frac{16}{3} \frac{C^3}{a} c_{\phi_{00}}^2 \right),$$

$$c_{S_3} \equiv \frac{\rho^3}{a^3} \left[c_{S_0} + \frac{8}{3} \left(3 + \frac{7}{2a} \right) C^3 c_{\phi_{00}}^2 \right],$$

$$c_{S_4} \equiv \frac{\rho^4}{a^4} \left[c_{S_0} + \frac{8}{3} \left(6 + \frac{5}{a} + 4 \frac{a}{\rho^2} \right) C^3 c_{\phi_{00}}^2 \right].$$

With this we satisfy (76), (88), (89) and (90).

Inequalities (77), (80) and (87) are satisfied with

$$c_R \equiv max\left\{128\frac{\rho^2}{a^3}C^3c_{\phi_{00}}^2, 6 + \frac{73}{36}\rho^2c_{\phi_{00}}^2\right\}.$$

With this definition for c_R we set

$$c_{\Omega_{00}} \equiv \frac{4}{3a}\left(c_R + 64C^2\frac{\rho^2}{a^2}c_{\phi_{00}}^2\right), \quad c_{\Omega_{11}} \equiv \frac{4}{3a}\left(c_R + 64C^3\frac{\rho^2}{a^2}c_{\phi_{00}}^2\right),$$

so (79) and (86) are respectively satisfied.

Using the previous definitions we set also

$$c_{\hat{\Gamma}_{0100}} \equiv \frac{2}{3a}c_{S_0}, \quad c_{\hat{\Gamma}_{0101}} \equiv \frac{1}{a}c_{S_1},$$

$$c_{\hat{\Gamma}_{0111}} \equiv \frac{1}{a}\left(c_{S_2} + \frac{1}{6}c_R\right), \quad c_{\hat{\Gamma}_{1100}} \equiv \frac{2}{a}c_{S_1},$$

$$c_{\hat{\Gamma}_{1101}} \equiv \frac{4}{a}\left(c_{S_2} + \frac{1}{12}c_R\right), \quad c_{\hat{\Gamma}_{1111}} \equiv \frac{4}{a}c_{S_3},$$

$$c_{\hat{e}^2\,01} \equiv \frac{1}{3a^2}c_{S_0}, \quad c_{\hat{e}^2\,11} \equiv \frac{2}{a^2}c_{S_1},$$

and (81), (82) and (83) are satisfied.

There are three inequalities that we have not yet considered, (78). These are now reduced to

$$4\rho^2\frac{c_{\phi_{00}}}{c_\phi} \leq a^3,$$

$$4\rho^2 \leq a^3,$$

$$4\rho^2\left[1 + \frac{1}{c_{S_0}}\frac{16}{3}C^3c_{\phi_{00}}^2\left(2 + \frac{1}{a}\right)\right] \leq a^3.$$

Taking into consideration now (67), (68), (69) we see that these inequalities can be satisfied if we define

$$\rho \equiv max\{\rho_\phi, \rho_{S_0}, \rho_{\phi_{00}}\} < \frac{1}{3},$$

$$a \equiv max\left\{\frac{1}{2}, (8\rho^2)^{\frac{1}{3}}\right\} < 1.$$

Now we choose some positive constants $c_\Omega, c_{\Omega_{01}}, c_{\hat{e}^1\,01}, c_{\hat{e}^1\,11}$, that are not restricted by the procedure.

Finally we choose r so large that

$$r > max\left\{r_\phi, r_{S_0}, r_{\phi_{00}}, C^3\left[c_\Omega + (c_\Omega^2 + 2c_\phi^2)^{\frac{1}{2}}\right]\right\}$$

and that all the A's are less or equal than 1. The induction proof is completed. $\square$

The following lemma states the convergence result. The proof follows as the one given in [8].

Lemma 6.7. *The estimates* (63) *for the derivatives of the functions f and the expansion types given in Lemma* 5.2 *imply that the associated Taylor series are absolutely convergent in the domain $|v| < \frac{1}{\alpha\rho}$, $|u| + |w| < \frac{\alpha^2}{r}$, for any real number α, $0 < \alpha \leq 1$. It follows that the formal expansion determined in Lemma* 5.1 *defines indeed a* (*unique*) *holomorphic solution to the conformal static vacuum field equations which induces the data ϕ, S_0 on W_0.*

7. The complete set of equations on $\hat{N}$

We have seen in Section 5 how to calculate a formal expansion for our fields using a subset of the conformal stationary vacuum field equations. In the previous section we have shown that these formal expansions are convergent in a neighborhood of infinity. In this section we shall show that these fields satisfy the complete system of conformal stationary vacuum field equations. First, we prove that the conformal stationary vacuum field equations are satisfied in the limit as $u \to 0$. Second, we derive a subsidiary system of equations, for which the first result provides the initial conditions, and which allows us to prove that the complete system is satisfied.

Lemma 7.1. *The functions $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ, ϕ_{AB}, Ω, Ω_{AB}, R, S_k, whose expansion coefficients are determined by Lemma* 5.1, *with expansions that converge on an open neighborhood of the point* 0, *neighborhood that we assume to coincide with $\hat{N}$, satisfy the complete set of conformal field equations on the set U_0 in the sense that the fields $t_{AB}{}^{CD}{}_{EF}$, R_{ABCDEF}, A_{AB}, Σ_{AB}, Φ_{AB}, Π_{AB}, Σ_{ABCD}, H_{ABCD} calculated from these functions on $\hat{N}\backslash U_0$ have vanishing limit as $u \to 0$.*

Proof. Taking into account which equations have already been used to determine the formal expansions, and the symmetries of the equations, it is left to show that $t_{01}{}^{EF}{}_{11}, R_{AB0111}, A_{01}, \Sigma_{01}, \Pi_{01}, \Sigma_{01CD}, H_{1(BCD)_{k=1,2,3}}$, have vanishing limit on $\hat{N}\backslash U_0$ as $u \to 0$, and that in the same limit $\Phi_{AB} = -\Phi_{BA}$.

Because $\langle \sigma^{AB}, e_{EF} \rangle = h^{AB}{}_{EF}$ then

$$t_{01}{}^{EF}{}_{11} = 2\Gamma_{01}{}^{(E}{}_1\epsilon_1{}^{F)} - 2\Gamma_{11}{}^{(E}{}_{(0}\epsilon_{1)}{}^{F)} - \sigma^{EF}{}_a \left(e^a{}_{11,b}e^b{}_{01} - e^a{}_{01,b}e^b{}_{11} \right),$$

and using the way in which the coordinates and the frame field were constructed, we see that

$$t_{01}{}^{EF}{}_{11} = O(u), \quad \text{as} \quad u \to 0.$$

We now consider

$$R_{AB0111} = -\frac{1}{2}\left(S_{AB11} - \frac{1}{6}R\epsilon_{A1}\epsilon_{B1} \right) + \frac{1}{2u}\partial_v\hat{\Gamma}_{11AB} - \frac{1}{u}\hat{\Gamma}_{111(A}\epsilon_{B)}{}^0$$
$$+ \epsilon_A{}^0\epsilon_B{}^0 \left(-\frac{1}{2u^2}\hat{e}^1{}_{11} + \frac{1}{u}\hat{\Gamma}_{0111} - \frac{1}{u}t_{01}{}^{01}{}_{11} \right) + O(u).$$

Using that

$$t_{01}{}^{01}{}_{11} = \hat{\Gamma}_{0111} - \frac{1}{2}\partial_v \hat{e}^2{}_{11} - \frac{1}{2u}\hat{e}^1{}_{11} + O(u^2)$$

we get

$$R_{AB0111} = \frac{1}{2u}\left[\partial_v \hat{\Gamma}_{11AB} - 2\hat{\Gamma}_{111(A}\epsilon_{B)}{}^0 + \epsilon_A{}^0\epsilon_B{}^0\left(-\frac{2}{u}\hat{e}^1{}_{11} - \partial_v \hat{e}^2{}_{11} + 4\hat{\Gamma}_{0111}\right)\right]$$
$$- \frac{1}{2}\left(S_{AB11} - \frac{1}{6}R\epsilon_{A1}\epsilon_{B1}\right) + O(u),$$

so that

$$\lim_{u\to 0}R_{AB0111} = \frac{1}{2}\left[\partial_u\partial_v\hat{\Gamma}_{11AB} - 2\partial_u\hat{\Gamma}_{111(A}\epsilon_{B)}{}^0\right.$$
$$+ \epsilon_A{}^0\epsilon_B{}^0(-\partial_u^2\hat{e}^1{}_{11} - \partial_u\partial_v\hat{e}^2{}_{11} + 4\partial_u\hat{\Gamma}_{0111})$$
$$\left.\left. - S_{AB11} + \frac{1}{6}R\epsilon_{A1}\epsilon_{B1}\right]\right|_{u=0}.$$

For the case $A = B = 0$ we get from the ∂_u-equations that

$$\partial_u^2\hat{e}^1{}_{11} = -2\partial_u\hat{\Gamma}_{1101}, \quad \partial_u\partial_v\hat{e}^2{}_{11} = \partial_u\partial_v\hat{\Gamma}_{1100}, \quad \partial_u\hat{\Gamma}_{0111} = \frac{1}{4}\left(S_2 - \frac{1}{6}R\right),$$

on U_0, and so $\lim_{u\to 0}R_{000111} = 0$.

Using the ∂_u-equations and that $\partial_v S_2 = 2S_3$ on U_0,

$$\partial_u\hat{\Gamma}_{1111} = S_3, \quad \partial_u\partial_v\hat{\Gamma}_{1101} = 2S_3,$$

on U_0, and so $\lim_{u\to 0}R_{010111} = 0$. As $\partial_v S_3 = S_4$ on U_0, $\lim_{u\to 0}R_{110111} = 0$.

We take now the limit of A_{01} as u goes to 0,

$$\lim_{u\to 0}A_{01} = \left[\frac{1}{2}\partial_u\partial_v\phi - \phi_{01}\right]\bigg|_{u=0}.$$

Using that $\phi_{01} = \frac{1}{2}\partial_v\phi_{00}$ on U_0 and that we have $A_{00} = 0$ as part of the ∂_u-equations we get $\lim_{u\to 0}A_{01} = 0$. With the same procedure we get $\lim_{u\to 0}\Sigma_{01} = 0$.

Now we consider Φ_{AB}. As $A_{AB} = 0$ on U_0 then

$$D^P{}_B\phi_{AP}|_{U_0} = -D^P{}_A\phi_{BP}|_{U_0},$$

so $\Phi_{AB}|_{U_0} = -\Phi_{BA}|_{U_0}$ and as we already have $\Phi_{10} = 0$ then $\Phi_{AB} = 0$ on U_0.

We now take the limit as u goes to 0 of the combination $\Pi_{01} - \frac{1}{2}\partial_v\Pi_{00}$. For the limits of the derivatives involved we have at $\{u = 0\}$

$$D_{00}\phi_{AB} = \partial_u\phi_{AB}\,,$$

$$D_{01}\phi_{01} = \frac{1}{2}\left(\partial_u\partial_v\phi_{01} - \partial_u\phi_{11}\right),$$

$$D_{01}\phi_{11} = \frac{1}{2}\partial_u\partial_v\phi_{11}\,,$$

$$D_{11}\phi_{11} = \partial_w\phi_{11}\,,$$

$$D_{00}R = \partial_uR\,,$$

$$D_{01}R = \frac{1}{2}\partial_u\partial_vR\,,$$

$$D_{11}R = \partial_wR\,.$$

We also use that on U_0

$$\partial_v\phi_k = (2-k)\phi_{k+1}\,,$$

$$\partial_vS_k = (4-k)S_{k+1}\,,$$

We have already used the equations $\Sigma_{11} = 0, \Sigma_{11CD} = 0$ restricted to U_0, finding that Ω, Ω_{A1} are zero on U_0.

Furthermore we use $\Phi_{00} = 0$, that says that on U_0

$$\partial_u\partial_v\phi_{00} = 4\partial_u\phi_{01}\,.$$

So we get for the limit

$$\lim_{u\to 0}\left(\Pi_{01} - \frac{1}{2}\partial_v\Pi_{00}\right) = 0\,.$$

Considering that from the ∂_u-equations we already have $\Pi_{00} = 0$ we get

$$\lim_{u\to 0}\Pi_{01} = 0\,.$$

We apply a similar procedure to the Σ_{01AB} equations. We take the limit as u goes to zero of the combinations

$$\Sigma_{0100} - \Sigma_{0001}\,,$$

$$2\Sigma_{0101} - \partial_v\Sigma_{0001} + \Sigma_{0011}\,,$$

$$2\Sigma_{0111} - \partial_v\Sigma_{0011}\,.$$

Using what has already been said together with the following limits at $\{u = 0\}$

$$D_{00}\Omega_{AB} = \partial_u\Omega_{AB}\,,$$

$$D_{01}\Omega_{01} = \frac{1}{2}\left(\partial_u\partial_v\Omega_{01} - \partial_u\Omega_{11}\right),$$

$$D_{01}\Omega_{11} = \frac{1}{2}\partial_u\partial_v\Omega_{11}\,,$$

$$D_{11}\Omega_{11} = \partial_w\Omega_{11}\,,$$

we see that the limits vanishes, which imply

$$\lim_{u \to 0} \Sigma_{01AB} = 0 \,.$$

Finally we consider the limit as u goes to zero of the combinations

$$4H_{1(ABC)_1} - \partial_v H_{1(ABC)_0} - \partial_v H_{0(ABC)_1} + 2H_{0(ABC)_2} \,,$$

$$12H_{1(ABC)_2} - \partial_v^2 H_{1(ABC)_0} - \partial_v^2 H_{0(ABC)_1} - 2\partial_v H_{0(ABC)_2} + 4H_{0(ABC)_3} \,,$$

$$24H_{1(ABC)_3} - \partial_v^3 H_{1(ABC)_0} - \partial_v^3 H_{0(ABC)_1} - 2\partial_v^2 H_{0(ABC)_2} - 8\partial_v H_{0(ABC)_3} \,,$$

and using what has been said together with:
the limits

$$D_{00}S_k = \partial_u S_k \,,$$

$$D_{01}S_k = \frac{1}{2}\left[\partial_u \partial_v S_k - (4-k)\partial_u S_{k+1}\right] \,,$$

$$D_{11}S_k = \partial_w S_k \,,$$

the equality on U_0

$$\partial_v S_k = (4-k)S_{k+1} \,,$$

and the equations

$$D_{00}\phi_{A1} = D_{A1}\phi_{00} \,,$$

we find that those limits are all zero, and considering the equations that we have
used to calculate the unknowns we get

$$\lim_{u \to 0} H_{1(ABC)_k} = 0 \,, \quad k = 1, 2, 3 \,.$$

This completes the proof that the complete system of conformal field equations
are satisfied in the limit as $u \to 0$. $\square$

Lemma 7.2. *The functions $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ, ϕ_{AB}, Ω, Ω_{AB}, R, S_k, corresponding to the expansions determined in Lemma* 5.1 *satisfy the complete set of conformal vacuum field equations on the set $\hat{N}$.*

Proof. We have to show that on $\hat{N}$ the quantities $t_{01}{}^{EF}{}_{11}$, R_{AB0111}, A_{01}, Σ_{A1}, Π_{A1}, Σ_{A1CD}, $H_{1(BCD)_{k=1,2,3}}$ vanish, and that $\Phi_{AB} = -\Phi_{BA}$. For this we derive a system of subsidiary equations for these fields. The values of the fields at U_0, given by Lemma 7.1, are the initial conditions for the subsidiary system of equations, and they are used throughout the proof.

Using the definitions of A_{AB} and Φ_{AB}:

$$D_{AB}A_{CD} - D_{CD}A_{AB} = -t_{AB}{}^{EF}{}_{CD}D_{EF}\phi + \epsilon_{AD}\Phi_{BC} + \epsilon_{BC}\Phi_{DA} \,,$$

and in particular

$$\left(\partial_u + \frac{1}{u}\right)A_{01} = 2\hat{\Gamma}_{0100}A_{01} \,,$$

which implies $A_{01} = 0$, and from that $A_{AB} = 0$. This also shows that $\Phi_{AB} = -\Phi_{AB}$, and as we already know that $\Phi_{10} = 0$ then $\Phi_{AB} = 0$.

Following the proof of Lemma 5.5 in [8] we find that

$$\left(\partial_u + \frac{1}{u}\right) t_{01}{}^{AB}{}_{11} = 2\hat{\Gamma}_{0100}t_{01}{}^{AB}{}_{11} + 2R^{(A}{}_{00111}\epsilon_0{}^{B)}, \tag{91}$$

which directly shows that $t_{01}{}^{11}{}_{11} = 0$.

Also following the proof of Lemma 5.5 in [8] and taking into account that S_{ABCD} and R satisfy the contracted Bianchi identity then

$$\left(\partial_u + \frac{1}{u}\right) R_{AB0111} = 2\hat{\Gamma}_{0100}R_{AB0111} - \frac{1}{2}\left(H_{1AB0} - \frac{1}{6}\Pi_{AB}\right), \tag{92}$$

from which $R_{000111} = 0$, which also gives $t_{01}{}^{01}{}_{11} = 0$.

It is still left to show that

$$t_{01}{}^{00}{}_{11}, \quad R_{A10111}, \quad \Sigma_{A1}, \quad \Pi_{A1}, \quad \Sigma_{A1CD}, \quad H_{1(BCD)_{k=1,2,3}} \tag{93}$$

vanish on $\hat{N}$.

Using the definitions of Σ_{AB} and Σ_{ABCD},

$$D_{AB}\Sigma_{CD} - D_{CD}\Sigma_{AB} = -t_{AB}{}^{EF}{}_{CD}D_{EF}\Omega - \Sigma_{ABCD} + \Sigma_{CDAB},$$

and from that

$$\partial_u\Sigma_{A1} + \frac{1}{u}\Sigma_{01}\epsilon_A{}^0 = 2\hat{\Gamma}_{A100}\Sigma_{01} + \Sigma_{A100}. \tag{94}$$

At this point the expressions became to long to be treated by hand, so we resort to a computer program for tensor manipulations.

For Σ_{ABCD} we obtain

$$
\begin{aligned}
D_{EF}\Sigma_{CDAB} - D_{CD}\Sigma_{EFAB} =\ & t_{CD}{}^{PQ}{}_{EF}D_{PQ}\Omega_{AB} - 2\Omega^P{}_{(A}R_{B)PCDEF} \\
& + \Omega\left(\epsilon_{DE}H_{FABC} + \epsilon_{CF}H_{DABE}\right) + S_{ABCD}\Sigma_{EF} - S_{ABEF}\Sigma_{CD} \\
& + \frac{R}{3}\left(h_{ABCD}\Sigma_{EF} - h_{ABEF}\Sigma_{CD}\right) + \frac{1}{3}\left(1 + \Omega\right)\left(h_{ABCD}\Pi_{EF} - h_{ABEF}\Pi_{CD}\right) \\
& + \frac{1}{6\left(1 + \Omega - \phi^2\right)}\Big\{3\left[1 - (1 - \Omega)\phi^2\right]\left[\Omega_{EF}\Sigma_{CDAB} - \Omega_{CD}\Sigma_{EFAB}\right. \\
& + \Omega_{AB}(\Sigma_{CDEF} - \Sigma_{EFCD})\big] \\
& - 6\Omega^2\phi\left[\phi_{EF}\Sigma_{CDAB} - \phi_{CD}\Sigma_{EFAB} + \phi_{AB}(\Sigma_{CDEF} - \Sigma_{EFCD})\right] \\
& + 4\left(2 + 3\Omega\right)\phi^2\left(h_{ABCD}\Sigma_{EFPQ}\Omega^{PQ} - h_{ABEF}\Sigma_{CDPQ}\Omega^{PQ}\right) \\
& - 8\left(1 + \Omega\right)\left(2 + 3\Omega\right)\phi\left(h_{ABCD}\Sigma_{EFPQ}\phi^{PQ} - h_{ABEF}\Sigma_{CDPQ}\phi^{PQ}\right)\Big\} \\
& + \frac{1}{\left(1 + \Omega - \phi^2\right)^2}\Big\{\frac{1}{2}\left(1 - \phi^2\right)^2\Omega_{AB}\left(\Omega_{CD}\Sigma_{EF} - \Omega_{EF}\Sigma_{CD}\right) \\
& + \Omega\phi\left(2 + \Omega - 2\phi^2\right)\left[\phi_{AB}(\Omega_{CD}\Sigma_{EF} - \Omega_{EF}\Sigma_{CD})\right. \\
& + \Omega_{AB}(\phi_{CD}\Sigma_{EF} - \phi_{EF}\Sigma_{CD})\big] \\
& - 2\Omega\left[2(1 + \Omega)^2 - (2 + 3\Omega)\phi^2\right]\phi_{AB}\left(\phi_{CD}\Sigma_{EF} - \phi_{EF}\Sigma_{CD}\right)
\end{aligned}
$$

$$-\frac{1}{3}\Big[\phi^2(-1+3\phi^2)\Omega^{PQ}\Omega_{PQ} + 4\phi\big[3(1+\Omega)^2 - (5+6\Omega)\phi^2\big]\Omega^{PQ}\phi_{PQ}$$

$$-4(1+\Omega)\big[(1+\Omega)(5+6\Omega) - (7+9\Omega)\phi^2\big]\phi^{PQ}\phi_{PQ}\Big]$$

$$(h_{ABCD}\Sigma_{EF} - h_{ABEF}\Sigma_{CD})\bigg\},$$

which implies

$$\partial_u\Sigma_{C1AB} + \frac{1}{u}\epsilon_C{}^0\Sigma_{01AB} = 2\hat{\Gamma}_{C100}\Sigma_{01AB} - \Omega\epsilon_{0C}H_{10AB} - S_{00AB}\Sigma_{1C} \qquad (95)$$

$$-\frac{R}{3}h_{00AB}\Sigma_{1C} - \frac{1}{3}(1+\Omega)h_{00AB}\Pi_{1C}$$

$$+\frac{1}{6(1+\Omega-\phi^2)}\Big\{3\big[1 - (1-\Omega)\phi^2\big](\Omega_{00}\Sigma_{1CAB} + \Omega_{AB}\Sigma_{1C00})$$

$$-6\Omega^2\phi\,(\phi_{00}\Sigma_{1CAB} + \phi_{AB}\Sigma_{1C00})$$

$$-4(2+3\Omega)\phi^2 h_{00AB}\Sigma_{C1PQ}\Omega^{PQ} + 8(1+\Omega)(2+3\Omega)\phi h_{00AB}\Sigma_{C1PQ}\phi^{PQ}\Big\}$$

$$+\frac{1}{(1+\Omega-\phi^2)^2}\bigg\{-\frac{1}{2}\left(1-\phi^2\right)^2\Omega_{AB}\Omega_{00}\Sigma_{1C}$$

$$-\Omega\phi\left(2+\Omega-2\phi^2\right)(\phi_{AB}\Omega_{00} + \Omega_{AB}\phi_{00})\Sigma_{1C}$$

$$+2\Omega\big[2\left(1+\Omega\right)^2 - (2+3\Omega)\phi^2\big]\phi_{AB}\phi_{00}\Sigma_{1C}$$

$$+\frac{1}{3}\Big[\phi^2\left(-1+3\phi^2\right)\Omega^{PQ}\Omega_{PQ} + 4\phi\big[3\left(1+\Omega\right)^2 - (5+6\Omega)\phi^2\big]\Omega^{PQ}\phi_{PQ}$$

$$-4\left(1+\Omega\right)\big[\left(1+\Omega\right)\left(5+6\Omega\right) - (7+9\Omega)\phi^2\big]\phi^{PQ}\phi_{PQ}\Big]h_{00AB}\Sigma_{1C}\bigg\}.$$

Now with Π_{AB}

$$D_{CD}\Pi_{AB} - D_{AB}\Pi_{CD} = t_{AB}{}^{EF}{}_{CD}D_{EF}R$$

$$+\frac{1}{1+\Omega-\phi^2}\bigg\{-2\left(4+7\Omega\right)\big[\phi\Omega^{GH} - 2(1+\Omega)\phi^{GH}\big]D_{EF}\phi_{GH}t_{AB}{}^{EF}{}_{CD}$$

$$-4\left(4+7\Omega\right)\big[\phi\Omega^{GH} - 2(1+\Omega)\phi^{GH}\big]\phi_{GE}R^E{}_{HABCD}$$

$$-\big[(3-3\phi^2+7\Omega\phi^2)\Omega^{EF} - 2\Omega\phi(4+7\Omega)\phi^{EF}\big](\epsilon_{BC}H_{DAEF} + \epsilon_{AD}H_{BCEF})$$

$$+\frac{1}{3}\phi\left(4+7\Omega\right)\big[\phi\left(\Omega_{CD}\Pi_{AB} - \Omega_{AB}\Pi_{CD}\right) - 2(1+\Omega)\left(\phi_{CD}\Pi_{AB} - \phi_{AB}\Pi_{CD}\right)\big]$$

$$+\frac{1}{3}\phi^2\left(4+7\Omega\right)R\left(\Sigma_{ABCD} - \Sigma_{CDAB}\right) + 2\left(4+7\Omega\right)\phi$$

$$\left(D_{CD}\phi^{EF}\Sigma_{ABEF} - D_{AB}\phi^{EF}\Sigma_{CDEF}\right)$$

$$+\left(3-3\phi^2+7\Omega\phi^2\right)\left(S_{CD}{}^{EF}\Sigma_{ABEF} - S_{AB}{}^{EF}\Sigma_{CDEF}\right)\bigg\}$$

$$+ \frac{1}{(1+\Omega-\phi^2)^2} \left\{ \frac{1}{6}\phi^2 \left(-12 + 40\phi^2 + 21\Omega\phi^2\right) \Omega^{EF}\Omega_{EF} \left(\Sigma_{ABCD} - \Sigma_{CDAB}\right)\right.$$

$$- \frac{2}{3}\phi\left[-18(1+\Omega) + (46 + 61\Omega + 21\Omega^2)\phi^2\right]\Omega^{EF}\phi_{EF} \left(\Sigma_{ABCD} - \Sigma_{CDAB}\right)$$

$$+ \frac{2}{3}(1+\Omega)\left[-24(1+\Omega) + (52 + 61\Omega + 21\Omega^2)\phi^2\right]\phi^{EF}\phi_{EF}$$
$$\left(\Sigma_{ABCD} - \Sigma_{CDAB}\right)$$

$$+ \frac{1}{3}\phi^2 \left(-12 + 40\phi^2 + 21\Omega\phi^2\right) \Omega^{EF} \left(\Omega_{CD}\Sigma_{ABEF} - \Omega_{AB}\Sigma_{CDEF}\right)$$

$$- \frac{2}{3}\phi\left[-18(1+\Omega) + (46 + 61\Omega + 21\Omega^2)\phi^2\right]\phi^{EF} \left(\Omega_{CD}\Sigma_{ABEF} - \Omega_{AB}\Sigma_{CDEF}\right)$$

$$- \frac{2}{3}\phi\left[12(1+\Omega) + (16 + 61\Omega + 21\Omega^2)\phi^2\right]\Omega^{EF} \left(\phi_{CD}\Sigma_{ABEF} - \phi_{AB}\Sigma_{CDEF}\right)$$

$$+ \frac{4}{3}(1+\Omega)\left[6(1+\Omega) + (22 + 61\Omega + 21\Omega^2)\phi^2\right]\phi^{EF}$$
$$\left(\phi_{CD}\Sigma_{ABEF} - \phi_{AB}\Sigma_{CDEF}\right)$$

$$+ \frac{1}{3}\phi^2 \left(-3 + 7\phi^2\right) R \left(\Omega_{AB}\Sigma_{CD} - \Omega_{CD}\Sigma_{AB}\right)$$

$$- \frac{2}{3}\phi\left[-7(1+\Omega)^2 + (11 + 14\Omega)\phi^2\right]R \left(\phi_{AB}\Sigma_{CD} - \phi_{CD}\Sigma_{AB}\right)$$

$$+ 2\phi\left(-3 + 7\phi^2\right)\Omega^{EF} \left(D_{AB}\phi_{EF}\Sigma_{CD} - D_{CD}\phi_{EF}\Sigma_{AB}\right)$$

$$- 4\left[-7(1+\Omega)^2 + (11 + 14\Omega)\phi^2\right]\phi^{EF} \left(D_{AB}\phi_{EF}\Sigma_{CD} - D_{CD}\phi_{EF}\Sigma_{AB}\right)$$

$$+ \left(-1 + \phi^2\right)\left(-3 + 7\phi^2\right)\Omega^{EF} \left(S_{EFAB}\Sigma_{CD} - S_{EFCD}\Sigma_{AB}\right)$$

$$\left. - 2\phi\left[-4 - 14\Omega - 7\Omega^2 + 2(2 + 7\Omega)\phi^2\right]\phi^{EF} \left(S_{EFAB}\Sigma_{CD} - S_{EFCD}\Sigma_{AB}\right)\right\}$$

$$+ \frac{1}{3(1+\Omega-\phi^2)^3} \left\{ \frac{1}{2}\phi^2\left[-24 + (59 + 21\Omega)\phi^2 + 21\phi^4\right]\Omega^{EF}\Omega_{EF}\right.$$
$$\left(\Omega_{AB}\Sigma_{CD} - \Omega_{CD}\Sigma_{AB}\right)$$

$$- 2\phi\left[-18(1+\Omega) + (13 + 19\Omega)\phi^2 + (61 + 42\Omega)\phi^4\right]\Omega^{EF}\phi_{EF}$$
$$\left(\Omega_{AB}\Sigma_{CD} - \Omega_{CD}\Sigma_{AB}\right)$$

$$+ 2\phi^2\left[-3(1+\Omega)(19 + 14\Omega + 7\Omega^2) + (113 + 164\Omega + 63\Omega^2)\phi^2\right]\phi^{EF}\phi_{EF}$$
$$\left(\Omega_{AB}\Sigma_{CD} - \Omega_{CD}\Sigma_{AB}\right)$$

$$- \phi\left[12(1+\Omega) + (-17 + 19\Omega)\phi^2 + (61 + 42\Omega)\phi^4\right]\Omega^{EF}\Omega_{EF}$$
$$\left(\phi_{AB}\Sigma_{CD} - \phi_{CD}\Sigma_{AB}\right)$$

$$+ 4\phi^2\left[-3(1+\Omega)(9 + 14\Omega + 7\Omega^2) + (83 + 164\Omega + 63\Omega^2)\phi^2\right]\Omega^{EF}\phi_{EF}$$
$$\left(\phi_{AB}\Sigma_{CD} - \phi_{CD}\Sigma_{AB}\right)$$

$$+ 4\phi(1+\Omega)\big[(1+\Omega)^2(61+42\Omega) - 3(39+75\Omega+28\Omega^2)\phi^2\big]\phi^{EF}\phi_{EF}$$
$$(\phi_{AB}\Sigma_{CD} - \phi_{CD}\Sigma_{AB})\bigg\},$$

and we get

$$\partial_u\Pi_{A1} + \frac{1}{u}\epsilon_A{}^0\Pi_{01} = 2\hat{\Gamma}_{A101} \tag{96}$$
$$+ \frac{1}{1+\Omega-\phi^2}\bigg\{\big[(3 - 3\phi^2 + 7\Omega\phi^2)\Omega^{EF} - 2\Omega\phi\,(4+7\Omega)\,\phi^{EF}\big]\epsilon_{0A}H_{10EF}$$
$$+ \frac{1}{3}\phi\,(4+7\Omega)\big[\phi\Omega_{00} - 2(1+\Omega)\phi_{00}\big]\Pi_{A1} + \frac{1}{3}\phi^2\,(4+7\Omega)\,R\Sigma_{A100}$$
$$+ 2\,(4+7\Omega)\,\phi\partial_u\phi^{EF}\Sigma_{A1EF} + (3 - 3\phi^2 + 7\Omega\phi^2)S_{00}{}^{EF}\Sigma_{A1EF}\bigg\}$$
$$+ \frac{1}{(1+\Omega-\phi^2)^2}\bigg\{\frac{1}{6}\phi^2(-12 + 40\phi^2 + 21\Omega\phi^2)\Omega^{EF}\Omega_{EF}\Sigma_{A100}$$
$$- \frac{2}{3}\phi\big[-18(1+\Omega) + (46 + 61\Omega + 21\Omega^2)\phi^2\big]\Omega^{EF}\phi_{EF}\Sigma_{A100}$$
$$+ \frac{2}{3}(1+\Omega)\big[-24(1+\Omega) + (52 + 61\Omega + 21\Omega^2)\phi^2\big]\phi^{EF}\phi_{EF}\Sigma_{A100}$$
$$+ \frac{1}{3}\phi^2(-12 + 40\phi^2 + 21\Omega\phi^2)\Omega^{EF}\Omega_{00}\Sigma_{A1EF}$$
$$- \frac{2}{3}\phi\big[-18(1+\Omega) + (46 + 61\Omega + 21\Omega^2)\phi^2\big]\phi^{EF}\Omega_{00}\Sigma_{A1EF}$$
$$- \frac{2}{3}\phi\big[12(1+\Omega) + (16 + 61\Omega + 21\Omega^2)\phi^2\big]\Omega^{EF}\phi_{00}\Sigma_{A1EF}$$
$$+ \frac{4}{3}(1+\Omega)\big[6(1+\Omega) + (22 + 61\Omega + 21\Omega^2)\phi^2\big]\phi^{EF}\phi_{00}\Sigma_{A1EF}$$
$$- \frac{1}{3}\phi^2(-3 + 7\phi^2)R\Omega_{00}\Sigma_{A1} + \frac{2}{3}\phi\big[-7(1+\Omega)^2 + (11+14\Omega)\phi^2\big]R\phi_{00}\Sigma_{A1}$$
$$- 2\phi(-3 + 7\phi^2)\Omega^{EF}\partial_u\phi_{EF}\Sigma_{A1}$$
$$+ 4\big[-7(1+\Omega)^2 + (11+14\Omega)\phi^2\big]\phi^{EF}\partial_u\phi_{EF}\Sigma_{A1}$$
$$- \left(-1 + \phi^2\right)\left(-3 + 7\phi^2\right)\Omega^{EF}S_{EF00}\Sigma_{A1}$$
$$+ 2\phi\big[-4 - 14\Omega - 7\Omega^2 + 2(2 + 7\Omega)\phi^2\big]\phi^{EF}S_{EF00}\Sigma_{A1}\bigg\}$$
$$- \frac{1}{3(1+\Omega-\phi^2)^3}\bigg\{\frac{1}{2}\phi^2\big[-24 + (59+21\Omega)\phi^2 + 21\phi^4\big]\Omega^{EF}\Omega_{EF}\Omega_{00}\Sigma_{A1}$$
$$- 2\phi\big[-18(1+\Omega) + (13+19\Omega)\phi^2 + (61+42\Omega)\phi^4\big]\Omega^{EF}\phi_{EF}\Omega_{00}\Sigma_{A1}$$
$$+ 2\phi^2\big[-3(1+\Omega)(19 + 14\Omega + 7\Omega^2) + (113 + 164\Omega + 63\Omega^2)\phi^2\big]$$
$$\phi^{EF}\phi_{EF}\Omega_{00}\Sigma_{A1}$$

$$- \phi \big[12(1+\Omega) + (-17+19\Omega)\phi^2 + (61+42\Omega)\phi^4 \big] \Omega^{EF}\Omega_{EF}\phi_{00}\Sigma_{A1}$$

$$+ 4\phi^2 \big[-3(1+\Omega)(9+14\Omega+7\Omega^2) + (83+164\Omega+63\Omega^2)\phi^2 \big]$$

$$\Omega^{EF}\phi_{EF}\phi_{00}\Sigma_{A1}$$

$$+ 4\phi(1+\Omega)\big[(1+\Omega)^2(61+42\Omega) - 3(39+75\Omega+28\Omega^2)\phi^2 \big]$$

$$\phi^{EF}\phi_{EF}\phi_{00}\Sigma_{A1} \bigg\}.$$

Finally, with H_{ABCD},

$$D^{EF}H_{EFCD} = -\frac{1}{2}t^{EFHI}{}_E{}^G D_{HI}S_{CDFG} - 2S_{E(FGC}R^E{}_{D)}{}^{HF}{}_H{}^G \tag{97}$$

$$+ \frac{1}{1+\Omega-\phi^2}\bigg\{ \big[\phi\Omega^{EF} - 2(1+\Omega)\phi^{EF} \big] \bigg[-\frac{1}{3}(1+\Omega)D_{GH}\phi_{EF}t^I{}_C{}^{GH}{}_{DI}$$

$$- \Omega D_{GH}\phi_{FI}t_E{}^{IGH}{}_{CD} - \frac{2}{3}(1+\Omega)\phi_{FG}R^G{}_E{}^H{}_{CDH} + \Omega\phi_{FG}R^{GH}{}_{EHCD}$$

$$- \Omega\phi^{GH}R_{GFEHCD} \bigg] + \frac{1}{3}\phi\Omega\big[\phi\Omega^{EF} - 2(1+\Omega)\phi^{EF} \big]H_{(CD)EF}$$

$$+ \frac{1}{2}\big[(1-\phi^2+\Omega\phi^2)\Omega^{EF} - 2\Omega^2\phi\phi^{EF} \big]H_{EFCD}$$

$$+ \frac{1}{18}(-2+\Omega)\phi\big[\phi\Omega^E{}_{(C}\Pi_{D)E} - 2(1+\Omega)\phi^E{}_{(C}\Pi_{D)E} \big]$$

$$+ \frac{1}{18}(-2+\Omega)\phi^2 R\Sigma_{E(CD)}{}^E + \Omega\phi D_{CD}\phi^{EF}\Sigma_{GEF}{}^G$$

$$- \frac{2}{3}(1+\Omega)\phi D^E{}_{(C}\phi^{FG}\Sigma_{D)EFG} + \frac{1}{2}(1-\phi^2+\Omega\phi^2)S_{CD}{}^{EF}\Sigma_{GEF}{}^G$$

$$- \frac{1}{3}\Omega\phi^2 S^{EFG}{}_{(C}\Sigma_{D)EFG} - 2\big[\phi\Omega_{CD} - 2(1+\Omega)\phi_{CD} \big]\phi^{EF}\Sigma_{GEF}{}^G$$

$$- 4\phi\Omega^{EF}\phi_{CD}\Sigma_{GEF}{}^G + 2\phi\Omega^{EF}\phi_F{}^G\Sigma_{EGCD} \bigg\}$$

$$+ \frac{1}{(1+\Omega-\phi^2)^2}\bigg\{ \frac{1}{18}\phi^2(3-10\phi^2+6\Omega\phi^2)\Omega^{EF}$$

$$\big(\Omega_{EF}\Sigma_{G(CD)}{}^G + 2\Omega^G{}_{(C}\Sigma_{D)GEF} \big)$$

$$+ \frac{2}{9}\phi^3(7+4\Omega-6\Omega^2)\phi^{EF}\big(\Omega_{EF}\Sigma_{G(CD)}{}^G + \Omega^G{}_{(C}\Sigma_{D)GEF} \big)$$

$$- \frac{2}{9}\phi\big[6(1+\Omega) - (13+4\Omega-6\Omega^2)\phi^2 \big]\Omega^{EF}\phi^G{}_{(C}\Sigma_{D)GEF}$$

$$- \frac{2}{9}(1+\Omega)\big[3(1+\Omega) + 2(2+2\Omega-3\Omega^2)\phi^2 \big]\phi^{EF}\phi_{EF}\Sigma_{G(CD)}{}^G$$

$$+ \frac{4}{9}(1+\Omega)^2(3-10\phi^2+6\Omega\phi^2)\phi^{EF}\phi^G{}_{(C}\Sigma_{D)GEF}$$

$$- \frac{1}{18}\phi^2(-3+\phi^2)R\Omega^E{}_{(C}\Sigma_{D)E} - \frac{1}{9}\phi\big[(1+\Omega)^2 + (1-2\Omega)\phi^2 \big]R\phi^E{}_{(C}\Sigma_{D)E}$$

$$+\frac{2}{3}\phi^3\Omega_{EF}D^G\,_{(C}\phi^{EF}\Sigma_{D)G} - \phi(1-\phi^2)\Omega^{EF}D_{CD}\phi_F\,^G\Sigma_{EG}$$

$$+\frac{4}{3}(1+\Omega)(1+\Omega-2\phi^2)\phi_{EF}D^G\,_{(C}\phi^{EF}\Sigma_{D)G} + 2\big[(1+\Omega)^2 - (1+2\Omega)\phi^2\big]$$

$$\phi^{EF}D_{CD}\phi_F\,^G\Sigma_{EG} - \frac{1}{3}(1-\phi^2)\phi^2\Omega^{EF}S^G\,_{EF(C}\Sigma_{D)G}$$

$$+\frac{1}{2}(1-\phi^2)^2\Omega^{EF}S^G\,_{FCD}\Sigma_{EG}$$

$$-\frac{2}{3}\phi\big[-(1+\Omega)^2 + (1+2\Omega)\phi^2\big]\phi^{EF}S^G\,_{EF(C}\Sigma_{D)G}$$

$$+\Omega\phi(2+\Omega-2\phi^2)\phi^{EF}S^G\,_{FCD}\Sigma_{EG} + 2\phi\,(\Omega_{CD} - 2\phi\phi_{CD})\,\Omega_E\,^F\phi^{EG}\Sigma_{FG}\bigg\}$$

$$+\frac{1}{9(1+\Omega-\phi^2)^3}\bigg\{-\phi^2\left(3-13\phi^3+3\Omega\phi^2+3\phi^4\right)\Omega^{EF}\Omega_{EF}\Omega^G\,_{(C}\Sigma_{D)G}$$

$$-4\phi^3(5+8\Omega+2\phi^2-6\Omega\phi^2)\Omega^{EF}\phi_{EF}\Omega^G\,_{(C}\Sigma_{D)G}$$

$$+4\phi^2\big[3(1+\Omega)^3+(4-2\Omega-9\Omega^2)\phi^2\big]\phi^{EF}\phi_{EF}\Omega^G\,_{(C}\Sigma_{D)G}$$

$$+2\phi\big[(1+\Omega)(3-8\phi^2)-2(1-3\Omega)\phi^4\big]\Omega^{EF}\Omega_{EF}\phi^G\,_{(C}\Sigma_{D)G}$$

$$+8(1+\Omega)\phi^2\big[3\Omega(2+\Omega)+(7-9\Omega)\phi^2\big]\Omega^{EF}\phi_{EF}\phi^G\,_{(C}\Sigma_{D)G}$$

$$-8(1+\Omega)\phi\big[2(1+\Omega)^2(-1+3\Omega)+3(3-4\Omega^2)\phi^2\big]\phi^{EF}\phi_{EF}\phi^G\,_{(C}\Sigma_{D)G}\bigg\}\,,$$

where the l.h.s. is

$$D^{EF}H_{EFCD} = \partial_u H_{11CD} + \frac{1}{u}\left(H_{11CD} + H_{110(C}\epsilon_{D)}\,^0\right)$$

$$-\left(\frac{1}{2u}\partial_v + \hat{e}^a\,_{01}\partial_a\right)H_{10CD} - 2\hat{\Gamma}_{0100}H_{11CD} - \hat{\Gamma}_{010C}H_{110D}$$

$$-\hat{\Gamma}_{010D}H_{110C} + \hat{\Gamma}_{011C}H_{100D} + \hat{\Gamma}_{011D}H_{100C} + \hat{\Gamma}_{1100}H_{10CD}\,.$$

Equations (91), (92), (94), (95), (96), (97) are the system of subsidiary equations for the quantities (93). The expressions on the right hand sides of these equations are homogeneous functions of the quantities (93). Together with Lemma 7.1 this implies that all the expansion coefficients of the quantities (93) vanish on U_0. As the functions (93) are necessarily holomorphic, this implies that they vanish on $\hat{N}$.   □

8. Analyticity at space-like infinity

Our gauge is singular and thus the holomorphic solution of Lemma 6.7 does not cover a full neighborhood of the point i. To show that we can indeed get a holomorphic solution in a hole neighborhood of i we go to a normal frame field based on the frame c_{AB} at i and the corresponding normal coordinates x^a. The argument follows with some modifications the line of the corresponding argument in [8].

The geodesic equation for $z^a(s) = (u(s), v(s), w(s))$, $D_{\dot{z}}\dot{z} = 0$, can be written in the form

$$\dot{z}^a = m^{AB}e^a{}_{AB},$$

$$\dot{m}^{AB} = -2m^{CD}\Gamma_{CD}{}^{(A}{}_B m^{B)E}.$$

The initial conditions for the geodesics to start at i are

$$u|_{s=0} = 0, \quad w|_{s=0} = 0,$$

and we have to prescribe

$$v_0 = v|_{s=0},$$

in order to determine the ∂_u-∂_w-plane where the tangent vector is.

The components of the tangent vector to the geodesic at i are given by $m^{AB}|_{s=0} = m_0^{AB}$, and by regularity and the geodesic equations we have

$$m_0^{00} = \dot{u}|_{s=0} \equiv \dot{u}_0, \quad m_0^{01} = 0, \quad m_0^{11} = \dot{w}|_{s=0} \equiv \dot{w}_0.$$

We can identify the frame e_{AB} with its projection into $T_i N_c$, then $m_0^{AB}e_{AB} = m^{*AB}c_{AB} = x^a c_\mathbf{a}$, where as defined $c_{AB} = \alpha^a{}_{AB}c_\mathbf{a}$, and we get

$$x^1 = \frac{1}{\sqrt{2}}\big(\dot{w}_0 + (v_0^2 - 1)\dot{u}_0\big), \quad x^2 = \frac{i}{\sqrt{2}}\big(\dot{w}_0 + (v_0^2 + 1)\dot{u}_0\big), \quad x^3 = \sqrt{2}v_0\dot{u}_0,$$

or, inverting the relations

$$\dot{u}_0(x^a) = -\frac{x^1 + ix^2}{\sqrt{2}}, \quad v_0(x^a) = -\frac{x^3}{x^1 + ix^2}, \quad \dot{w}_0 = \frac{\delta_{ab}x^a x^b}{\sqrt{2}(x^1 + ix^2)}.$$

Here we see that in order to have a well defined vector we need $x^1 + ix^2 \neq 0$, or, what is the same, $\dot{u}_0 \neq 0$. This correspond to the singular generator of $\mathcal{N}_i$ in the c_{AB}-gauge. The vectors $x^a c_\mathbf{a}$ cover all directions at i except those tangent to the complex null hyperplane $(c_\mathbf{1} + ic_\mathbf{2})^\perp = \{a(c_\mathbf{1} + ic_\mathbf{2}) + bc_\mathbf{3}|a, b \in \mathbb{C}\}$.

As we have used a frame formalism, we need also to determine the normal frame centered at i and based on the frame c_{AB} at i. As we already have the frame fields e_{AB}, we write the equation for the normal frame c_{AB}, $D_{\dot{x}}c_{AB} = 0$ as an equation for the transformation $t^A{}_B \in SL(2, \mathbb{C})$ that relates the frames e_{AB} and c_{AB}, $c_{AB} = t^C{}_A t^D{}_B e_{CD}$. The equation can be written as

$$\dot{t}^A{}_B = -m^{DE}\Gamma_{DE}{}^A{}_C t^C{}_B, \tag{98}$$

and the initial condition comes from having to take $e_{AB}|_i(v)$ to $c_{AB}|_i$,

$$t^A{}_B|_{s=0} = s^A{}_B(-v_0). \tag{99}$$

Following the proofs of Lemma 7.1, Lemma 7.2 and Lemma 7.3 in [8] we arrive at the following two lemmas.

Lemma 8.1. *For any given initial data $\dot{u}_0, v_0, \dot{w}_0$, with $\dot{u}_0 \neq 0$, there exist a number $t = t(\dot{u}_0, v_0, \dot{w}_0)$ and unique holomorphic solutions $z^a(s) = z^a(s, \dot{u}_0, v_0, \dot{w}_0)$ of the initial value problem for the geodesic equations with initial conditions as described above which is defined for $|s| < 1/t$. The functions $z^a(s, \dot{u}_0, v_0, \dot{w}_0)$ are in fact*

holomorphic functions of all four variables $(s, \dot{u}_0, v_0, \dot{w}_0)$ *in a certain* $P_{1/t}(0) \times U$, *where* U *is a compactly embedded subset of* $(\mathbb{C} \backslash \{0\}) \times \mathbb{C} \times \mathbb{C}$.

Lemma 8.2. *Along the geodesic corresponding to* $s \to z^a(s, \dot{u}_0, v_0, \dot{w}_0)$ *equations* (98) *have a unique holomorphic solution* $t^A{}_B(s) = t^A{}_B(s, \dot{u}_0, v_0, \dot{w}_0)$ *satisfying the initial conditions* (99). *The functions* $t^A{}_B(s) = t^A{}_B(s, \dot{u}_0, v_0, \dot{w}_0)$ *are holomorphic in all four variables in the domain where the* $z^a(s, \dot{u}_0, v_0, \dot{w}_0)$ *are holomorphic.*

Following the discussion in [8] it can be seen that, as $|x| \equiv \sqrt{\delta_{ab} \bar{x}^a x^b} \to 0$, $x^1 + ix^2 \neq 0$,

$$u(x^c) = -\frac{x^1 + ix^2}{\sqrt{2}} + \mathcal{O}(|x|^3),$$

$$v(x^c) = -\frac{x^3}{x^1 + ix^2} + \mathcal{O}(|x|^2),$$

$$w(x^c) = \frac{\delta_{ab} x^a x^b}{\sqrt{2}(x^1 + ix^2)} + \mathcal{O}(|x|^3).$$

This gives for the forms $\chi^{AB} = \chi^{AB}{}_c dx^c$ dual to the normal frame c_{AB}

$$\chi^{AB}(x^c) = \left(\alpha^{AB}{}_a + \hat{\chi}^{AB}{}_a\right) dx^a,$$

with holomorphic functions $\hat{\chi}^{AB}{}_a(x^c)$ which satisfy $\hat{\chi}^{AB}{}_a = \mathcal{O}(|x|^2)$ as $|x| \to 0$. Also the coefficients $c^a{}_{AB} = \langle dx^a, c_{AB} \rangle$ of the normal frame in the normal coordinates satisfy

$$c^a{}_{AB}(x^c) = \alpha^a{}_{AB} + \hat{c}^a{}_{AB},$$

with holomorphic functions $\hat{c}^a{}_{AB}(x^c)$ which satisfy $\hat{c}^a{}_{AB} = \mathcal{O}(|x|^2)$ as $|x| \to 0$.

The three 1-forms $\alpha^a{}_{AB} dx^a$ are linearly independent and thus for small $|x^c|$ the coordinate transformation $x^a \to z^a(x^c)$, where defined, is nondegenerate. This means that all the tensor fields entering the conformal stationary vacuum field equations can be expressed in terms of the normal coordinates x^c and the normal frame field c_{AB}.

Now we can derive our main result.

Proof of Theorem 1.1. The coordinates x^a cover a domain U in $\mathbb{C}^3$ on which the frame vector fields $c_{AB} = c^a{}_{AB} \partial / \partial_{x^a}$ exist, are linearly independent and holomorphic. Also in U the other tensor fields expressed in terms of the x^a and c_{AB} are holomorphic. However U does not contain the hypersurface $x^1 + ix^2 = 0$ but the boundary of U becomes tangent to this hypersurface at $x^a = 0$.

We want to see that the solution indeed cover a domain containing an open neighborhood of the origin.

We still have the gauge freedom to perform with some $t^A{}_B \in SU(2)$ a rotation $\delta^* \to \delta^* \cdot t$ of the spin frame. Whit this rotation is associated the rotation

$$c_{AB} \to c^t_{AB} = t^C{}_A t^D{}_B c_{CD}$$

of the frame c_{AB} at i. The construction of the submanifold $\hat{N}$ was done based on the frame c_{AB}, starting now with c^t_{AB} all the previous constructions and derivations

can be repeated as far as the estimates for the null data in the c_{AB}-gauge can be translated to the same type of estimates for the null data in the c^t_{AB}-gauge.

We will denote u', v', w' and e^t_{AB} the analogues in the new gauge of the coordinates u, v, w and the frame e_{AB}. The set $\mathcal{N}_i$ is invariant under this rotation. The sets $\{w = 0\}$ and $\{w' = 0\}$ are both lifts of the set $\mathcal{N}_i$ to the bundle of spin frames. The coordinates u and u' are both affine parameters on the null generators of $\mathcal{N}_i$, which vanish at i. The coordinates v, v' both label the null generators of $\mathcal{N}_i$. The frame vectors e_{00} and e^t_{00} are auto-parallel vector fields tangent to the null generators.

If v and v' label the same generator η of $\mathcal{N}_i$, then $e^t_{00}(v') = f^2 e_{00}(v)$ at i, with some $f \neq 0$. Furthermore, as e_{00} and e^t_{00} are auto-parallel, then $e^t_{00} = f^2 e_{00}$ must hold along η, with f constant along the geodesic. This means that at i

$$s^C{}_0(v')s^D{}_0(v')t^E{}_C t^F{}_D c_{EF} = f^2 s^C{}_0(v)s^D{}_0(v)c_{CD} \,,$$

and absorbing the undetermined sign in f,

$$t^E{}_C s^C{}_0(v') = f s^E{}_0(v) \,. \tag{100}$$

We can write $t^A{}_B \in SU(2)$ as

$$(t^A{}_B) = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}, \quad a, c \in \mathbb{C}, \quad |a|^2 + |c|^2 = 1 \,. \tag{101}$$

This gives with (100)

$$v' = \frac{-c + av}{\bar{a} + \bar{c}v}, \quad f = \frac{1}{\bar{a} + \bar{c}v}, \quad \text{resp.} \quad v = \frac{c + \bar{a}v'}{a - \bar{c}v'}, \quad f = a - \bar{c}v' \,. \tag{102}$$

As $\langle du, e_{00} \rangle = 1 = \langle du', e^t_{00} \rangle$ we have for the affine parameter along η

$$u = f^2 u' \,. \tag{103}$$

With (102), (103) holds $\eta(u', v') = \eta(u, v)$.

If $c \neq 0$ then $v \to \infty$ as $v' \to a/\bar{c}$. So the null generator in the c_{AB}-gauge, where we need information, is contained, excepting the origin, in the regular domain of the c^t_{AB}-gauge.

Let us consider now the abstract null data given in the c_{AB}-gauge $\hat{\mathcal{D}}^\phi_n$, $\hat{\mathcal{D}}^S_n$ satisfying estimates of the form (38), (39). In the c^t_{AB}-gauge we have $\hat{\mathcal{D}}^{\phi t}_n$, $\hat{\mathcal{D}}^{St}_n$, with terms given by

$$\psi^t_{A_m B_m \dots A_1 B_1} = t^{G_m}{}_{A_m} t^{H_m}{}_{B_m} \dots t^{G_1}{}_{A_1} t^{H_1}{}_{B_1} \psi_{G_m H_m \dots G_1 H_1} \,,$$

$$\Psi^t_{A_m B_m \dots A_1 B_1 CDEF}$$
$$= t^{G_m}{}_{A_m} t^{H_m}{}_{B_m} \dots t^{G_1}{}_{A_1} t^{H_1}{}_{B_1} t^I{}_C t^J{}_D t^K{}_E t^K{}_L \Psi_{G_m H_m \dots G_1 H_1 IJKL} \,.$$

Using the essential components of ψ and ψ^t

$$\psi^t_{(A_m B_m \ldots A_1 B_1)_n} = \sum_{j=0}^{2m} \binom{2m}{j} t^{(G_m}{}_{(A_m} t^{H_m}{}_{B_m} \ldots t^{G_1}{}_{A_1} t^{H_1)_j}{}_{B_1)_n} \psi_{(G_m H_m \ldots G_1 H_1)_j}$$

$$= \binom{2m}{n}^{-\frac{1}{2}} \sum_{j=0}^{2m} \binom{2m}{j}^{\frac{1}{2}} T_{2m}{}^j{}_n(t) \psi_{(G_m H_m \ldots G_1 H_1)_j} \,.$$

The numbers

$$T_{2m}{}^j{}_n(t) = \binom{2m}{n}^{\frac{1}{2}} \binom{2m}{j}^{\frac{1}{2}} t^{(G_m}{}_{(A_m} t^{H_m}{}_{B_m} \ldots t^{G_1}{}_{A_1} t^{H_1)_j}{}_{B_1)_n}$$

satisfy

$$|T_{2m}{}^j{}_n(t)| \le 1 \,, \quad m = 0, 1, 2, \ldots \,, \quad 0 \le j \le 2m \,, \quad 0 \le n \le 2m \,,$$

as they represent the matrix elements of a unitary representation of $SU(2)$. So we get

$$|\psi^t_{A_m B_m \ldots A_1 B_1}| \le \frac{m! M}{r'^m} \,, \quad m = 1, 2, 3, \ldots \,,$$

where $r' = r/4$.

In the same way we get

$$|\Psi^t_{A_m B_m \ldots A_1 B_1 CDEF}| \le \frac{m! M'}{r'^m} \,, \quad m = 0, 1, 2, 3, \ldots \,,$$

where $M' = 16M$.

So the estimates for the null data on the c_{AB}-gauge translate into the same type of estimates for the null data on the c^t_{AB}-gauge.

Assuming now $c \ne 0$ in (101), we have two possibilities for getting the solution in the c^t_{AB}-gauge:

i. Using the solution in the c_{AB}-gauge we can determine, where possible, the coordinate and frame transformation to the c^t_{AB}-gauge. In particular, the singular generator of $\mathcal{N}_i$ in the c^t_{AB}-gauge will coincide with the regular generator of $\mathcal{N}_i$ in the c_{AB} gauge on which $v = -\bar{a}/\bar{c}$. We are thus able to determine near the singular generator in the c^t_{AB}-gauge the expansion of the solution in terms of the coordinates u', v', w' and the frame field e^t_{AB}.

ii. Using the null data $\hat{\mathcal{D}}^{\phi t}_n$, $\hat{\mathcal{D}}^{St}_n$ in the c^t_{AB}-gauge, one can repeat all the steps of the previous sections to show the existence of a solution to the conformal stationary vacuum field equations in the coordinates u', v', w' of the c^t_{AB}-gauge. All the statements made about the solution in the c_{AB}-gauge apply also to this solution, in particular statements about domains of convergence.

The formal expansions of the fields in terms of u', v', w' are uniquely determined by the data $\hat{\mathcal{D}}^{\phi t}_n$, $\hat{\mathcal{D}}^{St}_n$, thus the solutions obtained by the two methods are holomorphically related to each other on certain domains by the gauge transformation obtained in (i). As done with the solution in the c_{AB}-gauge, the solution in the

c^t_{AB}-gauge can be expressed in terms of the normal coordinates x^a_t and the normal frame field c^t_{AB}. The x^a_t cover a certain domain $U_t \subset \mathbb{C}^3$ and the frame field c^t_{AB} is non-degenerate. All the tensor fields expressed in terms of x^a_t and c^t_{AB} are holomorphic on U_t. Then the solution in the c_{AB}-gauge and the solution in the c^t_{AB}-gauge are related on certain domains by the transformation

$$x^a_t = t^{-1a}{}_b x^b , \quad c^t_{AB} = t^C{}_A t^D{}_B c_{CD} ,$$

which gives the transformation corresponding to the rotation of normal coordinates. We can extend this as a coordinate and frame transformation to the solution obtained in (ii) to express all fields in terms of x^a and c_{AB}. With this extension all fields are defined and holomorphic on $t^{-1} U_t$. Then the solution obtained in the c_{AB}-gauge and the solution in the c^t_{AB}-gauge are genuine holomorphic extensions of each other, as one covers the singular generator of the other one away from the origin in a regular way.

Let now $x^a_* \neq 0$ be an arbitrary point in $\mathbb{C}^3$. We want to show that the solution extends in the coordinates x^a to a domain which covers the set $s x^a_*$ for $0 < s < \epsilon$ for some $\epsilon > 0$. That is the case in the c_{AB}-gauge as far as $x^a_* \neq (\alpha, i\alpha, \beta)$, $\alpha, \beta \in \mathbb{C}$. We need to see what happens if $x^a_* = (\alpha, i\alpha, \beta)$, with $\alpha \neq 0$ or $\beta \neq 0$.

If $x^a_* = (\alpha, i\alpha, \beta)$ and $\alpha \neq 0$, we consider the $c^{t'}_{AB}$-gauge, where t'_{AB} is given by (101) with $a = 0$, $c = 1$. The normal coordinates in the two gauges are related by

$$x^1_{t'} = -x^1 , \quad x^2_{t'} = x^2 , \quad x^3_{t'} = -x^3 .$$

The holomorphic transformation $(x^1_{t'}, x^2_{t'}, x^3_{t'}) \rightarrow (-x^1, x^2, -x^3)$ maps $U_{t'}$ onto a subset of $\mathbb{C}^3$, denoted by $t'^{-1} U_{t'}$, which has nonempty intersection with U. After the transformation the two solutions coincide on $t'^{-1} U_{t'} \cap U$.

Under this transformation, the singular set $\{x^1 + ix^2 = 0\}$ in the c_{AB}-gauge correspond to the set $\{x^1_{t'} - ix^2_{t'} = 0\}$, which is covered in a regular way in a neighborhood of i in the $c^{t'}_{AB}$-gauge. So the set $t'^{-1} U_{t'} \cup U$ admits a holomorphic extension of our solution in the coordinates x^a and the frame c_{AB}. In this extension there exist ϵ such that $s x^a_*$, $x^a_* = (\alpha, i\alpha, \beta)$ with $\alpha \neq 0$, is covered by the solution for $0 < s < \epsilon$.

We need also to consider the case $\alpha = 0$, that is, $x^a_* = (0, 0, \beta)$, $\beta \neq 0$. In this case we use the $c^{t''}_{AB}$-gauge, where t''_{AB} is given by (101) with $a = \frac{1}{\sqrt{2}}$, $c = \frac{i}{\sqrt{2}}$. The normal coordinates are related by

$$x^1_{t''} = x^1 , \quad x^2_{t''} = -x^3 , \quad x^3_{t''} = x^2 .$$

The argument follows the same lines as for the $\alpha \neq 0$ case.

Thus the set U can be extended so that the points $s x^a_*$ with $0 < s < \epsilon$ are covered by U and all fields are holomorphic on U in the coordinates x^a. Then it can be assumed U to contain a punctured neighborhood of the origin in which the solution is holomorphic in the normal coordinates x^a and the normal frame c_{AB}. Then the solution is in fact holomorphic on a full neighborhood of the origin $x^a = 0$, which represents the point i, as holomorphic functions in more than one dimension cannot have isolated singularities.

By Lemma 3.1 we have from null data satisfying the reality conditions a formal expansion of the solution with expansion coefficients satisfying the reality conditions. By the various uniqueness statements obtained in the lemmas, this expansion must coincide with the expansion in normal coordinates of the solution obtained above. This implies the existence of a 3-dimensional real slice on which the tensor fields satisfy the reality conditions. It is obtained by requiring the coordinates x^a to assume values in $\mathbb{R}^3$. □

9. Conclusions

We have seen how to determine a formal expansion of an asymptotically flat stationary vacuum solution to Einstein's field equations using a minimal set of freely specifiable data, the null data. These data are given by two sequences of symmetric trace free tensors at space-like infinity. We have obtained necessary and sufficient conditions on the null data for the formal expansion to be absolutely convergent, hence showing that the null data characterize all asymptotically flat, stationary vacuum solutions to the field equations with non-vanishing ADM mass. This work contains the static case as a particular case, and is a generalization of Friedrich's work [8] from the static to the stationary case.

Corvino and Schoen [6] and Chruściel and Delay [5] have shown that it is possible to produce vacuum initial data that is fairly general in the interior region and exactly static or stationary in the exterior region. The present work shows which are the possible exteriors.

It is a long standing conjecture that Hansen's multipoles [11], which are relevant because they have nice geometrical transformation properties under change of conformal factor, do characterize an asymptotically flat stationary vacuum solutions to the field equations in the way we have shown the null data do. This have been shown in the axisymmetric case [1] and some steps have been achieved in the general case, like showing that the multipoles determine a formal expansion of a solution [3, 13], or necessary bounds on the multipoles if the solution exist [2], but general conditions on the multipoles for the expansion to be convergent has not been found yet. As there is a bijective correspondence between the null data and Hansen's multipoles, although the relation is highly non linear, it would be nice if this correspondence could be exploited to get necessary and sufficient conditions on the multipoles to determine a convergent expansion.

References

[1] T. Bäckdahl, *Axisymmetric stationary solutions with arbitrary multipole moments*, Class. Quantum Grav. **24** (9) (2007), 2205–2215.

[2] T. Bäckdahl and M. Herberthson, *Calculation of, and bounds for, the multipole moments of stationary spacetimes*, Class. Quantum Grav. **23** (20) (2006), 5997–6006.

[3] R. Beig and W. Simon, *On the multipole expansion for stationay space-times*, Proc. R. Soc. Lond. A **376** (1765) (1981), 333–341.

[4] R. Beig and W. Simon, *The multipole structure of stationary space-times*, J. Math. Phys. **24** (5) (1983), 1163–1171.

[5] P. Chruściel and E. Delay, *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mémoires de la SMF **94** (2003), vi–103.

[6] J. Corvino and R. Schoen, *On the asymptotics for the vacuum Einstein constraint equations*, J. Differ. Geom. **73** (2) (2006), 185–217.

[7] S. Dain, *Initial data for stationary spacetimes near spacelike infinity*, Class. Quantum Grav. **18** (20) (2001), 4329–4338.

[8] H. Friedrich, *Static vacuum solutions from convergent null data expansions at spacelike infinity*, Ann. Henri Poincaré **8** (5) (2007), 817–884.

[9] R. Geroch, *Multipole moments. II. Curved space*, J. Math. Phys. **11** (8) (1970), 2580–2588.

[10] R. Geroch, *A method for generating solutions of Einstein's equations*, J. Math. Phys. **12** (6) (1971), 918–924.

[11] R. Hansen, *Multipole moments of stationary space-times*, J. Math. Phys. **15** (1) (1974), 46–52.

[12] D. Kennefick and N. Ó Murchadha, *Weakly decaying asymptotically flat static and stationary solutions to the Einstein equations*, Class. Quantum Grav. **12** (1) (1995), 149–158.

[13] P. Kundu, *On the analyticity of stationary gravitational fields at spatial infinity*, J. Math. Phys. **22** (9) (1981), 2006–2011.

[14] A. Lichnerowicz, *Sur les équations relativistes de la gravitation*, Bulletin de la S.M.F. **80** (1952), 237–251.

[15] O. Reula, *On existence and behaviour of asymptotically flat solutions to the stationary Einstein equations*, Commun. Math. Phys. **122** (4) (1989), 615–624.

[16] W. Simon, *Die Multipolstruktur stationärer Räume in der allgemeinen Relativitätstheorie*, PhD thesis, Formal- und Naturwissenschaftlichen Fakultät der Universität Wien, 1980.

[17] B. Xanthopoulos, *Multipole moments in general relativity*, J. Phys. A: Math. Gen. **12** (7) (1979), 1025–1028.

Andrés E. Aceña
Max Planck Institute for Gravitational Physics
Am Mühlenberg 1
D-14476 Golm
Germany
e-mail: `acena@aei.mpg.de`