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From Grassmann to maximal ($N = 8$) supergravity

Hermann Nicolai*

MPI für Gravitationsphysik (Albert-Einstein-Institut), Mühlenberg 1, 14476 Potsdam, Germany

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1 Introduction

Hermann Günther Grassmann, a citizen of Stettin/Szczecin during all of his life, was a deeply original thinker. Although a 'late starter', he created many concepts fundamental to modern mathematics. He was basically the inventor of *Linear Algebra* as it is taught nowadays (even in high schools), that is, the calculus of vectors (and tensors) as objects not only having a 'size' (length) but also a 'direction' (in some multi-dimensional space). Grassmann enriched mathematics also in other areas; most importantly for this talk, the fundamental idea of *anticommuting numbers* and the notion of an anticommuting algebra (Grassmann algebra) that is the basis of supersymmetry originated with him. Grassmann was active in many other areas of human and intellectual endeavour. He knew many languages, wrote a book on botany, conducted a choir and fathered eleven children. Equally remarkably (and perhaps out of frustration with the lack of

Fig. 1 Hermann Günther Grassmann (1809–1877).

* E-mail: Hermann.Nicolai@aei.mpg.de

recognition for his mathematical work, see below) he started learning Sanskrit in his forties, and produced the first translation of the Rigveda. As a result, Grassmann is not only famous as a mathematician, but also as a linguist and Indologist – his Sanskrit dictionary of 1873 is still in use today! However, his mathematical work was not appreciated for a long time due to his idiosyncratic style and terminology. For instance, the *princeps mathematicorum* C.F. Gauss simply refused to even take a look at Grassmann's *Opus Magnum*, the "Ausdehnungslehre", while the famous mathematician Eduard Kummer wrote of it

" ... this work will be ignored by mathematicians in the future as it has been until now; for, the effort it demands to understand it appears too great in comparison with the gain in knowledge that one would hope to get from it."

[*"... dass diese Schrift von den Mathematikern ferner ignoriert werden wird wie bisher; denn die Mühe, sich in dieselbe einzuarbeiten, erscheint zu gross in Beziehung auf den wirklichen Gewinn an Erkenntnis, welchen man aus derselben schöpfen zu können vermutet."*]

As a consequence all his applications for university professorships were turned down and he remained in Stettin/Szczecin as a "Gymnasium" teacher. It is hard to believe how hard he must have worked until the end of his life in order to accomplish all the things he did. His total dedication to his scientific work is perhaps best illustrated by a little anecdote: when the famous physician R. Virchow came from Berlin to Stettin for a 'Naturforscher' meeting, and afterwards wanted to chat with Grassmann over a beer, Grassmann did not even know where in his home town one could go for a beer! For us, who live in a time when it is common to expect instant recognition and gratification for whatever we do (also in theoretical physics!), it is quite amazing how Grassmann nevertheless kept up his spirits and persevered with his work in spite of all the negative reactions of his contemporaries. Perhaps it was the deep belief in the truth and importance of his work, beautifully expressed in the following sentences (from the Foreword of *Die Ausdehnungslehre: Vollständig und in strenger Form bearbeitet"*, second edition 1862):

"I remain completely confident that the labour I have expended on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers, will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into the actual development of science, still that time will come when it will be brought forth from the dust of oblivion and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me (as I have until now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich them further, yet there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments. For truth is eternal and divine and no phase of it ... can pass without a trace; it remains in existence even if the cloth in which weak mortals dress it disintegrates into dust."

In this contribution I would like to briefly discuss the impact of some of Grassmann's ideas today and on current developments in mathematics and physics. More specifically, we will concentrate on one specific example – the development of *supersymmetric (quantum) field theory*, one of the most important developments in mathematical physics over the last 40 years [1–3].[1] For sure, the discovery of supersymmetry at the Large Hadron Collider (LHC) would open many new avenues and revolutionize particle physics. But even if no supersymmetric particles are found, it appears that supersymmetry is here to stay: it has become an integral part of mathematics, having inspired several Fields Medalists, and will surely continue to play a key role in our search for a consistent (finite) theory of quantum gravity.

1 See also the textbooks [4, 5] for introductions.

2 Grassmann algebras

Grassmann algebras are essential for understanding and formulating modern quantum field theory with fermions. In particular, they are indispensable for supersymmetry and supergravity. They provide the mathematical framework for dealing with *"anticommuting c-numbers"*, which obey

$$\theta_i\theta_j + \theta_j\theta_i = 0 \quad \Rightarrow \quad \theta_i^2 = 0 \tag{2.1}$$

A *finitely generated Grassmann algebra* $\mathfrak{A} = \mathbb{K}\big[\theta_1, ..., \theta_n\big]$ is spanned by elements of the form

$$x = a_0 + a_i\,\theta_i + a_{ij}\,\theta_i\theta_j + \cdots + a_{1\dots n}\,\theta_1\cdots\theta_n, \quad a_0, a_i, \cdots \in \mathbb{K} \tag{2.2}$$

with completely antisymmetric coefficients $a_{ij} = -a_{ji}$, and so on; $\mathbb{K}$ is some field (in practice always $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Every Grassmann algebra splits into an *even* ('bosonic') and an *odd* ('fermionic') part:

$$\mathfrak{A} = \mathfrak{A}_+ \oplus \mathfrak{A}_- \qquad \text{with} \quad \dim(\mathfrak{A}_\pm) = 2^{n-1}, \tag{2.3}$$

Grassmann algebras can be regarded as *classical limits* of fermion systems, in the sense that the canonical anticommution relations of fermionic creation and annihilation operators $\psi_i^\dagger$ and ψ_i reduce to (2.1) in the limit of vanishing Planck's constant:

$$\{\psi_i, \psi_j^\dagger\} = \hbar\,\delta_{ij} \quad \to \quad \{\psi_i, \psi_j\} = 0 \quad \text{for} \quad \hbar \to 0. \tag{2.4}$$

For quantum field theory, one needs infinitely generated Grassmann algebras (with either a countable or an uncountable infinity of basis elements), but this case is formally no different from (2.4):

$$\{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \hbar\,\delta(\mathbf{x} - \mathbf{y}) \quad \to \quad \{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = 0 \quad \text{for} \quad \hbar \to 0 \tag{2.5}$$

Clearly, this is a *formal device*: anticommuting c-numbers are hard to 'visualize' (and impossible to 'measure'), but still extremely useful for description of fermions (fermionic Lagrangians, fermionic path integral). But this is no different from the use of complex numbers in quantum mechanics. The only thing that matters is that, at the end of the day, the formalism produces a *real number* that can be compared to the reading of a meter in some experiment.

There is a huge body of literature on the mathematics of Grassmann algebras. As every quantum field theorist knows[2] one can generalize many notions and results from 'commuting mathematics', and in particular develop linear algebra and a calculus over anticommuting c-numbers: for instance, differentiation and integration are defined as follows [6]:

$$\frac{\partial}{\partial\theta_i}(1) = 0\,, \quad \frac{\partial}{\partial\theta_i}\theta_j = \delta^i_j \quad , \qquad \int d\theta_i = 0\,, \int d\theta_i\,\theta_j = \delta_{ij}. \tag{2.6}$$

Note that these operations are defined *algebraically*, not via a limiting procedure as in real analysis. A further curious feature is that differentiation and integration are effectively the same operation!

3 Supersymmetry

Supersymmetry relates bosons and fermions, or in physical terms, it relates forces (carried by vector and tensor bosons) and matter (made up by spin-$\frac{1}{2}$ fermions, quarks and leptons). The very simplest example of a supersymmetric system is *supersymmetric quantum mechanics* [7, 8]: this is the bosonic harmonic (or

[2] This was not so in the early 1980's: when I moved to the University of Hamburg in 1988 I was told that one main reason why they had picked me was that I 'anticommute'.

anharmonic) oscillator with variables $q(t)$ augmented by a 'spin' (that is: a *two state system*), which is *classically* described by the anticommuting variables $\psi(t)\,,\ \bar{\psi}(t) \equiv (\psi(t))^*$. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}(W'(q))^2 + i\bar{\psi}\dot{\psi} - W''(q)\bar{\psi}\psi, \tag{3.1}$$

with some sufficiently differentiable function $W(q)$ (for $W(q) = \frac{1}{2}q^2$, we just recover the harmonic oscillator). Under supersymmetry variations with Grassmann-valued (anticommuting) parameter ε

$$\delta q(t) = \bar{\varepsilon}\psi(t) + \bar{\psi}(t)\varepsilon, \quad \delta\psi(t) = -\big[i\dot{q}(t) + W'(q(t))\big]\varepsilon, \tag{3.2}$$

we can see that the Lagrangian $\mathcal{L}$ is invariant modulo a total derivative:

$$\delta\mathcal{L} = \frac{1}{2}\frac{d}{dt}\Big[(\dot{q} - iW'(q))\bar{\varepsilon}\psi + \text{h.c.}\Big] \quad \Rightarrow \int dt\,\delta\mathcal{L} = 0. \tag{3.3}$$

This is a characteristic feature: in supersymmetric theories we never get $\delta\mathcal{L} = 0$, but always a total derivative (so one must pay attention to surface terms when looking at soliton-like solutions having a 'tail' at infinity). The function $W(q)$ is called the *superpotential*. The superpotential is a centerpiece of all supersymmetric model building because its specification fixes a good part of any supersymmetric Lagrangian and thus encapsulates most of its properties – this statement also applies to the latest version of the supersymmetric Standard Model!

A canonical treatment (*à la Dirac*) yields the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\big(W'(q)\big)^2 + W''(q)\bar{\psi}\psi. \tag{3.4}$$

which can now be quantized in the usual way by replacing Poisson (or Dirac) brackets by canonical commutators (for the bosons) and anticommutators (for the fermions). In particular, one then sees easily that the ψ and $\bar{\psi}$ become fermionic annihilation and creation operators which can be represented by 2-by-2 matrices in the standard fashion. As a result one arrives at a quantized anharmonic oscillator coupled to a spin-$\frac{1}{2}$ system. The basic supersymmetry relations are

$$H = \frac{1}{2}\{Q\,,\,Q^\dagger\} \quad \Rightarrow \quad [Q, H] = [Q^\dagger, H] = 0 \tag{3.5}$$

where the 'supercharge' Q is the canonical generator of supersymmetry transformations. These relations are at the heart of every supersymmetry algebra: a supersymmetric field theory simply consists of a (generally interacting) infinite assembly of such bosonic and fermionic oscillators. The second relation implies a degeneracy of the energy levels: with the possible exception of the groundstate, every bosonic state has a fermionic partner state of the same energy, and *vice versa* related to it by the action of the supercharge Q

$$Q|\text{boson}\rangle = |\text{fermion}\rangle \ , \qquad Q|\text{fermion}\rangle = |\text{boson}\rangle \tag{3.6}$$

In supersymmetric field theories, this implies in particular that, if supersymmetry is unbroken, every bosonic particle must have a fermionic partner of the same mass, and vice versa – a major headache for our model builder friends when trying to make predictions for LHC!

Supersymmetry Ward identities for correlators can be derived from (with *anti*-commuting f)

$$\big\langle\{Q\,,\,f(\psi,\bar{\psi},q)\}\big\rangle = \big\langle\{Q^\dagger\,,\,f(\psi,\bar{\psi},q)\}\big\rangle = 0. \tag{3.7}$$

Observe that these are relations between ordinary correlation functions (fermionic expectation values are ordinary numbers, too!), so at this point the anticommuting variables have done their duty, and can be dispensed with.

4 Supersymmetric quantum field theory

For (semi-)realistic field theories that can be applied to particle physics, one needs to 'marry' supersymmetry with other symmetries, *to wit*, the usual symmetries of relativistic quantum field theory, Lorentz and Poincaré invariance, as well as internal symmetries (isospin, flavor, color, *etc.*). The most general supersymmetric enlargement of these symmetries was found in [9]. Here we just quote the essential relations involving the translation operator P_μ and the supercharges, neglecting central charges. They read:

$$[P_\mu, P_\nu] = 0 \ , \ \ [P_\mu, Q^i_\alpha] = [P_\mu, \bar{Q}_{\dot{\alpha}i}] = 0 \ ,$$
$$\{Q^i_\alpha, Q^j_\beta\} = \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = 0 \ , \ \ \{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2\delta^i_j \sigma^\mu_{\alpha\dot{\beta}} P_\mu$$

where α and $\dot{\alpha}$ are $SL(2,\mathbb{C}) \cong SO(1,3)$ spinor indices, cf. [4]. The indices i, j run between 1 and N: when there are N supercharges one speaks of N*-extended supersymmetry*. This algebra generalizes the usual notion of a Lie algebra to a *Lie superalgebra* (or *graded Lie algebra*) in that it contains both commutators *and* anticommutators. For $N > 1$, the algebra admits $U(N)$ (or $SU(N)$) as a group of outer automorphisms; in this case, the algebra merges *spacetime and internal symmetries*, and this was a main reason for the original excitement about supersymmetry in the early 1970s. Note, however, that this unification of symmetries does *not* take place for the currently popular MSSM-type models with low energy supersymmetry, which is the reason why in these models every known particle must come with a (so far unobserved) supersymmetric partner of opposite statistics, but otherwise the *same* quantum numbers.

Irreducible representations of supersymmetry (that is: supersymmetric particle multiplets) can be constructed in the usual way by Wigner's method of induced representations [10]. Let us just briefly sketch how this is done for *massless* multiplets. In this case, one half of the supercharges generates zero norm states, and we need only consider one supercharge (call them Q^i) and its hermitean adjoint for each value of i. Starting from the state of highest helicity $|h\rangle$ in the multiplet one generates states of lower helicity by the successive application of the supercharge Q^i

$$|h\rangle \, , \ |h - \tfrac{1}{2}; i\rangle \equiv Q^i |h\rangle \, , \ |h-1; [ij]\rangle \equiv Q^i Q^j |h\rangle \, , \cdots , \ |h - \tfrac{1}{2}N; 1 \cdots N\rangle = Q^1 \cdots Q^N |h\rangle \quad (4.1)$$

where the lables $i, j, \ldots$ are always antisymmetrized. The CPT Theorem demands that these states must be supplemented by the CPT conjugate states

$$\mathrm{CPT}|h; \mathcal{R}\rangle = |-h, \bar{\mathcal{R}}\rangle \quad (4.2)$$

to make a full supermultiplet (where $\mathcal{R}$ denotes some representation of an internal symmetry group, and $\bar{\mathcal{R}}$ its conjugate). It is now obvious that the range of spins covered in a multiplet is the bigger the more supercharges there are. As a consequence, there is only a limited number of multiplets for any given maximum spin. For globally (or 'rigidly') supersymmetric models, one can only have spin $s \leq 1$ (that is, scalars, spin-$\frac{1}{2}$ fermions and vector bosons), and therefore rigidly supersymmetric models exists only for up to $N = 4$ supercharges. Naturally, the models become more and more restricted with increasing N, such that for the maximal value $N = 4$ with the supermultiplet (spin is indicated as $[s]$)

$$\boldsymbol{N = 4}\ \textbf{multiplet:} \qquad 1{\times}[1] \oplus 4{\times}\left[\tfrac{1}{2}\right] \oplus 6{\times}[0] \quad (4.3)$$

there is only one theory, the celebrated $N = 4$ super-Yang-Mills theory, a theory currently very popular in connection with the so-called AdS/CFT correspondence.

If we want to go higher in spin, we must admit spin-$\frac{3}{2}$ particles, which – as it turns out – is not possible unless one also includes spin-2 (the graviton). Then the bound on the number of supercharges increases to $s \leq 2 \ \leftrightarrow N \leq 8$, and the multiplet for the maximally supersymmetric theory is

$$\boldsymbol{N = 8}\ \textbf{multiplet:} \quad 1{\times}[2] \oplus 8{\times}\left[\tfrac{3}{2}\right] \oplus 28{\times}[1] \oplus 56{\times}\left[\tfrac{1}{2}\right] \oplus 70{\times}[0] \quad (4.4)$$

An important feature is that for the maximal theories the automorphism group $U(N)$ is reduced to $SU(N)$, because the maximal supermultiplets are CPT self-conjugate, so the CPT conjugate states need not be added 'by hand' as for non-maximal supermultiplets. The self-conjugacy is also necessary to get the right number of fields; for instance, the 70 spin-0 degrees of freedom of $N = 8$ supergravity are described by a *complex* antisymmetric four-index tensor ϕ^{ijkl} with $i, j, k, l = 1, \ldots, 8$, and to get the right count, we must impose the complex self-duality constraint

$$\phi^{ijkl} \equiv (\phi_{ijkl})^* = \frac{1}{24}\epsilon^{ijklmnpq}\phi_{mnpq} \tag{4.5}$$

which is only compatible with $SU(8)$, but not $U(8)$.

5 $N = 8$ supergravity

$N = 8$ Supergravity is a *unique* theory (modulo 'gauging'), and the *most symmetric* known field theoretic extension of the Einstein's relativity theory in four dimensions. The original 'ungauged' version was constructed in [11] while the gauged version with local $SO(8)$ was obtained in [12] (see there also for earlier references leading up to [11, 12]). It is a rather complicated theory! It therefore took a while to work out its complete Lagrangian with all the non-polynomial interactions, a task that could finally only be accomplished with the discovery by Cremmer and Julia of a hidden $E_{7(7)}$ invariance of its equations of motion [11]. I will spare readers the full details (which can be found in [11, 12]) and simply show the formula in a somewhat picturesque surrounding (cut out from a poster claiming a link with 'Vedic Science', which I only mention here because it relates back to one of Grassmann's research interests!).

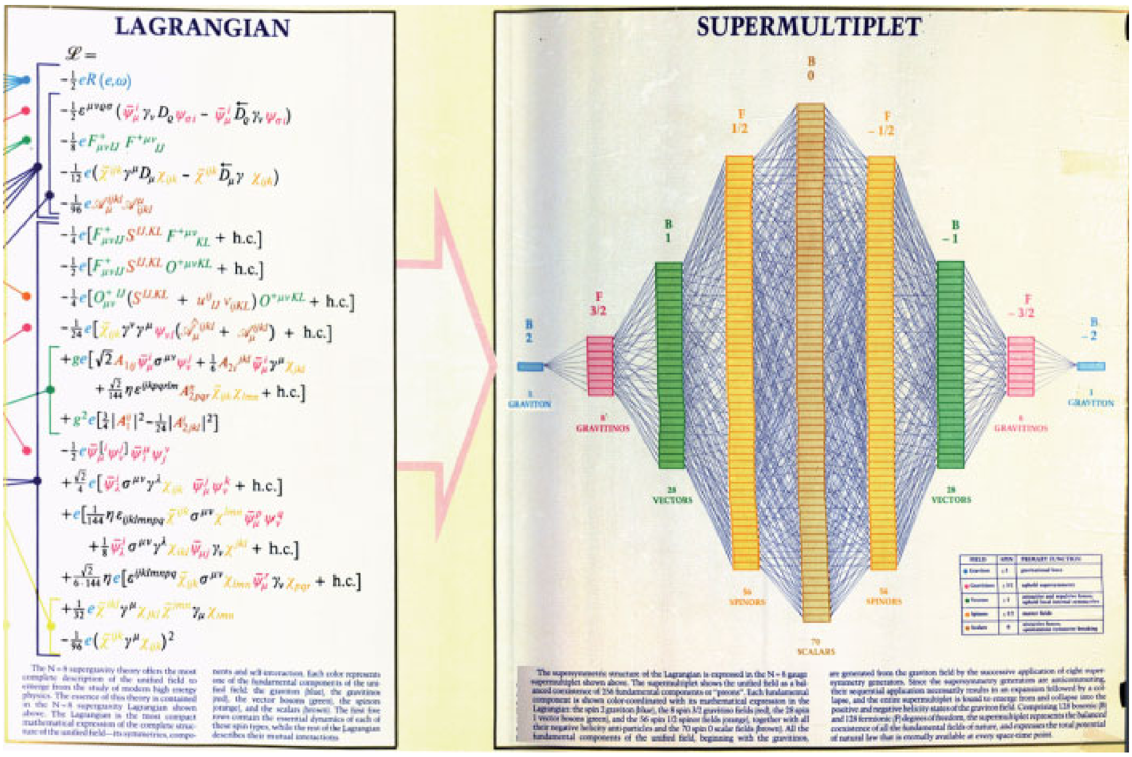

Fig. 2 (online colour at: www.ann-phys.org).

In the late 1970s $N = 8$ supergravity was thought to be a promising candidate for a unified theory of quantum gravity and matter interactions, but then fell into disfavor, not so much because of its evident complexity, but because it appeared that the two main hopes linked to this theory could not be fulfilled:

- The existence of supersymmetric counterterms suggested the appearance of non-renormalizable divergences from three loops onwards, and thus appeared to thwart all hope that maximal supersymmetry might cure the infinities of perturbatively quantized gravity; and
- the absence of chiral fermions and the appearance of a huge negative cosmological constant in the gauged theory appear to be in obvious conflict with observations; any attempt to directly connect this theory to particle physics thus appears doomed from the outset.

And not to forget: in the early 1980s superstring theory started to eclipse all other attempts to quantize gravity, particularly those based on *field theoretic* extensions of Einstein's theory. Indeed, it seemed to do much better in *both* regards!

6 Finiteness: to be or not to be?

Einstein's theory of gravity is *perturbatively non-renormalizable*. Although there were no concrete calculations available beyond one loop in the early 1980s, it was generally taken for granted that divergences that can appear actually do appear. This expectation was borne out shortly after by Goroff and Sagnotti's [13] (and later Van de Ven's [14]) calculation of the two-loop counterterm and the proof that its coefficient indeed does not vanish

$$\Gamma^{(2)}_{\rm div} = \frac{1}{\varepsilon}\,\frac{209}{2880}\,\frac{1}{(16\pi^2)^2}\int dV C_{\mu\nu\rho\sigma}C^{\rho\sigma\lambda\tau}C_{\lambda\tau}{}^{\mu\nu}, \tag{6.1}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl conformal tensor. This calculation was quite an achievement, because (at least in the original way of doing it [13]) the determination of the coefficient $\frac{209}{2880}$ required the consideration of $\mathcal{O}(100\,000)$ Feynman diagrams! Fortunately, this result was confirmed in [14], where the calculation was done with a different regulator and in a different gauge (which incidentally also reduced the number of Feynman diagrams considerably), and the coefficient still came out to be the same. Thus there was no more question at this point that perturbatively quantized Einstein gravity is doomed.

However, a glimmer of hope remained for supergravity: the counterterm (6.1) does not admit a supersymmetric extension. It therefore seemed reasonable to hypothesize that supersymmetry (for sufficiently high N) might actually exclude higher order counterterms, too. However, these hopes were soon shattered: at three loops a supersymmetric invariant does exist which can be built from the Bel-Robinson tensor [15]

$$\Gamma^{(3)}_{\rm div} \propto \int dV T_{\mu\nu\rho\sigma}T^{\mu\nu\rho\sigma} \quad \text{with} \quad T_{\mu\nu\rho\sigma} = R_\mu{}^\alpha{}_\nu{}^\beta R_{\rho\alpha\sigma\beta} + \tilde{R}_\mu{}^\alpha{}_\nu{}^\beta \tilde{R}_{\rho\alpha\sigma\beta} \tag{6.2}$$

indicating that, generically, supergravity should be expected to be UV infinite from three loops onwards.

Of course, for $N = 8$ supergravity,things are a bit more complicated. In order to be able to say anything concrete at all (beyond mere suppositions), one must rely on a superspace formulation of the theory. In that formulation, all on-shell degrees of freedom can be packaged into 'supervielbein' $E_M{}^A(x,\theta,\bar\theta)$ and a (non-geometric) superfield $W_{ijkl}(x,\theta,\bar\theta)$ [16–18] containing all the supergravity degrees of freedom

$$\phi_{ijkl} = W_{ijkl}\big|_{\theta=\bar\theta=0}\,,\ \chi_{\alpha jkl} = D^i_\alpha W_{ijkl}\big|_{\theta=\bar\theta=0}\,,\dots\,, C_{\alpha\beta\gamma\delta} = D^i_\alpha D^j_\beta D^k_\gamma D^l_\delta W_{ijkl}\big|_{\theta=\bar\theta=0} \tag{6.3}$$

where $C_{\alpha\beta\gamma\delta}$ is a chiral half of the Weyl tensor in $SL(2,\mathbb{C})$ spinor notation. Then

$$\mathcal{L}^{(3)} \propto \left(\int d^4x D^{[i_1\ldots i_4]\,[j_1\ldots j_4]}\bar{D}^{[k_1\ldots k_4]\,[l_1\ldots l_4]} \right.$$
$$\left. \times \left(W_{i_1\ldots i_4}W_{j_1\ldots j_4}W_{k_1\ldots k_4}W_{l_1\ldots l_4}\right)\Big|_{\mathbf{232\,848}} \right)_{\theta=\bar{\theta}=0} , \tag{6.4}$$

is a possible 3-loop counterterm, with

$$D^{[ijkl]\,[mnpq]} \equiv D^i_{(\alpha_1} \cdots D^l_{\alpha_4)} D^m_{(\beta_1} \cdots D^q_{\beta_4)}\, \varepsilon^{\alpha_1\beta_1} \cdots \varepsilon^{\alpha_4\beta_4} \tag{6.5}$$

where D^i_α is the usual superspace derivative, see [16]. The quartic product of the W's in (6.4) must be projected onto the $\mathbf{232\,848}$ representation of $SU(8)$ (the box-like Young tableau with four rows and columns); this projection is also needed in order to get the proper combination of Riemann (or Weyl) tensors 'in the middle' of (6.4).

Although widely taken as an indication that $N = 8$ supergravity would inevitably diverge at three loops, the above arguments were not entirely conclusive. The known superspace formulation of $N = 8$ supergravity, on which they are based, suffers from a severe drawback: it only works at the level of the equations of motion, that is, *on shell*, and does not allow to set up a scheme for computing Feynman diagrams. Furthermore the $E_{7(7)}$ symmetry of the equations of motion is realized only linearly by constant shifts of the scalar fields:

$$\delta W_{ijkl}(x, \theta, \bar{\theta}) = \Sigma_{ijkl} \tag{6.6}$$

Inconclusive as they were, these results appeared to leave the question of finiteness (or not) of $N = 8$ supergravity in the No Man's Land of undecidable propositions. A rough estimate shows that the computation of a 3-loop counterterm coefficient for $N = 8$ supergravity would require consideration of $\mathcal{O}(10^{20})$ (or even more) Feynman diagrams – an obviously hopeless task!

Or so it seemed for many years ... However, in a stunning development, Zvi Bern, Lance Dixon and collaborators recently have shown that widely held expectations concerning the UV behavior of $N = 8$ supergravity may need to be revised [19]. What enabled this breakthrough was a completely new technology for computing Feynman diagrams (see [19, 20] for a bibliography of earlier work leading up to this computation). However, to understand it (as far as this is possible at all for an outsider) one better forget just about everything one has learnt from quantum field theory textbook. Let me very briefly summarize the main idea. The key difficulty with ordinary Feynman diagrams is that they carry a lot of 'extra baggage'. Not only are the particles circulating in loops off-shell, but one must in addition sum over lots of unphysical polarizations. To compensate for the latter one must sum over extra ghost contributions. The Gordic knot is cut by entirely working with on-shell objects [20], that is, by

- Constructing on-shell field theory amplitudes as limits of on-shell string amplitudes;
- Using methods from S-matrix theory: unitarity, analyticity, cutting rules, dispersion relations (this is a concrete realization of the 'bootstrap' S-matrix program that never got anywhere in the 1960s!);
- Determining gravity amplitudes from 'squaring' Yang Mills amplitudes by means of KLT rules [21]

$$M_4^{\text{tree}}(1, 2, 3, 4) = -is_{12}A_4^{\text{tree}}(1, 2, 3, 4)A_4^{\text{tree}}(1, 2, 4, 3).$$

where A_4^{tree} is the 4-point open string amplitude, while M_4^{tree} is the corresponding amplitude for the closed string. The method therefore puts some flesh on an old idea: **Gravity = Yang-Mills** $\times$ **Yang-Mills**. This idea cannot work in any obvious way: no amount of fiddling with the Einstein-Hilbert action will reduce it to a square of a Yang-Mills action. In terms of vertices this can be seen by simple inspection of the 3-graviton vertex [20]

$$
\begin{aligned}
&G^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3)\\
&= \mathrm{Sym}\Big[-\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma})\\
&\qquad + \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + \frac{1}{2}P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_6(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma})\\
&\qquad - P_3(k_{1\beta}k_{1\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta})\\
&\qquad - P_3(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big].
\end{aligned}
\tag{6.7}
$$

There is evidently no way to factorize this into a product of two Yang Mills vertices. However, putting this expression on-shell (that is, putting momenta on-shell, and contracting the external legs with physical polarizations), the expression becomes the same as the one obtained from

$$
\propto \Big[\eta_{\mu\nu}(k_1-k_2)_\sigma + \text{cyclic}\Big] \times \Big[\eta_{\alpha\beta}(k_1-k_2)_\gamma + \text{cyclic}\Big], \tag{6.8}
$$

which is just the square of the color-stripped version of the Yang Mills amplitude [20]

$$
\propto f^{abc}\Big(\eta_{\mu\nu}(k_1-k_2)_\sigma + \text{cyclic}\Big) \tag{6.9}
$$

To establish the **four-loop finiteness of the 4-graviton amplitude of $N = 8$ supergravity**, one goes through the following steps [19]:

- Employ an on-shell formalism eliminating all higher point vertices so there remain only 3-point vertices;
- Use unitarity based arguments to reduce all amplitudes to integrals over products of tree amplitudes.

Amazingly, the calculation then reduces to the computation of $\mathcal{O}(50)$ 'Mondrian-like' diagrams (as well as some non-planar diagrams) schematically depicted in the middle diagram below (taken from [22]). This picture beautifully illustrates how real physics can metamorphose via mathematics into pure art!

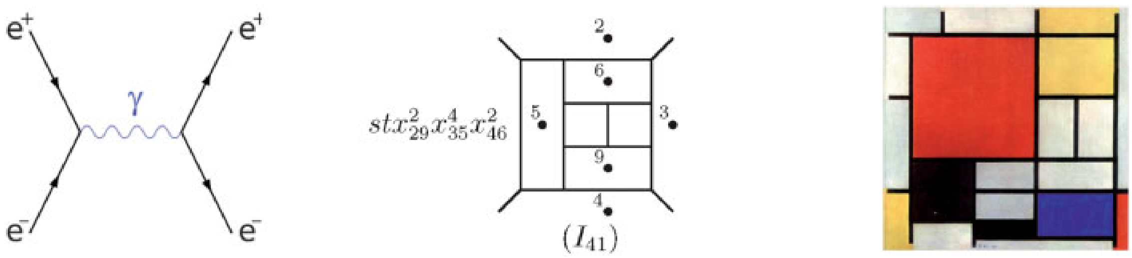

Fig. 3 (online colour at: www.ann-phys.org).

7 Outlook

The possible UV finiteness of $N = 8$ supergravity to all orders in perturbation theory remains a big mystery. As physicists, we have been brought up not to believe in miracles: if the theory is really finite then there must be a reason for it. Most plausibly, this reason has something to do with an as yet undiscovered hidden symmetry of the theory (my personal favorite being the hyperbolic Kac–Moody group E_{10}).

Concerning the present status of $N = 8$ supergravity, a poll some ten years ago would probably have produced the following 'majority opinion'

- $N = 8$ supergravity is perturbatively divergent;
- $N = 8$ supergravity is not viable for phenomenology.

While many (even in the superstring camp) would now agree that the first of these claims has become rather shaky, most would probably still subscribe to the second statement. Yet, there is one curious fact that I would like to mention at the end, and which could turn out to be either a deep truth or a complete mirage.[3] After breaking all eight supersymmetries of $N = 8$ supergravity one is left with 48 spin-$\frac{1}{2}$ fermions (eight of the 56 original fermions must be 'eaten' to render the eight gravitinos massive). This is just the right number if one includes massive neutrinos! What is more, gauged $N = 8$ supergravity has one stationary point with residual $SU(3) \times U(1)$ symmetry [23]. Then the assignments for the $48 = 3 \times 16$ quarks and leptons at the stationary point agree with those of $N = 8$ supergravity if one introduces a 'family symmetry' $SU(3)_{\rm family}$ linking the three generations of quarks and leptons, and [24, 25]

- identifies $SU(3)$ with the diagonal subgroup $\big[SU(3)_{\rm color} \times SU(3)_{\rm family}\big]_{\rm diag}$;
- shifts all $U(1)_{em}$ charges by a spurion charge $\frac{1}{6}$.

Of course, for this to work there would have to be new and *very* strange dynamics, where the weak interactions would have to be dynamically generated with composite W and Z bosons, while the gluons and the photon would be elementary. Furthermore, the family symmetry $SU(3)_{\rm family}$ does *not* commute with weak isospin $SU(2)_w$ in the above scheme. Looking at the diagonal subgroup may likewise appear a strange thing to do, but according to [26] such 'flavor color locking' may actually occur, and not only in strongly coupled QCD! So it is not completely excluded that an 'obviously wrong' theory (or some extension of it) could turn out to be right after all – just like QCD would have been considered 'obviously wrong' had it been proposed in 1950 as a theory of strong interactions, and without knowledge of its underlying dynamics! Let us therefore stick with Grassmann's attitude in the face of majority opinion: we are probably still far from knowing the true answers to our questions.

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[3] It would almost certainly be the second if LHC should find genuinely new fundamental spin-$\frac{1}{2}$ degrees of freedom (as, for instance, predicted by models of TeV scale supersymmetry). If not, all bets are open!

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