Moduli spaces of G_2 manifolds

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November 11, 2009

Abstract

This paper is a review of current developments in the study of moduli spaces of G_2 manifolds. G_2 manifolds are 7-dimensional manifolds with the exceptional holonomy group G_2. Although they are odd-dimensional, in many ways they can be considered as an analogue of Calabi-Yau manifolds in 7 dimensions. They play an important role in physics as natural candidates for supersymmetric vacuum solutions of M-theory compactifications. Despite the physical motivation, many of the results are of purely mathematical interest. Here we cover the basics of G_2 manifolds, local deformation theory of G_2 structures and the local geometry of the moduli spaces of G_2 structures.

1 Introduction

Ever since antiquity there has been a very close relationship between physics and geometry. Originally, in *Timaeus*, Plato related four of the five Platonic solids - tetrahedron, hexahedron, octahedron, icosahedron to the elements fire, earth, air and water, respectively, while the fifth solid, the dodecahedron was the *quintessence* of which the cosmos itself is made. Later, Isaac Newton's Laws of Motion and Theory of Gravitation gave a precise mathematical framework in which the motion of objects can be calculated. However Albert Einstein's General Relativity made it very explicit that the physics of spacetime is determined by its geometry. More recently, this fundamental relationship has been taken to a new level with the development of String and M-theory. Over the past 25 years, superstring theory has emerged as a successful candidate for the role of a theory that would unify gravity with other interactions. It was later discovered that all five superstring theories can be obtained as special limits of a more general eleven-dimensional theory known as M-theory and moreover, the low energy limit of which is the eleven-dimensional supergravity [40, 42]. The complete formulation of M-theory is, however, not known yet.

One of the key features of String and M-theory is that these theories are formulated in ten- and eleven-dimensional spacetimes, respectively. One of the techniques to relate this to the visible four-dimensional world is to assume that the remaining six or seven dimensions are curled up as a small, compact, so-called *internal space*. This is known as *compactification*. Such a procedure also leads to a remarkable interrelationship between physics and geometry, since the effective physical content of the resulting four-dimensional theory is determined by the geometry of the internal space. Usually the full multidimensional spacetime is regarded as a direct product $M_4 \times X$, where M_4 is a 4-dimensional non-compact manifold with Lorentzian signature $(-+++)$ and X is a compact six or seven dimensional Riemannian manifold. In general, the parameters that define the geometry of the internal space give rise to massless scalar fields known as *moduli*, and the properties of the moduli space are determined by the class of spaces used in the compactification.

The properties of the internal space in String and M-theory compactifications are governed by physical considerations. A key ingredient of these theories is *supersymmetry* [41]. Supersymmetry is a physical symmetry between particles the spin of which differs by $\frac{1}{2}$ - that is, between integer spin *bosons* and half-integer spin *fermions*. Mathematically, bosons are represented as functions or tensors and fermions as spinors. When looking for a supersymmetric vacuum solution, that is a Ricci-flat solution that is invariant under supersymmetry transformations, it turns out that a necessary requirement is the existence of covariantly constant, or parallel, spinor. That is, there must exist a non-trivial spinor η on the Riemannian manifold X that satisfies

$$\nabla \eta = 0 \tag{1.1}$$

where ∇ is the relevant spinor covariant derivative [8]. This condition implies that η is invariant under parallel transport.

Properties of parallel transport on a Riemannian manifold are closely related to the concept of *holonomy*. Consider a vector v at some point x on X. Using the natural Levi-Civita connection that comes from the Riemannian metric, we can parallel transport v along paths in X. In particular, consider a closed contractible path γ based at x. As shown in Figure 1, if we parallel transport v along γ, then the new vector v' which we get will necessarily have the same magnitude as the original vector v, but otherwise it does not have to be the same. This gives the notion of *holonomy group*. Below we give the precise definition.

Definition 1 *Let* (X, g) *be a Riemannian manifold of dimension* n *with metric* g *and corresponding Levi-Civita connection* ∇*, and fix point* $x \in X$*. Let* $\gamma : [0,1] \longrightarrow X$ *be a closed loop based at* x*, that is* $\gamma(0) = \gamma(1) = x$*. The parallel transport map* $P_\gamma : T_xX \longrightarrow T_xX$ *is then an invertible linear map which lies in* $SO(n)$*. Define the Riemannian* holonomy group $Hol_x(X, g)$ *of* ∇ *based at* x *to be*

$$Hol_x(X, g) = \{P_\gamma : \gamma \textit{ is a loop based at } x\} \subset SO(n)$$

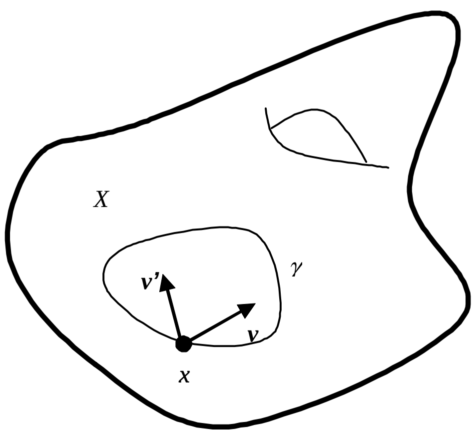

Figure 1: Parallel transport of a vector

If the manifold X is connected, then it is trivial to see that the holonomy group is independent of the base point, and can hence be defined for the whole manifold. Parallel transport is initially defined for vectors, but can then be naturally extended to other objects like tensors and spinors, with the holonomy group acting on these objects via relevant representations.

Geometry	**Holonomy**	**Dimension**
Kähler	$U(k)$	$2k$
Calabi-Yau	$SU(k)$	$2k$
HyperKähler	$Sp(k)$	$4k$
Exceptional	G_2	7
Exceptional	$Spin(7)$	8

Figure 2: List of special holonomy groups

Now going back to the covariantly constant spinor η, (1.1) implies that η is invariant under the action of the holonomy group. This shows that the spinor representation of $Hol(X,g)$ must contain the trivial representation. For $Hol(X,g) = SO(n)$, this is not possible since the spinor representation is reducible, so $Hol(X,g) \subset SO(n)$. Hence the condition (1.1) implies a reduced holonomy group. Thus, Ricci-flat special holonomy manifolds occur very naturally in string and M-theory.

As shown by Berger [9], the list of possible special holonomy groups is very limited. In particular, if X is simply-connected, and neither locally a product nor symmetric, the only possibilities are given in Figure 2.When we have a 10-dimensional theory, we need X to be 6-dimensional in order to reduce to 4 dimension, and it thus has to be a Calabi-Yau manifold. For an 11-dimensional theory, X has to be 7-dimensional, so it has to have G_2 holonomy.

We have thus seen that even rather simple physical requirement restrict the geometry of the manifold X to rather special classes. In particular, the study of Calabi-Yau manifolds has been crucial in the development of String Theory, and in fact some very important discoveries in the theory of Calabi-Yau have been made thanks to advances in the physics. One such major discovery is Mirror Symmetry [36, 26]. This symmetry first appeared in String Theory where evidence was found that conformal field theories (CFTs) related to compactifications on a Calabi-Yau manifold with Hodge numbers $(h_{1,1}, h_{2,1})$ are equivalent to CFTs on a Calabi-Yau manifold with Hodge numbers $(h_{2,1}, h_{1,1})$. Mirror symmetry is currently a powerful tool both for calculations in String Theory and in the study of the Calabi-Yau manifolds and their moduli spaces.

In mathematical literature G_2 holonomy first appeared in Berger's list of special holonomy groups in 1955 [9]. In 1966 Bonan has shown that manifolds with G_2 holonomy are Ricci. It was known from general theory that a having a holonomy group G is equivalent to having a torsion-free G-structure. So it was natural to study G_2 structures on manifolds to get a better understanding of G_2 holonomy. The different classes of G_2 structures have been explored by Fernández and Gray in their 1982 paper [17]. In particular they have shown that a torsion-free G_2 structure is equivalent to the G_2-invariant 3-form φ being closed and co-closed.

It was not known whether the group G_2 (or indeed $Spin(7)$ for that matter) does actually appear as a non-symmetric holonomy group until in 1987 Bryant [12] proved the existence of metrics with G_2 and $Spin(7)$ holonomy. In a later paper, Bryant and Salamon [11] constructed complete metrics with G_2 holonomy. However the first *compact* examples of G_2 holonomy manifolds have been constructed by Joyce in 1996 [27]. These examples are based on quotients T^7/Γ where Γ is a finite group. Such quotient spaces usually exhibit singularities, and Joyce has shown that it is possible to resolve these singularities in such a way as to get a smooth, compact manifold with G_2 holonomy. Since then, a number of other types of constructions have been found, in particular the construction by Kovalev [32] where a compact G_2 manifold is obtained by gluing together two non-compact asymptotically cylindrical Riemannian manifolds with holonomy $SU(3)$.

In the G_2 holonomy compactification approach to M-theory, the physical content of the four-dimensional theory is given by the moduli of G_2 holonomy manifolds. Such a compactification of M-theory is in many ways analogous to Calabi-Yau compactifications in String Theory, where much progress has been made through the study of the Calabi-Yau moduli spaces. In particular, as it was shown in [14] and [35], the moduli space of complex structures and the complexified moduli space of Kähler structures are both in fact, Kähler manifolds. Moreover, both have a *special geometry*: that is, both have a line bundle whose first Chern class coincides with the Kähler class. However, until recently, the structure of the moduli space of G_2 holonomy manifolds has not been studied in that much detail. Generally, it turns out that the study of G_2 manifolds is quite difficult. Unlike the study of Calabi-Yau manifolds where the machinery of algebraic geometry has been used with great success, in the case of G_2 manifolds there is no analogue, so analytical rather than algebraic study is needed.

In this review, we aim to give an overview of what is currently known about G_2 moduli spaces and corresponding deformations of G_2 structures. We first give an introduction to the properties of the group G_2 - definitions and representations. Then we look at general properties of G_2 structures. Finally we move on to properties of G_2 moduli spaces.

2 The group G_2

2.1 Automorphisms of octonions

The group G_2 is the smallest of the 5 exceptional Lie groups, the others being F_4, E_6, E_7 and E_8. Surprisingly, all of these Lie groups are related to the octonions, but G_2 is especially close. So let us first give a few facts about the octonions. The eight-dimensional algebra of octonions, denoted by $\mathbb{O}$, is the largest possible normed division algebra. The others of course are the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$. Following Baez [6], it turns out that division algebras can be defined using the notion of *triality*. Given three real vector spaces U, V, W, then a triality is a non-degenerate trilinear map

$$t : U \times V \times W \longrightarrow \mathbb{R}.$$

Non-degenerate here means that for any fixed non-zero elements of U and V, the induced functional on W is non-zero. Hence, t also defines a bilinear map m

$$m : U \times V \longrightarrow W^*.$$

For each fixed element of U, this map defines an isomorphism between V and W^*, and for each fixed element of V, an isomorphism between U and W^*. Hence these three spaces are isomorphic to each, and if we choose to identify non-zero elements $e_1 \in U$, $e_2 \in V$, and $e_1e_2 \in W^*$, we can identify the spaces U, V, W with each other, and we can say that m now defines multiplication on U with identity element $e = e_1 = e_2 = e_1e_2$. Note that in particular, the existence of a non-degenerate trilinear map implies that the original vector spaces U,V,W are all of the same dimension.

Due to the non-degeneracy of the original triality, multiplication by a fixed element is an isomorphism, so in fact, U is a division algebra! Assuming further that U, V, W are inner product spaces, if the triality map satisfies

$$|t(u, v, w)| \leq \|u\| \|v\| \|w\|$$

then similarly we get a normed division algebra. The converse is also true - any division algebra defines a triality.

As discussed in detail by Baez [6], on $\mathbb{R}^n$ it is possible to construct bilinear maps m_n involving the vector and spinor representations of $SO(n)$

$$m_n \quad : \quad V_n \times S_n^{\pm} \longrightarrow S_n^{\mp} \text{ for } n = 0, 4 \mod 8 \qquad (2.1aa)$$
$$m_n \quad : \quad V_n \times S_n \longrightarrow S_n \text{ otherwise} \qquad (2.1ab)$$

where V_n is the vector representation of $SO(n)$, $S_n^{(\pm)}$ are the (left- and right-handed) spinor representations.

The spinor representations in (2.1) are self-dual, so in principle, by dualizing the maps in (2.1), we could obtain trilinear maps into $\mathbb{R}$. However, in order to obtain trialities, these maps have to be non-degenerate, and hence the dimensions of the relevant representations must agree. This happens only for $n = 1, 2, 4, 8$, and each of these trialities gives a normed division algebra of the corresponding dimension:

$$\begin{array}{ll} t_1 : V_1 \times S_1 \times S_1 \longrightarrow \mathbb{R} & \Longrightarrow \mathbb{R} \\ t_2 : V_2 \times S_2 \times S_2 \longrightarrow \mathbb{R} & \Longrightarrow \mathbb{C} \\ t_4 : V_4 \times S_4^+ \times S_4^- \longrightarrow \mathbb{R} & \Longrightarrow \mathbb{H} \\ t_8 : V_8 \times S_8^+ \times S_8^- \longrightarrow \mathbb{R} & \Longrightarrow \mathbb{O} \end{array} \qquad (2.2)$$

This way, via the trialities we obtain all of the normed division algebras.

In general, suppose we have a triality $t : U_1 \times U_2 \times U_3 \longrightarrow \mathbb{R}$. Then to define a normed division algebra from t, we fix two vectors in the two of the three spaces. Hence the automorphism of the division algebra is the subgroup of the automorphism of the triality that fixes these two vectors. For t_8 the automorphism group of the triality turns out to be $Spin(8)$, while G_2 is defined as the automorphism group of the corresponding octonion algebra. Thus we have

Definition 2 *The group G_2 is the automorphism group of the octonion algebra.*

Since G_2 is the automorphism group of octonions, it is the subgroup of $Spin(8)$ (the automorphism group of the triality t_8) that preserves unit vectors in V_8 and S_8^+. As explained by Baez in [6], the subgroup of $Spin(8)$ that fixes a unit vector in V_8 is $Spin(7)$. Moreover, if the representation S_8^+ is restricted to $Spin(7)$, we get the spinor representation S_7. Therefore, G_2 is the subgroup of $Spin(7)$ that fixes a unit vector in S_7. In this representation, $Spin(7)$ acts transitively on the unit sphere S^7, so we have

$$Spin(7)/G_2 = S^7. \qquad (2.3)$$

Hence we have the following result.

Proposition 3 *The group G_2 has dimension 14.*

Proof. From (2.3),

$$\dim G_2 = \dim(Spin(7)) - \dim S^7 = 21 - 7 = 14.$$

■

The automorphism group fixes the identity, so in fact G_2 acts non-trivially on octonions that are orthogonal to the identity - the imaginary octonions, denoted by $\mathrm{Im}(\mathbb{O})$ and thus we get a natural 7-dimensional representation of G_2. A closer look at this representation reveals another description of G_2. Using octonion multiplication, we can define a cross product on $\mathrm{Im}(\mathbb{O})$ by

$$a \times b = \mathrm{Im}(ab) = \frac{1}{2}(ab - ba). \qquad (2.4)$$

But G_2 preserves octonion multiplication, hence any element of G_2 preserves the 7-dimensional cross product. Alternatively, (2.4) can be written as

$$a \times b = ab + \langle a, b \rangle \tag{2.5}$$

where $\langle , \rangle$ is the octonionic inner product, in general defined by

$$\langle a, b \rangle = \frac{1}{2} \left(a^* b + b a^* \right).$$

Also, it can be shown that

$$\langle a, b \rangle = -\frac{1}{6} \operatorname{Tr} \left(a \times \left(b \times \cdot \right) \right) \tag{2.6}$$

Therefore, from (2.5), multiplication of imaginary octonions can be defined in terms of the cross product, hence any transformation preserving the cross product preserves multiplication on $\operatorname{Im}(\mathbb{O})$, and is thus in G_2. So, G_2 is precisely the group that preserves the 7-dimensional cross product.

Moreover, from the cross product we can form a "scalar triple product" on $\operatorname{Im}(\mathbb{O})$ given by

$$\varphi_0 \left(a, b, c \right) = \langle a, b \times c \rangle = \langle a, bc \rangle . \tag{2.7}$$

This defines φ_0 as an anti-symmetric trilinear functional - that is, a 3-form on $\mathbb{R}^7$. Equivalently, for a basis e_i of $\operatorname{Im}(\mathbb{O})$,

$$e_i \times e_j = \varphi_0{}^k{}_{ij} e_k. \tag{2.8}$$

So in this description, the components of φ_0 are essentially the structure constants of the algebra of imaginary octonions.

A well-known way to encode the multiplication rules for the octonions is the *Fano plane* [6]. It is shown in Figure 3. In the diagram, the vertices $e_1, ..., e_7$ are the seven square roots of -1. Multiplication follows along the six straight lines (sides of the triangle and the altitudes) and along the central circle in the direction of the arrows. So if e_i, e_j, e_k are in this order on a straight line, then $e_i e_j = e_k$ and $e_j e_i = -e_k$.

However, from (2.8) we see that φ_0 encodes precisely the same information as the Fano plane. Suppose $x^1, ..., x^7$ are coordinates on $\mathbb{R}^7$ and let $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$, then just reading off from the Fano plane, φ_0 can be written as

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{2.9}$$

Note that in order to keep the same convention for φ_0 as Joyce [28], in the Fano plane we have a different numbering for the octonions compared to Baez [6].

With this choice of coordinates, the inner product on $\operatorname{Im}(\mathbb{O}) \cong \mathbb{R}^7$ is given by the standard Euclidean metric

$$g_0 = \left(dx^1 \right)^2 + ... + \left(dx^7 \right)^2 . \tag{2.10}$$

As seen from (2.6), G_2 preserves the inner product on $\operatorname{Im}(\mathbb{O})$, so it clearly preserves g_0 and is hence a subgroup of $SO(7)$.

Since φ_0 defines the 7-dimensional cross product, and G_2 is the symmetry group of this cross product, G_2 is the stabilizer of φ_0 in $GL(7, \mathbb{R})$. So we can state :

Theorem 4 (Bryant, [12]) *The subgroup of* $GL(7, \mathbb{R})$ *that preserves the* 3*-form* φ_0 *is* G_2*.From the metric* g_0 *we can define the Hodge star* $*_0$ *on* $\mathbb{R}^7$*, and using this, the dual* 4*-form* $\psi_0 = *_0 \varphi_0$ *which is given by*

$$\psi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \tag{2.11}$$

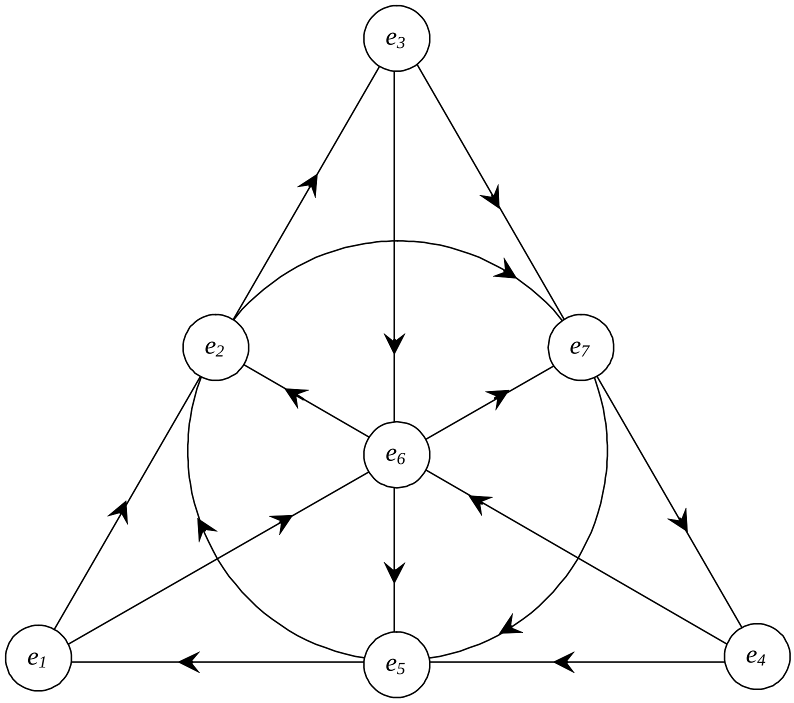

Figure 3: Fano plane

This is a key property of G_2 and as such this is often taken as the definition of the group G_2, in particular in [28]. As we have seen, G_2 preserves both φ_0 and g_0, so it also preserves ψ_0. In particular, φ_0 and ψ_0 give alternate descriptions of the trivial 1-dimensional representation of G_2.

It also turns out that ψ_0 is closely related to the *associator* on $\mathrm{Im}(\mathbb{O})$. As the octonions are non-associative, we can define a non-trivial associator map

$$[\cdot,\cdot,\cdot] : \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \longrightarrow \mathrm{Im}(\mathbb{O})$$

given by

$$[a,b,c] = a(bc) - (ab)c. \tag{2.12}$$

Just as φ_0 is defined as a dualization of the cross product using the inner product to obtain the map

$$\varphi_0 : \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \longrightarrow \mathbb{R}$$

so it turns out that up to a constant multiple the map

$$\psi_0 : \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \longrightarrow \mathbb{R}$$

is a dualization of the associator, given by

$$\psi_0(a,b,c,d) = \frac{1}{2}\langle [a,b,c], d\rangle. \tag{2.13}$$

It is possible to show that φ_0 and ψ_0 satisfy various contraction identities. In particular, from [13, 20, 30], we have

Proposition 5 *The* 3*-form* φ_0 *and the corresponding* 4*-form* ψ_0 *satisfy the following identities:*

$$\varphi_{0abc}\varphi_{0mn}{}^{c} = g_{0am}g_{0bn} - g_{0an}g_{0bm} + \psi_{0abmn} \tag{2.14a}$$

$$\varphi_{0abc}\psi_{0mnp}{}^{c} = 3\left(g_{0a[m}\varphi_{0np]b} - g_{0b[m}\varphi_{0np]a}\right) \tag{2.14b}$$

$$\psi_{0abcd}\psi_0^{mnpq} = 24\delta_a^{[m}\delta_b^n\delta_c^p\delta_d^{q]} + 72\psi_{0[ab}{}^{[mn}\delta_c^p\delta_{d]}^{q]} - 16\varphi_{0[abc}\varphi_0{}^{[mnp}\delta_{d]}^{q]} \tag{2.14c}$$

where $[m\ n\ p]$ *denotes antisymmetrization of indices and* δ_a^b *is the Kronecker delta, with* $\delta_b^a = 1$ *if* $a = b$ *and* 0 *otherwise.*

The above identities can be of course further contracted - the details can be found in [20, 30]. These identities and their contractions are crucial whenever any calculations involving φ_0 and ψ_0 have to be done. In particular, these are very useful when studying G_2 manifolds.

2.2 Representations of G_2

As we will see in section 3, a crucial role in the study of G_2 structures is played by the representations of G_2. Since G_2 is a subgroup of $SO(7)$, it has a fundamental vector representation on $\mathbb{R}^7$. In the study of G_2 manifolds, it is very important to understand the representations of G_2 on p-forms. So let us consider first the representations of G_2 on antisymmetric tensors in $\mathbb{R}^7$. For brevity let $V = \mathbb{R}^7$. Following Bryant [13], we first look at the the Lie algebra $\mathfrak{so}(7)$, which is the space of antisymmetric 7×7 matrices on V. For a vector $\omega \in V$, define the map

$$\rho_\varphi : V \longrightarrow \mathfrak{so}(7) \text{ given by } \quad \rho_\varphi(\omega) = \omega \lrcorner \varphi_0 \tag{2.15}$$

which is clearly injective. Conversely, define the map

$$\tau_\varphi : \mathfrak{so}(7) \longrightarrow V \text{ given by } \tau_\varphi(\alpha_{ab})^c = \frac{1}{6}\varphi_0^c{}_{ab}\alpha^{ab}. \tag{2.16}$$

From (2.14), we get that

$$\tau_\varphi(\rho_\varphi(\omega)) = \omega,$$

so that τ_φ is a partial inverse of ρ_φ. Thus we get a decomposition

$$\mathfrak{so}(7) = \ker\tau_\varphi \oplus \rho_\varphi(V) \tag{2.17}$$

where $\dim\rho_\varphi(V) = 7$ and $\dim\ker\tau_\varphi = 14$. It turns out that $\ker\tau_\varphi$ is in fact a Lie algebra with respect to the matrix commutator. This is the Lie algebra bracket on $\mathfrak{so}(7)$ and satisfies the Jacobi identity. It is hence only necessary to show that for $\alpha, \beta \in \ker\tau_\varphi$, we have $[\alpha, \beta] \in \ker\tau_\varphi$. This is an exercise in applying the contractions for φ. Thus we get a 14-dimensional Lie subalgebra of $\mathfrak{so}(7)$. However, this is precisely the Lie algebra $\mathfrak{g}_2$ [30], that is

$$\mathfrak{g}_2 = \ker\tau_\varphi = \left\{\alpha \in \mathfrak{so}(7) : \varphi_{0abc}\alpha^{bc} = 0\right\}. \tag{2.18}$$

This further implies that we get the following decomposition of $\mathfrak{so}(7)$:

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \rho_\varphi(V). \tag{2.19}$$

The group G_2 acts via the adjoint representation on the 14-dimensional vector space $\mathfrak{g}_2$ and via the fundamental vector representation on the 7-dimensional space $\rho_\varphi(V)$. This is a G_2-invariant irreducible decomposition of $\mathfrak{so}(7)$ into the representations **7** and **14**. Hence we get the following result:

Theorem 6 (Bryant, [12]) *The space Λ^2 of 2-forms on V decomposes as*

$$\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2. \tag{2.20}$$

with the components Λ_7^2 and Λ_{14}^2 given by:

$$\Lambda_7^2 = \{\omega \lrcorner \varphi : \omega \text{ a vector}\} \tag{2.21a}$$

$$\Lambda_{14}^2 = \left\{\alpha = \frac{1}{2}\alpha_{ab} e^a \wedge e^b : (\alpha_{ab}) \in \mathfrak{g}_2\right\} \tag{2.21b}$$

An alternative, but fully equivalent, description of Λ_7^2 and Λ_{14}^2 presents them as eigenspaces of the operator

$$T_\psi : \Lambda^2 \longrightarrow \Lambda^2 \text{ given by } T_\psi (\alpha_{ab}) = \psi_{0abcd}\alpha^{cd} \tag{2.22}$$

With this description, we have [30]:

$$\Lambda_7^2 = \left\{\alpha \in \Lambda^2 : T_\psi \alpha = 4\alpha\right\} \tag{2.23a}$$

$$\Lambda_{14}^2 = \left\{\alpha \in \Lambda^2 : T_\psi \alpha = -2\alpha\right\}. \tag{2.23b}$$

Correspondingly, the description of the **7** and **14** pieces of Λ^5 is obtained from (2.21a) and (2.21b) via Hodge duality.

Let us now look at 3-forms in more detail. Consider $\mathrm{Sym}^2(V^*)$ - the space of symmetric 2-tensors on V, and define a map

$$\mathrm{i}_\varphi : \mathrm{Sym}^2(V^*) \longrightarrow \Lambda^3 \text{ given by } \mathrm{i}_\varphi (h)_{abc} = h_{[a}^d \varphi_{0bc]d}. \tag{2.24}$$

We can decompose $\mathrm{Sym}^2(V^*) = \mathbb{R}g_0 \oplus \mathrm{Sym}_0^2(V^*)$ where $\mathbb{R}g_0$ is the set of symmetric tensors proportional to the metric g_0 and $\mathrm{Sym}_0^2(V^*)$ is the set of traceless symmetric tensors. This is a G_2-invariant irreducible decomposition of $\mathrm{Sym}^2(V^*)$ into 1-dimensional and 27-dimensional representations. We clearly have

$$\mathrm{i}_\varphi (g_0)_{abc} = \varphi_{0abc},$$

so the map i_φ is also G_2-invariant and is injective on each summand of this decomposition. Looking at the first summand, we get that $\mathrm{i}_\varphi(\mathbb{R}g_0) = \Lambda_1^3$ - the one-dimensional singlet representation of G_2. Now look at the second summand and consider $\mathrm{i}_\varphi\left(\mathrm{Sym}_0^2(V^*)\right)$. This is 27-dimensional and irreducible, so it gives a 27-dimensional representation of G_2 on 3-forms:

$$\mathrm{i}_\varphi\left(\mathrm{Sym}_0^2(V^*)\right) = \Lambda_{27}^3(V^*).$$

Now, Λ^3 is 35-dimensional, and we have accounted for $1+27 = 28$ dimensions. Thus we still have 7 dimensions left unaccounted for in Λ^3. So let us extend the map i_φ to Λ^2 - the antisymmetric 2-tensors on $\mathbb{R}^7$. Suppose $\beta \in \Lambda_7^2$. Then $\beta = \omega \lrcorner \varphi_0$, for some vector $\omega \in V$ so

$$\mathrm{i}_\varphi (\beta)_{abc} = \varphi_0{}^d{}_{[a|e|}\varphi_{0bc]d}\omega^e = \psi_{0abcd}\omega^d \tag{2.25}$$

where we have used (2.14). This defines a G_2-invariant map from V to Λ^3 and hence gives Λ_7^3.

So overall we thus have a decomposition of 3-forms into irreducible representations of G_2:

Theorem 7 (Bryant, [13]) *The space Λ^3 of 3-forms on V decomposes as*

$$\Lambda^3 = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3 \tag{2.26}$$

where

$$\Lambda_1^3 = \left\{\chi \in \Lambda^3 : \chi_{abc} = f\varphi_{0abc} \text{ for scalar } f\right\} \tag{2.27a}$$

$$\Lambda_7^3 = \{\omega \lrcorner \psi_0 : \omega \text{ a vector}\}. \tag{2.27b}$$

$$\Lambda_{27}^3 = \left\{\chi \in \Lambda^3 : \chi_{abc} = h_{[a}^d \varphi_{0bc]d} \text{ for } h_{ab} \text{ traceless, symmetric}\right\}. \tag{2.27c}$$

From the identities for contraction of φ_0 and ψ_0, it is possible to see that an equivalent description of Λ^3_{27} is

$$\Lambda^3_{27} = \left\{\chi \in \Lambda^3 : \chi \wedge \varphi_0 = 0 \text{ and } \chi \wedge \psi_0 = 0\right\}.$$

A similar decomposition of 4-forms is again obtained via Hodge duality.

Suppose we have $\chi \in \Lambda^3$, then define π_1, π_7 and π_{27} to be projections of χ onto Λ^3_1, Λ^3_7 and Λ^3_{27}, respectively. Using contraction identities for φ and ψ, we get the following relations [20]:

Proposition 8 *Given a 3-form $\chi \in \Lambda^3$, the projections of χ onto the components (2.26) of Λ^3 are given by:*

$$\pi_1(\chi) = a\varphi_0 \text{ where } a = \frac{1}{42}\left(\chi_{abc}\varphi_0^{abc}\right) = \frac{1}{7}\langle\chi,\varphi_0\rangle \text{ with } |\pi_1(\chi)|^2 = 7a^2 \quad (2.28a)$$

$$\pi_7(\chi) = \omega\lrcorner\psi_0 \text{ where } \omega^a = -\frac{1}{24}\chi_{mnp}\psi_0^{mnpa} \text{ with } |\pi_7(\chi)|^2 = 4|\omega|^2 \quad (2.28b)$$

$$\pi_{27}(\chi) = \mathrm{i}_\varphi(h) \text{ where } h_{ab} = \frac{3}{4}\chi_{mn\{a}\varphi_{0b\}}^{\ \ mn} \text{ with } |\pi_{27}(\chi)|^2 = \frac{2}{9}|h|^2. \quad (2.28c)$$

Here $\{a\ b\}$ denotes the traceless symmetric part.

Note that similar projections can be defined for 4-forms as well.

3 G_2 structures

3.1 Definition

As we shall see, the notion of holonomy is closely related to *G-structures* on manifolds. Let us give the definition of a G-structure

Definition 9 *Let X be a manifold of dimension n. The* frame bundle *of X is a principal bundle over X with fibre $GL(n,\mathbb{R})$. Let G be a Lie subgroup of $GL(n,\mathbb{R})$. Then a* G-structure *on X is a principal subbundle P of F with fibre G.*

The framework of G-structures is very powerful, and a number of geometrical structures can be reformulated in this language. In particular, a Riemannian metric on a manifold is equivalent to a $O(n)$ structure. Given a Riemannian metric, a unique torsion-free Levi-Civita connection can always be defined, hence all $O(n)$ structures are torsion-free. On a complex manifold with complex dimension, a integrable complex structure is equivalent to a torsion-free $GL(m,\mathbb{C})$ structure. A Kähler structure is then equivalent to a torsion-free $U(m)$-structure. From [28] we have a key result that relates torsion-free structures and holonomy:

Proposition 10 *Let (X,g) be a Riemannian manifold of dimension n, with $O(n)$-structure P corresponding to g. Let G be a Lie subgroup of $O(n)$. Then $Hol(g) \subseteq G$ if and only if X admits a torsion-free G-structure Q that is a subbundle of P.*

As Proposition 10 shows, the study of holonomy is equivalent to studying torsion-free G-structures. Hence in order to study G_2 holonomy manifolds we will first consider G_2 structures.

Now suppose X is a smooth, oriented 7-dimensional manifold. Following Joyce [28], define a 3-form φ to be *positive* if we locally can choose coordinates such that φ is written in the form (2.9) - that is for every $p \in X$ there is an oriented isomorphism q_p between T_pX and $\mathbb{R}^7$ such that $\varphi|_p = \varphi_0$. For each $p \in X$ define $\mathcal{P}^3_pX$ to be set of such 3-forms. To each positive φ we can associate a metric g and a Hodge dual $*\varphi$ which are identified with g_0 and ψ_0 under the q_p and the associated metric is written (2.10).

Since φ_0 is preserved by G_2 and $GL(7,\mathbb{R})_+$ acts transitively on $\mathcal{P}^3_p X$ it follows that

$$\mathcal{P}^3_p X \cong GL(7,\mathbb{R})_+ / G_2$$

and hence $\dim \mathcal{P}^3_p X = \dim GL(7,\mathbb{R})_+ - \dim G_2 = 49 - 14 = 35$. This is equal to the dimension of $\Lambda^3 T^*_p X$, hence $\mathcal{P}^3_p X$ is an open subset of $\Lambda^3 T^*_p X$. Moreover if we consider the bundle $\mathcal{P}^3 X$ over X with fibre $\mathcal{P}^3_p X$, it will be an open subbundle of $\Lambda^3 T^* X$.

Given a positive 3-form φ on X, consider at each point p the set Q_p of isomorphisms q_p between $T_p X$ and $\mathbb{R}^7$ such that $\varphi|_p = \varphi_0$. It is then easy to see that $Q_p \cong G_2$ and that the bundle Q over X with fibre Q_p is in fact a principal subbundle of the frame bundle F. So in fact, Q is a G_2 structure. The converse is also true - given an oriented G_2 structure Q, we can uniquely define a positive 3-form φ and associated metric g and 4-form ψ that correspond to φ_0,g_0 and ψ_0 respectively. We thus have a key result:

Theorem 11 (Joyce, [28]) *Let X be an oriented 7-dimensional manifold. There exists a $1-1$ correspondence between positive 3-forms on X and oriented G_2-structures Q on X. Moreover, to each positive 3-form φ we can associate a Riemannian metric g and a corresponding 4-form $*_\varphi \varphi = \psi$ such for each $p \in X$, under the isomorphism $q_p : T_p X \longrightarrow \mathbb{R}^7$, these quantities are identified with φ_0,g_0 and ψ_0 respectively.*

So given a positive 3-form φ on X, it is possible to define a metric g associated to φ. This metric then defines the Hodge star, which we denote by $*_\varphi$ to emphasize the dependence on φ. Given the Hodge star, we can in turn define the 4-form $\psi = *_\varphi \varphi$. Thus in fact both the metric g and the 4-form ψ are functions of φ. By definition, at point $p \in X$ there is an isomorphism that identifies φ with φ_0, ψ with ψ_0 and g with g_0. Therefore, properties of φ_0 and ψ_0 such as the contraction identities (2.14) that we encountered in Section 2.1 also hold for the differential forms φ and ψ.

In general, any G-structure on a manifold X induces a splitting of bundles of p-forms into subbundles corresponding to irreducible representations of G. The same is of course true for G_2 structures. The decomposition of p-forms on $\mathbb{R}^7$ carries over to any manifold with a G_2 structure, so from the previous section we have the following decomposition of the spaces of p-forms Λ^p:

$$\Lambda^1 = \Lambda^1_7 \tag{3.1a}$$
$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \tag{3.1b}$$
$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27} \tag{3.1c}$$
$$\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \tag{3.1d}$$
$$\Lambda^5 = \Lambda^5_7 \oplus \Lambda^5_{14} \tag{3.1e}$$
$$\Lambda^6 = \Lambda^6_7 \tag{3.1f}$$

Here each Λ^p_k corresponds to the k-dimensional irreducible representation of G_2. Moreover, for each k and p, Λ^p_k and Λ^{7-p}_k are isomorphic to each other via Hodge duality, and also Λ^p_7 are isomorphic to each other for $n = 1, 2, ..., 6$.

Define the standard inner product on Λ^p, so that for p-forms α and β,

$$\langle \alpha, \beta \rangle = \frac{1}{p!} \alpha_{a_1 \dots a_p} \beta^{a_1 \dots a_p}. \tag{3.2}$$

This is related to the Hodge star, since

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \,\mathrm{vol} \tag{3.3}$$

where vol is the invariant volume form given locally by

$$\mathrm{vol} = \sqrt{\det g} dx^1 \wedge ... \wedge dx^7. \tag{3.4}$$

Then the decompositions (3.1) are orthogonal with respect to (3.2). Note that $\langle \varphi, \varphi \rangle = 7$, so in fact we have

$$V = \frac{1}{7} \int \varphi \wedge *\varphi \tag{3.5}$$

where V is the volume of the manifold X.

We know that the metric g is defined by the 3-form φ and we can use some of the results from Section 2.1 to find a direct relationship between the two quantities.

Proposition 12 *Given a positive* 3*-form* φ *on a* 7*-manifold* X*, the associated metric* g *is given by*

$$g_{ab} = (\det s)^{-\frac{1}{9}} s_{ab}. \tag{3.6}$$

with

$$s_{ab} = \frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \hat{\varepsilon}^{mnpqrst} \tag{3.7}$$

where $\hat{\varepsilon}^{mnpqrst}$ *is the alternating symbol with* $\hat{\varepsilon}^{12...7} = +1$*. Alternatively, for u,v vector fields on* X*,*

$$\langle u, v \rangle \, \mathrm{vol} = \frac{1}{6} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \tag{3.8}$$

where $\lrcorner$ *denotes interior multiplication:* $(u \lrcorner \varphi)_{bc} = u^a \varphi_{abc}$.

Proof. Consider the quantity P_{ab} given by

$$P_{ab} = \varphi_{amn} \varphi_{bpq} \psi^{mnpq}$$

Using identities (2.14) to contract φ and ψ, this gives

$$P_{ab} = 24 g_{ab}.$$

Expanding ψ^{mnpq} in terms of φ and the Levi-Civita tensor we get

$$P_{ab} = \frac{1}{6} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \varepsilon^{mnpqrst}.$$

If we write $\hat{\varepsilon}^{mnpqrst}$ for the alternating symbol with $\hat{\varepsilon}^{12...7} = +1$, then we get

$$g_{ab} \sqrt{\det g} = \frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \hat{\varepsilon}^{mnpqrst}. \tag{3.9}$$

Alternatively, let u and v be vector fields on X. Then

$$\langle u, v \rangle \sqrt{\det g} = \frac{1}{144} (u^a \varphi_{amn}) \left(v^a \varphi_{bpq} \right) \varphi_{rst} \hat{\varepsilon}^{mnpqrst}.$$

Hence we get (3.8). Now define

$$s_{ab} = \frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \hat{\varepsilon}^{mnpqrst}$$

so that then, after taking the determinant of (3.9) we get (3.6). ■

Thus we see that even though given the 3-form φ we can define the metric g, this relationship is rather complicated and non-linear. In particular, this also shows that $\psi = *_\varphi \varphi$ depends on φ in an even more non-trivial fashion, since the Hodge star depends itself on the metric.

Here we need to say a few words about the notation used for the G_2 3-form φ and the associated 4-form ψ. The notation that we use here is due to Karigiannis - where the Hodge dual of φ is denoted by ψ and was first introduced in [29]. In Figure 4 we summarize the different notations used by other authors: where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $e^{ijkl} = e^i \wedge e^j \wedge e^k \wedge e^l$ for basis covectors e^i.

Authors	3-form	Dual 4-form	References
Beasley and Witten Gukov, Yau and Zaslow	Φ	$*\Phi$	[7, 21]
Bryant	$\phi = \frac{1}{6}\varepsilon_{ijk}e^{ijk}$	$*_\phi\phi = \frac{1}{24}\varepsilon_{ijkl}e^{ijkl}$	[12, 13]
Hitchin; Lee and Leung	Ω	$\Theta = *\Omega$	[25, 33]
Joyce	φ	$\Theta(\varphi) = *\varphi$	[27, 28]
Karigiannis; Karigiannis and Leung Grigorian and Yau	φ	$\psi = *_\varphi\varphi$	[20, 29, 30, 31]

Figure 4: Notation that is used by different authors

3.2 Torsion-free structures

The definition of a G_2 structure only defines the algebraic properties of φ, and in general does not address the analytical properties of φ. Using the associated metric g we can define the Levi-Civita connection ∇ on X. Then it is natural to ask what are the properties of $\nabla\varphi$. This quantity is known as the *torsion* of the G_2 structure. Originally the torsion of G_2 structures was studied by Fernández and Gray [17], and their analysis revealed that there are in fact a total of 16 torsion classes of G_2 structures. Later on, Karigiannis reproduced their results using simple computational arguments [30].

Following [30], consider the 3-form $\nabla_X\varphi$ for some vector field X. We know that 3-forms split as $\Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, so consider the projections π_1,π_7 and π_{27} of $\nabla_X\varphi$ onto these components. Using (2.28), we have

$$\pi_1(\nabla_X\varphi) = a\varphi$$

where

$$\begin{aligned} a &= X^a(\nabla_a\varphi_{bcd})\varphi^{bcd} = X^a\nabla_a\left(\varphi_{bcd}\varphi^{bcd}\right) - \varphi_{bcd}X^a\nabla_a\varphi^{bcd} \\ &= -X^a(\nabla_a\varphi_{bcd})\varphi^{bcd} \\ &= 0. \end{aligned}$$

Hence we see that the Λ^3_1 component vanishes. Similarly, for Λ^3_{27} we have

$$\pi_{27}(\nabla_X\varphi) = \mathrm{i}_\varphi(h)$$

where

$$\begin{aligned} h_{ab} &= \frac{3}{4}\left(X^c\nabla_c\varphi_{mn\{a}\right)\varphi_{b\}}{}^{mn} = \frac{3}{4}X^c\nabla_c\left(\varphi_{mn\{a}\varphi_{b\}}{}^{mn}\right) - \frac{3}{4}\varphi_{mn\{a}X^c\nabla_c\varphi_{b\}}{}^{mn} \\ &= -\frac{3}{4}\left(X^c\nabla_c\varphi_{mn\{a}\right)\varphi_{b\}}{}^{mn} \\ &= 0. \end{aligned}$$

Here we have used the fact that $\varphi_{mna}\varphi_b{}^{mn} = 6g_{ab}$, the traceless part of which vanishes. Therefore, the Λ^3_{27} part of $\nabla_X\varphi$ also vanishes. Now consider the Λ^3_7 component. In this case,

$$\pi_7(\nabla_X\varphi) = \omega\lrcorner\psi$$

where

$$\omega^a = -\frac{1}{24}X^c(\nabla_c\varphi_{mnp})\psi^{mnpa} = \frac{1}{24}X^a\left(\nabla_a\psi^{bcde}\right)\varphi_{bcd}.$$

This quantity does not vanish in general, so we can conclude that

$$\nabla_X \varphi \in \Lambda^3_7 \tag{3.10}$$

and thus overall,

$$\nabla \varphi \in W = \Lambda^1_7 \otimes \Lambda^3_7. \tag{3.11}$$

Further classification of torsion classes depends on the decomposition of W into components according to irreducible representations of G_2. Given (3.11), we can write

$$\nabla_a \varphi_{bcd} = T_a{}^e \psi_{ebcd} \tag{3.12}$$

where T_{ab} is the *full torsion tensor.* This 2-tensor full defines $\nabla\varphi$ since pointwise, it has 49 components and the space W is also 49-dimensional (pointwise). In general we can split T_{ab} as

$$T = \tau_1 g + \tau_7 + \tau_{14} + \tau_{27} \tag{3.13}$$

where τ_1 is a function, and gives the **1** component of T, $\tau_7 \in \Lambda^2_7$ and hence gives the **7** component, $\tau_{14} \in \Lambda^2_{14}$ gives the **14** component and τ_{27} is traceless symmetric, giving the **27** component. Note that the normalization of these components is different from [30]. Hence we can split W as

$$W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}. \tag{3.14}$$

The 16 torsion classes arise as the subsets of W which $\nabla\varphi$ belongs to. Moreover, as shown in [30], the torsion components τ_i relate directly to the expression for $d\varphi$ and $d\psi$. In fact, in our notation,

$$d\varphi = 4\tau_1\psi + 3\tau_7 \wedge \varphi - *\tau_{27} \tag{3.15a}$$
$$d\psi = 4\tau_7 \wedge \psi - 2 * \tau_{14}. \tag{3.15b}$$

Now suppose $d\varphi = d\psi = 0$. Then this means that all four torsion components vanish and hence $T = 0$, and as a consequence $\nabla\varphi = 0$. The converse is trivially true, since d and $d*$ can both be expressed in terms of the covariant derivative. This result is due to Fernández and Gray [17]. If we add the fact that $Hol\,(g)$ is a subgroup of G if and only if X admits a torsion-free G structure from Proposition 10, then we get the following important result.

Theorem 13 (Joyce, [28, Prop. 10.1.3]) *Let X be a 7-manifold with a G_2 structure defined by the 3-form φ and equipped with the associated Riemannian metric g. Then the following are equivalent:*

1. *The G_2-structure is torsion-free*
2. *$Hol\,(g) \subseteq G_2$ and φ is the induced 3-form*
3. *$\nabla\varphi = 0$ on X where ∇ is the Levi-Civita connection of g*
4. *$d\varphi = d\psi = 0$ where $\psi = *\varphi$ with the Hodge star defined by g*

Different torsion classes of the G_2 structure also restrict the curvature of the manifold. Consider the curvature tensor R_{abcd}. Then for fixed a,b, we have

$$(R_{ab})_{cd} \in \Lambda^2,$$

so we can decompose it as

$$(R_{ab})_{cd} = (\pi_7 R_{ab})_{cd} + (\pi_{14} R_{ab})_{cd}\,. \tag{3.16}$$

Following Karigiannis [30], consider the operator T_ψ (2.22) acting on R_{abcd}. Then we have

$$\begin{aligned} g^{ad} T_\psi R_{abcd} &= R_{abef}\psi^{ef}{}_{cd} g^{ad} \\ &= -\left(R_{beaf} + R_{eabf}\right)\psi^{ef}{}_{cd} g^{ad} \\ &= -R_{beaf}\psi^{e}{}_{c}{}^{af} + R_{fbea}\psi^{eaf}{}_{c} \\ &= -2g^{ad} T_\psi R_{abcd} \\ &= 0 \end{aligned}$$

where we have used the cyclic identity for R_{abef}. Hence, from (2.23) we get

$$Ric_{bd} = 3\left(\pi_7 R_{ab}\right)_{cd} g^{ac} = \frac{3}{2}\left(\pi_{14} R_{ab}\right)_{cd} g^{ac} \tag{3.17}$$

where Ric_{bd} is the Ricci tensor. However, in general, by the Ambrose-Singer holonomy theorem [5], if $Hol\,(g) \subseteq G$, then $R_{abcd} \in \mathrm{Sym}^2(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G. Therefore, in the G_2 case, if the G_2 structure is torsion-free and hence $Hol\,(g) \subseteq G_2$, then $R_{abcd} \in \mathrm{Sym}^2(\mathfrak{g}_2)$. This however implies that in (3.16), the π_7 component vanishes, and thus from (3.17), we have the following result:

Theorem 14 (Bonan, [10]) *Let X be a Riemannian 7-manifold with metric g. If $Hol\,(g) \subseteq G_2$, then X is Ricci-flat.*

In fact, this result can also be derived without invoking the general Ambrose-Singer theorem. In [30], Karigiannis expressed the Λ^2_7 component of the curvature tensor in terms of the torsion tensor T_{ab}, so that when the torsion vanishes, the curvature tensor is fully contained in Λ^2_{14}, thus directly confirming the Ambrose-Singer theorem in the G_2 case. The original proof of Theorem 14 due to Bonan [10]. relied on the fact that the Lie algebra structure of $\mathfrak{g}_2$ imposes strong conditions on the Riemann tensor, and that these imply that the Ricci tensor cannot be non-vanishing.

Given a compact manifold with a torsion-free G_2 structure, the decompositions (3.1) carry over to de Rham cohomology [28], so that we have

$$H^1(X,\mathbb{R}) = H^1_7 \tag{3.18a}$$
$$H^2(X,\mathbb{R}) = H^2_7 \oplus H^2_{14} \tag{3.18b}$$
$$H^3(X,\mathbb{R}) = H^3_1 \oplus H^3_7 \oplus H^3_{27} \tag{3.18c}$$
$$H^4(X,\mathbb{R}) = H^4_1 \oplus H^4_7 \oplus H^4_{27} \tag{3.18d}$$
$$H^5(X,\mathbb{R}) = H^5_7 \oplus H^5_{14} \tag{3.18e}$$
$$H^6(X,\mathbb{R}) = H^6_7 \tag{3.18f}$$

Define the refined Betti numbers $b^p_k = \dim\left(H^p_k\right)$. Clearly, $b^3_1 = b^4_1 = 1$ and we also have $b^1 = b^k_7$ for $k = 1, ..., 6$. Moreover, it turns out that if $Hol\,(X,g) = G_2$ then $b^1 = 0$. Therefore, in this case the H^k_7 component vanishes in (3.18). It can be easily shown that on a Ricci-flat manifold, any harmonic 1-form must be parallel. However this happens if and only if $Hol\,(g)$ has an invariant 1-form. However the only G_2-invariant forms are φ and ψ. Therefore there are no non-trivial harmonic 1-forms when $Hol\,(g) = G_2$ and thus $b^1 = 0$.

An example of a construction of a manifold with a torsion-free G_2 structure is to consider $X = Y \times S^1$ where Y is a Calabi-Yau 3-fold. Define the metric and a 3-form on X as

$$g_X = d\theta^2 \times g_Y \tag{3.19}$$
$$\varphi = d\theta \wedge \omega + \operatorname{Re}\Omega \tag{3.20}$$

where θ is the coordinate on S^1. This then defines a torsion-free G_2 structure, with

$$*\,\varphi = \frac{1}{2}\omega \wedge \omega - d\theta \wedge \operatorname{Im}\Omega. \tag{3.21}$$

However, the holonomy of X in this case is $SU(3) \subset G_2$. From the Künneth formula we get the following relations between the refined Betti numbers of X and the Hodge numbers of Y

$$\begin{aligned} b_7^k &= 1 \quad \text{for } k = 1, ..., 6 \\ b_{14}^k &= h^{1,1} - 1 \quad \text{for } k = 2, 5 \\ b_{27}^k &= h^{1,1} + 2h^{2,1} \quad \text{for } k = 3, 4. \end{aligned}$$

In [27] and [28], Joyce describes a possible construction of a smooth manifold with holonomy equal to G_2 from a Calabi-Yau manifold Y. So suppose Y is a Calabi-Yau 3-fold as above. Then suppose $\sigma : Y \longrightarrow Y$ is an antiholomorphic isometric involution on Y, that is, χ preserves the metric on Y and satisfies

$$\sigma^2 = 1 \tag{3.22a}$$
$$\sigma^*(\omega) = -\omega \tag{3.22b}$$
$$\sigma^*(\Omega) = \bar{\Omega}. \tag{3.22c}$$

Such an involution σ is known as a *real structure* on Y. Define now a quotient given by

$$Z = \left(Y \times S^1\right)/\hat{\sigma} \tag{3.23}$$

where $\hat{\sigma}$:$Y \times S^1 \longrightarrow Y \times S^1$ is defined by $\hat{\sigma}(y, \theta) = (\sigma(y), -\theta)$. The 3-form φ defined on $Y \times S^1$ by (3.20) is invariant under the action of $\hat{\sigma}$ and hence provides Z with a G_2 structure. Similarly, the dual 4-form $*\varphi$ given by (3.21) is also invariant. Generically, the action of σ on Y will have a non-empty fixed point set N, which is in fact a special Lagrangian submanifold on Y [28]. This gives rise to orbifold singularities on Z. The singular set is two copies of Z. It is conjectured that it is possible to resolve each singular point using an ALE 4-manifold with holonomy $SU(2)$ in order to obtain a smooth manifold with holonomy G_2, however the precise details of the resolution of these singularities are not known yet. We will therefore consider only free-acting involutions, that is those without fixed points.

Manifolds defined by (3.23) with a freely acting involution were called *barely G_2 manifolds* by Harvey and Moore in [24]. The cohomology of barely G_2 manifolds is expressed in terms of the cohomology of the underlying Calabi-Yau manifold Y:

$$H^2(Z) = H^2(Y)^+ \tag{3.24a}$$
$$H^3(Z) = H^2(Y)^- \oplus H^3(Y)^+ \tag{3.24b}$$

Here the superscripts $\pm$ refer to the $\pm$ eigenspaces of σ^*. Thus $H^2(Y)^+$ refers to two-forms on Y which are invariant under the action of involution σ and correspondingly $H^2(Y)^-$ refers to two-forms which are odd under σ. Wedging an odd two-form on Y with $d\theta$ gives an invariant 3-form on $Y \times S^1$, and hence these forms, together with the invariant 3-forms $H^3(Y)^+$ on Y, give the three-forms on the quotient space Z. Also note that $H^1(Z)$ vanishes, since the 1-form on S^1 is odd under $\hat{\sigma}$. Now, given a 3-form on Y, its real part will be invariant under σ, hence $H^3(Y)^+$ is essentially the real part of $H^3(Y)$. Therefore the Betti numbers of Z in terms of Hodge numbers of Y are

$$b^1 = 0 \tag{3.25a}$$
$$b^2 = h^+_{1,1} \tag{3.25b}$$
$$b^3 = h^-_{1,1} + h_{2,1} + 1 \tag{3.25c}$$

A class of barely G_2 manifolds that are constructed from complete intersection Calabi-Yau manifolds has recently been considered in [19], where the Betti numbers of all such manifolds have been calculated explicitly.

Note that barely G_2 manifolds have holonomy $SU(3) \ltimes \mathbb{Z}_2$ while the first Betti number still vanishes. This shows that vanishing first Betti number is not a necessary and sufficient condition for $Hol(g) = G_2$. In fact, as shown by Joyce in [27], $Hol(g) = G_2$ if and only if the fundamental group $\pi_1(X)$ is finite.

Let us briefly describe Joyce's construction of compact torsion-free manifolds with $Hol(g) = G_2$. Here we follow [28]. On T^7 we can define a flat G_2 structure (φ_0, g_0), similarly as on $\mathbb{R}^7$. Now suppose that Γ is a finite group acting on T^7 that preserves the G_2 structure. Then we can define the orbifold T^7/Γ. The key to resolving the orbifold singularities is to consider appropriate *Quasi Asymptotically Locally Euclidean* (QALE) G_2 manifolds. These are 7-manifolds with a torsion-free G_2 structure that is asymptotic to the G_2 structure on $\mathbb{R}^7/G$ where G is a finite subgroup of G_2. The orbifold T^7/Γ is then resolved to obtain a smooth compact manifold. However on the resolution, the resulting G_2-structure is not necessarily torsion-free, so it is shown that it can be deformed to a torsion-free G_2 structure (φ, g). Further, the fundamental group is calculated, and if it is finite, then $Hol(g) = G_2$. Using this method, Joyce found 252 topologically distinct G_2 holonomy manifolds with unique pairs of Betti numbers (b^2, b^3).

4 Moduli space

4.1 Deformations of G_2 structures

One of the interesting directions in the study of G_2 holonomy manifolds is the structure of the *moduli space*. Essentially, the idea is to consider the space of all torsion-free G_2 structures modulo diffeomorphisms on a manifold with fixed topology. The moduli space itself has an interesting geometry that may give further information about G_2 manifolds.

Currently, we can only say something about the very local structure of the G_2 moduli space. For this, we take a fixed G_2 structure and deform it slightly. The space of these deformations is the local moduli space. To study it, we thus need to understand the deformations of G_2 structures. Although, we are mostly interested in deformations of torsion-free G_2 structures, many of the results are valid for any G_2 structures.

Our aim is to consider infinitesimal deformations of φ of the form

$$\varphi \longrightarrow \varphi + \varepsilon\chi \tag{4.1}$$

for some 3-form χ. As we already know, the G_2 structure on X and the corresponding metric g are all determined by the invariant 3-form φ. Hence, deformations of φ will induce deformations of the metric. These deformations of metric will then also affect the deformation of $\psi = *\varphi$. Theoretically, "large" deformations could also be considered, and in fact, as we shall see below in some cases closed expressions can be obtained for large deformations. However in that case, it is difficult to determine the resulting torsion class of the new G_2 structure [29]. In order for the deformed φ to define a new G_2 structure, the new φ must also be a positive form (as per the definition of a G_2 structure). However it is known [28] that the bundle of positive 3-forms on X is an open subbundle of $\Lambda^3 T^* X$, so we can always find ε small enough in order for the deformed φ to be positive.

Using the decomposition of 3-forms (3.1c), we can split χ into Λ^3_1, Λ^3_7 and Λ^3_{27} parts, and at first let us consider each one separately. As shown by Karigiannis in [29], metric deformations can be made explicit when the 3-form deformations are either in Λ^3_1 or Λ^3_7. Let us first review some of these results. First suppose

$$\tilde{\varphi} = f\varphi \tag{4.2}$$

We will also use the notation $\tilde{\psi} = \tilde{*}\tilde{\varphi}$ where $\tilde{*}$ is the Hodge star derived from the metric $\tilde{g}$ corresponding to $\tilde{\varphi}$. Then from (3.9) we get

$$\begin{aligned} \tilde{g}_{ab}\sqrt{\det \tilde{g}} &= \frac{1}{144}\tilde{\varphi}_{amn}\tilde{\varphi}_{bpq}\tilde{\varphi}_{rst}\hat{\varepsilon}^{mnpqrst} \\ &= f^3 g_{ab}\sqrt{\det g} \end{aligned} \tag{4.3}$$

After taking the determinant on both sides, we obtain

$$\det \tilde{g} = f^{\frac{14}{3}} \det g. \tag{4.4}$$

Substituting (4.4) into (4.3), we finally get

$$\tilde{g}_{ab} = f^{\frac{2}{3}} g_{ab}. \tag{4.5}$$

and hence

$$\tilde{\psi} = f^{\frac{4}{3}}\psi. \tag{4.6}$$

So, a scaling of φ gives a conformal transformation of the metric. Hence deformations of φ in the direction Λ^3_1 also give infinitesimal conformal transformation. Suppose $f = 1 + \varepsilon a$, then to third order in ε, we can write

$$\tilde{\psi} = \left(1 + \frac{4}{3}a\varepsilon + \frac{2}{9}a^2\varepsilon^2 - \frac{4}{81}a^3\varepsilon^3 + O\left(\varepsilon^4\right)\right)\psi. \tag{4.7}$$

Given a torsion-free G_2 structure, $d\varphi = d\psi = 0$, so if we want the deformed structure to be also torsion-free, f must be constant.

Now, suppose in general that $\tilde{\varphi} = \varphi + \varepsilon\chi$ for some $\chi \in \Lambda^3$. Then using (3.8) for the definition of the metric associated with $\tilde{\varphi}$, after some manipulations, we get:

$$\begin{aligned} \widetilde{\langle u, v\rangle}\widetilde{\mathrm{vol}} &= \frac{1}{6}\left(u\lrcorner\varphi\right)\wedge\left(v\lrcorner\varphi\right)\wedge\varphi \\ &+\frac{1}{2}\varepsilon\left[\left(u\lrcorner\chi\right)\wedge *\left(v\lrcorner\varphi\right) + \left(v\lrcorner\chi\right)\wedge *\left(u\lrcorner\varphi\right)\right] \\ &+\frac{1}{2}\varepsilon^2\left(u\lrcorner\chi\right)\wedge\left(v\lrcorner\chi\right)\wedge\varphi \\ &+\frac{1}{6}\varepsilon^3\left(u\lrcorner\chi\right)\wedge\left(v\lrcorner\chi\right)\wedge\chi. \end{aligned} \tag{4.8}$$

Rewriting (4.8) in local coordinates, we get

$$\tilde{g}_{ab}\frac{\sqrt{\det \tilde{g}}}{\sqrt{\det g}} = g_{ab} + \frac{1}{2}\varepsilon\chi_{mn(a}\varphi_{b)}{}^{mn} + \frac{1}{8}\varepsilon^2\chi_{amn}\chi_{bpq}\psi^{mnpq} + \frac{1}{24}\varepsilon^3\chi_{amn}\chi_{bpq}\left(*\chi\right)^{mnpq} \tag{4.9}$$

Now suppose the deformation is in the Λ^3_7 direction. This implies that

$$\chi = \omega\lrcorner\psi \tag{4.10}$$

for some vector field ω. Look at the first order term in (4.9). From (2.28) we see that this is essentially a projection onto $\Lambda^3_1 \oplus \Lambda^3_{27}$ - the traceless part gives the Λ^3_{27} component and the trace gives the Λ^3_1 component. Hence this term vanishes for $\chi \in \Lambda^3_7$. For the third order term, it is more convenient to study at it in (4.8). By looking at

$$\omega \lrcorner ((u \lrcorner \omega \lrcorner \psi) \wedge (v \lrcorner \omega \lrcorner \psi) \wedge \psi) = 0$$

we immediately see that the third order term vanishes. So now we are left with

$$\begin{aligned} \tilde{g}_{ab}\sqrt{\det \tilde{g}} &= \left(g_{ab} + \frac{1}{8}\varepsilon^2 \omega^c \omega^d \psi_{camn} \psi_{dbpq} \psi^{mnpq} \right) \sqrt{\det g} \\ &= \left(g_{ab}\left(1 + \varepsilon^2 |\omega|^2\right) - \varepsilon^2 \omega_a \omega_b \right) \sqrt{\det g} \end{aligned} \tag{4.11}$$

where we have used a contraction identity for ψ twice. Taking the determinant of (4.11) gives

$$\sqrt{\det \tilde{g}} = \left(1 + \varepsilon^2 |\omega|^2\right)^{\frac{2}{3}} \sqrt{\det g}. \tag{4.12}$$

Eventually we have the following result:

Theorem 15 (Karigiannis, [29]) *Given a deformation of a G_2 structure (4.1) with $\chi = \omega \lrcorner \psi \in \Lambda^3_7$, then the new metric $\tilde{g}_{ab}$ is given by*

$$\tilde{g}_{ab} = \left(1 + \varepsilon^2 |\omega|^2\right)^{-\frac{2}{3}} \left(\left(g_{ab}\left(1 + \varepsilon^2 |\omega|^2\right) - \varepsilon^2 \omega_a \omega_b \right) \right) \tag{4.13}$$

and the deformed 4-form $\tilde{\psi}$ is given by

$$\tilde{\psi} = \left(1 + \varepsilon^2 |\omega|^2\right)^{-\frac{1}{3}} \left(\psi + *\varepsilon (\omega \lrcorner \psi) + \varepsilon^2 \omega \lrcorner * (\omega \lrcorner \varphi)\right). \tag{4.14}$$

One of the key reasons why it is possible to get these closed form expressions for modified g and ψ is because as shown by Karigiannis in [29], the determinant of (4.11) can be calculated in a closed form. Notice that to first order in ε, both $\sqrt{\det g}$ and g_{ab} remain unchanged under this deformation. Now let us examine the last term in (4.14) in more detail. Firstly, we have

$$\omega \lrcorner * (\omega \lrcorner \varphi) = * \left(\omega^\flat \wedge (\omega \lrcorner \varphi)\right)$$

and

$$\begin{aligned} \left(\omega^\flat \wedge (\omega \lrcorner \varphi)\right)_{mnp} &= 3\omega_{[m} \omega^a \varphi_{|a|np]} \\ &= 3\mathrm{i}_\varphi (\omega \circ \omega) \end{aligned} \tag{4.15}$$

where $(\omega \circ \omega)_{ab} = \omega_a \omega_b$. Therefore, in (4.14), this term gives Λ^4_1 and Λ^4_{27} components. So, can write (4.14) as

$$\tilde{\psi} = \left(1 + \varepsilon^2 |\omega|^2\right)^{-\frac{1}{3}} \left(\left(1 + \frac{3}{7}\varepsilon^2 |\omega|^2\right) \psi + *\varepsilon (\omega \lrcorner \psi) + \varepsilon^2 * \mathrm{i}_\varphi \left((\omega \circ \omega)_0\right) \right). \tag{4.16}$$

Here $(\omega \circ \omega)_0$ denotes the traceless part of $\omega \circ \omega$, so that $\mathrm{i}_\varphi ((\omega \circ \omega)_0) \in \Lambda^3_{27}$ and thus, in (4.16), the components in different representations are now explicitly shown.

To first order, we thus have the deformations

$$\begin{aligned} \tilde{\varphi} &= \varphi + \varepsilon (\omega \lrcorner \psi) \\ \tilde{\psi} &= \psi + *\varepsilon (\omega \lrcorner \psi). \end{aligned}$$

If originally $d\varphi = d\psi = 0$, that is, the G_2 structure is torsion-free, then for the deformed structure to be torsion-free to first order we need

$$d\left(\omega \lrcorner \psi\right) = d * \left(\omega \lrcorner \psi\right) = 0.$$

By expanding $d\left(\omega \lrcorner \psi\right)$ in terms of the decomposition of Λ^4, and setting each term individually to 0, we find that the symmetric part of $\nabla_a \omega_b$ and the Λ^2_7 part of $d\omega^\flat$ must vanish. Furthermore, by expanding $*d*\left(\omega \lrcorner \psi\right)$ in terms of the decomposition of Λ^2 we find that the Λ^2_{14} part of $d\omega^\flat$ must also vanish. Hence we get that $\nabla\omega = 0$. If $Hol\left(g\right) = G_2$, then we know that in this case $\omega = 0$, so there are no interesting small Λ^3_7 deformations of manifolds with holonomy equal to G_2.

As we have seen above, in the cases when the deformations were in Λ^3_1 or Λ^3_7 directions, there were some simplifications, which make it possible to write down all results in a closed form. In the case of deformations in Λ^3_{27} the only known way to get results for deformations of the metric and the 4-form ψ is to consider the deformations order by order in ε. This analysis has been carried out in [20], and here we will review those results. So suppose we have a deformation

$$\tilde{\varphi} = \varphi + \varepsilon\chi$$

where $\chi \in \Lambda^3_{27}$. Now let us set up some notation. Define

$$\begin{aligned} \tilde{s}_{ab} &= \frac{1}{144}\frac{1}{\sqrt{\det g}}\tilde{\varphi}_{amn}\tilde{\varphi}_{bpq}\tilde{\varphi}_{rst}\hat{\varepsilon}^{mnpqrst} && (4.18) \\ &= \tilde{g}_{ab}\sqrt{\frac{\det\tilde{g}}{\det g}} && (4.19) \end{aligned}$$

From (3.9), the untilded s_{ab} is then just equal to g_{ab}. We can rewrite (4.19) as

$$\tilde{g}_{ab} = \sqrt{\frac{\det g}{\det\tilde{g}}}\left(g_{ab} + \delta s_{ab}\right) \tag{4.20}$$

where δg_{ab} is the deformation of the metric and δs_{ab} is the deformation of s_{ab}, which from (4.9) is given by

$$\delta s_{ab} = \frac{1}{2}\varepsilon\chi_{mn(a}\varphi_{b)}{}^{mn} + \frac{1}{8}\varepsilon^2\chi_{amn}\chi_{bpq}\psi^{mnpq} + \frac{1}{24}\varepsilon^3\chi_{amn}\chi_{bpq}\left(*\chi\right)^{mnpq}. \tag{4.21}$$

Also introduce the following short-hand notation

$$s_k = \mathrm{Tr}\left(\left(\delta s\right)^k\right) \tag{4.22}$$

where the trace is taken using the original metric g. From (4.21), note that since $\chi \in \Lambda^3_{27}$, when taking the trace the first order term vanishes, and hence s_1 is at least second-order in ε. Clearly, for $k > 1$, s_k are at least of order k in ε. Similarly as before, take the determinant of (4.18):

$$\left(\frac{\det\tilde{g}}{\det g}\right)^{\frac{9}{2}} = \frac{\det\left(g + \delta s\right)}{\det\left(g\right)}. \tag{4.23}$$

Unlike in the case of Λ^3_7 deformations, we cannot compute $\det\left(g + \delta s\right)$ in closed form, so we have to calculate it order by order in ε. From the standard expansion of $\det\left(I + X\right)$, we find

$$\frac{\det\left(g + \delta s\right)}{\det g} = 1 + s_1 + \frac{1}{2}\left(s_1^2 - s_2\right) + \frac{1}{6}\left(s_1^3 - 3s_1 s_2 + 2s_3\right) + O\left(\varepsilon^4\right) \tag{4.24}$$

However, as we noted above, s_1 is second-order in ε, so this expression actually simplifies:

$$\frac{\det(g+\delta s)}{\det g} = 1 + \left(s_1 - \frac{1}{2}s_2\right) + \frac{1}{3}s_3 + O\left(\varepsilon^4\right). \tag{4.25}$$

Raising this to the power of $-\frac{1}{9}$, and expanding again to fourth order in ε, we get

$$\sqrt{\frac{\det g}{\det \tilde{g}}} = 1 + \left(\frac{1}{18}s_2 - \frac{1}{9}s_1\right) - \frac{1}{27}s_3 + O\left(\varepsilon^4\right). \tag{4.26}$$

Using this and (4.20) we can immediately get the deformed metric, but the expressions using the current form of δs_{ab} are not very useful. So far, the only property of Λ^3_{27} that we have used is that it is orthogonal to φ, thus in fact, up to this point everything applies to Λ^3_7 as well. Now however, let χ be of the form

$$\chi_{abc} = h^d_{[a}\varphi_{bc]d} \tag{4.27}$$

where h_{ab} is traceless and symmetric, so that $\chi \in \Lambda^3_{27}$. Let us first introduce some further notation. Let h_1, h_2, h_3, h_4 be traceless, symmetric matrices, and introduce the following shorthand notation

$$(\varphi h_1 h_2 \varphi)_{mn} = \varphi_{abm} h_1^{ad} h_2^{be} \varphi_{den} \tag{4.28a}$$
$$\varphi h_1 h_2 h_3 \varphi = \varphi_{abc} h_1^{ad} h_2^{be} h_3^{cf} \varphi_{def} \tag{4.28b}$$
$$(\psi h_1 h_2 h_3 \psi)_{mn} = \psi_{abcm} \psi_{defn} h_1^{ad} h_2^{be} h_3^{cf} \tag{4.28c}$$
$$\psi h_1 h_2 h_3 h_4 \psi = \psi_{abcm} \psi_{defn} h_1^{ad} h_2^{be} h_3^{cf} h_4^{mn} \tag{4.28d}$$

It is clear that all of these quantities are symmetric in the h_i and moreover $(\varphi h_1 h_2 \varphi)_{mn}$ and $(\psi h_1 h_2 h_3 \psi)_{mn}$ are both symmetric in indices m and n. Then, it can be shown that

$$\begin{aligned}
\chi_{(a|mn|}\varphi_{b)}{}^{mn} &= \frac{4}{3}h_{ab} \\
\chi_{amn}\chi_{bpq}\psi^{mnpq} &= -\frac{4}{7}|\chi|^2 g_{ab} + \frac{16}{9}\left(h^2\right)_{\{ab\}} - \frac{4}{9}(\varphi h h \varphi)_{\{ab\}} \\
\chi_{amn}\chi_{bpq} * \chi^{mnpq} &= \frac{32}{189}\operatorname{Tr}\left(h^3\right) g_{ab} - \frac{8}{9}\left(\varphi h h^2 \varphi\right)_{\{ab\}}
\end{aligned}$$

where as before $\{a\ b\}$ denotes the traceless symmetric part. Using this and (4.21), we can now express δs_{ab} in terms of h:

$$\begin{aligned}
\delta s_{ab} = {} & \frac{2}{3}\varepsilon h_{ab} + g_{ab}\left(-\frac{1}{63}\varepsilon^2 \operatorname{Tr}\left(h^2\right) + \frac{4}{567}\varepsilon^3 \operatorname{Tr}\left(h^3\right)\right) + \varepsilon^2\left(\frac{2}{9}\left(h^2\right)_{\{ab\}} - \frac{1}{18}(\varphi h h \varphi)_{\{ab\}}\right) \\
& -\frac{\varepsilon^3}{27}\left(\varphi h h^2 \varphi\right)_{\{ab\}}
\end{aligned} \tag{4.29}$$

and hence

$$s_1 = \operatorname{Tr}(\delta s) = -\frac{1}{9}\varepsilon^2 \operatorname{Tr}\left(h^2\right) + \frac{4}{81}\varepsilon^3 \operatorname{Tr}\left(h^3\right) \tag{4.30a}$$
$$s_2 = \operatorname{Tr}\left(\delta s^2\right) = \frac{4}{9}\varepsilon^2 \operatorname{Tr}\left(h^2\right) + \varepsilon^3\left(\frac{8}{27}\operatorname{Tr}\left(h^3\right) - \frac{2}{27}(\varphi h h h \varphi)\right) \tag{4.30b}$$
$$s_3 = \operatorname{Tr}\left(\delta s^3\right) = \frac{8}{27}\varepsilon^3 \operatorname{Tr}\left(h^3\right) \tag{4.30c}$$

Substituting these expressions into (4.26) and (4.20), we can get the full expression for the deformed metric (up to third order in ε) and correspondingly the expression for the deformed 4-form ψ :

Theorem 16 (Grigorian and Yau, [20]) *Given a deformation of a G_2 structure (4.1) with $\chi_{abc} = h^d_{[a}\varphi_{bc]d} \in \Lambda^3_{27}$, then the new metric $\tilde{g}_{ab}$ is given to third order in ε by*

$$\begin{aligned}\tilde{g}_{ab} &= \left(1 + \frac{1}{18}\varepsilon^2 \operatorname{Tr}\left(h^2\right) + \frac{1}{81}\varepsilon^3 \operatorname{Tr}\left(h^3\right) - \frac{1}{243}\varepsilon^3 \left(\varphi hhh\varphi\right)\right) g_{ab} + \frac{2}{3}\varepsilon h_{ab} \\ &\quad +\varepsilon^2 \left(\frac{2}{9}\left(h^2\right)_{(ab)} - \frac{1}{18}\left(\varphi hh\varphi\right)_{ab}\right) + \frac{2}{81}\varepsilon^3 h_{ab} \operatorname{Tr}\left(h^2\right) - \frac{\varepsilon^3}{27}\left(\varphi h h^2 \varphi\right)_{ab} + O\left(\varepsilon^4\right)\end{aligned} \tag{4.31}$$

and correspondingly, the deformed 4-form $\tilde{\psi}$ is given by

$$\begin{aligned}\tilde{\psi} &= \psi - \varepsilon * \chi + \varepsilon^2 \left(-\frac{1}{189}\operatorname{Tr}\left(h^2\right)\psi + \frac{1}{6} * \mathrm{i}_\varphi\left(\left(\phi hh\phi\right)_0\right)\right) \\ &\quad +\varepsilon^3 \left(-\frac{2}{1701}\left(\varphi hhh\varphi\right)\psi - \frac{5}{108}\operatorname{Tr}\left(h^2\right) * \chi + \frac{1}{18} * \mathrm{i}_\varphi\left(h_0^3\right) - \frac{1}{36} * \mathrm{i}_\varphi\left(\left(\psi hhh\psi\right)_0\right) + \frac{1}{324}\alpha \wedge \varphi\right) \\ &\quad +O\left(\varepsilon^4\right)\end{aligned} \tag{4.32}$$

where $(\phi hh\phi)_0$, h_0^3 and $(\psi hhh\psi)_0$ denote the traceless parts of $(\phi hh\phi)_{ab}$, $\left(h^3\right)_{ab}$ and $(\psi hhh\psi)_{ab}$, respectively, and

$$\alpha_a = \psi_{amnp}\varphi_{rst}h^{mr}h^{ns}h^{pt} \tag{4.33}$$

In general if such a deformation is performed on a torsion-free G_2 structure, then it is not known what conditions must h satisfy in order for the torsion class to be preserved. If we restrict our analysis only to first order deformations, then it is easier to see these conditions.

Suppose we have $d\varphi = d\psi = 0$ and we apply a deformation (4.1) with $\chi = \mathrm{i}_\varphi(h)$ for traceless and symmetric. Then to first order the conditions for $d\tilde{\varphi} = d\tilde{\psi} = 0$ are

$$d\chi = d * \chi = 0.$$

Hence the deformation must be a form that is closed and co-closed. For a compact manifold this is thus equivalent to χ being harmonic. We can also find what this condition means in terms of h. By decomposing $d\chi$ into Λ^4_1, Λ^4_7 and Λ^4_{27} components, we find that we must have

$$\nabla_r h^r{}_a = 0 \tag{4.34a}$$

$$\nabla_m h_{a(b}\varphi^{ma}{}_{c)} = 0 \tag{4.34b}$$

Further, if we decompose $*d*\chi$ into Λ^2_7 and Λ^2_{14} components, we again get(4.34a) and moreover get a new constraint

$$\nabla_m h_{a[b}\varphi^{ma}{}_{c]} = 0 \tag{4.35a}$$

Thus overall, for h traceless and symmetric, $\chi = \mathrm{i}_\varphi(h)$ being closed and co-closed is equivalent to

$$\nabla_r h^r{}_a = 0 \quad \text{and} \quad \nabla_m h_{ab}\varphi^{ma}{}_c = 0.$$

On a compact manifold χ being closed and co-closed is equivalent to χ being harmonic. It also turns out [2] that, if χ is defined as above, then

$$\Delta\chi = 0 \iff \Delta_L h = 0$$

where Δ_L is the Lichnerowicz operator given by

$$\Delta_L h_{ab} = \nabla^2 h_{ab} + 2R_{acbd}h^{cd}. \tag{4.36}$$

Therefore to preserve the torsion-free G_2 structure, we have to limit our attention to zero modes of the Lichnerowicz operator. Note that, to linear order, traceless deformations of the metric which preserve the Ricci tensor are also precisely the Lichnerowicz zero modes, and this is consistent with (4.31) where the linear term in the metric deformation is proportional to h.

Let us compare what happens here to what happens on Calabi-Yau manifolds [14]. In that case, deformations of the metric δg_{mn} split into deformations of mixed type $\delta g_{\mu\bar{\nu}}$ and deformations of pure type $\delta g_{\mu\nu}$ and $\delta g_{\bar{\mu}\bar{\nu}}$. From the mixed type deformations we can define a real $(1,1)$-form

$$i\delta g_{\mu\bar{\nu}} dx^{\mu} \wedge dx^{\bar{\nu}} \tag{4.37}$$

and given the holomorphic 3-form Ω, we can use the mixed type deformation to define a real $(2,1)$-form

$$\Omega_{\kappa\lambda}{}^{\bar{\nu}} \delta g_{\bar{\mu}\bar{\nu}} dx^{k} \wedge dx^{\lambda} \wedge dx^{\bar{\mu}}. \tag{4.38}$$

In order to preserve the Calabi-Yau structure, the metric deformation must preserve the vanishing Ricci curvature, and hence δg_{mn} must satisfy the Lichnerowicz equation:

$$\Delta_L \delta g_{mn} = 0$$

However, the Lichnerowicz equation for δg_{mn} becomes equivalent to both the $(1,1)$-form (4.37) and the $(2,1)$-form (4.38) being harmonic. Note that the definition (4.38) is very similar to $\chi_{abc} = h^{d}_{[a}\varphi_{bc]d}$ in the G_2 case with φ playing the role of Ω and h the role of $\delta g_{\bar{\mu}\bar{\nu}}$.

4.2 Geometry of the moduli space

In the theory of Calabi-Yau moduli spaces, one of the key results is that the local moduli space of complex structure deformations is isomorphic to an open set in $H^{m-1,1}(X)$ where X is a Calabi-Yau m-fold. Moreover, as it has been shown by Tian and Todorov [38, 39], any infinitesimal deformation can be in fact lifted to a full deformation. For the moduli spaces of G_2 manifolds however, we can only replicate the results about local moduli space. First let us define the moduli space of torsion-free G_2 structures. Let $\mathcal{X}$ be the set of of positive 3-forms $\varphi \in \mathcal{P}^3 X$ such that $d\varphi = d *_{\varphi} \varphi = 0$. Here we use $*_{\varphi}$to emphasize that the Hodge star is defined using the G_2 holonomy metric that is defined by φ itself. Then $\mathcal{X}$ gives the set of all 3-forms that correspond to oriented, torsion-free G_2 structures. However we do not want to distinguish between 3-forms that are related by a diffeomorphism. Hence, let $\mathcal{D}$ be the group of all diffeomorphisms of X isotopic to the identity. This group then acts naturally on 3-forms. The *moduli space* of torsion-free G_2 structures is then defined as the quotient $\mathcal{M} = \mathcal{X}/\mathcal{D}$. The key result by Joyce is that $\mathcal{M}$ is that locally $\mathcal{M}$ is diffeomorphic to an open set of $H^3(X,\mathbb{R})$:

Theorem 17 (Joyce, [27]) *Define a map $\Xi : \mathcal{X} \longrightarrow H^3(X,\mathbb{R})$ by $\Xi(\varphi) = [\varphi]$. Then Ξ is invariant under the action of $\mathcal{D}$ on $\mathcal{X}$. Moreover, Ξ induces a diffeomorphism between neighbourhoods of $\varphi\mathcal{D} \in \mathcal{M}$ and $[\varphi] \in H^3(X,\mathbb{R})$.*

Since the dimension of $H^3(X,\mathbb{R})$ is $b^3(X)$, this result implies that $\dim \mathcal{M} = b^3(X)$. The full proof of this result can be found either in [27] or [28]. This result covers the basic local properties of the G_2 moduli space, but we do not yet know anything about the global structure of $\mathcal{M}$. So anything we can say about the moduli space only holds in a small neighbourhood.

Looking back at the study of Calabi-Yau moduli spaces, we know that the complex structure moduli space admits a Kähler structure, and the Kähler structure moduli space admits a Hessian structure [14]. It turns out that on the G_2 moduli space we can also define a Hessian structure. First let us define the notion of a *Hessian manifold* [34]

Definition 18 *Let M be a smooth manifold and suppose D is a flat, torsion-free connection on M. A Riemannian metric G on a flat manifold (M,D) is called* Hessian *if G can be locally expressed as*

$$G = D^2 H \tag{4.39}$$

that is,

$$G_{ij} = \frac{\partial^2 H}{\partial x^i \partial x^j} \tag{4.40}$$

where $\{x^1, ..., x^n\}$ *is an affine coordinate system with respect to* D*. Then* H *is called the Hessian potential.*

Note that this is the closest analogue to a Kähler structure that can be defined on a real manifold. In fact, as shown by Shima [34], if we define a complex structure on the manifold TM, then the straightforward extension of G onto TM is Kähler if and only if G is a Hessian metric on (M, D). Thus the complexification of a Hessian manifold is Kähler.

In the case of the G_2 moduli space $\mathcal{M}$, we know that $\mathcal{M}$ is isomorphic to an open set in $H^3(X, \mathbb{R})$. Suppose we choose a basis $[\varphi_0], ..., [\varphi_n]$ on $H^3(X, \mathbb{R})$ where $n = b^3(X) - 1$. Taking the unique harmonic representatives of the basis elements, we can expand $\varphi \in \mathcal{M}$ as

$$\varphi = \sum_{N=0}^{n} s^N \phi_N. \tag{4.41}$$

Since $H^3(X, \mathbb{R})$ is a vector space, $s_0, ..., s_n$ give an affine coordinate system, which in turn defines a flat connection $D = d$ on $\mathcal{M}$. It is trivial to check that this connection is well-defined [31].

In order to define a metric on $\mathcal{M}$, we have to choose a Hessian potential function on $\mathcal{M}$. The only natural function on $\mathcal{M}$ is the volume function $V(\varphi)$ given by (3.5):

$$V(\varphi) = \frac{1}{7}\int_X \varphi \wedge \psi.$$

Note that as before, $\psi = *_\varphi \varphi$ is itself a function of φ. So we can consider V or some function of V as potential candidates for a Hessian potential. Let us calculate the Hessian of V. Note that under a scaling $s^M \longrightarrow \lambda s^M$, φ scales as $\varphi \longrightarrow \lambda\varphi$ and from (4.6), $*\varphi$ scales as $*\varphi \longrightarrow \lambda^{\frac{4}{3}} * \varphi$, and so V scales as

$$V \longrightarrow \lambda^{\frac{7}{3}} V.$$

So V is homogeneous of order $\frac{7}{3}$ in the s^M, and hence

$$\begin{aligned} s^M \frac{\partial V}{\partial s^M} &= \frac{7}{3} V \\ &= \frac{1}{3}\int s^M \phi_M \wedge *\varphi \end{aligned}$$

and thus,

$$\frac{\partial V}{\partial s^M} = \frac{1}{3}\int \phi_M \wedge *\varphi. \tag{4.42}$$

Using our results on deformations of G_2 structures from Section 4.1, we can deduce that

$$\partial_N (*\varphi) = \frac{4}{3} * \pi_1(\phi_N) + *\pi_7(\phi_N) - *\pi_{27}(\phi_N). \tag{4.43}$$

Hence differentiating (4.42) again, we find that

$$\begin{aligned} \frac{\partial V}{\partial s^M \partial s^N} &= \frac{4}{9}\int \pi_1(\varphi_M) \wedge *\pi_1(\varphi_N) + \frac{1}{3}\int \pi_7(\varphi_M) \wedge *\pi_7(\varphi_N) \\ &\quad -\frac{1}{3}\int \pi_{27}(\varphi_M) \wedge *\pi_{27}(\varphi_N) \end{aligned} \tag{4.44}$$

Note that in the case when $b^1(X) = 0$ (which in particular is true when $Hol(g) = G_2$), since $H^3_7 = H^1$, the H^3_7 component of $H^3(X, \mathbb{R})$ is empty. Therefore, the second term in (4.44) vanishes, and we find that the signature of this metric is Lorentzian - $(1, b_3 - 1)$. Up to a constant factor, this definition of the moduli space metric has been been used in mathematical literature - in particular by Hitchin in [25] and Karigiannis and Leung in [31]. However in physics literature, in particular by Beasley and Witten in [7] and by Gutowski and Papadopoulos in [22], the potential K given by

$$K = -3 \log V \tag{4.45}$$

has been used instead.

The motivation for using this modified potential is twofold. Firstly, this is more in line with the logarithmic Kähler potentials on Calabi-Yau moduli spaces. Secondly, and perhaps most importantly is that the metric that arises from this potential appears as the target space metric of the effective theory in four dimensions when the action for the 11-dimensional supergravity is reduced to four dimensions on a G_2 manifold. We will hence define the moduli space metric G_{MN} as

$$G_{MN} = \frac{\partial^2 K}{\partial s^M \partial s^N}.$$

Using the definition of K and (4.44), we get

$$\begin{aligned} \frac{\partial^2 K}{\partial s^M \partial s^N} &= \frac{1}{V}\left(\int \pi_1(\varphi_M) \wedge *\pi_1(\varphi_N) - \int \pi_7(\varphi_M) \wedge *\pi_7(\varphi_N)\right. \\ &\quad \left. + \int \pi_{27}(\varphi_M) \wedge *\pi_{27}(\varphi_N)\right) \end{aligned} \tag{4.46}$$

In this case, if $b^1(X) = 0$, we get

$$G_{MN} = \frac{1}{V}\int_X \phi_M \wedge *\phi_N. \tag{4.47}$$

This metric is then in fact Riemannian. In the physics setting, apart from the G_2 3-form, there is another 3-form C and when the 11-dimensional supergravity action is reduced to four dimensions, the parameters of φ and C naturally combine to give a complexification of the G_2 moduli space. The extension of the metric G_{MN} to this complex space is then Kähler [7, 20, 22]. However since the metric on the complexified space does not depend on C, there is not much difference in treating the moduli space as a complexified Kähler manifold or a real Hessian manifold. Here we will treat $\mathcal{M}$ as a real Hessian manifold.

Now that we have fixed a metric on $\mathcal{M}$, we can proceed various other geometrical quantities. For this we will need to use higher derivatives of ψ. In what follows we will assume that $b^1(X) = 0$, so that there no harmonic forms in H^3_7. Let us introduce local special coordinates on $\mathcal{M}$. Let $\phi_0 = a\varphi$ and $\phi_\mu \in \Lambda^3_{27}$ for $\mu = 1, ..., b^3_{27}$, so that s^0 defines directions parallel to φ and s^μ define directions in H^3_{27}. Then, from the deformations of ψ in Section 4.1, we can extract the higher derivatives of ψ in these directions:

$$\partial_0\partial_0\psi = \frac{4}{9}a^2\psi \qquad \partial_0\partial_0\partial_0\psi = -\frac{8}{27}a^3\psi \tag{4.48a}$$

$$\partial_0\partial_\mu\psi = -\frac{1}{3}a * \phi_\mu \qquad \partial_0\partial_0\partial_\mu\psi = \frac{2}{9}a^2 * \phi_\mu \tag{4.48b}$$

$$\partial_\mu\partial_\nu\psi = -\frac{2}{189}\operatorname{Tr}(h_\mu h_\nu)\psi + \frac{1}{3} * \mathrm{i}_\varphi\left((\varphi h_\mu h_\nu \varphi)_0\right) \tag{4.48c}$$

$$\partial_0\partial_\mu\partial_\nu\psi = \frac{4}{567}a\operatorname{Tr}(h_\mu h_\nu)\psi - \frac{2}{9}a * \mathrm{i}_\varphi\left((\varphi h_\mu h_\nu \varphi)_0\right) \tag{4.48d}$$

$$\begin{aligned} \partial_\mu\partial_\nu\partial_\kappa\psi &= -\frac{5}{18}\operatorname{Tr}(h_\mu h_\nu) * \phi_\kappa + \frac{1}{3} * \mathrm{i}_\varphi\left((h_\mu h_\nu h_\kappa)_0\right) \\ &\quad -\frac{1}{6} * \mathrm{i}_\varphi\left((\psi h_\mu h_\nu h_\kappa \psi)_0\right) - \frac{4}{567}(\varphi h_\mu h_\nu h_\kappa \varphi)\psi \end{aligned} \tag{4.48e}$$

where h_μ,h_ν and h_κ are traceless symmetric matrices corresponding to the 3-forms ϕ_μ,φ_ν and ϕ_κ, respectively. On a Hessian manifold, there is a natural symmetric 3-tensor given by the derivative of the metric, or equivalently the third derivative of the Hessian potential. We will denote this tensor A_{MNP}. By analogy with similar quantities on Calabi-Yau moduli spaces, this tensors is called the *Yukawa coupling*. Using these expressions, following [20] we can now write down all the components of A_{MNR}:

$$A_{000} = -14a^3 \tag{4.49a}$$
$$A_{00\mu} = 0 \tag{4.49b}$$
$$A_{0\mu\nu} = -\frac{2a}{V}\int \phi_\mu \wedge *\phi_\nu = -2aG_{\mu\nu} \tag{4.49c}$$
$$A_{\mu\nu\rho} = -\frac{2}{27V}\int (\varphi h_\mu h_\nu h_\rho \varphi)\, dV \tag{4.49d}$$

The full Riemann curvature on a Hessian manifold is then defined by

$$\mathcal{R}^M{}_{NPQ} = \frac{1}{4}\left(A^M{}_{QR}A^R{}_{NP} - A^M_{PR}A^R_{NQ}\right). \tag{4.50}$$

Note that since the fourth derivative of K is fully symmetric, the fourth derivative terms vanish here. However, we can also define the *Hessian curvature* tensor by

$$\mathcal{Q}_{KLMN} = \partial_M\partial_N\partial_L\partial_K K - A_{KMR}A^R{}_{LN}. \tag{4.51}$$

This tensor is the equivalent of the Kähler curvature, and carries more information than the actual Riemann tensor (4.50). The Riemann curvature tensor is obtained from $\mathcal{Q}$ by

$$\mathcal{R}_{MNPQ} = \frac{1}{2}\left(\mathcal{Q}_{MNPQ} - \mathcal{Q}_{NMPQ}\right). \tag{4.52}$$

From (4.48), we can calculate the fourth derivatives of K, and hence get all the components of $\mathcal{Q}$:

Theorem 19 (Grigorian and Yau, [20]) *The components of the Hessian curvature tensor $\mathcal{Q}$ corresponding to the metric (4.47) on the local moduli space of torsion-free G_2 structures are given by:*

$$\mathcal{Q}_{0000} = 14a^4 \tag{4.53a}$$
$$\mathcal{Q}_{000\mu} = 0 \tag{4.53b}$$
$$\mathcal{Q}_{00\mu\nu} = 2a^2G_{\mu\nu} \tag{4.53c}$$
$$\mathcal{Q}_{0\mu\nu\rho} = -A_{\mu\nu\rho}a \tag{4.53d}$$
$$\mathcal{Q}_{\kappa\mu\nu\rho} = \frac{1}{3}\left(G_{\mu\nu}G_{\kappa\rho} + G_{\mu\kappa}G_{\nu\rho} - \frac{5}{7}G_{\mu\rho}G_{\kappa\nu}\right) - G^{\tau\sigma}A_{\mu\tau\rho}A_{\kappa\nu\sigma} + \frac{1}{V}\int\left(-\frac{2}{27}\operatorname{Tr}(h_\kappa h_\mu h_\nu h_\rho) + \frac{1}{27}(\psi h_\kappa h_\mu h_\nu h_\rho\psi) + \frac{5}{81}\operatorname{Tr}\left(h_{(\kappa}h_\mu\right)\operatorname{Tr}\left(h_\nu h_{\rho)}\right)\right)\text{vol} \tag{4.53e}$$

Let us look at more detail at the expression for $A_{\mu\nu\rho}$. If we define $h^a_\mu = h_\mu{}^a{}_m dx^m$, then we get

$$A_{\mu\nu\rho} = -\frac{4}{9V}\int \varphi_{abc}h^a_\mu \wedge h^b_\nu \wedge h^c_\rho \wedge \psi. \tag{4.54}$$

Expressions for the G_2 Yukawa coupling has been derived by different authors - in particular by Lee and Leung, [33], de Boer, Naqvi and Shomer [16], and Karigiannis [30]. Similarly, we can

Quantity	G_2 moduli in Λ^3_{27}	Complex structure moduli
Form	φ, ψ	Ω
Deformation space	H^3_{27}	$H^{(2,1)}$
Metric deformation	$\frac{2}{3}h_{\mu\nu}$	$\delta g_{\bar{\mu}\bar{\nu}}$
Form deformation	$\chi_{abc} = h^d_{[a}\varphi_{bc]d}$	$\chi_{\alpha\beta\bar{\gamma}} = -\frac{1}{2}\Omega_{\alpha\beta}{}^{\delta}\delta g_{\bar{\gamma}\bar{\delta}}$
Kähler potential	$K = -3\log\left(\int\varphi\wedge\psi\right)$	$K = -\log\left(i\int\Omega\wedge\bar{\Omega}\right)$
Moduli space metric	$G_{\mu\nu} = \frac{1}{V}\int\phi_M\wedge *\phi_N$	$G_{\mu\bar{\nu}} = -\frac{\int\chi_\mu\wedge\bar{\chi}_{\bar{\nu}}}{\int\Omega\wedge\bar{\Omega}}$
Yukawa coupling	$A_{\mu\nu\rho} = -\frac{4}{9V}\int\varphi_{abc}h^a_\mu\wedge h^b_\nu\wedge h^c_\rho\wedge\psi$	$\kappa_{\mu\nu\rho} = -\int\Omega_{\alpha\beta\gamma}\chi^\alpha_\mu\wedge\chi^\beta_\nu\wedge\chi^\gamma_\rho\wedge\Omega$
Curvature	$\mathcal{Q}_{\kappa\mu\nu\rho}$ as in (4.55)	$\mathcal{R}_{\mu\bar{\nu}\rho\bar{\sigma}} = G_{\mu\bar{\nu}}G_{\rho\bar{\sigma}} + G_{\mu\bar{\sigma}}G_{\rho\bar{\nu}}$ $-e^{2K_C}\kappa_{\mu\nu}{}^{\bar{\tau}}\kappa_{\bar{\nu}\bar{\sigma}\bar{\tau}}$

Figure 5: Comparison of G_2 moduli space and Calabi-Yau complex structure moduli space

rewrite (4.53e) as

$$\begin{aligned}\mathcal{Q}_{\kappa\mu\nu\rho} &= \frac{1}{3}\left(G_{\mu\nu}G_{\kappa\rho} + G_{\mu\kappa}G_{\nu\rho} - \frac{5}{7}G_{\mu\rho}G_{\kappa\nu}\right) - G^{\tau\sigma}A_{\mu\tau\rho}A_{\kappa\nu\sigma} \\ &+\frac{8}{9}\frac{1}{V}\int\psi_{abcd}h^a_\kappa\wedge h^b_\mu\wedge h^c_\nu\wedge h^d_\rho\wedge\varphi \\ &+\frac{1}{81}\frac{1}{V}\int\left(5\,\mathrm{Tr}\left(h_{(\kappa}h_\mu\right)\mathrm{Tr}\left(h_\nu h_{\rho)}\right) - 6\,\mathrm{Tr}\left(h_\kappa h_\mu h_\nu h_\rho\right)\right)\mathrm{vol}\end{aligned} \tag{4.55}$$

As we have mentioned previously, by complexifying the G_2 moduli space, it is possible to turn the Hessian structure into a Kähler structure. Similarly, the Hessian curvature $\mathcal{Q}$ becomes Kähler curvature. On Calabi-Yau manifolds, the complex structure moduli space is naturally a complex manifold, and admits a Kähler structure, while the Kähler structure moduli space is natural a Hessian manifold, but can be complexified to become Kähler itself. We compare the various quantities on G_2 moduli space and on the Calabi-Yau complex structure moduli space in Figure 5. We can see that there are a number of similarities. This leads to a speculation that perhaps the G_2 moduli space possesses more structures than it is currently known. One of the key features of Calabi-Yau moduli spaces is the *special geometry*, that is, both have a line bundle whose first Chern class coincides with the Kähler class [18, 35]. From physics point of view, special geometry relates to the effective theory having $\mathcal{N}=2$ supersymmetry. M-theory compactified on G_2 manifolds only gives $\mathcal{N}=1$ supersymmetry, so from this point of view it is perhaps unlikely that the (complexified) G_2 moduli space would admit precisely this structure. Moreover, it was shown by Alekseevsky and Cortés in [4] that a so-called *special real* structure on a Hessian manifold corresponds to special Kähler structure on the tangent bundle. A special real manifold is a Hessian manifold on which the cubic form DG (with D being the flat connection, and G the Hessian metric) is parallel with respect to D. In our terms, this would mean that the derivative of the Yukawa coupling A vanishes. This is a rather strong condition which is not necessarily fulfilled in our case. So perhaps instead there is some intermediate structure that could be defined on the G_2 moduli space or its complexification.

5 Concluding remarks

In this paper we have reviewed the developments in the study of G_2 moduli spaces. Currently only the local picture of the moduli space is known, so in the future it is natural to try and obtain at least some information on the global structure of the G_2 moduli space. On Calabi-Yau manifolds, the extension to the global moduli space was originally done by Tian and Todorov [38, 39]. We have seen that there are a number of similarities in the local structure of Calabi-Yau moduli spaces and G_2 moduli spaces, so it is feasible that it could also be possible to derive similar global properties of G_2 moduli spaces. However torsion-free G_2 structures are very non-linear in some aspects - in particular, the metric depends non-linearly on φ and hence the differential equation $\nabla\varphi = 0$ for a torsion-free structure is also non-linear. Therefore, it is not clear how to extend infinitesimal deformations of a G_2 structure to large deformations, apart from considering deformations order by order. However even such expansions quickly get very complicated.

Another possible topic for study would be to further develop approaches to mirror symmetry on G_2 holonomy manifolds [21]. One possible direction for further research is to look at G_2 manifolds in a slightly different way. Suppose we have type IIA superstrings on a non-compact Calabi-Yau 3-fold with a special Lagrangian submanifold which is wrapped by a $D6$ brane which also fills M_4. Then, as explained in [3], from the M-theory perspective this looks like a S^1 bundle over the Calabi-Yau which is degenerate over the special Lagrangian submanifold, but this 7-manifold is still a G_2 manifold. The moduli space of this manifold will be then determined by the Calabi-Yau moduli and the special Lagrangian moduli. This possibly could provide more information about mirror symmetry on Calabi-Yau manifolds [37].

One more direction is to look at G_2 manifolds with singularities. So far in this work we have considered only smooth G_2 manifolds, however, from a physical point of view, G_2 manifolds with singularities are even more interesting, as they yield more realistic matter content [1]. Also, the moduli spaces which we studied are for manifolds with fixed topology. By allowing topological transitions through singularities [15], it may be possible to find some relations between the different moduli spaces. Understanding these questions would improve our grasp of both the geometry and physics of G_2 moduli spaces and the interplay between them.

References

[1] B. Acharya and E. Witten, *Chiral fermions from manifolds of G(2) holonomy*, hep-th/0109152.

[2] B. S. Acharya and S. Gukov, *M theory and Singularities of Exceptional Holonomy Manifolds*, *Phys. Rept.* **392** (2004) 121–189 [hep-th/0409191].

[3] M. Aganagic, A. Klemm and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, *Z. Naturforsch.* **A57** (2002) 1–28 [hep-th/0105045].

[4] D. V. Alekseevsky and V. Cortes, *Geometric construction of the r-map: from affine special real to special k
"ahler manifolds*, 0811.1658.

[5] W. Ambrose and I. M. Singer, *A theorem on holonomy*, *Trans. Am. Math. Soc.* **75** (1953) 428–443.

[6] J. Baez, *The Octonions*, *Bull. Amer. Math. Soc. (N.S.)* **39** (2002) 145–205.

[7] C. Beasley and E. Witten, *A note on fluxes and superpotentials in M-theory compactifications on manifolds of G(2) holonomy*, *JHEP* **07** (2002) [hep-th/0203061].

[8] K. Becker, M. Becker and J. H. Schwarz, *String theory and M-theory: A modern introduction.* Cambridge University Press, 2007.

[9] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, *Bull. Soc. Math. France* **83** (1955).

[10] E. Bonan, *Sur les variétés riemanniennes à groupe d'holonomie g_2 our spin(7)*, *C. R. Acad. Sci. Paris* **262** (1966) 127–129.

[11] R. Bryant and S. Salamon, *On construction of some complete metrics with exceptional holonomy*, *Duke Math. J.* **58** (1989) 829–850.

[12] R. L. Bryant, *Metrics with exceptional holonomy*, *Ann. of Math. (2)* **126** (1987), no. 3 525–576.

[13] R. L. Bryant, *Some remarks on G_2-structures*, math/0305124.

[14] P. Candelas and X. de la Ossa, *Moduli space of Calabi-Yau manifolds*, *Nucl. Phys.* **B355** (1991) 455–481.

[15] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, *M-theory conifolds*, *Phys. Rev. Lett.* **88** (2002) 121602 [hep-th/0112098].

[16] J. de Boer, A. Naqvi and A. Shomer, *The topological G(2) string*, hep-th/0506211.

[17] M. Fernández and A. Gray, *Riemannian manifolds with structure group G_2*, *Ann. Mat. Pura Appl. (4)* **132** (1982) 19–45 (1983).

[18] D. S. Freed, *Special Kaehler manifolds*, *Commun. Math. Phys.* **203** (1999) 31–52 [hep-th/9712042].

[19] S. Grigorian, *Betti numbers of a class of barely G2 manifolds*, 0909.4681.

[20] S. Grigorian and S.-T. Yau, *Local geometry of the G2 moduli space*, *Commun. Math. Phys.* **287** (2009) 459–488 [0802.0723].

[21] S. Gukov, S.-T. Yau and E. Zaslow, *Duality and fibrations on G(2) manifolds*, hep-th/0203217.

[22] J. Gutowski and G. Papadopoulos, *Moduli spaces and brane solitons for M theory compactifications on holonomy G(2) manifolds*, *Nucl. Phys.* **B615** (2001) 237–265 [hep-th/0104105].

[23] F. R. Harvey, *Spinors and Calibrations.* Academic Press, 1990.

[24] J. A. Harvey and G. W. Moore, *Superpotentials and membrane instantons*, hep-th/9907026.

[25] N. J. Hitchin, *The geometry of three-forms in six and seven dimensions*, math/0010054.

[26] K. Hori *et. al.*, *Mirror symmetry.* AMS - Providence, USA, 2003.

[27] D. D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2. I, II*, *J. Differential Geom.* **43** (1996), no. 2 291–328, 329–375.

[28] D. D. Joyce, *Compact manifolds with special holonomy.* Oxford Mathematical Monographs. Oxford University Press, 2000.

[29] S. Karigiannis, *Deformations of G_2 and Spin(7) Structures on Manifolds*, *Canadian Journal of Mathematics* **57** (2005) 1012 [math/0301218].

[30] S. Karigiannis, *Geometric Flows on Manifolds with G_2 Structure, I*, math/0702077.

[31] S. Karigiannis and N. C. Leung, *Hodge theory for G2-manifolds: Intermediate Jacobians and Abel-Jacobi maps*, 0709.2987.

[32] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, math/0012189.

[33] J.-H. Lee and N. C. Leung, *Geometric structures on G(2) and Spin(7)-manifolds*, math/0202045.

[34] H. Shima, *The geometry of Hessian structures.* World Scientific Publishing - Hackensack, NJ, 2007.

[35] A. Strominger, *Special geometry*, *Commun. Math. Phys.* **133** (1990) 163–180.

[36] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, *Nucl. Phys.* **B479** (1996) 243–259 [hep-th/9606040].

[37] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, *Nucl. Phys.* **B479** (1996) 243–259 [hep-th/9606040].

[38] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, in *Mathematical aspects of string theory (San Diego, Calif., 1986)*, vol. 1 of *Adv. Ser. Math. Phys.*, pp. 629–646. World Sci. Publishing, 1987.

[39] A. Todorov, *The Weil-Petersson geometry of the moduli space of SU* $(n \geq 3)$ *(Calabi-Yau) manifolds I*, *Commun. Math. Phys.* **126** (1989) 325–346.

[40] P. K. Townsend, *The eleven-dimensional supermembrane revisited*, *Phys. Lett.* **B 350** (1995) 184–187 [hep-th/9501068].

[41] P. C. West, *Introduction to supersymmetry and supergravity.* World Scientific Publishing - Singapore, 1990.

[42] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85–126 [hep-th/9503124].