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On a generalization of Jentzsch's theorem

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Abstract

Let E be a compact subset of $\mathbb{C}$ with connected, regular complement $\Omega = \overline{\mathbb{C}} \setminus E$ and let $G(z)$ denote Green's function of Ω with pole at ∞. For a sequence $(p_n)_{n\in\Lambda}$ of polynomials with $\deg p_n = n$, we investigate the value-distribution of p_n in a neighbourhood U of a boundary point z_0 of E if $G(z)$ is an exact harmonic majorant of the subharmonic functions

$$\frac{1}{n}\log|p_n(z)|, \quad n \in \Lambda$$

in $\overline{\mathbb{C}} \setminus E$. The result holds for partial sums of power series, best polynomial approximations, maximally convergent polynomials and can be extended to rational functions with a bounded number of poles.

1. Introduction

The classical theorem of Jentzsch [1] concerns the limiting behaviour of the zeros of the partial sums of a power series. More precisely, if

$$s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu, \quad n \in \mathbb{N}_0,$$

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are the partial sums of a power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence 1, then Jentzsch has proved that each point on the circle of convergence $C_1 := \{z \in \mathbb{C} : |z| = 1\}$ is a limit point of zeros of the polynomials $s_n(z)$, $n = 0, 1, 2, \ldots$. Moreover, substituting the series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ by

$$\sum_{\nu=0}^{\infty} a_\nu z^\nu - \alpha, \quad \alpha \in \mathbb{C}.$$

Jentzsch's theorem immediately yields that any point of C_1 is also a limit point of α-points.

Luh [2] has given a new interpretation of Jentzsch's result: If $z_0 \in C_1$ and $\delta > 0$, then there exist infinitely many $n \in \mathbb{N}$ such that the image domains $s_n(B_\delta(z_0))$, where $B_\delta(z_0) := \{z : |z - z_0| < \delta\}$, contain the origin and rather big disks $B_{\rho_n}(0)$.

Theorem A (*[2]*). *Let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence 1. Suppose that $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers with*

$$\limsup_{n\to\infty} \rho_n^{1/n} \leq 1$$

and let $z_0 \in C_1$, $\delta > 0$ arbitrary. Then there exist infinitely many $n \in \mathbb{N}$ such that the image domains $s_n(B_\delta(z_0))$ contain the disks $B_{\rho_n}(0)$.

Since the partial sums are best least-square approximants to the power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ on any circle $C_r := \{z \in \mathbb{C} : |z| = r\}$, $r < 1$, it is natural to ask if other sequences of approximating polynomials possess the Jentzsch property. Walsh [3] studied this question for polynomials converging maximally to a function $f(z)$ analytic on a compact set $E \subset \mathbb{C}$. Blatt and Saff [4] considered the case of best approximating polynomials with respect to the maximum norm. Moreover, in recent years the focus of investigations turned to the limiting distribution of the zeros in the sense of Szegő [5] (cf. [6]). In this context, Lorentz [7] was interested in sharpening discrepancy results of Kadec [8] for the distribution of alternation points in polynomial Chebyshev approximation.

In this paper, we present a result of the type described in Theorem A that includes polynomials of best approximation, maximally convergent polynomials or Faber expansions and rational best approximations.

2. Main result

Let E be a compact set in $\mathbb{C}$ that has *positive* logarithmic capacity cap E. Moreover, we assume that the complement $\Omega = \overline{\mathbb{C}} \setminus E$ is connected and *regular* in the sense that Ω has a Green's function with pole at ∞, i. e.

(a) $G(z)$ is harmonic in $\Omega \setminus \{\infty\}$,
(b) $\lim_{z\to\infty}(G(z) - \log|z|) = -\log \operatorname{cap} E$,
(c) $G(z) \to 0$ as $z \to \xi \in \partial E = \partial\Omega$ $(z \in \Omega)$.

Hence, $G(z)$ can be continuously extended to $\overline{\Omega}$ with $G(\xi) = 0$ for $\xi \in \partial E = \partial\Omega$.

We denote by $\mathcal{P}_n$ the set of algebraic polynomials of degree at most n and by

$$\|f\|_E = \max_{z\in E} |f(z)|$$

the uniform norm for continuous f on E.

Let Λ be a subsequence of $\mathbb{N}$ and let $(p_n)_{n\in\Lambda}$ be a sequence of polynomials with $p_n \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$. Then Bernstein–Walsh's lemma implies

$$\frac{1}{n}\log|p_n(z)| - G(z) \le \frac{1}{n}\log\|p_n\|_E \tag{2.1}$$

for $z \in \Omega$ (cf. [9], lemma in Section 4.6). The inequality (2.1) also holds for $z \in \partial E$.

Let E° denote the interior of E. If $(p_n)_{n\in\Lambda}$ is bounded on E, then Montel's theorem tells us that for any connected component E' of E° there exists a subsequence $(p_n)_{n\in\Lambda'}(\Lambda' = \Lambda'(E') \subset \Lambda)$ that converges uniformly on any compact set of E'. In the following theorem we assume that $(p_n)_{n\in\Lambda}$ converges locally uniformly on any connected component of E° to a function f which is consequently analytic on E°.

In polynomial best approximation, the generalization of Jentzsch's theorem is only proved for boundary points $z_0 \in \partial E \setminus \partial S$ where

$$S := \{x \in E^\circ : f(z) \equiv 0 \text{ in a neighbourhood of } x\}$$

(cf. [4], Theorem 2.2). Such exceptional points may also occur if Theorem A is extended. This possible exceptional point set can be described by the limit function f of $(p_n)_{n\in\Lambda}$ on E°. For this reason, we set T as the union of all connected components of E° on which the function f is constant, i. e.

$$T := \{z \in E^\circ : f \text{ is constant in some neighbourhood of } z\}. \tag{2.2}$$

Then we can formulate our main result in the following theorem.

Theorem 1. *Let E be compact in $\mathbb{C}$ with connected, regular complement, $\Lambda \subset \mathbb{N}$ and let $(p_n)_{n\in\Lambda}$ be a sequence of polynomials with $n = \deg p_n$ that converges locally uniformly to the analytic function f in E°. Moreover, we assume that the sequence $(p_n)_{n\in\Lambda}$ satisfies the following conditions:*

(1) $\limsup_{n\in\Lambda, n\to\infty} \frac{1}{n}\log\|p_n\|_E \le 0$,
(2) *there exists a compact set $S \subset \Omega$ with*

$$\liminf_{n\in\Lambda, n\to\infty} \max_{z\in S}\left[\frac{1}{n}\log|p_n(z)| - G(z)\right] \ge 0.$$

If $(\rho_n)_{n\in\Lambda}$ is a sequence of positive numbers with

$$\limsup_{n\in\Lambda, n\to\infty} \rho_n^{1/n} \le 1, \tag{2.3}$$

then for any point z_0 in the closure of $\partial E \setminus \partial T$ and any neighbourhood U of z_0 there exists $n_0 = n_0(U)$ such that for all $n \ge n_0$, $n \in \Lambda$,

$$p_n(U) \supset B_{\rho_n}(0).$$

We remark that by Bernstein–Walsh's lemma, (1) and (2) are equivalent to:

(1′) $\lim_{n\in\Lambda, n\to\infty} \frac{1}{n}\log\|p_n\|_E = 0$

and

(2′) $\lim_{n\in\Lambda, n\to\infty} \max_{z\in S}\left[\frac{1}{n}\log|p_n(z)| - G(z)\right] = 0.$

Moreover, in the notation of Walsh [10] the condition (2′) is referred to as: "*The sequence* $(1/n)\log|p_n(z)|$, $n \in \Lambda$, *has Green's function* $G(z)$ *as exact harmonic majorant*".

It follows from the Bernstein–Walsh inequality (2.1) that in Theorem 1 the condition $\limsup_{n\in\Lambda,n\to\infty} \rho_n^{1/n} \le 1$ cannot be replaced by $\limsup_{n\in\Lambda,n\to\infty} \rho_n^{1/n} > 1$.

For the proof of Theorem 1 we use the following modification of condition (2).

Lemma 1. *If the conditions (1) and (2) of Theorem 1 hold then the inequality (2) is true for any continuum* $\widetilde{S} \subset \Omega$.

For a proof of this lemma we refer to Remark 1.3 in section 2.1 of [6], resp. Grothmann [11].

Proof of Theorem 1. Let $z_0 \in \partial E \setminus \partial T$ and assume, to the contrary, that there exist a bounded neighbourhood U of z_0, a subsequence $\Lambda_1 \subset \Lambda$ and a sequence $(w_n)_{n\in\Lambda_1}$ with $|w_n| < \rho_n$ such that

$$w_n \notin p_n(U) \quad \text{for } n \in \Lambda_1.$$

Since $T \subset E^\circ$ and $z_0 \in \partial E \setminus \partial T$, we can choose U such that

$$U \cap T = \emptyset. \tag{2.4}$$

For $z \in U$ and $n \in \Lambda_1$ we define the single-valued analytic function

$$g_n(z) := (p_n(z) - w_n)^{1/n} = \exp\left(\frac{1}{n}\log(p_n(z) - w_n)\right) \tag{2.5}$$

by taking the branch of $\log(p_n(z) - w_n)$ for which

$$-\pi < \operatorname{Im}\log(p_n(z_0) - w_n) \le \pi.$$

By the lemma of Bernstein–Walsh we obtain

$$|g_n(z)| \le (\|p_n\|_E + |w_n|)^{1/n} \mathrm{e}^{G(z)}$$

for $z \in U \cap \overline{\Omega}$. Then we get with (1) and (2.3) that the functions $(g_n)_{n\in\Lambda_1}$ are uniformly bounded in U. Hence, there exists a subsequence of $(g_n)_{n\in\Lambda_1}$, say $(g_n)_{n\in\Lambda_2}$, $\Lambda_2 \subset \Lambda_1$, that converges locally uniformly to an analytic function $g(z)$ in U.

Let us fix $\widetilde{z} \in U \cap \Omega$. Then

$$\kappa := \frac{1}{4} G(\widetilde{z}) > 0. \tag{2.6}$$

We choose a continuum S in $U \cap \Omega$ with $\widetilde{z} \in S$, $\min_{z\in S} G(z) \ge 2\kappa$ and

$$|g(z) - g(\widetilde{z})| \le (\mathrm{e}^\kappa - 1)/2 \quad \text{for all } z \in S. \tag{2.7}$$

Because of Lemma 1, there exists $n_1 \in \mathbb{N}$ such that

$$\max_{z\in S}\left[\frac{1}{n}\log|p_n(z)| - G(z)\right] \ge -\kappa$$

for $n \in \Lambda_2$, $n \ge n_1$, and therefore

$$\max_{z\in S}\frac{1}{n}\log|p_n(z)| \ge \min_{z\in S} G(z) - \kappa \ge \kappa \tag{2.8}$$

for all $n \in \Lambda_2$, $n \ge n_1$. Fix $\xi_n \in S$ with

$$\log|p_n(\xi_n)| = \max_{z\in S}\log|p_n(z)|.$$

Then by (2.8)

$$|p_n(\xi_n)| \geq e^{n\kappa} \quad \text{for } n \in \Lambda_2, n \geq n_1 \tag{2.9}$$

and (2.3) implies

$$\lim_{n\in\Lambda_2, n\to\infty} \left|\frac{w_n}{p_n(\xi_n)}\right| = 0. \tag{2.10}$$

Since S is compact, we can choose a subsequence $\Lambda_3 \subset \Lambda_2$ such that

$$\lim_{n\in\Lambda_3, n\to\infty} \xi_n = \xi \in S.$$

The functions $(g_n)_{n\in\Lambda_3}$ are equicontinuous on S, hence

$$\begin{aligned}|g(\xi)| &= \lim_{n\in\Lambda_3, n\to\infty} |g_n(\xi_n)| = \lim_{n\in\Lambda_3, n\to\infty} |p_n(\xi_n)|^{1/n} \left|1 - \frac{w_n}{p_n(\xi_n)}\right|^{1/n} \\ &= \lim_{n\in\Lambda_3, n\to\infty} |p_n(\xi_n)|^{1/n} \geq e^{\kappa} > 1.\end{aligned}$$

Because of (2.7), we obtain

$$\begin{aligned}|g(\widetilde{z})| &\geq |g(\xi)| - |g(\xi) - g(\widetilde{z})| \\ &\geq e^{\kappa} - \frac{e^{\kappa} - 1}{2} = \frac{1}{2}(e^{\kappa} + 1) > 1.\end{aligned}$$

Hence, for all $z \in U \cap \Omega$ we have

$$|g(z)| > 1. \tag{2.11}$$

On the other hand, by (1) and (2.3) we obtain for all $z \in E \cap U$

$$\begin{aligned}|g(z)| &= \lim_{n\in\Lambda_2, n\to\infty} |g_n(z)| \\ &\leq \limsup_{n\in\Lambda_2, n\to\infty} [|p_n(z)| + |w_n|]^{1/n} \leq 1.\end{aligned} \tag{2.12}$$

Since $E \cap U \neq \emptyset$, it follows that the function $g(z)$ is not constant in U and so the open set

$$V := \{z \in U : |g(z)| < 1\}$$

is nonempty. Because of (2.11), $V \subset E^\circ$.

Fix $z \in V$. Then there exist $\varepsilon > 0$ and $n_2 \in \mathbb{N}$ such that

$$|g_n(z)| < 1 - \varepsilon \quad \text{for all } n \in \Lambda_2, n \geq n_2.$$

For such n we have

$$|g_n(z)|^n = |p_n(z) - w_n| < (1 - \varepsilon)^n$$

and consequently

$$0 = \lim_{n\in\Lambda_2, n\to\infty} |p_n(z) - w_n| = \lim_{n\in\Lambda_2, n\to\infty} |f(z) - w_n|$$

or for all $z \in V$,

$$f(z) = \lim_{n\in\Lambda_2, n\to\infty} w_n,$$

i. e. f is constant on V. Since $V \subset E^\circ$, we have $V \subset T$, which contradicts (2.4). The theorem is proved for all $z \in \partial E \setminus \partial T$, and consequently for all points z belonging to the closure of $\partial E \setminus \partial T$. $\square$

Remark. If we introduce the subharmonic function

$$h_n(z) := \frac{1}{n} \log |p_n(z)| - G(z) \tag{2.13}$$

for $z \in \Omega$, where $h_n(\infty) := \lim_{z\to\infty} h_n(z)$, then the lemma of Bernstein–Walsh is just the maximum principle:

$$h_n(z) \le \max_{z\in\partial E} h_n(z).$$

Moreover, we can summarize the conditions (1) and (2) (resp. (1′) and (2′)) of Theorem 1 as follows:

There exists a compact set $S \subset \Omega$ with

$$0 = \lim_{n\in\Lambda, n\to\infty} \max_{z\in\partial E} h_n(z) = \lim_{n\in\Lambda, n\to\infty} \max_{z\in S} h_n(z). \tag{2.14}$$

3. Applications

3.1. Faber series

Let E be a continuum with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ and let $\Phi : \Omega \to \Delta$ be the Riemann mapping function normalized by

$$\Phi(\infty) = \infty, \qquad \Phi'(\infty) = \lim_{z\to\infty} \frac{\Phi(z)}{z} > 0,$$

where $\Delta := \{z \in \overline{\mathbb{C}} : |z| > 1\}$. Then Green's function of Ω with pole at ∞ is given by

$$G(z) = \log |\Phi(z)|.$$

Suppose f is analytic on E, but not entire. Then there exists a maximal $\rho > 1$ such that f has an analytic extension to

$$E_\rho = \{z \in \Omega : G(z) < \log \rho\} \cup E, \tag{3.1}$$

and f can be expanded into a series of *Faber polynomials*

$$f(z) = \sum_{n=0}^{\infty} a_n \Phi_n(z), \tag{3.2}$$

where the Faber polynomials $\Phi_n(z)$ are defined by the generating function

$$\frac{w\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{\Phi_n(z)}{w^n}, \quad z \in E,$$

and $\Psi(w) = \Phi^{-1}(w)$, $|w| > 1$.

The Faber coefficients a_n are defined by

$$a_n = \frac{1}{2\pi i} \int_{|w|=r} f(\Psi(w)) w^{-n-1} dw, \quad 1 < r < \rho,$$

and satisfy

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{\rho}.$$

The Faber polynomial Φ_n is of degree n,

$$\Phi_n(z) = (\operatorname{cap} E)^{-n} z^n + \cdots$$

and the expansion (3.2) converges locally uniformly in E_ρ.

Let us consider the partial sums

$$p_n(z) = \sum_{\nu=0}^{n} a_\nu \Phi_\nu(z)$$

of the expansion (3.2) and define for $z \in \Omega \setminus E_\rho$

$$\widetilde{h}_n(z) := \frac{1}{n} \log |p_n(z)| - \widetilde{G}(z)$$

where $\widetilde{G}(z) = G(z) - \log \rho$ is Green's function of $\overline{\mathbb{C}} \setminus \overline{E}_\rho$ with pole at ∞, and $\widetilde{h}_n(\infty) = \lim_{z\to\infty} \widetilde{h}_n(z)$. If we choose the subsequence $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n\in\Lambda, n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{\rho}$$

then

$$\lim_{n\in\Lambda, n\to\infty} \widetilde{h}_n(\infty) = 0.$$

Hence, for $(p_n)_{n\in\Lambda}$, E replaced by $\overline{E}_\rho$ and $S = \{\infty\}$ all conditions of Theorem 1 are satisfied and we get

Corollary 1. *Let f be analytic on the continuum E with connected complement and let the maximal ρ, such that f is analytic in E_ρ, be finite. Suppose $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers with*

$$\limsup_{n\to\infty} \rho_n^{1/n} \le 1 \tag{3.3}$$

and let p_n denote the nth partial sum in the Faber expansion (3.2). Then there exists a subsequence Λ of $\mathbb{N}$ with the following property: For any boundary point z_0 of E_ρ and any neighbourhood U of z_0, there exists $n_0 = n_0(U) \in \mathbb{N}$ such that

$$p_n(U) \supset B_{\rho_n}(0) \quad \text{for } n \in \Lambda, n \ge n_0.$$

If E is the unit disk, then the Faber polynomials are just $\Phi_n(z) = z^n$ and we obtain the theorem of Jentzsch and Luh [2] resp.

3.2. Polynomials of best approximation

Let E be compact in $\mathbb{C}$ with connected and regular complement Ω. Let f be analytic in the interior E° of E and continuous on E. We denote by p_n^* the polynomial in $\mathcal{P}_n$ of best uniform approximation to f on E. On writing

$$p_n^*(z) = a_n^* z^n + \cdots$$

it is shown in [4] that

$$\limsup_{n\to\infty} |a_n^*|^{1/n} = \frac{1}{\operatorname{cap} E}$$

if and only if f is not analytic on E. Then

$$\limsup_{n\to\infty} h_n(\infty) = 0,$$

where

$$h_n(z) = \frac{1}{n}\log|p_n^*(z)| - G(z).$$

Hence, if we choose the subsequence $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n\in\Lambda, n\to\infty} |a_n^*|^{1/n} = \frac{1}{\operatorname{cap} E}, \tag{3.4}$$

and apply Theorem 1 with $S = \{\infty\}$, we get

Corollary 2. *Let E be compact with connected and regular complement, f continuous on E and analytic in E°, but not analytic on E. Let $(\rho_n)_{n\in\mathbb{N}}$ be as in (3.3) and z_0 in the closure of $\partial E \setminus \partial T$, where T is the union of all connected components of E° on which f is constant. Then for any neighbourhood U of z_0, there exists $n_0 = n_0(U) \in \mathbb{N}$ such that the polynomials p_n^* of best uniform approximation to f satisfy $p_n^*(U) \supset B_{\rho_n}(0)$ for $n \in \Lambda$, $n \ge n_0$, where Λ is defined by (3.4).*

3.3. Maximally convergent polynomials

Let E be as above in Section 3.2, f analytic on E, but not entire. As in Section 3.1 choose $\rho > 1$ maximal such that f can be extended analytically to E_ρ.

Then a sequence of polynomials $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, is said to *converge maximally* to f on E if

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} = \frac{1}{\rho}. \tag{3.5}$$

For example, polynomials p_n^* of best uniform approximation on E are maximally convergent (cf. Walsh [10]). Walsh [3] proved that a boundary point z_0 of E_ρ is a limit point of zeros of p_n if z_0 is a limit of points in E_ρ on which $f(z)$ is not zero.

Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of polynomials converging maximally to f on E. We want to apply Theorem 1 for $(p_n)_{n\in\mathbb{N}}$ with E replaced by $\overline{E}_\rho$. Therefore, we define analogously to (2.13) the functions

$$\widetilde{h}_n(z) := \frac{1}{n}\log|p_n(z)| - \widetilde{G}(z) \tag{3.6}$$

where $\widetilde{G}(z)$ is Green's function of $\widetilde{\Omega} = \overline{\mathbb{C}} \setminus \overline{E}_\rho$ with pole at ∞, i. e. $\widetilde{G}(z) = G(z) - \log\rho$. The role of S will be played by some level line Γ_r of Green's function $G(z)$,

$$\Gamma_r := \{z \in \Omega : G(z) = \log r\}, \quad r \ge 1. \tag{3.7}$$

Now, Bernstein–Walsh's lemma implies together with (3.5) that

$$\limsup_{n\to\infty} \|p_n - p_{n+1}\|_{\Gamma_r}^{1/n} \le \frac{r}{\rho} \quad \text{for all } r \ge 1. \tag{3.8}$$

Hence, the sequence $(p_n)_{n\in\mathbb{N}}$ converges locally uniformly to f on E_ρ. Moreover, using $\widetilde{h}_n(z)$ of (3.6) we have

$$0 = \lim_{n\to\infty} \max_{z\in\partial E} \widetilde{h}_n(z) \geq \limsup_{n\to\infty} \max_{z\in\Gamma_\sigma} \widetilde{h}_n(z) \quad \text{for } 1 < \rho < \sigma. \tag{3.9}$$

Lemma 2. *Let f be analytic on E and ρ maximal such that f is analytic on E_ρ, ρ finite. Then for any fixed σ with $\sigma > \rho$*

$$\limsup_{n\to\infty} \max_{z\in\Gamma_\sigma} \widetilde{h}_n(z) \geq 0. \tag{3.10}$$

Lemma 2 is essentially due to Walsh [3]. For completeness, we include a proof.

Proof. Let us assume, contrary to (3.10), that

$$\limsup_{n\to\infty} \max_{z\in\Gamma_\sigma} \widetilde{h}_n(z) < 0 \tag{3.11}$$

for some fixed σ with $\sigma > \rho$. Then,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|p_n\|_{\Gamma_\sigma} < \log \frac{\sigma}{\rho}$$

and consequently

$$b := \limsup_{n\to\infty} \frac{1}{n} \log \|p_n - p_{n+1}\|_{\Gamma_\sigma} < \log \frac{\sigma}{\rho}.$$

Define a harmonic function g on $E_\sigma \setminus E$ that tends to 0 on ∂E and to $b - \log(\sigma/\rho) < 0$ on ∂E_σ. By the maximum principle, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log |(p_n - p_{n+1})(z)| \leq g(z) + G(z) - \log \rho$$

for $z \in E_\rho \setminus E$. Since $g(z)$ is strictly negative on $E_\sigma \setminus E$, we obtain

$$\limsup_{n\to\infty} \|p_n - p_{n+1}\|_{\Gamma_\rho}^{1/n} < 1.$$

It follows that $(p_n)_{n\in\mathbb{N}}$ converges in $E_{\widetilde{r}}$ for some $\widetilde{r} > \rho$, which is impossible because of the maximality of ρ. Hence, our assumption (3.11) is false and the lemma is proved. □

Combining (3.9) with Lemma 2 and the fact that $(p_n)_{n\in\mathbb{N}}$ converges locally uniformly on E_ρ, we obtain by Theorem 1.

Corollary 3. *Let E be compact with connected regular complement, f analytic on E but not entire, ρ maximal such that f is analytic in E_ρ, $\rho > 1$, and $(p_n)_{n\in\Lambda}$ a sequence of polynomials converging maximally to f. Then there exists a subsequence $\Lambda \subset \mathbb{N}$ with the property: If $z_0 \in \Gamma_\rho$ is a boundary point of a component of E_ρ where $f(z)$ is not constant, and if U is a neighbourhood of z_0, then there exists $n_0 = n_0(U)$ such that $p_n(U) \supset B_{\rho_n}(0)$ for $n \in \Lambda$, $n \geq n_0$, where $(\rho_n)_{n\in\mathbb{N}}$ is a sequence with $\lim_{n\to\infty} \rho_n^{1/n} \leq 1$.*

4. Jentzsch's theorem for $\mathcal{R}_{n,N}$ (N fixed)

Let E be again compact with connected regular complement $\Omega = \overline{\mathbb{C}} \setminus E$ and Green's function $G(z)$. We consider the classes of rational functions

$$\mathcal{R}_{n,N} = \left\{ r_n(z) = \frac{p_n(z)}{q_{n,N}(z)} : p_n \in \mathcal{P}_n, q_{n,N} \in \mathcal{P}_N \right\} \tag{4.1}$$

where $N \in \mathbb{N}$ is fixed and $n = 0, 1, 2, \ldots$.

Theorem 2. *Let $\Lambda \subset \mathbb{N}$, $c > 0$ and $(r_n)_{n\in\Lambda}$ be a sequence of rational functions with*

$$r_n = \frac{p_n}{q_{n,N}} \in \mathcal{R}_{n,N} \setminus \mathcal{R}_{n-1,N} \quad \text{and} \quad \|q_{n,N}\|_E \le c$$

for $n \in \Lambda$. We assume that $(r_n)_{n\in\Lambda}$ converges locally uniformly in E° to the analytic function f. Moreover, the sequence $(r_n)_{n\in\Lambda}$ satisfies the following conditions:

(1) $\lim_{n\in\Lambda, n\to\infty} \frac{1}{n} \log \|r_n\|_E \le 0$,
(2) *there exists a compact set $S \subset \Omega$ with*

$$\liminf_{n\in\Lambda, n\to\infty} \max_{z\in S} \left[\frac{1}{n} \log |p_n(z)| - G(z) \right] \ge 0.$$

If $(\rho_n)_{n\in\Lambda}$ is a sequence of positive numbers with $\limsup_{n\in\Lambda,n\to\infty} \rho_n^{1/n} \le 1$, then for any point z_0 in the closure of $\partial E \setminus \partial T$ (T as in (2.2)) and any neighbourhood U of z_0 there exists $n_0 = n_0(U)$ such that

$$r_n(U) \supset B_{\rho_n}(0) \quad \text{for } n \in \Lambda, n \ge n_0.$$

Proof. Let $z_0 \in \partial E \setminus \partial T$ and assume that there exist a bounded neighbourhood of z_0, a subsequence $\Lambda_1 \subset \Lambda$ and a sequence $(w_n)_{n\in\Lambda_1}$ with $|w_n| < \rho_n$ such that

$$w_n \notin r_n(U) \quad \text{for } n \in \Lambda_1.$$

Since $\mathcal{P}_N$ is finite-dimensional and $\|q_{n,N}\|_E \le c$, we may assume that $(q_{n,N})_{n\in\Lambda_1}$ converges to $q_N \in \mathcal{P}_N$ and $\|q_N\|_E \le c$. Fix z_0 with $q_N(z_0) \ne 0$. Because of $T \subset E^\circ$ and $z_0 \in \partial E \setminus \partial T$, we can choose U and Λ_1 such that

$$U \cap T = \emptyset \tag{4.2}$$

and

$$|q_{n,N}(z)| \ge \alpha > 0 \quad \text{for } n \in \Lambda_1, z \in U. \tag{4.3}$$

Next, we follow the lines in the proof of Theorem 1:

For $z \in U$ and $n \in \Lambda_1$ we define the single-valued analytic function

$$g_n(z) := (r_n(z) - w_n)^{1/n} = \exp\left(\frac{1}{n} \log(r_n(z) - w_n) \right)$$

by fixing a branch of $\log(r_n(z) - w_n)$ at the point z_0. Using Bernstein–Walsh's lemma we get for $z \in U \cap \overline{\Omega}$

$$|g_n(z)| \le (|r_n(z)| + |w_n|)^{1/n} = \left(\left| \frac{p_n(z)}{q_{n,N}(z)} \right| + |w_n| \right)^{1/n}$$
$$\le \left(\frac{e^{nG(z)} \|p_n\|_E}{\alpha} + |w_n| \right)^{1/n}.$$

Hence, the functions $(g_n)_{n\in\Lambda_1}$ are uniformly bounded in U and we can choose a subsequence $\Lambda_2 \subset \Lambda_1$ such that $(g_n)_{n\in\Lambda_2}$ converges locally uniformly to an analytic function g in U.

Let us fix $\widetilde{z} \in U \cap \Omega$. Following the proof of Theorem 1, we choose a continuum $S \subset U \cap \Omega$ with $\widetilde{z} \in S$, and a constant $\kappa > 0$ such that

$$|g(z) - g(\widetilde{z})| \leq (e^{\kappa} - 1)/2 \quad \text{for } z \in S \tag{4.4}$$

and

$$\max_{z\in S} \frac{1}{n} \log |p_n(z)| \geq \kappa \quad \text{for } n \in \Lambda_2. \tag{4.5}$$

Fix $\xi_n \in S$ with

$$\log |p_n(\xi_n)| = \max_{z\in S} \log |p_n(z)|,$$

then

$$|p_n(\xi_n)| \geq e^{n\kappa} \tag{4.6}$$

and

$$\lim_{n\in\Lambda_2, n\to\infty} \left| \frac{w_n}{r_n(\xi_n)} \right| \leq \lim_{n\in\Lambda_2, n\to\infty} \left| \rho_n \frac{q_{n,N}(\xi_n)}{p_n(\xi_n)} \right| = 0. \tag{4.7}$$

Since S is compact, we can choose a subsequence $\Lambda_3 \subset \Lambda_2$ such that $\lim_{n\in\Lambda_3, n\to\infty} \xi_n = \xi \in S$. The functions g_n, $n \in \Lambda_3$, are equicontinuous on S, hence

$$\begin{aligned} g(\xi) &= \lim_{n\in\Lambda_3, n\to\infty} g_n(\xi_n) = \lim_{n\in\Lambda_3, n\to\infty} |r_n(\xi_n)|^{1/n} \left| 1 - \frac{w_n}{r_n(\xi_n)} \right|^{1/n} \\ &= \lim_{n\in\Lambda_3, n\to\infty} |r_n(\xi_n)|^{1/n} \geq e^{\kappa} > 1. \end{aligned}$$

Because of (4.4),

$$|g(\widetilde{z})| \geq |g(\xi)| - |g(\xi) - g(\widetilde{z})| > 1.$$

Hence, we have got

$$|g(z)| > 1 \quad \text{for } z \in U \cap \Omega. \tag{4.8}$$

On the other hand, the condition (1) implies

$$\begin{aligned} |g(z)| &= \lim_{n\in\Lambda_2, n\to\infty} g_n(z) \\ &\leq \limsup_{n\in\Lambda_2, n\to\infty} [|r_n(z)| + |w_n|]^{1/n} \leq 1 \end{aligned}$$

for $z \in E \cap U$. Since $E \cap U \neq \emptyset$, it follows that the function g is not constant in U and so the open set

$$V := \{z \in U : |g(z)| < 1\}$$

is not empty. Because of (4.8), $V \subset E^{\circ}$.

Fix $z \in V$. Then there exists $\varepsilon > 0$ and $n_2 \in \mathbb{N}$ such that

$$|g_n(z)| < 1 - \varepsilon \quad \text{for } n \in \Lambda_2, n \geq n_2,$$

and for such n

$$|g_n(z)|^n = |r_n(z) - w_n| < (1-\varepsilon)^n$$

and therefore

$$0 = \lim_{n\in\Lambda_2, n\to\infty} |r_n(z) - w_n| = \lim_{n\in\Lambda_2, n\to\infty} |f(z) - w_n|$$

or

$$f(z) = \lim_{n\in\Lambda_2, n\to\infty} w_n,$$

i. e. f is constant in V. Since $V \subset E^\circ$, we have $V \subset T$ which contradicts (4.2). Hence, the theorem is proved for all $z \in \partial E \setminus \partial T$ where $q_N(z) \neq 0$, and consequently for all points in the closure of $\partial E \setminus \partial T$. □

Finally, we want to apply Theorem 2 to the approximation of f by rational functions.

Corollary 4. *Let E and f be as in Corollary 2, $(\rho_n)_{n\in\mathbb{N}}$ a sequence with $\limsup_{n\to\infty} \rho_n^{1/n} \leq 1$. For fixed $N \in \mathbb{N}$ and $n = 0, 1, \ldots$, let $r^*_{n,N}(z)$ denote a rational function in $\mathcal{R}_{n,N}$ of best uniform approximation to f. Then there exists a subsequence $\Lambda \subset \mathbb{N}$ with the following property: For any point z_0 in the closure of $\partial E \setminus \partial T$, where T is the union of all components of E° on which f is constant, there exists $n_0 = n_0(U)$ such that*

$$r^*_{n,N}(U) \supset B_{\rho_n}(0) \quad \text{for } n \in \Lambda, n \geq n_0.$$

Proof. Choose $1 < \sigma < \infty$ such that the region E_σ (defined as in (3.1)) contains the origin. Let $s_j^{(n)}$, $j = 1, 2, \ldots, k_n$, denote the poles of $r^*_{n,N}(z)$ in E_σ and let $t_j^{(n)}$, $j = 1, 2, \ldots, l_n$ denote the poles in the complement of E_σ. We write

$$r^*_{n,N}(z) = \frac{p_n(z)}{q_{n,N}(z)} = \frac{a_n z^n + \cdots}{q_{n,N}(z)}$$

where

$$q_{n,N}(z) = \prod_{j=1}^{k_n} (z - s_j^{(n)}) \prod_{j=1}^{l_n} (1 - z/t_j^{(n)})$$

and the product is 1 if the number of factors is empty. Then Blatt, Saff, Simkani [12] (Proof of Theorem 4.1) have shown that there exists a subsequence $\Lambda = \Lambda(f) \subset \mathbb{N}$ such that

$$\lim_{n\in\Lambda, n\to\infty} |a_n|^{1/n} = \frac{1}{\operatorname{cap} E}.$$

Hence, the sequence $(r^*_{n,N})_{n\in\Lambda}$ with $r^*_{n,N} = p_n/q_{n,N}$ satisfies the conditions (1) and (2) of Theorem 2, where we choose $S = \{\infty\}$. Then the statement of Corollary 4 follows from Theorem 2. □

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