Existence of a Solution to a Vector-valued Allen-Cahn Equation with a Three Well Potential

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ABSTRACT. In this paper we prove the existence of a vector-valued solution v to

$$-\Delta v + \frac{\nabla_v W(v)}{2} = 0,$$
$$\lim_{r\to\infty} v(r\cos\theta, r\sin\theta) = c_i \quad \text{for } \theta \in (\theta_{i-1}, \theta_i),$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative function that attains its minimum 0 at $\{c_i\}_{i=1}^3$, and the angles θ_i are determined by the function W. This solution is an energy minimizer.

1. INTRODUCTION

In this paper we establish the existence of a vector-valued solution $v : \mathbb{R}^2 \to \mathbb{R}^2$ to the following elliptic problem:

$$-\Delta v + \frac{\nabla_v W(v)}{2} = 0, \tag{1.1}$$

$$\lim_{r\to\infty} v(r\cos\theta, r\sin\theta) = c_i \quad \text{for } \theta \in (\theta_{i-1}, \theta_i), \tag{1.2}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a positive function with three local minima, given by $\{c_i\}_{i=1}^3$, and the angles θ_i, with $\theta_3 = 2\pi + \theta_0$, are determined by the potential W (for a more precise description on how these angles are determined we refer the reader to definitions (1.7) and (1.8)).

In [22] an analogous result was proved by P. Sternberg when W has two minima. Later on, Bronsard, Gui and Schatzman [6] considered potentials with three minima that were equivariant under the symmetry group of the equilateral triangle. Under these conditions they proved existence of a solution to (1.1)–(1.2). The system of equations given by (1.1) was also studied in [8], but the domains considered were bounded and Neumann boundary condition was imposed. In that paper, under appropriate assumptions over the potential W, Flores, Padilla and Tonegawa established the existence of solutions that join the three minima (c_1, c_2 and c_3); however, no precise description of the triple junction was provided. Recently, potentials with four minima were studied in [11], establishing (under several assumptions over the potential W) the existence of solutions to (1.1) that connect all the four wells.

Our interest in this problem is originated in some models of three-boundary motion. Material scientists working on the theory of transition layers have found that the motion of grain boundaries is governed by its local mean curvature (see [15],[16] for example). These models naturally arise as the singular limit of the parabolic Allen-Cahn equation (see [2]). The expected relation between grain boundaries motion and the parabolic Allen-Cahn equation can be described as follows: Consider a positive potential $W : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ with a finite number of minima $\{c_i\}_{i=i}^m$. Let $u_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ be a solution to

$$\frac{du_\varepsilon}{dt} - \Delta u_\varepsilon + \frac{\nabla_v W(u_\varepsilon)}{2\varepsilon^2} = 0. \tag{1.3}$$

As $\varepsilon \to 0$ the solutions u_ε will converge almost everywhere to one of the constants c_i (see [12], [18]). For every t, this creates a partition of $\Omega = \bigcup_{i=1}^m \Omega_i(t)$, where $\Omega_i(t) = \{x \in \Omega : u_\varepsilon(x,t) \to c_i \text{ as } \varepsilon \to 0\}$. The interface between these sets corresponds to the grain boundaries, which evolve under its curvature. When $n = 2$ and $m = 3$ the solution will describe a "three-phase" boundary motion that might present "triple-points", namely points where these 3 boundaries meet. Bronsard and Reitich [7], via a formal asymptotic expansion, predicted that at points that are away from the triple junctions and close to the interface between c_i and c_j the solutions to (1.3) should be approximated by $\zeta_{ij}(d_{ij}(x,t)/\varepsilon)$, where d_{ij} is the distance function to this interface and ζ_{ij} is a solution to the equation

$$\zeta_{ij}''(\lambda) + \frac{\nabla W(\zeta_{ij}(\lambda))}{2} = 0, \tag{1.4}$$

$$\lim_{\tau \to -\infty} \zeta_{ij}(\tau) = c_i, \quad \lim_{\tau \to \infty} \zeta_{ij}(\tau) = c_j. \tag{1.5}$$

On the other hand, the analysis performed by Bronsard and Reitich also predicted the behavior of solutions to (1.3) at points where triple junctions form. More precisely, they proposed that, after rescaling at one of this points, solutions to (1.3) will be modeled after a solution to (1.1)–(1.2). However, the existence of such solution has not been established in the general case before. This is the main goal of this paper.

Based on the previous discussion, in order to match the expected behavior of solutions to (1.3) near double junctions and the one close to triple junctions, we expect that solutions to (1.1)–(1.2) satisfy an extra condition at infinity. Namely, solutions to (1.1)–(1.2) should resemble solutions to (1.4)–(1.5) near the half-lines of direction θ_i. We will implicitly impose this condition throughout the paper. Therefore, we briefly discuss the existence of solutions to (1.4)–(1.5): For potentials with two wells the existence of such curves was proved by P. Sternberg in [22]. However, the problem is more subtle when considering arbitrary three-well potentials, even if conditions analogous to the ones imposed in [22] hold. In [1] Alikakos, Betelú and Chen provided some examples of potentials where solutions to (1.4)–(1.5) did not exist for certain i, j. On the other hand, in several simple cases (such as in the symmetric case studied in [6]) the existence of such solution curves is known. Furthermore in [1], the authors established appropriate conditions under which all these solutions in fact do exist. In what follows we will assume we are in the latter case. Namely, we assume the existence of ζ_{ij} for every i and j. This and other technical assumptions on the potential W will be discussed in detail in the following section. At the moment we state the main theorem of this paper:

Theorem 1.1. *Let $W : \mathbb{R}^2 \to \mathbb{R}$ be a proper C^3 function that satisfies*

(a) *W has only three local minima c_1, c_2 and c_3 and $W(c_i) = 0$;*
(b) *The matrix $\partial^2 W(u)/\partial u_i \partial u_j$ is positive definite at $\{c_i\}_{i=1}^3$, that is, the minima are non-degenerate.*
(c) *The hessian of the function $W(u)$ (which we denote by W'') is positive semidefinite for $|u| > K$, where $K > 0$ is a fixed real number;*
(d) *There exist positive constants K_1, K_2 and m, and a number $p \geq 2$ such that*

$$K_1|u|^p \leq W(u) \leq K_2|u|^p \quad \text{for } |u| \geq m;$$

(e) *Hypothesis 2.1 holds (see the next section for a description of this hypothesis). In particular, there are solutions to (1.4)–(1.5) for every i and j.*

Define

$$\Gamma(\zeta_1, \zeta_2) = \inf\left\{ \int_0^1 W^{1/2}(\gamma(\lambda))|\gamma'(\lambda)|\, d\lambda : \gamma \in C^1([0,1], \mathbb{R}^2),\ \gamma(0) = \zeta_1 \text{ and } \gamma(1) = \zeta_2 \right\}. \tag{1.6}$$

Consider $\{\alpha_i\}_{i=1}^3 \in [0, 2\pi)$ such that

$$\frac{\sin\alpha_1}{\Gamma(c_2, c_3)} = \frac{\sin\alpha_2}{\Gamma(c_1, c_3)} = \frac{\sin\alpha_3}{\Gamma(c_1, c_2)}. \tag{1.7}$$

Then for $\theta_i \in [0, 2\pi)$ such that

$$\alpha_i = \theta_i - \theta_{i-1} \tag{1.8}$$

there is a solution v to (1.1)–(1.2). Moreover, there exists a differentiable function φ satisfying (1.2) such that for

$$\mathcal{G}(w) = \int_{\mathbb{R}^2} (|Dw|^2 + W(w) - |D\varphi|^2 - W(\varphi))\, dx,$$

we have

$$\mathcal{G}(v) = \inf\{\mathcal{G}(w) : w \in \mathcal{V}\},$$

for

$$\mathcal{V} = \left\{ w \in C^1 : \int_{\mathbb{R}^2} |Dw - D\varphi|\, dx,\ \int_{\mathbb{R}^2} |w - \varphi|\, dx < \infty \right\}.$$

We would like to remark that the function φ in Theorem 1.1 will be defined explicitly in the coming section (more specifically in Subsection 2.2) and it will capture the behavior at infinity of the solution u to (1.1)–(1.2). In the construction of this function, hypothesis (e) is required. Relaxations of this hypothesis are possible, but we will skip them in order to keep the presentation simpler. We also want to point out that, as discussed in [7], the definitions of α_i and Γ_i imply that $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$ and $\theta_3 = 2\pi + \theta_0$.

Before proceeding to the coming sections, we would like to briefly outline our proof of Theorem 1.1 and its organization through the paper. The basic idea is the following: Let B_R denote the ball of radius R and let v_R solve equation (1.1) in B_R with Dirichlet boundary condition $v_R|_{\partial B_r} = \varphi$ (the function φ is defined in equation (2.10) and captures the desired behaviour at infinity, as it is discussed in Remark 2.2 below). The proof of Theorem 1.1 will be equivalent to show convergence of the solutions v_R in an appropriate norm.

In order to prove the convergence result we use the following key observation: In the unit ball we define the function

$$u_R(x) = v_R(Rx);$$

then u_R satisfies

$$-\Delta u_R + \frac{R^2 \nabla_v W(u_R)}{2} = 0 \quad \text{for } x \in B_1.$$

Hence for $\varepsilon = 1/R$, the function u_ε satisfies

$$-\Delta u_\varepsilon + \frac{\nabla_v W(u_\varepsilon)}{2\varepsilon^2} = 0. \tag{1.9}$$

As $R \to \infty$ (or equivalently as $\varepsilon \to 0$) we expect v_R to converge to the solution v to (1.1)–(1.2) (this will be proved in Section 5), and correspondingly, we expect the limiting solution u_ε to (1.9) to capture the behavior of v at infinity. Equation (1.9) has been largely studied (see for example [5] and [17]). This motivates us to analyze in Section 3 some existing results for (1.9) that apply in our context and provide useful information for our problem. More precisely, combining results in [3], [13] and [23] and using Γ-convergence techniques we prove that the rescaled u_ε converge to a function u_0 in the L^1 norm in the unit ball. Moreover, the function u_0 equals c_i in the the angular sectors defined by $\theta \in (\theta_{i-1}, \theta_i)$ and it is minimizing for an appropriate functional (eventually, this property will imply the minimizing result in Theorem 1.1). Hypotheses (d) and (e) are essential in this section. However, we would like to point out that it is not clear whether they are just technical conditions (which may be removed) or not. On the other hand, hypotheses (a) and (b) (which are also used in this section) are natural in the context of the problem.

In order to finish the proof, in Section 4 we show that the convergence holds in a norm stronger than L^1. The main idea in this computation is to use the parabolic version of equation (1.9) to interpolate between an approximate solutions to (1.1) in the ball (which we will denote by $U_{\vec{q}}$) and the real solution. More precisely, we consider a function $\tilde{h}_\varepsilon$ that is a solution to

$$\begin{aligned} \frac{d\tilde{h}_\varepsilon}{dt} - \Delta\tilde{h}_\varepsilon + \frac{\nabla_v W(\tilde{h}_\varepsilon)}{2\varepsilon^2} &= 0 \quad \text{for } x \in B_1,\ t \in (0,\infty), \\ \tilde{h}_\varepsilon(x,t) &= \varphi_\varepsilon(x) \quad \text{for } x \in \partial B_1, \\ \tilde{h}_\varepsilon(x,0) &= U_{\vec{q}}(x) \quad \text{for } x \in B_1. \end{aligned}$$

The "approximate solution" $U_{\vec{q}}(x)$ depends on ε, satisfies $U_{\vec{q}}(x) = \varphi_\varepsilon(x)$ for $x \in \partial B_1$ and $((-\Delta U_{\vec{q}} + \nabla_v W(U_{\vec{q}}))/(2\varepsilon^2))(x) \to 0$ as $\varepsilon \to 0$ point-wise in B_1. Using Theorem 4.1 we prove that in fact $\tilde{h}_\varepsilon$ and $U_{\vec{q}}$ remain appropriately close (with respect to the sup norm) in time. We conclude by observing that, as $t \to \infty$, it holds that $\sup_{x\in\mathbb{R}^2} |\tilde{h}_\varepsilon(x,t) - u_\varepsilon(x)| \to 0$. This will imply that in fact u_ε is ε close to $U_{\vec{q}}$ in the sup norm. Also in that section, we use similar techniques to control the convergence in compact domains of the sequence $v_\varepsilon : B_{1/\varepsilon} \to \mathbb{R}^2$ given by $v_\varepsilon(x) = u(\varepsilon x)$. The proof of Theorem 1.1 can be easily finished by combining the elements described above. This is achieved in Section 5.

We would like to remark that the techniques presented in this paper were already used by the author in similar problems (see [20] and [21]). In general, the method can be extended as long as the solutions to (1.1) converge to minima of W as $\varepsilon \to 0$ and that approximate solutions with the desired characteristics (such as $U_{\vec{q}}$ in this case) can be constructed.

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2. DEFINITIONS AND PRELIMINARY LEMMAS

We divide this section into three subsections. The first one is devoted to several definitions that will be used in the analysis performed in Section 3. The main objective of the second subsection is to construct the function φ used in Theorem 1.1. In the final subsection we summarize a collection of existing results that will be used throughout this paper.

2.1. General definitions In this subsection we will address several general definitions that will simplify the notation in the coming sections.

Define the function $g_i : \mathbb{R}^2 \to \mathbb{R}$ for any $p \in \mathbb{R}^2$ as

$$g_i(p) = \Gamma(c_i, p), \tag{2.1}$$

where the function Γ is defined by (1.6). Notice that Γ can be regarded as a degenerate distance function. Hence $g_i(p)$ represents the distance of a point p (with respect to the distance function Γ) to the critical point c_i.

Inspired in [22] we consider the following assumption:

Hypothesis 2.1. *Suppose that for every $u \in \mathbb{R}^2$, there exists a curve $\gamma_u^i : [-1,1] \to \mathbb{R}^2$ such that $\gamma_u^i(-1) = c_i$, $\gamma_u^i(1) = u$ and*

$$g_i(u) = \int_{-1}^{1} \sqrt{W(\gamma_u^i(t))}\, |(\gamma_u^i)'(t)|\, dt. \tag{2.2}$$

The function g_i is Lipschitz continuous and satisfies

$$|Dg_i(u)| = \sqrt{W(u)} \quad \text{a.e.} \tag{2.3}$$

For potentials with two wells the existence of such curves was proved by P. Sternberg in [22]. He also proved that, when considering a curve that joins the minima of W, it can be re-parametrized by a curve $\beta_{ij} : (-\infty, \infty) \to (-1,1)$ such that the curves defined by

$$\zeta_{ij}(\tau) = \gamma_{c_j}^i(\beta_{ij}(\tau))$$

satisfy

$$2g_i(c_j) = \int_{-\infty}^{\infty} W(\zeta_{ij}) + |\zeta'|^2\, d\tau, \tag{2.4}$$

as well as (1.4) and (1.5) (where the limits in (1.5) are attained at an exponential rate). In our situation, if we assume Hypothesis 2.1, the previous construction can also be carried out (see [7] and [22] for details on this computation). Hence, in what follows we will work under Hypothesis 2.1 and, in particular, we assume that for any pair of minima c_i, c_j there is a solution to (1.4)–(1.5). We would like to remark that Hypothesis 2.1 holds for several potentials W (see [1] and [5] for some explicit examples).

As mentioned in the introduction, we want to relate equation (1.1)–(1.2) with the following equation in the unit ball:

$$-\Delta u_\varepsilon + \frac{\nabla_v W(u_\varepsilon)}{\varepsilon^2} = 0 \quad \text{for } x \in B_1, \tag{2.5}$$

$$u_\varepsilon(x) = \varphi_\varepsilon(x) \quad \text{for } x \in \partial B_1. \tag{2.6}$$

where φ_ε will be properly defined in the coming subsection. This equation motivates us to define the following functional:

$$\mathcal{I}_\varepsilon(u) = \begin{cases} \displaystyle\int_{B_1} \varepsilon |Du|^2 + \frac{1}{\varepsilon} W(u)\, dy & \text{if } u \in H^1(B_1) \text{ and } u|_{\partial B_1}(x) = \varphi_\varepsilon(x), \\ \infty & \text{otherwise.} \end{cases} \tag{2.7}$$

where $u : B_1 \to \mathbb{R}^2$, $\varphi_\varepsilon : \partial B_1 \to \mathbb{R}^2$. It is easy to check that weak solutions to (2.5)–(2.6) can be regarded as critical points of (2.7).

We are interested in studying the limiting problem as $\varepsilon \to 0$. More specifically, we expect the limit of the solutions u_ε to (2.5) will capture the behavior at infinity of the function v which satisfies (1.1)–(1.2). In particular, we want to show that it is possible to obtain, as the limit of the functions u_ε, a function u_0 that satisfies

$$u_0(r\cos\theta, r\sin\theta) = c_i \quad \text{for } \theta \in (\theta_{i-1}, \theta_i), \tag{2.8}$$

where $\alpha_i = \theta_i - \theta_{i-1}$ satisfy (1.7). Without loss of generality we are going to assume that $\theta_0 = 0$ and $\theta_3 = 2\pi$.

In order to study the limit of the functions u_ε we define the following limit functional (that we will show corresponds to the Γ-limit of the functionals $\mathcal{I}_\varepsilon$):

$$\mathcal{I}_0(u) = \begin{cases} \displaystyle\sum_{i,j=1}^{3} \Gamma(c_i, c_j) H_1(\partial_{B_1}\Omega_i(u) \cap \partial_{B_1}\Omega_{i+1}(u)) \\ \qquad \displaystyle + \sum_{i,j=1}^{3} \Gamma(c_i, c_j) H_1((\partial\Omega_j(u) \cap \partial B_1) \setminus \Phi_i) \\ \qquad\qquad \text{if } g_i(u) \in BV(B_1) \text{ and } u \in \{c_i\}_{i=0}^{3}, \\ \infty \qquad\qquad \text{otherwise,} \end{cases} \tag{2.9}$$

where $\Omega_i(u) = \{x \in B_1 : u(x) = c_i\}$, $\varphi_0(x) = \lim_{\varepsilon\to 0} \varphi_\varepsilon(x)$, $\Phi_i = \{x \in \partial B_1 : \varphi_0(x) = c_i\}$ and H_1 is the one dimensional Hausdorff measure.

2.2. The function φ As described in the introduction, the function φ should represent the boundary condition at infinity, that is, it should satisfy (1.2). In particular, we expect the sequence of functions φ_ε (defined by $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$) to converge to c_i as $\varepsilon \to 0$ in the angular sectors of B_1 defined by $\theta \in (\theta_{i-1}, \theta_i)$ (where the angles θ_i are defined by (1.7)–(1.8)). Moreover, we will construct a function φ that, away from the triple point, approximates a solution to (1.9) (we will make this statement more precise in Section 4).

More precisely, let L_i be the half-lines starting at the origin, with direction θ_i. Away from L_i, the function φ is defined by one of the constants c_j (that is, one of the minima of W). Notice that in fact c_j are solutions to (2.5). Near the half-lines L_i, the function φ will be equal to an appropriate solution to (1.4) (that we denote ζ_{ij}), evaluated at the distance to L_i. These functions are approximate solutions in a sense to be discussed in Section 4.

We summarize the description above with the following equations: Consider a smooth function $\eta : \mathbb{R}^2 \to \mathbb{R}$ such that $\eta(x) \equiv 1$ when $|x| \le \frac{1}{2}$ and $\eta(x) \equiv 0$ for $|x| \ge 1$, the distance

$$d_i(x) = d(x, L_i),$$

and a partition of unity $\{\eta_i\}_{i=1}^6$ associated to the family of intervals $\{\mathcal{A}_j\}_{j=1}^6$, where

$$\begin{aligned}\mathcal{A}_{2i} &= (\theta_i - \delta,\ \theta_i + \delta),\\ \mathcal{A}_{2i+1} &= \left(\theta_i + \frac{\delta}{2},\ \theta_{i+1} - \frac{\delta}{2}\right).\end{aligned}$$

Now we define

$$\begin{aligned}\varphi(x) = (1 - \eta(x))\Big(&\eta_5(\theta) c_3 + \eta_6(\theta)\zeta_{31}(d_0(x))\\ &+ \sum_{i=1}^{2}(\eta_{2i}(\theta)\zeta_{ii+1}(d_i(x)) + \eta_{2i-1}(\theta)c_i)\Big)\end{aligned} \tag{2.10}$$

and

$$\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right). \tag{2.11}$$

Notice that since L_i is a half-line, we have that $d_i(x/\varepsilon) = d_i(x)/\varepsilon$.

Remark ***2.2***. The functions φ_ε are not only well defined on the boundary of B_1, but also in the interior. Moreover, under these definitions we have that

$$\varphi_0(x) := \lim_{\varepsilon \to 0} \varphi_\varepsilon(x) = u_0(x) \quad \text{almost everywhere.}$$

Furthermore, in Section 4 will be shown that near the boundary (more precisely, for $|x| > \varepsilon^\alpha$) the function φ_ε is an "approximate solution" to the equation (1.1), in the sense that for every x there holds $(-\Delta\varphi_\varepsilon + (\nabla_v W(\varphi_\varepsilon))/(2\varepsilon^2))(x) \to 0$ as $\varepsilon \to 0$. We will prove that in fact for every $\alpha < 1$, $\sup_{\varepsilon^\alpha<|x|<1} |u_\varepsilon - \varphi_\varepsilon| \to 0$ as $\varepsilon \to 0$. Correspondingly, for $v_\varepsilon : B_{1/\varepsilon} \to \mathbb{R}^2$ defined by $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ there holds $\sup_{\varepsilon^{\alpha-1}<|x|<(1/\varepsilon)} |v_\varepsilon - \varphi| \to 0$ as $\varepsilon \to 0$.

On the other hand, it is not expected that the functions φ_ε are good approximations to the solution inside the ball of radius ε^α (or correspondingly, φ is not a good approximation of v_ε in the ball of radius $\varepsilon^{\alpha-1}$). This can be illustrated as follows: The choice of the functions φ_ε in (2.11) flexible as long as the features described above are preserved (namely, for $|x| > \varepsilon^\alpha$ they approach u_0 and they are an approximated solution to the equation). For example, it is possible to consider $\tilde{\varphi}_\varepsilon(x) = \varphi(x/\varepsilon + \ln(\varepsilon)x_0)$. In fact, for every $k \in \mathbb{N}$ there holds $\sup_{|x|>\varepsilon^\alpha} |D^k\varphi_\varepsilon - D^k\tilde{\varphi}_\varepsilon| \to 0$ as $\varepsilon \to 0$. However, for every $\sigma < \varepsilon^{1-\alpha}$ we have $\min c_i < \sup_{|x|<\varepsilon^\alpha} |\tilde{\varphi}_\varepsilon(x) - \tilde{\varphi}_\sigma(\sigma x/\varepsilon)|$, which contrasts with the second inequality in Theorem 4.1. In particular, it is clear that $\tilde{\varphi}$ cannot be a good approximation of the solution inside the ball of radius ε^α. Similarly, it is not expected that φ_ε approximates the solution u_ε inside the ball of radius ε^α (or that the corresponding function v_ε would be approximated by φ inside the ball of radius $\varepsilon^{\alpha-1}$).

2.3. Technical lemmas Now we state some technical lemmas. The first one was originally proved in [19]:

Lemma 2.3. *Let $u_\varepsilon(x) \in C^2$ satisfy (2.5)–(2.6), where $W : \mathbb{R}^2 \to \mathbb{R}$ is a proper function in C^2 bounded below, with a finite number of critical points (that we label as $\{c_i\}_{i=1}^m$), and such that the Hessian of $W(u)$ is positive semidefinite for $|u| \geq K$ for some real number K. Suppose that the functions φ_ε are uniformly bounded. Then there is a constant C depending only on uniform bounds over φ_ε and W, but not on ε, such that*

$$\sup |u_\varepsilon| \leq C,$$

where C only depends on uniform bounds over φ_ε and W.

Proof. Consider $\omega_\varepsilon(x) = W(u_\varepsilon)(x)$; then

$$\begin{aligned} -\Delta\omega_\varepsilon &= -\sum_i (\nabla_v W(u_\varepsilon) \cdot (u_\varepsilon)_{x_i})_{x_i} \\ &= -(W''(u_\varepsilon)Du_\varepsilon) \cdot Du_\varepsilon - \nabla_v W(u_\varepsilon) \cdot \Delta u_\varepsilon, \end{aligned}$$

where W'' denotes the Hessian matrix of W and the dot product between two 2×2 matrices is the standard dot product in $\mathbb{R}^4$. Since u_ε satisfies (2.5), this becomes

$$-\Delta\omega_\varepsilon + \frac{|W'(u_\varepsilon)|^2}{2\varepsilon^2} + (W''(u_\varepsilon)Du) \cdot Du_\varepsilon = 0. \tag{2.12}$$

If the maximum of ω_ε is attained at the boundary, then it is bounded by the maximum of $W(\varphi_\varepsilon(x))$.

Suppose that ω_ε has an interior maximum at x_0 and $|u_\varepsilon(x_0)| \geq K$. Since x_0 is a maximum for ω_ε, it holds that $\Delta\omega_\varepsilon(x_0) \leq 0$. We also have by hypothesis that $W''(u)$ is positive semidefinite for $|u| \geq K$, hence

$$-\Delta\omega_\varepsilon + \frac{|D_u W(u_\varepsilon)|^2}{\varepsilon^2} + (W''(u_\varepsilon)Du_\varepsilon) \cdot Du_\varepsilon \geq 0.$$

The inequality is strict (which contradicts (2.12)) unless

$$\frac{|D_u W(u_\varepsilon)|^2}{\varepsilon^2} = (W''(u_\varepsilon)Du_\varepsilon) \cdot Du_\varepsilon = 0.$$

If $\nabla_v W(u_\varepsilon(x_0)) = 0$, we would have $u_\varepsilon(x_0) = c_i$ for some i and this implies (since the maximum is attained at this point) that $W(u_\varepsilon(x,t)) \leq W(c_i)$. Hence we have $\omega_\varepsilon \leq \max\{\sup_{|u|\leq K} W(u_\varepsilon), W(\varphi_\varepsilon), \max_{i=1,\ldots,m} W(c_i)\}$.

Since W is a proper function, we conclude the result of the lemma. $\square$

We will also use Lemma A.1 and Lemma A.2 in [4]. We restate them here without proof:

Lemma 2.4 (Lemma A.1 in [4]). *Assume that u satisfies*

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^n.$$

Then

$$|Du(x)|^2 \leq C\left(\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} + \frac{1}{\operatorname{dist}^2(x,\partial\Omega)}||u||^2_{L^\infty(\Omega)}\right) \quad \forall\, x \in \Omega, \tag{2.13}$$

where C is a constant that depends only on n.

Lemma 2.5 (Lemma A.2 in [4]). *Assume that u satisfies*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a smooth bounded domain. Then it holds

$$||Du||^2_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)}, \tag{2.14}$$

where C is a constant depending only on Ω.

3. CONVERGENCE IN L^1

In this section we show that solutions u_ε to equation (2.5)–(2.6) converge in L^1. More precisely, we prove the following result.

Proposition 3.1. *Let u_0 be defined by (2.8). Consider $\mathcal{I}_\varepsilon$ and $\mathcal{I}_0$ defined by (2.7) and (2.9) respectively. For φ_ε defined by (2.10)–(2.11) there exists a sequence of minimizers u_ε of $\mathcal{I}_\varepsilon$, such that $\mathcal{I}_\varepsilon(u_\varepsilon) \to \mathcal{I}_0(u_0)$ and $u_\varepsilon \to u_0$ in L^1.*

As stated in [23], when considering the Neumman boundary condition problem, Proposition 3.1 follows from results in [3], [13] and [23]. In what follows we are going to state these results and point out the necessary modifications in our setting.

Theorem 3.2 ([23])**.** *Let u_0 be defined by (2.8) and $C_i = \{x \in \Omega : u_0(x) = c_i\}$. Consider a domain Ω and partition (E, F, G) of Ω. Define*

$$\mathcal{F}(E,F,G) = \Gamma(c_1,c_2)H_1(\partial_\Omega E \cap \partial_\Omega G) + \Gamma(c_1,c_3)H_1(\partial_\Omega E \cap \partial_\Omega F) + \Gamma(c_3,c_2)H_1(\partial_\Omega F \cap \partial_\Omega G).$$

Then the partition formed by C_1, C_2 and C_3 is an isolated local minimizer of $\mathcal{F}$, that is

$$\mathcal{F}(C_1,C_2,C_3) = \min \mathcal{F}(E,F,G), \tag{3.1}$$

where the minimum is taken over all the partitions (E, F, G) of Ω satisfying the condition

$$|C_1 \Delta E| + |C_2 \Delta F| + |C_3 \Delta G| \le \delta, \tag{3.2}$$

where δ is some small positive number.

*Remark **3.3**.* The proof of Lemma 3.1 in [23] implies that this δ can be uniformly chosen for balls of all radii.

Theorem 3.4 (Theorem 2.5 in [3])**.** *Let*

$$\tilde{\mathcal{I}}_{\varepsilon,\Omega}(u) = \begin{cases} \displaystyle\int_\Omega \varepsilon|Du|^2 + \frac{1}{\varepsilon}W(u)\,\mathrm{d}y & \text{if } u \in H^1(\Omega) \text{ and } \displaystyle\int_\Omega u(x)\,\mathrm{d}x = m, \\ \infty & \text{otherwise.} \end{cases} \tag{3.3}$$

and

$$\tilde{\mathcal{I}}_{0,\Omega}(u) = \begin{cases} \displaystyle\sum_{i,j=1}^{3} \Gamma(c_i,c_j)H_1(\partial_{B_1}\Omega_i(u) \cap \partial_{B_1}\Omega_j(u)) & \\ \qquad \text{if } g_i(u) \in BV(\Omega) \text{ for } i \in \{1,2,3\}, & \\ \qquad W(u(x)) = 0 \text{ a.e. and } \displaystyle\int_\Omega u(x)\,\mathrm{d}x = m, & \\ \infty \qquad \text{otherwise.} \end{cases} \tag{3.4}$$

It holds for every $\varepsilon_h \to 0$ *that*

- *For every* $u_{\varepsilon_h} \to u$ *in* $L^1(\Omega)$ *we have that*

$$\tilde{\mathcal{I}}_0(u) \leq \liminf_{h\to\infty} \tilde{\mathcal{I}}_{\varepsilon_h}(u_{\varepsilon_h}).$$

- *There is* $u_{\varepsilon_h} \to u$ *in* $L^1(\Omega)$ *such that*

$$\tilde{\mathcal{I}}_0(u) \geq \limsup_{h\to\infty} \tilde{\mathcal{I}}_{\varepsilon_h}(u_{\varepsilon_h}).$$

Proposition 3.5 (Proposition 2.2 in [3])**.** *The function* g_i *is locally Lipschitz-continuous. Moreover, if* $u \in H^1(\Omega) \cup L^\infty(\Omega)$*, then* $g_i(u) \in W^{1,1}(\Omega)$ *and the following inequality holds:*

$$\int_\Omega |D(g_i(u))|\,\mathrm{d}x \leq \int_\Omega \sqrt{W(u)}|Du|\,\mathrm{d}x. \tag{3.5}$$

*Remark **3.6**.* Following the proof of Theorem 3.4 in [3] it is easy to see that the restriction $\int_\Omega u(x)\,\mathrm{d}x = m$, imposed by Baldo in his work, can be removed from Theorem 3.4 without modifying the proof.

Theorem 3.7. [13] *Suppose that a sequence of functionals* $\{\mathcal{I}_\varepsilon\}$ *and a functional* $\mathcal{I}_0$ *satisfy the following conditions:*

(1) *if* $w_\varepsilon \to w_0$ *in* $L^1(\Omega)$ *as* $\varepsilon \to 0$*, then* $\liminf \mathcal{I}_\varepsilon(w_\varepsilon) \geq \mathcal{I}_0(w_0)$*;*
(2) *for any* $w_0 \in L^1(\Omega)$ *there is a family* $\{\rho_\varepsilon\}_{\varepsilon>0}$ *with* $\rho_\varepsilon \to w_0$ *in* $L^1(\Omega)$ *and* $\mathcal{I}_\varepsilon(\rho_\varepsilon) \to \mathcal{I}_0(w_0)$*;*
(3) *any family* $\{w_\varepsilon\}_{\varepsilon>0}$ *such that* $\mathcal{I}_\varepsilon(w_\varepsilon) \leq C < \infty$ *for all* $\varepsilon > 0$ *is compact in* $L^1(\Omega)$*;*
(4) *there exists an isolated* L^1*-local minimizer* u_0 *of* $\mathcal{I}_0$*; that is,* $\mathcal{I}_0(u_0) < \mathcal{I}_0(w)$ *whenever* $0 < \|u_0 - w\|_{L^1(\Omega)} \leq \delta$ *for some* $\delta > 0$*.*

Then there exist an $\varepsilon_0 > 0$ *and a family* $\{u_\varepsilon\}$ *for* $\varepsilon < \varepsilon_0$ *such that* u_ε *is an* L^1*-local minimizer of* $\mathcal{I}_\varepsilon$ *and* $u_\varepsilon \to u_0$ *in* $L^1(\Omega)$*.*

Theorem 3.4 establishes conditions (1) and (2) of Theorem 3.7 for $\tilde{\mathcal{I}}_{\varepsilon,\Omega}$ (defined by (3.3)) and $\tilde{\mathcal{I}}_0$ (defined by (3.4)). Theorem 3.2 establishes that u_0 is a local minimizer for $\tilde{\mathcal{I}}_{0,\Omega}$ (condition (4) of Theorem 3.7). We need to show that these theorems imply that the conditions of Theorem 3.7 also hold for $\mathcal{I}_\varepsilon$ and $\mathcal{I}_0$ (defined by (2.7) and (2.9), respectively). In addition, we need to prove that condition (3) holds for these functionals.

Lemma 3.8. *Theorem 3.2 implies that* u_0 *is a local minimizer for* $\mathcal{I}_0$*.*

Proof. Let $C_i = \{x \in B_1 : u_0(x) = c_i\}$ and for any w let $\Omega_i(w) = \{x \in B_1 : w(x) = c_i\}$. Consider δ for B_1 as in Theorem 3.2. We are going to show by contradiction that for every w such that $w(x) \in \{c_i\}_{i=1}^3$ almost everywhere and

$$|C_1 \Delta \Omega_1(w)| + |C_2 \Delta \Omega_2(w)| + |C_3 \Delta \Omega_3(w)| \le \delta$$

there holds that

$$\mathcal{I}_0(u_0) \le \mathcal{I}_0(w).$$

Suppose that there is a w such that

$$|C_1 \Delta \Omega_1(w)| + |C_2 \Delta \Omega_2(w)| + |C_3 \Delta \Omega_3(w)| \le \delta \tag{3.6}$$

and

$$\mathcal{I}_0(u_0) > \mathcal{I}_0(w). \tag{3.7}$$

Consider $\sigma > 0$ and $B_{1+\sigma}$. Define

$$\mathcal{I}_\varepsilon^\sigma(u) = \tilde{\mathcal{I}}_{\varepsilon, B_{1+\sigma}}(u).$$

Notice first that u_0 (given by (2.8)) is well defined for every $x \in \mathbb{R}^2$. In particular, it is well defined for every $x \in B_{1+\sigma}$ for any $\sigma > 0$. Hence, we can define

$$w^\sigma(x) = \begin{cases} w(x) & \text{if } x \in \bar{B}_1, \\ \\ u_0(x) & \text{if } x \in B_{1+\sigma} \setminus B_1. \end{cases} \tag{3.8}$$

Let

$$\tilde{C}_i = \{x \in B_{1+\sigma} : u_0(x) = c_i\},$$
$$\tilde{\Omega}_i(w) = \{x \in B_{1+\sigma} : w^\sigma(x) = c_i\}.$$

Using definition (3.8) and equation (3.6) we also have

$$|\tilde{C}_1 \Delta \tilde{\Omega}_1(w)| + |\tilde{C}_2 \Delta \tilde{\Omega}_2(w)| + |\tilde{C}_3 \Delta \tilde{\Omega}_3(w)| \le \delta. \tag{3.9}$$

Notice that every subset on the boundary where w does not agree with u_0 becomes an interior boundary term for w^σ in $B_{1+\sigma}$. By the definition of $\mathcal{I}_0^\sigma$ we have that

$$\mathcal{I}_0^\sigma(w^\sigma) = \mathcal{I}_0(w^\sigma) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j),$$
$$\mathcal{I}_0^\sigma(u_0) = \mathcal{I}_0(u_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j).$$

Inequality (3.7) implies that

$$\mathcal{I}_0^\sigma(w^\sigma) < \mathcal{I}_0^\sigma(u_0), \tag{3.10}$$

which together with (3.9) contradicts the local minimality of u_0 given by Theorem 3.2. ◻

Proof of Proposition 3.1. In what follows, we are going to show that Theorem 3.4 and Proposition 3.5 imply conditions (1) and (2) of Theorem 3.7 for the functionals defined by (2.7) and (2.9).

Recall that φ_ε is given by (2.11), $\varphi_0 = \lim_{\varepsilon\to 0}\varphi_\varepsilon$, and $\varphi_0 = u_0$ a.e.

Proof of condition (1). Let

$$w_\varepsilon \to w_0 \quad \text{in } L^1. \tag{3.11}$$

As in the proof of Lemma 3.8, consider $\sigma > 0$ and define

$$\mathcal{I}_\varepsilon^\sigma(u) = \tilde{\mathcal{I}}_{\varepsilon,B_{1+\sigma}}(u), \tag{3.12}$$

$$w_\varepsilon^\sigma(x) = \begin{cases} w_\varepsilon(x) & \text{if } x \in \bar{B}_1, \\ \varphi_\varepsilon(x) & \text{if } x \in B_{1+\sigma}\setminus B_1, \end{cases} \tag{3.13}$$

$$w_0^\sigma(x) = \begin{cases} w_0(x) & \text{if } x \in \bar{B}_1, \\ \varphi_0(x) & \text{if } x \in B_{1+\sigma}\setminus B_1. \end{cases} \tag{3.14}$$

Notice that, again, the boundary portions of w_0 that do not agree with φ_0 become interior boundaries of w_0^σ. Hence, as before, if $\mathcal{I}_0^\sigma(w_0) \neq \infty$, we have that

$$\mathcal{I}_0^\sigma(w_0) = \mathcal{I}_0(w_0) + \sigma\sum_{i,j=1}^{3}\Gamma(c_i,c_j). \tag{3.15}$$

Using (3.11) and definitions (3.13) and (3.14) we have that

$$w_\varepsilon^\sigma \to w_0^\sigma \quad \text{in } L^1.$$

Theorem 3.4 and Remark 3.6 imply that

$$\mathcal{I}_0^\sigma(w_0^\sigma) \le \liminf_{\varepsilon\to 0}\mathcal{I}_\varepsilon^\sigma(w_\varepsilon^\sigma). \tag{3.16}$$

We can explicitly compute that

$$\tilde{\mathcal{I}}_{\varepsilon,B_{1+\sigma}\setminus B_1}(\varphi_\varepsilon) \to \sigma\sum_{i,j=1}^{3}\Gamma(c_i,c_j). \tag{3.17}$$

It is also easy to check that

$$\mathcal{I}_\varepsilon^\sigma(w_\varepsilon) = \mathcal{I}_\varepsilon(w_\varepsilon) + \tilde{\mathcal{I}}_{\varepsilon, B_{1+\sigma}\setminus B_1}(\varphi_\varepsilon). \tag{3.18}$$

Equations (3.17) and (3.18) imply that

$$\mathcal{I}_\varepsilon^\sigma(w_\varepsilon) \to \infty \quad \text{if and only if} \quad \mathcal{I}_\varepsilon(w_\varepsilon) \to \infty.$$

We can assume that $\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(w_\varepsilon) < \infty$ (otherwise the result is trivial). Equations (3.15), (3.16), (3.18) and (3.17) imply that

$$\begin{aligned}
\mathcal{I}_0(w_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j) &= \mathcal{I}_0^\sigma(w_0) \\
&\le \liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon^\sigma(w_\varepsilon^\sigma) \\
&= \liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(w_\varepsilon) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j).
\end{aligned}$$

This implies

$$\mathcal{I}_0(w_0) \le \liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(w_\varepsilon),$$

which proves the result. □

Proof of condition (2). The proof of condition (2) follows directly from the proof in [3] of the equivalent statement. Hence, we are going to follow Baldo's proof, use some of his constructions and point out the necessary modifications in our setting. For more details, we refer the reader to [3].

As in the proof of condition (1), let $\mathcal{I}_\varepsilon^\sigma$ be defined by (3.12), that is

$$\mathcal{I}_\varepsilon^\sigma(u) = \tilde{\mathcal{I}}_{\varepsilon, B_{1+\sigma}}(u).$$

Consider $w_0 \in \{c_i\}_{i=1}^3$, such that $\mathcal{I}_0(w_0) < \infty$ (otherwise the result is trivial). As before, we extend the domain to $B_{1+\sigma}$, for some $\sigma > 0$, and we extend w_0 by φ_0 outside the unit ball. We label this extension as w_0^σ.

Let ρ_ε^σ be the sequence of functions given by Theorem 3.4 that satisfy $\rho_\varepsilon^\sigma \to w_0^\sigma$ in L^1 and $\mathcal{I}_\varepsilon^\sigma(\rho_\varepsilon) \to \mathcal{I}_0^\sigma(w_0^\sigma)$.

We can write $w_0 = \sum_{i=1}^3 c_i 1_{\Omega_i}$. The functions ρ_ε^σ constructed by Baldo in [3] are uniformly bounded functions, that ε-near the boundaries $\partial\Omega_i \cap \partial\Omega_j \cap B_{1+\sigma}$ are equal to the geodesic ζ_{ij}. In the interior of Ω_i, ρ_ε^σ approaches c_i uniformly. In particular, we have that $\rho_\varepsilon \to w_0$ almost everywhere and it is uniformly bounded. By dominated convergence theorem we have that the restriction of ρ_ε^σ to B_1, that we will label as ρ_ε, converges to w_0 in the L^1 norm.

As in the proof of (1), we have

$$\mathcal{I}_0^\sigma(w_0^\sigma) = \mathcal{I}_0(w_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j). \tag{3.19}$$

By the definitions of $\mathcal{I}_\varepsilon^\sigma$, $\mathcal{I}_\varepsilon$, ρ_ε^σ and ρ_ε, for every $\sigma > 0$ holds that

$$\mathcal{I}_\varepsilon^\sigma(\rho_\varepsilon^\sigma) \geq \mathcal{I}_\varepsilon(\rho_\varepsilon). \tag{3.20}$$

Combining (3.19), (3.20) and Theorem 3.4 we have

$$\begin{aligned}
\mathcal{I}_0(w_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j) &= \mathcal{I}_0^\sigma(w_0^\sigma) \\
&= \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon^\sigma(\rho_\varepsilon^\sigma) \\
&\geq \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\rho_\varepsilon).
\end{aligned}$$

Taking $\sigma \to 0$, it follows that

$$\mathcal{I}_0(w_0) \geq \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\rho_\varepsilon).$$

Combining this equation and condition (1) (that we proved above) we conclude that

$$\mathcal{I}_0(w_0) = \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\rho_\varepsilon),$$

which finishes the proof. □

Proof of condition (3). We will follow the proof in [22]. Suppose that $\mathcal{I}_\varepsilon(w_\varepsilon) \leq C < \infty$ for some family $\{w_\varepsilon\}_{\varepsilon>0}$.

Define

$$G_\varepsilon(x) = g_1(w_\varepsilon(x)).$$

Proposition 3.5 implies that

$$\begin{aligned}
\int_{B_1} |DG_\varepsilon(x)|\,\mathrm{d}x &\leq \int_{B_1} \sqrt{W(w_\varepsilon)}|Dw_\varepsilon|\,\mathrm{d}x \\
&\leq \varepsilon \int_{B_1} |Dw_\varepsilon|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{B_1} W(w_\varepsilon)\,\mathrm{d}x \\
&\leq C.
\end{aligned}$$

Hypothesis (d) of Theorem 1.1 implies that the functions w_ε are uniformly bounded in $L^p(B_1)$ for some p. Hence, G_ε are uniformly bounded in $L^1(B_1)$ and

$$\|G_\varepsilon\|_{BV(B_1)} \leq C.$$

Since bounded sequences in BV are compact in L^1(see [9]), there is a subsequence G_ε convergent to G_0 in L^1. This function G_0 takes the form

$$G_0(x) = \begin{cases} 0 & \text{if } x \in C_1, \\ g_1(c_2) & \text{if } x \in C_2, \\ g_1(c_3) & \text{if } x \in C_3. \end{cases}$$

Since c_1 is the only value x such that $g_1(x) = 0$ and g_1 is continuous, we have that there is a subsequence $\{w_{\varepsilon_j}\}$ that converges in measure to c_1 on C_1. The uniform bounds in L^p (provided by hypothesis (d)) imply that $\{w_{\varepsilon_j}\}$ converge on C_1 also in the L^1 norm. The proof can be finished by repeating the same argument for g_2 and g_3. ☐

Using Theorem 3.7 and that Lemma 3.8 implies condition (4), we conclude the result of Proposition 3.1. ☐

From Theorem 3.7 we conclude the following corollary:

Corollary 3.9. *Let u_0 be defined as in Theorem 3.7. Then there is a subsequence of the family $\{u_\varepsilon\}$ that converges point-wise almost everywhere to u_0.*

4. Uniform Convergence

In this section we focus on improving the convergence bounds proved in the previous section. Namely, we prove the following result.

Theorem 4.1. *Fix $0 < \alpha < 1$. Let $0 < \sigma \leq \varepsilon^{1-\alpha}$; then for every $m > 0$ there is a constant C (that might depend on α and m) such that*

- $\sup_{|x|\geq\varepsilon^\alpha} |u_\varepsilon - \varphi_\varepsilon| \leq C\varepsilon^m$.
- $\sup_{|x|\leq\varepsilon^\alpha/2} |u_\varepsilon(x) - u_\sigma(\sigma x/\varepsilon)| \leq C\varepsilon^m$.

There are two main ingredients in the proof of this theorem. The first is the construction of a function $U_{\vec{q}}$ that satisfies $U_{\vec{q}}(x) = \varphi_\varepsilon(x)$ for $x \in B_1 \setminus B_{\varepsilon^\alpha}$, $U_{\vec{q}}(x) = u_\varepsilon(\sigma x/\varepsilon)$ for $x \in B_{\varepsilon^\alpha/2}$ and $|-\Delta U_{\vec{q}} + \nabla_v W(U_{\vec{q}})/(2\varepsilon^2)|(x) \to 0$ pointwise; the second one is Theorem 4.3. The idea is the following: We consider $U_{\vec{q}}$ as the initial condition for the parabolic equation (1.3) in the unit ball. Since $U_{\vec{q}}$ is almost a solution to this equation, we expect that the actual solution to (1.3) will stay close $U_{\vec{q}}$. This assertion it is ensured by Theorem 4.3. However, in order to apply that theorem, it is necessary to consider solutions to an equation with 0 boundary condition. For this reason, instead of considering equation (1.3) we take (4.15)–(4.16)–(4.17) (which correspond to subtracting the function $U_{\vec{q}}$ from the solution to (1.3)). We finally conclude Theorem 4.1 by observing that our solution to (1.3) converges to u_ε as $t \to \infty$.

We would also like to remark that the minimizing property of solutions u_ε will not be used in this section. In fact, the construction presented here would work for any type of critical point of the functional $\mathcal{I}_\varepsilon$ with the appropriate boundary values. However, the minimizing property will be used again in Section 5 in order to show the minimizing statement of Theorem 1.1.

Now we proceed with the construction of the function $U_{\vec{q}}$. Since this function depends also on other parameters besides ε (such as α above and σ, which will be shortly introduced), we use the subindex $\vec{q}$, which stands for $\vec{q} = (\varepsilon, \sigma, \alpha)$.

Let

$$v_\varepsilon(x) = u_\varepsilon(\varepsilon x) \quad \text{and} \quad u_\sigma^\varepsilon(x) = u_\sigma\left(\frac{\sigma x}{\varepsilon}\right).$$

Consider a positive function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(x) = 0$ for $|x| \leq \frac{1}{2}$ and $\eta(x) = 1$ for $|x| \geq 1$. Fix $0 < \alpha < 1$ and

$$E = 2\varepsilon^\alpha - \varepsilon^{2m+4-\alpha}. \tag{4.1}$$

Then define for $y \in \mathbb{R}^2$ the function

$$\eta_\alpha(y) = \eta\left(\frac{\varepsilon}{2E}|y| + 1 - \frac{\varepsilon^\alpha}{2E}\right).$$

Notice that the function $\eta_\alpha(y)$ satisfies $\eta_\alpha(y) = 0$ for $|y| \leq \varepsilon^{\alpha-1} - E/\varepsilon$ and $\eta_\alpha(x) = 1$ for $|y| \geq \varepsilon^{\alpha-1}$. Moreover, defining

$$\eta_\alpha^\varepsilon(y) = \eta_\alpha\left(\frac{y}{\varepsilon}\right)$$

it satisfies $\eta_\alpha^\varepsilon(y) = 0$ for $|y| \leq \varepsilon^\alpha - E$ (where E is defined by (4.1)) and $\eta_\alpha^\varepsilon(y) = 1$ for $|y| \geq \varepsilon^\alpha$.

We will denote by $\mathcal{H}_\Omega$ the heat kernel in $\Omega \subset \mathbb{R}^2$. A more detailed description and some properties of the heat kernel can be found in Appendix A.

Let

$$\mathcal{Q} = \{(\varepsilon, \sigma, \alpha) \in (0,1] \times (0,1] \times [0,1] : \sigma \leq \varepsilon^{1-\alpha}\}. \tag{4.2}$$

Define for $\vec{q} = (\varepsilon, \sigma, \alpha) \in \mathcal{Q}$ the function

$$V_{\vec{q}}(y) = \eta_\alpha(y)\varphi(y) + (1 - \eta_\alpha(y))v_\sigma(y).$$

Now we take

$$U_{\vec{q}}(y) = V_{\vec{q}}\left(\frac{y}{\varepsilon}\right).$$

Let us denote by C_S the set of continuous functions from S to $\mathbb{R}^2$. For $\vec{q}$ as above consider the function $F_{\vec{q}} : C_{B_{1/\varepsilon}\times[0,T]} \times C_{B_{1/\varepsilon}} \to C_{B_{1/\varepsilon}\times[0,T]}$ defined by

$$F_{\vec{q}}(h,\psi)(x,t) = \int_0^t \int_{B_{1/\varepsilon}} \mathcal{H}_{B_{1/\varepsilon}}(x,y,t-s)(-\nabla_v W(h+V_{\vec{q}})(y,s) + \Delta V_{\vec{q}})\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_{B_{1/\varepsilon}} \mathcal{H}_{B_{1/\varepsilon}}(x,y,t)\psi(y)\,\mathrm{d}y\,.$$

Notice that, for a given ψ, Duhamel's formula implies that, if there is a fixed point $h_{\vec{q},\psi}$ of $F_{\vec{q}}(\cdot,\psi)$, it would satisfy

$$\frac{dh_{\vec{q},\psi}}{dt} - \Delta h_{\vec{q},\psi} + \frac{\nabla_v W(h_{\vec{q},\psi} + V_{\vec{q}})}{2} = \Delta V_{\vec{q}} \quad \text{in } B_{1/\varepsilon}, \tag{4.3}$$

$$h_{\vec{q},\psi}(x,t) = 0 \quad \text{on } \partial B_{1/\varepsilon}, \tag{4.4}$$

$$h_{\vec{q},\psi}(x,0) = \psi(x) \quad \text{in } B_{1/\varepsilon}. \tag{4.5}$$

The next lemma shows the existence of such a fixed point.

Lemma 4.2. *Fix a uniformly bounded continuous function ψ_ε and $\vec{q} \in \mathcal{Q}$, where $\mathcal{Q}$ is defined by (4.2). The function $F_{\vec{q}}(\cdot,\psi) : C_{B_1\times[0,T]} \to C_{B_1\times[0,T]}$ has a unique fixed point that we label $h_{\vec{q},\psi}$. Moreover, for $K > 0$ and functions $w_{\vec{q}}$ satisfying $|w_{\vec{q}}| \le K$, there are constants M and β (that might depend on K), such that for every $T \ge 0$ there holds*

$$\sup_{B_{1/\varepsilon}\times[T,T+2\beta/M]} |w_{\vec{q}} - h_{\vec{q},\psi}| \le \frac{1}{1-\beta}\Big(2 \sup_{B_{1/\varepsilon}\times[T,T+2\beta/M]} |F_{\vec{q}}(w_{\vec{q}},\psi) - w_{\vec{q}}| + \sup_{x\in B_{1/\varepsilon}} |w_{\vec{q}} - h_{\vec{q},\psi}|(x,T)\Big). \tag{4.6}$$

We postpone the proof of this lemma to Appendix A.

From Lemma 4.2 we can prove the following theorem (which provides one of the essential tools in the proof of Theorem 4.1):

Theorem 4.3. *Under the hypothesis of Lemma 4.2, one of the two following alternatives holds:*

(1) $\lim_{n\to\infty} \sup_{B_{1/\varepsilon_n}\times[0,T_n]} |w_n - h_{\vec{q}_n,\psi_n}| = 0$, *or*
(2) *there is a constant C, independent of $\vec{q}_n$ and T_n such that*

$$\sup_{B_{1/\varepsilon}\times[0,T_n]} |w_n - h_{\vec{q}_n,\psi_n}| \le C \sup_{B_{1/\varepsilon}\times[0,T_n]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n|. \tag{4.7}$$

Remark ***4.4.*** In Theorem 4.3 it is possible to choose $T_n = \infty$ for every n.

Proof of Theorem 4.3. Consider sequences of continuous functions ψ_n, w_n satisfying

$$\sup_{B_1} |\psi_n|, \sup_{B_1\times[0,T_n)} |w_n| \le K, \text{ and } \vec{q}_n \in \mathcal{Q}.$$

Suppose that neither (1) nor (2) hold. Then there are subsequences such that

$$\lim_{n\to\infty} \sup_{B_{1/\varepsilon_n}\times[0,T_n]} |w_n - h_{\vec{q}_n,\psi_n}| \neq 0 \tag{4.8}$$

and

$$\sup_{B_{1/\varepsilon_n}\times[0,T_n]} |w_n - h_{\vec{q}_n,\psi_n}| = n \sup_{B_{1/\varepsilon_n}\times[0,T_n]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n|. \tag{4.9}$$

The a priori bounds shown in Theorem A.3 and the boundedness hypothesis imply that there is a constant independent of n such that $|w_n - h_{\vec{q}_n,\psi_n}| \le C$. Then, (4.9) implies

$$\sup_{B_{1/\varepsilon_n}\times[0,T_n]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n| \to 0. \tag{4.10}$$

Applying inequality (4.6) recursively we have that for every $0 \le T < \infty$ there is a constant that depends on T (but independent of $\vec{q}_n$) such that

$$\sup_{B_{1/\varepsilon_n}\times[0,T]} |w_n - h_{\vec{q}_n,\psi_n}| \le C(T) \sup_{B_{1/\varepsilon_n}\times[0,T]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n|. \tag{4.11}$$

Therefore if the T_n are uniformly bounded, case (2) holds trivially, which contradicts (4.9). Hence we may assume $T_n \to \infty$. We will show that in this case

$$\lim_{n\to\infty} \sup_{B_{1/\varepsilon_n}\times[0,T_n]} |w_n - h_{\vec{q}_n,\psi_n}| = 0,$$

contradicting (4.8).

Let

$$\tau = \Big\{(S_n)_{n\in\mathbb{N}} : 0 \le S_n \le T_n,\ \lim_{n\to\infty} \sup_{B_{1/\varepsilon_n}\times[0,S_n]} |w_n - h_{\vec{q}_n,\psi_n}| = 0\Big\}.$$

For the set of sequences in $\mathbb{R}_+$ we consider the topology defined on the basis of open sets given by $B_\sigma((S_n)_{n\in\mathbb{N}}) = \{(\tilde{S}_n)_{n\in\mathbb{N}} : \tilde{S}_n \ge 0 \text{and } \sup_{n\in\mathbb{N}} |S_n - \tilde{S}_n| \le \sigma\}$ for any $\sigma > 0$. Notice that in particular (4.11) implies that τ is a non-empty set, since at least $S_n = \inf_n T_n \in \tau$.

Claim 4.5. *τ is open.*

Proof. Consider $(S_n)_n \in \tau$. Let $\tilde{S}_n = \min\{S_n + 2\beta/M, T_n\}$. Using inequality (4.6) we have

$$\begin{aligned}&\sup_{B_{1/\varepsilon_n}\times[S_n,\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| \\ &\quad \le \frac{1}{1-\beta}\Big(2 \sup_{B_{1/\varepsilon_n}\times[S_n,\tilde{S}_n]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n| + \sup_{x\in B_{1/\varepsilon_n}} |w_n - h_{\vec{q}_n,\psi_n}|(x,S_n)\Big).\end{aligned}$$

Since $\tilde{S}_n \le T_n$ and $S_n \in \tau$, taking $n \to \infty$ we have that

$$\lim_{n\to\infty} \sup_{B_{1/\varepsilon_n}\times[S_n,\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| = 0,$$

and $B_{2\beta/M} \cap \tau \subset \tau$. Hence τ is open. ◻

Claim 4.6. *τ is closed.*

Proof. Suppose that $S^k = (S^k_n)_n \in \tau$ satisfy $S^k \to \tilde{S} = (\tilde{S}_n)_n$ as $k \to \infty$. By the definition of the topology we have that there is a k_0 such that for every $n \in \mathbb{N}$ and $k \ge k_0$ holds $|S^k_n - \tilde{S}_n| \le 2\beta/M$. Using inequality (4.6) we have

$$\sup_{B_{1/\varepsilon_n}\times[S^{k_0}_n,\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| \le \frac{1}{1-\beta}\Big(2 \sup_{B_{1/\varepsilon_n}\times[S^{k_0}_n,\tilde{S}_n]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n| + \sup_{x\in B_{1/\varepsilon_n}} |w_n - h_{\vec{q}_n,\psi_n}|(x, S^{k_0}_n)\Big).$$

Using that $(S^{k_0}_n)_n \in \tau$ and (4.10), when $n \to \infty$ we have

$$\sup_{B_{1/\varepsilon_n}\times[0,\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| = \max\Big\{ \sup_{B_{1/\varepsilon_n}\times[0,S^{k_0}_n]} |w_n - h_{\vec{q}_n,\psi_n}|, \sup_{B_{1/\varepsilon_n}\times[S^{k_0}_n,\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}|\Big\} \to 0.$$

Therefore $\tilde{S} \in \tau$ and τ is closed. ◻

Since τ is open, closed and non-empty we conclude that $\tau = \{(S_n)_{n\in\mathbb{N}} : 0 \le S_n \le T_n\}$. In particular $(T_n)_n \in \tau$, which contradicts (4.8) and proves the theorem. ◻

Following the proof of Theorem 4.3 we obtain the next result:

Corollary 4.7. *Consider the sequences ψ_n, w_n, $\vec{q}_n \in \mathcal{Q}$, and $T_n > 0$ as in Theorem 4.3. Assume in addition that there are constants C, m such that*

$$\sup_{B_{1/\varepsilon_n}\times[0,T_n]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n| \le C\varepsilon_n^m.$$

Then for every $\tilde{m} < m$ there holds either

(1) $\lim_{n\to\infty} \sup_{B_{1/\varepsilon_n}\times[0,T_n]} (|w_n - h_{\vec{q}_n,\psi_n}|/\varepsilon_n^{\tilde{m}}) = 0$, *or*

(2) *there is a constant C, independent of $\vec{q}_n$ and T_n such that*

$$\sup_{B_{1/\varepsilon_n}\times[0,T_n]} \frac{|w_n - h_{\vec{q}_n,\psi_n}|}{\varepsilon^{\tilde{m}}} \le C \sup_{B_{1/\varepsilon_n}\times[0,T_n]} \frac{|F_{\vec{q}_n}(w_n,\psi_n) - w_n|}{\varepsilon_n^{\tilde{m}}}. \tag{4.12}$$

In particular, there is a constant C such that

$$\sup_{B_{1/\varepsilon_n}\times[0,T_n]} |w_n - h_{\vec{q}_n,\psi_n}| \leq C\varepsilon_n^{\bar{m}}.$$

Now we would like to rescale the estimates of the previous theorem and corollary to the unit ball. Namely, instead of considering the function $h_{\vec{q},\psi^{\varepsilon}_{\varepsilon}}$: $B_{1/\varepsilon}\times[0,T/\varepsilon^2]\to\mathbb{R}^2$ we define the function $k_{\vec{q},\psi^{\varepsilon}_{\varepsilon}}: B_1\times[0,T]\to\mathbb{R}^2$ by

$$k_{\vec{q},\psi^{\varepsilon}_{\varepsilon}}(x,t) = h_{\vec{q}}\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon^2}\right). \tag{4.13}$$

Notice that, under this definition, for every $\varepsilon>0$ we can write the left hand side of equation (4.7) as

$$\sup_{B_{1/\varepsilon}\times[0,T/\varepsilon^2]} |h_{\vec{q},\psi^{\varepsilon}_{\varepsilon}}(x,t) - w_\varepsilon(x,t)| = \sup_{B_1\times[0,T]} |k_{\vec{q}}(x,t) - w^{\varepsilon}_{\varepsilon}(x,t)|,$$

where $w^{\varepsilon}_{\varepsilon}(x,t) = w_\varepsilon(x/\varepsilon, t/\varepsilon^2)$.

Now we would like to rescale the right hand side of inequality (4.7). Notice that by applying the function $F_{\vec{q}}$ to any pair of continuous functions w_ε, φ_ε we obtain a continuous function $F_{\vec{q}}(w^{\varepsilon}_{\varepsilon},\psi^{\varepsilon}_{\varepsilon}): B_{1/\varepsilon}\times[0,T/\varepsilon^2]\to\mathbb{R}^2$, which satisfies (via Duhamel's formula) the following equation:

$$\frac{dF_{\vec{q}}(w_\varepsilon,\psi_\varepsilon)}{dt} - \Delta F_{\vec{q}}(w_\varepsilon,\psi_\varepsilon) + \frac{\nabla_v W(w_\varepsilon + V_{\vec{q}})}{2} = \Delta V_{\vec{q}} \quad \text{in } B_{1/\varepsilon}\times\left[0,\frac{T}{\varepsilon^2}\right],$$

$$\begin{aligned} F_{\vec{q}}(w_\varepsilon,\psi_\varepsilon)(x,t) &= 0 && \text{for } x\in\partial B_{1/\varepsilon}\,,\\ F_{\vec{q}}(w_\varepsilon,\psi_\varepsilon)(x,0) &= \psi_\varepsilon(x) && \text{for } x\in B_{1/\varepsilon}\,. \end{aligned}$$

Let us define the function $\mathcal{L}_{\vec{q}}: C_{B_1\times[0,T]}\times C_{B_1}\to C_{B_1\times[0,T]}$ as

$$\mathcal{L}_{\vec{q}}(w^{\varepsilon}_{\varepsilon},\psi^{\varepsilon}_{\varepsilon})(x,t) = F_{\vec{q}}(w_\varepsilon,\psi_\varepsilon)\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon^2}\right),$$

where as before $w^{\varepsilon}_{\varepsilon}(x,t) = w_\varepsilon(x/\varepsilon,t/\varepsilon^2)$ and similarly $\psi^{\varepsilon}_{\varepsilon}(x,t) = \psi_\varepsilon(x/\varepsilon,t/\varepsilon^2)$. A simple computation shows that for any $w^{\varepsilon}_{\varepsilon}$, $\psi^{\varepsilon}_{\varepsilon}$ the function obtained by evaluating $\mathcal{L}_{\vec{q}}$ at $(w^{\varepsilon}_{\varepsilon},\psi^{\varepsilon}_{\varepsilon})$, denoted by $\mathcal{L}_{\vec{q}}(w^{\varepsilon}_{\varepsilon},\psi^{\varepsilon}_{\varepsilon})$, satisfies

$$\frac{d\mathcal{L}_{\vec{q}}(w^{\varepsilon}_{\varepsilon},\psi^{\varepsilon}_{\varepsilon})}{dt} - \Delta\mathcal{L}_{\vec{q}}(w^{\varepsilon}_{\varepsilon},\psi^{\varepsilon}_{\varepsilon}) + \frac{\nabla_v W(w^{\varepsilon}_{\varepsilon} + U_{\vec{q}})}{2\varepsilon^2} = \Delta U_{\vec{q}} \quad \text{in } B_1\times[0,T],$$

$$\begin{aligned} \mathcal{L}_{\vec{q}}(w^{\varepsilon}_{\varepsilon},\psi^{\varepsilon}_{\varepsilon})(x,t) &= 0 && \text{for } x\in\partial B_1\,,\\ \mathcal{L}_{\vec{q}}(w^{\varepsilon}_{\varepsilon},\psi^{\varepsilon}_{\varepsilon})(x,0) &= \psi^{\varepsilon}_{\varepsilon}(x) && \text{for } x\in B_1\,, \end{aligned}$$

Using again Duhamel's formula we conclude that

$$\mathcal{L}_{\vec{q}}(w_\varepsilon^\xi, \psi_\varepsilon^\xi)(x,t) \tag{4.14}$$

$$= \int_0^t \int_{B_1} \mathcal{H}_{B_1}(x,y,t-s) \left(- \frac{\nabla_v W(w_\varepsilon^\xi + U_{\vec{q}})(y,s)}{\varepsilon^2} + \Delta U_{\vec{q}}(y) \right) \mathrm{d}y\, \mathrm{d}s + \int_{B_1} \mathcal{H}_{B_1}(x,y,t) \psi_\varepsilon^\xi(y)\, \mathrm{d}y\,.$$

In particular, we have that $k_{\vec{q},\psi}$ defined by (4.13) is a fixed point of $\mathcal{L}_{\vec{q}}(\cdot\,,\psi)$.

Hence, the right hand side of equation (4.7) reads

$$\sup_{B_{1/\varepsilon}\times[0,T/\varepsilon^2]} |F_{\vec{q}}(w_\varepsilon, \psi_\varepsilon) - w_\varepsilon| = \sup_{B_1\times[0,T]} |\mathcal{L}_{\vec{q}}(w_\varepsilon^\xi, \psi_\varepsilon) - w_\varepsilon^\xi|.$$

In this context we can re-formulate Theorem 4.3 (dropping the super-indices to simplify the notation) as follows.

Theorem 4.8. *Let $k_{\vec{q}_n,\psi}$ be defined by (4.13). Then is the unique fixed point of $\mathcal{L}_{\vec{q}_n}(\cdot\,,\psi)$. Moreover, for any fixed $K > 0$ and sequences of continuous functions ψ_n, w_n satisfying $\sup|\psi_n|$, $\sup|w_n| \le K$ and vectors $\vec{q}_n \in \mathcal{Q}$ and $T_n > 0$ there holds either*

(1) $\lim_{n\to\infty} \sup_{B_1\times[0,T_n]} |k_{\vec{q}_n,\psi_n}(x,t) - w_n(x,t)| = 0$ *or*

(2) *there is a constant C, independent of n, $\vec{q}_n$ and T_n such that*

$$\sup_{B_1\times[0,T_n]} |k_{\vec{q}_n,\psi_n}(x,t) - w_n(x,t)| \le C \sup_{B_1\times[0,T_n]} |\mathcal{L}_{\vec{q}_n}(w_n,\psi_n) - w_n|,$$

where $\vec{q}_n = (\varepsilon_n, \sigma_n, \alpha_n)$.

Now we can devote ourselves to prove Theorem 4.1. We divide the proof into two steps: Lemma 4.9 and Lemma 4.10.

Notice first that the function $k_{\vec{q},\psi}$ defined by (4.13) is a solution to the following equation:

$$Pk_{\vec{q},\psi} + \frac{\nabla_v W(k_{\vec{q},\psi} + U_{\vec{q}})}{2\varepsilon^2} = \Delta U_{\vec{q}} \quad \text{in } B_1, \tag{4.15}$$

$$k_{\vec{q},\psi}(x,t) = 0 \quad \text{on } \partial B_1\,, \tag{4.16}$$

$$k_{\vec{q},\psi}(x,0) = \psi \quad \text{in } B_1\,, \tag{4.17}$$

where $Pk_{\vec{q},\psi} = dk_{\vec{q},\psi}/dt - \Delta k_{\vec{q},\psi}$. In order to simplify the notation, when $\psi \equiv 0$ we will simply denote this solution by $k_{\vec{q}}$ (instead of $k_{\vec{q},0}$). In Lemma 4.9 we show that

$$\lim_{\varepsilon\to 0} \sup_{B_1\times[0,\infty]} |k_{\vec{q}}(x,t)| = 0.$$

In order to do this computation we will use several estimates from Appendix A. Thereafter we will conclude the proof of Theorem 4.1 by showing in Lemma 4.10 that for every fixed ε there is a sequence $0 < t_n \nearrow \infty$ satisfying

$$\lim_{n\to\infty} \sup_{B_1} |k_{\vec{q}}(x, t_n) - u_\varepsilon + U_{\vec{q}}| = 0.$$

Lemma 4.9. *Let $k_{\vec{q}}$ be the solution to (4.15)–(4.16)–(4.17), for $\psi = 0$. Then*

$$\lim_{\varepsilon\to 0} \sup_{B_1\times[0,\infty]} |k_{\vec{q}}(x,t)| = 0.$$

Proof. Suppose that

$$\sup_{B_1\times[0,\infty)} |k_{\vec{q}}| \not\to 0.$$

Theorem 4.8 implies that (by choosing $w_\varepsilon = \psi_\varepsilon = 0$)

$$\sup_{B_1\times[0,\infty)} |k_{\vec{q}}| \le C \sup_{B_1\times[0,\infty)} |\mathcal{L}_{\vec{q}}(0,0)|. \tag{4.18}$$

Set $S_\varepsilon = \sup_{B_1\times[0,\infty)} |\mathcal{L}_{\vec{q}}(0,0)|$ (possibly infinity). Fix $\delta > 0$ and notice that, by definition of supremum, there is a t_ε such that

$$\sup_{x\in B_1} |\mathcal{L}_{\vec{q}}(0,0)(x,t_\varepsilon) - S_\varepsilon| \le \delta$$

(or when $S_\varepsilon = \infty$ pick t_ε such that $\sup_{x\in B_1} |\mathcal{L}_{\vec{q}}(0,0)(x,t_\varepsilon)| \ge \delta^{-1}$).

We will show that, independently of δ, holds $\sup_{x\in B_1} |\mathcal{L}_{\vec{q}}(0,0)|(x,t_\varepsilon) \to 0$ as $\varepsilon \to 0$ (notice that this immediately contradicts $S_\varepsilon = \infty$). Recall first that

$$\begin{aligned}\mathcal{L}_{\vec{q}}(0,0)(x,t) \\ = \int_0^t \int_{B_1} \mathcal{H}_{B_1}(x,y,t-s)\left(-\frac{\nabla_v W(U_{\vec{q}})(y,s)}{\varepsilon^2} + \Delta U_{\vec{q}}(y)\right) \mathrm{d}y\, \mathrm{d}s.\end{aligned} \tag{4.19}$$

Notice that for $|x| \le \varepsilon^\alpha - E$ we have

$$\frac{-\nabla_v W(U_{\vec{q}})}{\varepsilon^2} + \Delta U_{\vec{q}} = \frac{-\nabla_v W(u^\varepsilon_\sigma)}{\varepsilon^2} + \Delta u^\varepsilon_\sigma = 0.$$

Hence, (4.19) implies

$$|\mathcal{L}_{\vec{q}}(0,0)|(x,t) \le I_1(x,t) + I_2(x,t), \tag{4.20}$$

where

$$I_1(x,t) = \int_0^t \int_{\{|y|\geq \varepsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s) \left| \frac{-\nabla_v W(\varphi_\varepsilon)}{\varepsilon^2} + \Delta\varphi_\varepsilon \right| (y,s)\, dy\, ds,$$

$$I_2(x,t) = \int_0^t \int_{\{\varepsilon^\alpha - E\leq |y|\leq \varepsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s)$$
$$\times \left| \frac{-\nabla_v W(U_{\vec{q}})}{\varepsilon^2} + \eta_\alpha^\varepsilon \Delta\varphi_\varepsilon + \Delta(\eta_\alpha^\varepsilon)(h_\sigma^\varepsilon - \varphi_\varepsilon) + \nabla(\eta_\alpha^\varepsilon)\cdot D(u_\sigma^\varepsilon - \varphi_\varepsilon) \right|$$
$$(y,s)\, dy\, ds\,.$$

Now we find bounds for I_1 and I_2. For each of these integrals we will consider two ranges for the variable t, namely $t \leq T$ and $t \geq T$, where $T > 0$ is any fixed positive constant.

Bounds over I_1: Since $\varepsilon < \varepsilon^\alpha$ (when $\varepsilon < 1$) we have that for every $|x| \geq \varepsilon^\alpha$ the function $\eta(x) \equiv 0$, and for such x we have

$$\Delta\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\Bigg(\varepsilon^2 \Delta\eta_6 \zeta_{31}\left(d_0\left(\frac{x}{\varepsilon}\right)\right) + \eta_6 \zeta_{31}''\left(d_0\left(\frac{x}{\varepsilon}\right)\right)$$
$$+ 2\varepsilon \nabla\eta_6 \cdot \nabla d_0\left(\frac{x}{\varepsilon}\right)\zeta_6'\left(d_0\left(\frac{x}{\varepsilon}\right)\right) + \varepsilon^2\Delta\eta_5 c_i$$
$$+ \sum_{i=1}^{3} \varepsilon^2 \Delta\eta_{2i}\zeta_{ii+1}\left(d_i\left(\frac{x}{\varepsilon}\right)\right) + \eta_{2i}\zeta_{ii+1}''\left(d_i\left(\frac{x}{\varepsilon}\right)\right)$$
$$+ 2\varepsilon\nabla\eta_{2i}\cdot\nabla d_i\left(\frac{x}{\varepsilon}\right)\zeta_{ii+1}'\left(d_i\left(\frac{x}{\varepsilon}\right)\right) + \varepsilon^2 \Delta\eta_{2i-1}c_i\Bigg).$$

Since the functions η_j depend only on the angle θ, we have that

$$\Delta\eta_j = \frac{\eta_j''}{r^2} \quad \text{and} \quad |\nabla\eta_j| \leq |\eta_j'|.$$

In particular for $|x| \geq \varepsilon^\alpha$

$$|\Delta\eta_j| \leq \frac{4|\eta_j''|}{\varepsilon^{2\alpha}} \quad \text{and} \quad |\nabla\eta_j| \leq |\eta_j'|.$$

Recall that for $\theta \in [\theta_i - \delta/2, \theta_i + \delta/2]$ we have $\eta_{2i} \equiv 1$ and $\eta_j \equiv 0$ for every $j \neq 2i$. Then

$$\frac{\nabla_v W(\varphi_\varepsilon)}{\varepsilon^2} + \Delta\varphi_\varepsilon = 0 \quad \text{for } \theta \in \left[\theta_i - \frac{\delta}{2}, \theta_i + \frac{\delta}{2}\right]. \tag{4.21}$$

Now we need to find bounds for $\theta \in [\theta_i + \delta/2, \theta_{i+1} - \delta]$. Notice first that

$$|\Delta\eta(\theta)| = \left|\frac{\eta''(\theta)}{r^2}\right| \leq \frac{K}{r^2} \leq \frac{K}{\varepsilon^{2\alpha}} \quad \text{for } |x| \geq \varepsilon^\alpha, \tag{4.22}$$

$$|\nabla\eta| = \left|\frac{\eta'}{r}\right| \leq \frac{K}{r} \leq \frac{K}{\varepsilon^\alpha} \quad \text{for } |x| \geq \varepsilon^\alpha. \tag{4.23}$$

Notice also that $\eta_j \neq 0$ only for $j = 21, 2i-1$ and $\eta_{2i} + \eta_{2i-1} = 1$. Hence

$$\begin{aligned}
\Delta\varphi_\varepsilon = \frac{1}{\varepsilon^2}\Big(&\varepsilon^2\Delta\eta_{2i}(\theta)\zeta_{ii+1}\left(d_i\left(\frac{x}{\varepsilon}\right)\right) + \eta_{2i}(\theta)\zeta''_{ii+1}\left(d_i\left(\frac{x}{\varepsilon}\right)\right) \\
&+ 2\varepsilon\nabla\eta_{2i}(\theta)\cdot\nabla d_i\left(\frac{x}{\varepsilon}\right)\zeta'_{ii+1}\left(d_i\left(\frac{x}{\varepsilon}\right)\right) + \varepsilon^2\Delta\eta_{2i-1}(\theta)c_i\Big) \\
= \Delta\eta_{2i}(\theta)&\left(\zeta_{ii+1}\left(d_i\left(\frac{x}{\varepsilon}\right)\right) - c_i\right) + \eta_{2i}(\theta)\frac{-\nabla_v W(\zeta_{ii+1})}{\varepsilon^2}\left(d_i\left(\frac{x}{\varepsilon}\right)\right) \\
&+ 2\frac{1}{\varepsilon}\nabla\eta_{2i}(\theta)\cdot\nabla d_i\left(\frac{x}{\varepsilon}\right)\zeta'_{ii+1}\left(d_i\left(\frac{x}{\varepsilon}\right)\right).
\end{aligned}$$

Using Hypothesis 2.1 we have that there are constants $K, c > 0$ such that

$$\left|\frac{\nabla_v W(\varphi_\varepsilon)}{\varepsilon^2} + \Delta\varphi_\varepsilon\right| \leq K\frac{e^{-cd_i/\varepsilon}}{\varepsilon^2} \quad \text{for } |x| \geq \varepsilon^\alpha \text{ and } \theta \in \left[\theta_i + \frac{\delta}{2}, \theta_{i+1} - \frac{\delta}{2}\right]. \tag{4.24}$$

Furthermore, for $|x| > \varepsilon^\alpha$ and $\theta \in [\theta_i+\delta/2, \theta_{i+1}-\delta/2]$ we have $|d_i| \geq \varepsilon^\alpha \sin\delta$. Hence,

$$\left|\frac{\nabla_v W(\varphi_\varepsilon)}{\varepsilon^2} + \Delta\varphi_\varepsilon\right| \leq K\frac{e^{-(c\varepsilon^\alpha\sin\delta)/\varepsilon}}{\varepsilon^2} \quad \text{for } |x| > \varepsilon^\alpha \text{ and } \theta \in \left[\theta_i + \frac{\delta}{2}, \theta_{i+1} - \frac{\delta}{2}\right]. \tag{4.25}$$

Now we proceed to find bounds in two different cases:

(1) Suppose that $t \leq T$. Equations (4.21) and (4.25) imply

$$I_1(x,t) \leq K\frac{e^{-(c\varepsilon^\alpha\sin\delta)\varepsilon}}{\varepsilon^2}\int_0^t\int_{\{|x|\geq\varepsilon^\alpha\}}\mathcal{H}_{B_1}(x,y,t-s)\,dy\,ds.$$

Using Lemma A.1 we have

$$I_1(x,t) \leq KT\frac{e^{-(c\varepsilon^\alpha\sin\delta)/\varepsilon}}{\varepsilon^2} \quad \text{for every } x \in B_1 \text{ and } 0 \leq t \leq T. \tag{4.26}$$

(2) Suppose that $t \geq T$. Let

$$f_\varepsilon = \left| \frac{\nabla_v W(\varphi_\varepsilon)}{\varepsilon^2} + \Delta\varphi_\varepsilon \right|$$

and fix $\delta > 0$. Now we divide I_1 in the three following integrals:

$$I_{11}(x,t) = \int_0^{t-\delta} \int_{\{|y|\geq\varepsilon^\alpha\}\cap\{|x-y|\leq\sqrt{t-s}/t\}} \mathcal{H}_{B_1}(x,y,t-s) f_\varepsilon(y,s)\, \mathrm{d}y\, \mathrm{d}s,$$

$$I_{12}(x,t) = \int_0^{t-\delta} \int_{\{|y|\geq\varepsilon^\alpha\}\cap\{|x-y|\geq\sqrt{t-s}/t\}} \mathcal{H}_{B_1}(x,y,t-s) f_\varepsilon(y,s)\, \mathrm{d}y\, \mathrm{d}s,$$

$$I_{13}(x,t) = \int_{t-\delta}^{\delta} \int_{\{|y|\geq\varepsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s) f_\varepsilon(y,s)\, \mathrm{d}y\, \mathrm{d}s.$$

Then

$$I_1 = I_{11} + I_{12} + I_{13}.$$

By Theorem A.2 we have that $|\mathcal{H}_{B_1}(x,y,t-s)| \leq C/(t-s)$; then

$$\begin{aligned} I_{11}(x,t) &\leq C \int_0^{t-\delta} \frac{\sup_{|y|\geq\alpha} f_\varepsilon}{(t-s)} \frac{(t-s)}{t^2} \pi\, \mathrm{d}s \\ &= \frac{\sup_{|y|\geq\alpha} f_\varepsilon}{t^2}(t-\delta) \\ &\leq C \frac{e^{-cd_i/\varepsilon}}{t\varepsilon^2}. \end{aligned}$$

Using again Theorem A.2, for $|x-y| \geq \sqrt{t-s}/t$ we have

$$|\mathcal{H}_{B_1}(x,y,t-s)| = O([1/t]^{-\infty}).$$

In particular, there is a constant C such that $|\mathcal{H}_{B_1}(x,y,t-s)| \leq C/t$; then

$$\begin{aligned} I_{12}(x,t) &\leq \int_0^{t-\delta} \frac{C}{t} \int_{B_1} f_\varepsilon(y)\, \mathrm{d}y \\ &\leq t\frac{C}{t} \int_{B_1} f_\varepsilon(y)\, \mathrm{d}y \\ &\leq C \frac{e^{-cd_i/\varepsilon}}{\varepsilon^2}. \end{aligned}$$

Finally, using Lemma A.1 we have

$$I_{13}(x,t) \leq \delta \sup f_\varepsilon \leq C \frac{e^{-cd_i/\varepsilon}}{\varepsilon^2}.$$

Combining the previous estimates we obtain

$$I_1(x,t) \le C\frac{e^{-cd_i/\varepsilon}}{\varepsilon^2} \quad \text{for every } x \in B_1 \text{ and } t \ge T. \tag{4.27}$$

Bounds over I_2: Using the definitions of $U_{\vec{q}}$, φ_ε, Theorem A.3, and Lemma 2.4, we have

$$\left| \frac{-\nabla_v W(U_{\vec{q}})}{\varepsilon^2} + \eta_\alpha^\varepsilon \Delta\varphi_\varepsilon \right| \le \frac{C}{\varepsilon^2},$$

$$|\Delta(\eta_\alpha^\varepsilon)\,(h_\sigma^\varepsilon - \varphi_\varepsilon)| \le \frac{C}{E^2},$$

$$|\nabla(\eta_\alpha^\varepsilon)\cdot D(u_\sigma^\varepsilon - \varphi_\varepsilon)| \le \frac{C}{E\varepsilon}.$$

Hence:

(1) For $t \le T$

$$I_2(x,t) \le C\int_0^t \int_{\varepsilon^\alpha - E \le |x| \le \varepsilon^\alpha} \mathcal{H}(x,y,t-s)\left(\frac{1}{\varepsilon^2} + \frac{1}{E^2} + \frac{1}{E\varepsilon}\right) dy\, ds.$$

Theorem A.2 implies that for $t-s \ge \varepsilon^{m+2}$ there is a constant C independent of x, y such that $|\mathcal{H}(x,y,t-s)| \le C/\varepsilon^{m+2}$. Moreover, by definition $\varepsilon^\alpha \le E = \varepsilon^\alpha(2-\varepsilon^{2m+4}) \le 2\varepsilon^\alpha$. Hence

$$\begin{aligned} I_2 \le& \int_0^{t-\varepsilon^{m+2}} \int_{\varepsilon^\alpha - E\le|x|\le\varepsilon^\alpha} \frac{C}{\varepsilon^{m+2}}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon^{2\alpha}}+\frac{1}{\varepsilon^{1+\alpha}}\right) dy\, ds \\ &+ \int_{t-\varepsilon^{m+2}}^t \int_{\varepsilon^\alpha - E\le|x|\le\varepsilon^\alpha} \mathcal{H}_{B_1}(x,y,t-s)\frac{1}{\varepsilon^2}\left(1+\varepsilon^{2-2\alpha}+\varepsilon^{1-\alpha}\right) dy\, ds \\ \le& \frac{C}{\varepsilon^{m+4}}\int_0^{t-\varepsilon^{m+2}} (1+\varepsilon^{2-2\alpha}+\varepsilon^{1-\alpha})\pi(\varepsilon^{2\alpha}-(\varepsilon^\alpha - E)^2)\, ds \\ &+ \frac{C}{\varepsilon^2}\int_{t-\varepsilon^{m+2}}^t \int_{B_1} \mathcal{H}(x,y,t-s)\, dy\, ds. \end{aligned}$$

Using that $t \le T$, Lemma A.1 and the definition of E, we conclude

$$\begin{aligned} I_2(x,t) &\le \frac{C}{\varepsilon^{m+4}}E(2\varepsilon^\alpha - E) + \frac{C}{\varepsilon^2}\varepsilon^{m+2} \le \frac{C}{\varepsilon^{m+4}}\varepsilon^\alpha\varepsilon^{2m+4-\alpha} + C\varepsilon^m \\ &\Rightarrow\ I_2(x,t) \le C\varepsilon^m \quad \text{for } x \in B_1 \text{ and } 0 \le t \le T. \end{aligned} \tag{4.28}$$

(2) For $t \geq T$, the previous estimates show that the integrand of I_2 can be bounded by C/ε^2. Dividing up the integral as we did for I_1, we obtain

$$\begin{aligned} I_2 \leq & \int_0^{t-\varepsilon^{m+2}} \int_{\{\varepsilon^\alpha - E \leq |y| \leq \varepsilon^\alpha\} \cap \{|x-y| \leq \sqrt{t-s}/t\}} \mathcal{H}_{B_1}(x,y,t-s) \frac{C}{\varepsilon^2} \, dy \, ds \\ & + \int_0^{t-\varepsilon^{m+2}} \int_{\{\varepsilon^\alpha - E \leq |y| \leq \varepsilon^\alpha\} \cap \{|x-y| \geq \sqrt{t-s}/t\}} \mathcal{H}_{B_1}(x,y,t-s) \frac{C}{\varepsilon^2} \, dy \, ds \\ & + \int_{t-\varepsilon^{m+2}}^{t} \int_{\{\varepsilon^\alpha - E \leq |y| \leq \varepsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s) \frac{C}{\varepsilon^2} \, dy \, ds. \end{aligned}$$

Using Hölder's inequality in the first integral for $p < 2$ we get

$$\begin{aligned} I_2 \leq & \int_0^{t-\varepsilon^{m+2}} \left(\int_{\{\varepsilon^\alpha - E \leq |y| \leq \varepsilon^\alpha\}} \frac{C}{\varepsilon^{2p}} \, dy \right)^{1/p} \\ & \qquad \times \left(\int_{\{|x-y| \leq \sqrt{t-s}/t\}} \mathcal{H}^q_{B_1}(x,y,t-s) \, dy \right)^{1/q} ds \\ & + \int_0^{t-\varepsilon^{m+2}} \int_{\{\varepsilon^\alpha - E \leq |y| \leq \varepsilon^\alpha\} \cap \{|x-y| \geq \sqrt{t-s}/t\}} \mathcal{H}_{B_1}(x,y,t-s) \frac{1}{\varepsilon^2} \, dy \, ds \\ & + C \frac{\varepsilon^{m+2}}{\varepsilon^2}. \end{aligned}$$

As before, Theorem A.2 implies $|\mathcal{H}_{B_1}| \leq C/(t-s)$ and that for $|x-y| \geq (t-s)/t$ holds $\mathcal{H}_{B_1}(x,y,t-s) = O((1/t)^{-\infty})$, therefore

$$\begin{aligned} I_2 \leq C \Bigg[& \int_0^{t-\varepsilon^{m+2}} \left(\frac{C\varepsilon^{2m+4}}{\varepsilon^{2p}} \right)^{1/p} \frac{1}{t-s} \left(\frac{t-s}{t^2} \right)^{1/q} ds \\ & + \int_0^{t-\varepsilon^{m+2}} \int_{\{\varepsilon^\alpha - E \leq |y| \leq \varepsilon^\alpha\}} \frac{C}{t} \frac{\varepsilon^{2m+4}}{\varepsilon^2} + C \frac{\varepsilon^{m+2}}{\varepsilon^2} \Bigg] \\ \leq & \, C(\varepsilon^{2m+4-2p})^{1/p} \frac{t^{1/q} - \varepsilon^{(m+2)/q}}{t^{2/q}} + t \frac{C}{t} \varepsilon^{2m+2} + C\varepsilon^m. \end{aligned}$$

Therefore, for $t \geq T$ and $p < 2$ there holds

$$I_2(x,t) \leq C \left(\frac{\varepsilon^{(2m+4-2p)/p}}{T^{1/q}} + \varepsilon^2 + \varepsilon^2 \right) \leq C\varepsilon^2. \tag{4.29}$$

Now we can conclude the result of lemma by combining (4.26), (4.27), (4.28) in (4.29) in (4.20) and (4.18). More precisely:

$$\sup_{B_1 \times [0,\infty)} |k_{\vec{q}}| \leq C \frac{e^{-(c\varepsilon^\alpha \sin\delta)/\varepsilon}}{\varepsilon^2} + C\varepsilon^m \leq C\varepsilon^m,$$

where C depends on α and m. This implies the desired lemma. □

To finish the proof of Theorem 4.1 we need the following lemma.

Lemma 4.10. *Fix $\varepsilon > 0$ and let $k_{\vec{q}}$ be the solution of* (4.15)–(4.16)–(4.17). *Then, there is a sequence of times $t_n \nearrow \infty$ such that*

$$\lim_{n\to\infty} \sup_{B_1} |k_{\vec{q}}(x, t_n) - u_\varepsilon(x) + U_{\vec{q}}(x)| = 0.$$

Proof. Corollary A.8 in the appendix shows that for every $t > 0$ there is a constant C such that $|Dk_{\vec{q}}(x,t)| \le C/\varepsilon$. Similarly, by taking derivatives on the equation, we can find bounds over the second and third space derivatives (these bounds will depend on ε). Since ε is fixed, using Arzela-Ascoli's theorem we conclude that for every sequence $t_n \nearrow \infty$ there is a subsequence $k_{\vec{q}}(x, t_n)$ that converges in C^2. Let us denote this limit by $k_{\vec{q}}^\infty(x)$ and the convergent subsequence $\{t_n\}_{n\in\mathbb{N}}$ as well.

We will show that $k_{\vec{q}}^\infty(x)$ satisfies

$$\Delta k_{\vec{q}}^\infty(x) = \frac{\nabla_v W(k_{\vec{q}}^\infty + U_{\vec{q}})}{\varepsilon^2} - \Delta U_{\vec{q}} \quad \text{for } x \in B_1, \tag{4.30}$$

$$k_{\vec{q}}^\infty(x) = 0 \quad \text{for } x \in \partial B_1. \tag{4.31}$$

First we need to show that for every $\tau > 0$ the sequence $k_{\vec{q}}(x, t_n + \tau)$ also converges in C^2 to $k_{\vec{q}}^\infty(x)$. Define

$$\mathcal{J}(t) = \int_{B_1} \left(\frac{|\nabla k_{\vec{q}}|^2}{2} + \frac{W(k_{\vec{q}} + U_{\vec{q}})}{\varepsilon^2} - k_{\vec{q}} \cdot \Delta U_{\vec{q}} \right)(x,t)\, dx.$$

Using Theorem A.3 and the definition of $U_{\vec{q}}$, it is easy to see that $\mathcal{J}(t)$ is bounded below for every t. Moreover, taking the time derivative we have

$$\begin{aligned}
\frac{d\mathcal{J}}{dt} &= \int_{B_1} \left(\nabla k_{\vec{q}} \cdot \nabla (k_{\vec{q}})_t + \frac{\nabla W(k_{\vec{q}} + U_{\vec{q}})}{\varepsilon^2} \cdot (k_{\vec{q}})_t - \Delta U_{\vec{q}} \cdot (k_{\vec{q}})_t \right)(x,t)\, dx \\
&= \int_{B_1} \left[\left(-\Delta k_{\vec{q}} + \frac{\nabla W(k_{\vec{q}} + U_{\vec{q}})}{\varepsilon^2} - \Delta U_{\vec{q}} \right) \cdot (k_{\vec{q}})_t \right](x,t)\, dx \\
&= -\int_{B_1} |(k_{\vec{q}})_t|^2 (x,t)\, dx.
\end{aligned}$$

Therefore $\mathcal{J}$ is bounded below and decreasing, hence it converges. Moreover, for every fixed $\tau > 0$

$$\int_{t_n}^{t_n+\tau} \int_{B_1} |k_{\vec{q}}|_t^2 (x,s)\, dx\, ds = \mathcal{J}(t_n) - \mathcal{J}(t_n + \tau) \to 0.$$

Since for every fixed x we can write $k_{\vec{q}}(x,t_n+\tau)-k_{\vec{q}}(x,t_n)=\int_{t_n}^{t_n+\tau}(k_{\vec{q}})_t(x,s)\,\mathrm{d}s$, we have that

$$\int_{B_1}|k_{\vec{q}}(x,t_n+\tau)-k_{\vec{q}}(x,t_n)|\,\mathrm{d}x \le \int_{t_n}^{t_n+\tau}\int_{B_1}|(k_{\vec{q}})_t|(x,s)\,\mathrm{d}x\,\mathrm{d}s$$
$$\le C\left(\int_{t_n}^{t_n+\tau}\int_{B_1}|(k_{\vec{q}})_t|^2(x,s)\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}\to 0$$
$$\text{as } n\to\infty\,.$$

Hence $k_{\vec{q}}(x,\ t_n+\tau)-k_{\vec{q}}(x,t_n)$ converges to 0 almost everywhere. Let us show that this convergence is also uniform. Suppose that $\sup_{x\in B_1}|k_{\vec{q}}(x,t_n+\tau)-k_{\vec{q}}(x,t_n)|\not\to 0$ as $n\to\infty$. Then there is a $\delta>0$ and a subsequence of times such that

$$\sup_{x\in B_1}|k_{\vec{q}}(x,t_n+\tau)-k_{\vec{q}}(x,t_n)|\ge\delta. \tag{4.32}$$

As before, there is subsequence of these $\{t_n\}$ such that $k_{\vec{q}}(x,t_n+\tau)-k_{\vec{q}}(x,t_n)$ converges uniformly. Since it converges almost everywhere to 0, the uniform limit must be 0 contradicting (4.32).

Since $\mathcal{J}(t_n)-\mathcal{J}(t_n+\tau)\to 0$, from the definition for $\mathcal{J}$ and the previous estimate we can see that

$$\int_{B_1}(|\nabla k_{\vec{q}}|^2(x,t_n)-|\nabla k_{\vec{q}}|^2(x,t_n+\tau))\,\mathrm{d}x\to 0\quad\text{as } n\to\infty.$$

As above, we can conclude that this convergence is almost everywhere and uniform. Standard parabolic estimates imply also that $k_{\vec{q}}(x,t_n+\tau)-k_{\vec{q}}(x,t_n)$ converges in the C^2 norm.

Now we can prove that $k_{\vec{q}}^{\infty}$ is a solution to the elliptic equation (4.30). Since $k_{\vec{q}}$ solves equation (4.15)–(4.16)–(4.17), we have that for any $\varphi\in C^\infty(B_1)$

$$\int_{B_1}(k_{\vec{q}}(y,t_n+1)-k_{\vec{q}}(y,t_n))\varphi(y)\,\mathrm{d}y$$
$$=\int_{t_n}^{t_n+1}\int_{B_1}\left(\Delta k_{\vec{q}}(y,t_n+\tau)-\frac{\nabla_v W(k_{\vec{q}}^{\infty})}{\varepsilon^2}(y,t_n+\tau)-\Delta U_{\vec{q}}\right)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\tau.$$

Letting $n\to\infty$, we get

$$\int_{B_1}\left(\Delta k_{\vec{q}}^{\infty}-\frac{\nabla_v W(k_{\vec{q}}^{\infty})}{\varepsilon^2}-\Delta U_{\vec{q}}\right)\varphi(y)\,\mathrm{d}y=0.$$

Moreover, since for every t there holds $k_{\vec{q}}(x,t)|_{\partial B_1} = 0$ it must hold that $k^{\infty}_{\vec{q}}|_{\partial B_1} = 0$. Uniqueness of solutions implies that necessarily $k^{\infty}_{\vec{q}} \equiv u_\varepsilon - U_{\vec{q}}$, which proves the lemma. □

Now the proof of Theorem 4.1 is direct.

Proof of Theorem 4.1. Fix $\varepsilon > 0$ and $m > 0$. Consider t_n as in Lemma 4.10, then

$$\begin{aligned}\sup_{B_1}|u_\varepsilon - U_{\vec{q}}| &\leq \sup_{B_1}|u_\varepsilon(x) - U_{\vec{q}}(x) - k_{\vec{q}}(x,t_n)| + \sup_{B_1\times[0,\infty)}|k_{\vec{q}}(x,t)| \\ &\leq \sup_{B_1}|u_\varepsilon(x) - U_{\vec{q}}(x) - k_{\vec{q}}(x,t_n)| + C\varepsilon^m.\end{aligned}$$

Taking $t_n \to \infty$, we have

$$\sup_{B_1}|u_\varepsilon - U_{\vec{q}}| \leq C\varepsilon^m.$$

Recalling the definition of $U_{\vec{q}}$ we have the result. □

It is easy to see that the size of the radius of the inner ball in Theorem 4.1 (that is the ball where $u_\varepsilon(x) - u_\sigma(\sigma x/\varepsilon)$ converges to 0) can be increased to ε^α. Namely, we let

$$\tilde{U}_{\vec{q}}(y) = \tilde{\eta}^\varepsilon_\alpha(y)\varphi_\varepsilon(y) + (1 - \tilde{\eta}^\varepsilon_\alpha(y))u_\sigma(y),$$

where $\tilde{\eta} : \mathbb{R} \to \mathbb{R}$ is a positive function such that $\tilde{\eta}(x) = 0$ for $|x| \leq 1$ and $\tilde{\eta}(x) = 2$ for $|x| \geq 1$ and

$$\tilde{\eta}^\varepsilon_\alpha(y) = \tilde{\eta}\left(\frac{1}{2\tilde{E}}|y| + 2 - \frac{2\varepsilon^\alpha}{2\tilde{E}}\right),$$

with $\tilde{E} = 4\varepsilon^\alpha - \varepsilon^{2m+4-\alpha}$. As before, $\alpha > 0$.

Notice that

$$\tilde{U}_{\vec{q}}(x) = \begin{cases} \varphi_\varepsilon(x) & \text{for } |x| \geq 2\varepsilon^\alpha, \\ u_\sigma\left(\dfrac{\sigma x}{\varepsilon}\right) & \text{for } |x| \leq 2\varepsilon^\alpha - E. \end{cases}$$

Hence, following the proof of Theorem 4.1, but changing $U_{\vec{q}}$ for $\tilde{U}_{\vec{q}}$ we have the following result.

Corollary 4.11. *Fix $0 < \alpha < 1$. Let $0 < \sigma \leq 2\varepsilon^{1-\alpha}$. Then for every $m > 0$ there is a constant C (that might depend on α and m) such that*

- $\sup_{|x|\geq 2\varepsilon^\alpha}|u_\varepsilon - \varphi_\varepsilon| \leq C\varepsilon^m$.
- $\sup_{|x|\leq \varepsilon^\alpha}|u_\varepsilon(x) - u_\sigma(\sigma x/\varepsilon)| \leq C\varepsilon^m$.

Using Lemmas 2.4 and 2.5 we can also prove the following result.

Corollary 4.12. *Fix $0 < \alpha < 1$. Let $0 < \sigma \leq 2\varepsilon^{1-\alpha}$. Then for every $m > 0$ there is a constant C (that might depend on α and m) such that*

- $\sup_{|x|\geq\varepsilon^\alpha} |Du_\varepsilon - D\varphi_\varepsilon| \leq C\varepsilon^m$.
- $\sup_{|x|\leq\varepsilon^\alpha} |Du_\varepsilon(x) - (\sigma/\varepsilon)Du_\sigma(\sigma x/\varepsilon)| \leq C\varepsilon^m$.

Proof. We start by proving the first inequality of the corollary. To prove this inequality we estimate separately in two different sets, namely we first prove the inequality for $x \in B_{1-\varepsilon^\alpha/2} \setminus B_{\varepsilon^\alpha}$ and then for $x \in B_1 \setminus B_{1-\varepsilon^\alpha/2}$ (in fact, in the second step we find a bound in a larger set: $B_1 \setminus B_{3/4}$).

We consider the function $u_\varepsilon - \varphi_\varepsilon$ in the domain $B_1 \setminus B_{\varepsilon/2}$. Then

$$\Delta(u_\varepsilon - \varphi_\varepsilon) = \frac{\nabla W(u_\varepsilon) - \nabla W(\varphi_\varepsilon)}{\varepsilon^2} - \Delta\varphi_\varepsilon + \frac{\nabla W(\varphi_\varepsilon)}{\varepsilon^2}.$$

Using Lemma 2.4 we have for every $x \in B_{1-\varepsilon^\alpha/2} \setminus B_{\varepsilon^\alpha}$ that

$$\begin{aligned}
&|D(u_\varepsilon - \varphi_\varepsilon)|^2(x)\\
&\quad\leq C\Bigg(\sup_{|x|\geq\varepsilon^\alpha/2} |u_\varepsilon - \varphi_\varepsilon| \sup_{|x|\geq\varepsilon^\alpha/2} \left| \frac{\nabla W(u_\varepsilon) - \nabla W(\varphi_\varepsilon)}{\varepsilon^2} - \Delta\varphi_\varepsilon + \frac{\nabla W(\varphi_\varepsilon)}{\varepsilon^2} \right|\\
&\qquad\qquad + \frac{1}{\varepsilon^\alpha} \sup_{|x|\geq\varepsilon^\alpha/2} |u_\varepsilon - \varphi_\varepsilon|^2 \Bigg)\\
&\quad\leq C\Bigg(\frac{M}{\varepsilon^2} \sup_{|x|\geq\varepsilon^\alpha/2} |u_\varepsilon - \varphi_\varepsilon|^2 + |u_\varepsilon - \varphi_\varepsilon| \sup_{|x|\geq\varepsilon^\alpha/2} \left| - \Delta\varphi_\varepsilon + \frac{\nabla W(\varphi_\varepsilon)}{\varepsilon^2} \right|\\
&\qquad\qquad + \frac{1}{\varepsilon^\alpha} \sup_{|x|\geq\varepsilon^\alpha/2} |u_\varepsilon - \varphi_\varepsilon|^2 \Bigg).
\end{aligned}$$

Using Theorem 4.1 and the estimates for $|-\Delta\varphi_\varepsilon + \nabla W(\varphi_\varepsilon)/\varepsilon^2|$ in its proof we have for $m > 0$ a constant C (that depends on m and α) such that

$$|D(u_\varepsilon - \varphi_\varepsilon)|^2(x) \leq C\varepsilon^m,$$

for $x \in B_{1-\varepsilon^\alpha/2} \setminus B_{\varepsilon^\alpha}$.

In order to find bounds for $x \in B_1 \setminus B_{1-\varepsilon^\alpha/2}$ we consider a smooth function η such that $\eta(x) \equiv 1$ for $x \geq \frac{3}{4}$ and $\eta \equiv 0$ for $x \leq \frac{1}{2}$ and we consider the function $\eta(u_\varepsilon - \varphi_\varepsilon)$ (notice that in fact this will provide bounds in a larger set, namely $B_1 \setminus B_{3/4}$). Then $\eta(u_\varepsilon - \varphi_\varepsilon)$ satisfies

$$\begin{aligned}
\Delta(\eta(u_\varepsilon - \varphi_\varepsilon)) &= \Delta\eta(u_\varepsilon - \varphi_\varepsilon) + \nabla\eta\nabla(u_\varepsilon - \varphi_\varepsilon)\\
&\qquad + \eta\left(\frac{\nabla W(u_\varepsilon) - \nabla W(\varphi_\varepsilon)}{\varepsilon^2} - \Delta\varphi_\varepsilon + \frac{\nabla W(\varphi_\varepsilon)}{\varepsilon^2} \right).
\end{aligned}$$

Lemma 2.5, Theorem 4.1 and the previous estimates imply that

$$|D(\eta(u_\varepsilon - \varphi_\varepsilon))|^2(x) = |D(u_\varepsilon - \varphi_\varepsilon)|^2(x) \le C\varepsilon^m \quad \text{for } \frac{3}{4} \le |x| \le 1,$$

finishing the proof of the first inequality.

Now we need to prove the second inequality. Let $u_\sigma^\varepsilon(x) = u_\sigma(\sigma x/\varepsilon)$. To prove the second estimate we consider $u_\varepsilon(x) - u_\sigma^\varepsilon(x)$ in $B_{3\varepsilon^\alpha/2}$. Since

$$\Delta(u_\varepsilon - u_\sigma^\varepsilon) = \frac{\nabla W(u_\varepsilon) - \nabla W(u_\sigma^\varepsilon)}{\varepsilon^2},$$

Lemma 2.4 implies for every $x \in B_{\varepsilon^\alpha}$

$$\begin{aligned} &|D(u_\varepsilon - u_\sigma^\varepsilon)|^2 \\ &\le C\left(\sup_{|x|\le\varepsilon^\alpha/2}|u_\varepsilon - u_\sigma^\varepsilon| \sup_{|x|\le\varepsilon^\alpha/2}\left|\frac{\nabla W(u_\varepsilon) - \nabla W(u_\sigma^\varepsilon)}{\varepsilon^2}\right| + \frac{1}{\varepsilon^\alpha}\sup_{|x|\le\varepsilon^\alpha/2}|u_\varepsilon - u_\sigma^\varepsilon|^2\right) \\ &\le C\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^\alpha}\right)\sup_{|x|\le\varepsilon^\alpha/2}|u_\varepsilon - u_\sigma^\varepsilon|^2. \end{aligned}$$

Corollary 4.11 implies that for every $m > 0$ there is a constant C such that

$$|D(u_\varepsilon - u_\sigma^\varepsilon)|^2 \le C\varepsilon^m,$$

which finishes the proof. □

5. PROOF OF THEOREM 1.1

Let

$$v_\varepsilon(x) = u_\varepsilon(\varepsilon x). \tag{5.1}$$

It holds

$$\begin{aligned} -\Delta v_\varepsilon + \nabla_v W(v_\varepsilon) = 0 &\quad \text{for } x \in B_{1/\varepsilon}, \\ v_\varepsilon(x) = \varphi(x) &\quad \text{for } x \in \partial B_{1/\varepsilon}. \end{aligned}$$

We define the following sequence of continuous functions $\tilde{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$

$$\tilde{v}_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{for } |x| \le 1/\varepsilon, \\ \varphi(x) & \text{if } |x| \ge 1/\varepsilon. \end{cases} \tag{5.2}$$

We will divide the proof of Theorem 1.1 into two different theorems: Theorem 5.1 and Theorem 5.2. First we prove the following result.

Theorem 5.1. *There is a subsequence of $\tilde{v}_\varepsilon$ such that $\tilde{v}_\varepsilon \to v$ uniformly on compact sets as $\varepsilon \to 0$ and v satisfies*

$$-\Delta v + \nabla_v W(v) = 0 \quad \text{for } x \in \mathbb{R}^2, \tag{5.3}$$

$$\lim_{|x|\to\infty} |v(x) - \varphi(x)| = 0. \tag{5.4}$$

Proof. Recall first that $\tilde{v}_\varepsilon$ is given by (5.2). We will use the following strategy to prove Theorem 5.1:

(1) Using the results of Section 4, we show that $\tilde{v}_\varepsilon$ is a Cauchy sequence in the sup norm. Therefore, $\tilde{v}_\varepsilon$ has a uniform limit v.
(2) Using the definition of $\tilde{v}_\varepsilon$ and the first step, we show that the limit v satisfies (5.4).
(3) Finally, we represent v_ε via Green's formula in compact sets. Taking limits, we conclude that v satisfies (5.3).

Now we prove these steps:

Proof of Step 1: $\{\tilde{v}_\varepsilon\}$ is a Cauchy sequence in the sup norm. Consider $\delta > 0$ and take $0 < \sigma < \varepsilon < 1$. We will show that there is an ε_0 such that for every $0 < \sigma < \varepsilon < \varepsilon_0$

$$|\tilde{v}_\varepsilon(x) - \tilde{v}_\sigma(x)| \le \delta \quad \text{for every } x \in \mathbb{R}^2.$$

We will mainly use Theorem 4.1 and Corollary 4.11 with $\alpha = \frac{1}{2}$.

- If $|x| \le \varepsilon^{-1/2}$:
  Notice first, that also holds $|x| \le \sigma^{-1/2}$ (since $\sigma < \varepsilon$). By the definitions of $\tilde{v}_\varepsilon$ and v_ε we have that

$$\begin{aligned}\tilde{v}_\varepsilon(x) - \tilde{v}_\sigma(x) &= u_\varepsilon(\varepsilon x) - u_\sigma(\sigma x)\\ &= u_\varepsilon(y) - u_\sigma\left(\frac{\sigma y}{\varepsilon}\right),\end{aligned}$$

where $y = \varepsilon x$. Notice that $|y| = \varepsilon|x| \le \varepsilon^{1/2}$. Corollary 4.11 implies that there is a ε_0 such that for every $\varepsilon < \varepsilon_0$

$$|\tilde{v}_\varepsilon(x) - \tilde{v}_\sigma(x)| \le \delta \quad \text{for } |x| \le \varepsilon^{-1/2}. \tag{5.5}$$

- If $|x| \ge \varepsilon^{-1/2}$ and $|x| \ge \sigma^{-1/2}$:
  By the definition of φ and φ_ε we have that

$$\varphi(x) = \varphi_\varepsilon(\varepsilon x) = \varphi_\sigma(\sigma x).$$

This implies

$$|\tilde{v}_\varepsilon(x) - \tilde{v}_\sigma(x)| \le |\tilde{v}_\varepsilon(x) - \varphi_\varepsilon(\varepsilon x)| + |\varphi_\sigma(\sigma x) - \tilde{v}_\sigma(x)|. \tag{5.6}$$

If $|x| \geq \varepsilon^{-1}$, by definition $\tilde{v}_\varepsilon(x) = \varphi_\varepsilon(\varepsilon x)$, hence

$$|\tilde{v}_\varepsilon(x) - \varphi_\varepsilon(\varepsilon x)| = 0. \tag{5.7}$$

For $\varepsilon^{-1/2} \leq |x| \leq \varepsilon^{-1}$, by definition $\tilde{v}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$. It also holds that $|\varepsilon x| \geq \varepsilon\varepsilon^{-1/2}$. Therefore, Theorem 4.1 implies that there is an ε_1 such that for every $\varepsilon < \varepsilon_1$

$$|\tilde{v}_\varepsilon(x) - \varphi_\varepsilon(\varepsilon x)| \leq \frac{\delta}{2} \quad \text{for } x \in B_1. \tag{5.8}$$

Combining (5.7) and (5.8) we have that for $\varepsilon < \varepsilon_1$

$$|\tilde{v}_\varepsilon(x) - \varphi_\varepsilon(\varepsilon x)| \leq \frac{\delta}{2} \quad \text{for } |x| \geq \varepsilon^{-1/2}. \tag{5.9}$$

Since $\sigma < \varepsilon < \varepsilon_1$ it also holds that

$$|\tilde{v}_\sigma(x) - \varphi_\sigma(\sigma x)| \leq \frac{\delta}{2} \quad \text{for } |x| \geq \sigma^{-1/2}. \tag{5.10}$$

Equations (5.6), (5.9) and (5.10) imply that

$$|\tilde{v}_\varepsilon(x) - \tilde{v}_\sigma(x)| \leq \delta \quad \text{for } |x| \geq \varepsilon^{-1/2},\ |x| \geq \sigma^{-1/2}. \tag{5.11}$$

- If $\varepsilon^{-1/2} \leq |x| \leq \sigma^{-1/2}$:
  Let us fix any x in this range and define $\bar{\sigma} = 1/|x|^2$. As before,

$$\varphi(x) = \varphi_\varepsilon(\varepsilon x) = \varphi_{\bar{\sigma}}(\bar{\sigma} x).$$

Then, we have

$$|\tilde{v}_\varepsilon(x) - \tilde{v}_\sigma(x)| \leq |\tilde{v}_\varepsilon(x) - \varphi_\varepsilon(\varepsilon x)| + |\varphi_{\bar{\sigma}}(\bar{\sigma} x) - u_{\bar{\sigma}}(\bar{\sigma} x)| + \left| u_{\bar{\sigma}}(y) - u_\sigma\left(\frac{\sigma y}{\bar{\sigma}}\right)\right|, \tag{5.12}$$

where $y = \bar{\sigma} x$. As before if $|x| \geq \varepsilon^{-1}$, by definition

$$|\tilde{v}_\varepsilon(x) - \varphi_\varepsilon(\varepsilon x)| = 0. \tag{5.13}$$

If $\varepsilon^{-1/2} \leq |x| \leq \varepsilon^{-1}$, by definition $\tilde{v}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$. Hence, Theorem 4.1 implies that there is a ε_2 such that for every $\varepsilon < \varepsilon_2$

$$|\tilde{v}_\varepsilon(x) - \varphi_\varepsilon(\varepsilon x)| \leq \frac{\delta}{3}. \tag{5.14}$$

Combining (5.13) and (5.14) we have for $\varepsilon < \varepsilon_2$

$$|\tilde{v}_\varepsilon(x) - \varphi_\varepsilon(\varepsilon x)| \leq \frac{\delta}{3} \quad \text{for } \varepsilon^{-1/2} \leq |x| \leq \sigma^{-1/2}. \tag{5.15}$$

By the definition of $\tilde{\sigma}$ we have that $|\tilde{\sigma}x| = 1/|x| = \tilde{\sigma}^{1/2}$ and $\tilde{\sigma} \leq \varepsilon$. Hence, using Theorem 4.1 for $\tilde{\sigma} \leq \varepsilon < \varepsilon_2$ we have

$$|\varphi_{\tilde{\sigma}}(\tilde{\sigma}x) - u_{\tilde{\sigma}}(\tilde{\sigma}x)| \leq \frac{\delta}{3} \quad \text{for } \varepsilon^{-1/2} \leq |x| \leq \sigma^{-1/2}. \tag{5.16}$$

Finally, as $|\tilde{\sigma}x| = \tilde{\sigma}^{1/2}$ and $\sigma \leq \tilde{\sigma} \leq \tilde{\sigma}^{1/2}$, Corollary 4.11 implies that there is an ε_3 such that

$$\left| u_{\tilde{\sigma}}(\tilde{\sigma}x) - \tilde{v}_\sigma\left(\frac{\tilde{\sigma}x}{\tilde{\sigma}}\right)\right| \leq \frac{\delta}{3} \quad \text{for } \varepsilon^{-1/2} \leq |x| \leq \sigma^{-1/2}. \tag{5.17}$$

Equations (5.15), (5.16) and (5.17) in (5.12) imply that

$$|\tilde{v}_\varepsilon(x) - \tilde{v}_\sigma(x)| \leq \delta \quad \text{for } \varepsilon^{-1/2} \leq |x| \leq \sigma^{-1/2}. \tag{5.18}$$

Combining equations (5.5), (5.11) and (5.18) we conclude that $\tilde{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ is a Cauchy sequence in the sup norm, hence there is a continuous function $v(x)$ such that $\tilde{v}_\varepsilon \to v$ uniformly in $\mathbb{R}^2$ as $\varepsilon \to 0$. □

Proof of Step 2: v satisfies (5.4). Consider any sequence of points x_n such that $|x_n| \to \infty$. Showing that $\lim_{n\to\infty} |v(x_n) - \varphi(x_n)| = 0$ is equivalent to (5.4). Let $\varepsilon_n = 1/|x_n|$. Then for any $\beta > 0$ the definition of $\tilde{v}_{\varepsilon_n}$ implies:

$$\begin{aligned} |v(x_n) - \varphi(x_n)| &= |v(x_n) - \tilde{v}_{\varepsilon_n}(x_n)| \\ &\leq \sup_{\mathbb{R}^2} |v(x) - \tilde{v}_{\varepsilon_n}(x)|. \end{aligned}$$

Taking $n \to \infty$, Step (1) implies that

$$\lim_{n\to\infty} |v(x_n) - \varphi(x_n)| \to 0,$$

which finishes the proof. □

Proof of Step 3: v satisfies (5.3). Let us fix a ball of radius ρ in $\mathbb{R}^2$. In every fixed ball B_ρ we can use Green's formula to represent v_ε. We have for $\varepsilon \leq 1/\rho$ that

$$v_\varepsilon(x) = -\int_{\partial B_\rho} v_\varepsilon(y) \frac{\partial \mathcal{K}}{\partial \nu}(x,y)\, ds(y) + \int_{B_\rho} \nabla_v W(v_\varepsilon)(y) \mathcal{K}(x,y)\, dy,$$

where $\mathcal{K}$ is the Green's function in the ball. Since in B_ρ we have $v_\varepsilon \to v$ uniformly as $\varepsilon \to 0$, the function v satisfies

$$v(x) = -\int_{\partial B_\rho} v(y)\frac{\partial \mathcal{K}}{\partial \nu}(x,y)\,\mathrm{d}s(y) + \int_{B_\rho} \nabla_v W(v)(y)\mathcal{K}(x,y)\,\mathrm{d}y.$$

Hence,

$$-\Delta v + \nabla_v W(v) = 0 \quad \text{for } x \in B_\rho.$$

□

Since this is true for arbitrary x and ρ, we have that v satisfies (5.3) for every $x \in \mathbb{R}^2$, which concludes the proof of the theorem. □

Now we finish the Proof of Theorem 1.1 by showing the following result.

Theorem 5.2. *Let*

$$\mathcal{V} = \left\{w \in C^1 : \int_{\mathbb{R}^2} |Dw - D\varphi|\,\mathrm{d}x,\ \int_{\mathbb{R}^2} |w-\varphi|\,\mathrm{d}x < \infty\right\}.$$

Define the energy functional

$$\mathcal{G}(w) = \begin{cases} \int_{\mathbb{R}^2} (|Dw|^2 + W(w) - |D\varphi|^2 - W(\varphi))\,\mathrm{d}y & \text{if } w \in \mathcal{V}, \\ \infty & \text{otherwise.} \end{cases} \tag{5.19}$$

The energy $\mathcal{G}$ is bounded below and the solution v described by Theorem 5.1 minimizes $\mathcal{G}$. That is

$$\mathcal{G}(v) = \inf_{w \in C^1} \mathcal{G}(w).$$

Proof. Define

$$\tilde{\mathcal{G}}_\varepsilon(w) = \begin{cases} \int_{B_{\varepsilon^{-1}}} |Dw|^2 + W(w)\,\mathrm{d}y & \text{if } w \in H^1(B_{\varepsilon^{-1}}) \text{ and } \\ & \quad w\big|_{\partial B_{\varepsilon^{-1}}}(x) = \varphi_\varepsilon(x), \\ \infty & \text{otherwise.} \end{cases} \tag{5.20}$$

and consider v_ε as in the previous theorem. We will divide the proof of Theorem 5.2 into the following steps:

(i) v_ε is a minimizer for $\tilde{\mathcal{G}}_\varepsilon$ among $w_\varepsilon \in H^1(B_{\varepsilon^{-1}})$. This implies that v_ε minimizes $\mathcal{G}\varepsilon(w) = \tilde{\mathcal{G}}_\varepsilon(w) - \tilde{\mathcal{G}}_\varepsilon(\varphi)$ in the same class of functions.

(ii) The sequence $\mathcal{G}\varepsilon(v_\varepsilon)$ is convergent.

(iii) $v \in \mathcal{V}$.

(iv) For every w in $\mathcal{V}$ there exists a sequence w_ε such that $w_\varepsilon \in H^1(B_{\varepsilon^{-1}})$, $w_\varepsilon|_{\partial B_{\varepsilon^{-1}}}(x) = \varphi(x)$ and $\mathcal{G}\varepsilon(w_\varepsilon) \to \mathcal{G}(w)$.
(v) $\mathcal{G}_\varepsilon(v_\varepsilon) \to \mathcal{G}(v)$.
(vi) Conclude the result using the previous steps.

Proof of Step (i). Notice first that for every $w_\varepsilon \in H^1(B_{\varepsilon^{-1}})$ satisfying the condition $w_\varepsilon|_{\partial B_{\varepsilon^{-1}}} = \varphi(x)$ there holds that $w_\varepsilon^\varepsilon(x) = w_\varepsilon(x/\varepsilon) \in H^1(B_1)$ and $w_\varepsilon^\varepsilon|_{\partial B_1} = \varphi_\varepsilon(x)$. Recall that u_ε is a minimizer for $\mathcal{I}_\varepsilon$ (defined by (2.7)), that is, for every $w_\varepsilon^\varepsilon \in H^1(B_1)$ satisfying $w_\varepsilon^\varepsilon|_{\partial B_1} = \varphi_\varepsilon(x)$ there holds

$$\mathcal{I}_\varepsilon(u_\varepsilon) \le \mathcal{I}_\varepsilon(w_\varepsilon^\varepsilon).$$

Dividing by ε and changing variables we obtain

$$\begin{aligned}\frac{1}{\varepsilon}\mathcal{I}_\varepsilon(u_\varepsilon) &= \int_{B_{1/\varepsilon}} (|Dv_\varepsilon|^2 + W(v_\varepsilon))\,\mathrm{d}y \le \frac{1}{\varepsilon}\mathcal{I}_\varepsilon(w_\varepsilon^\varepsilon)\\ &= \int_{B_{1/\varepsilon}} (|Dw_\varepsilon|^2 + W(w_\varepsilon))\,\mathrm{d}y,\end{aligned}$$

or equivalently

$$\tilde{\mathcal{G}}_\varepsilon(v_\varepsilon) \le \tilde{\mathcal{G}}_\varepsilon(w_\varepsilon), \quad \text{for every } w_\varepsilon \in H^1(B_{1/\varepsilon}).$$

By subtracting $\tilde{\mathcal{G}}_\varepsilon(\varphi)$ we get

$$\mathcal{G}_\varepsilon(v_\varepsilon) \le \mathcal{G}_\varepsilon(w_\varepsilon), \quad \text{for every } w_\varepsilon \in H^1(B_{1/\varepsilon}).$$

Proof of Step (ii). Fix $0 < \varepsilon < \sigma$. We need to show that $\mathcal{G}_\varepsilon(v_\varepsilon)$ is a Cauchy sequence. Namely, we prove that for every $\delta > 0$ there is an ε_0 such that

$$|\mathcal{G}_\varepsilon(v_\varepsilon) - \mathcal{G}_\sigma(v_\sigma)| \le \delta \quad \text{for } \varepsilon, \sigma \le \varepsilon_0.$$

We will study separately two cases: $\sigma \ge \sqrt{\varepsilon}$ and $\sigma < \sqrt{\varepsilon}$.

- $\sigma \ge \sqrt{\varepsilon}$:

$$\begin{aligned}&|\mathcal{G}_\varepsilon(v_\varepsilon) - \mathcal{G}_\sigma(v_\sigma)|\\ &\quad\le \left| \int_{B_{1/\varepsilon}\setminus B_{1/\sigma}} (|Dv_\varepsilon|^2 - |D\varphi|^2 + W(v_\varepsilon) - W(\varphi))\,\mathrm{d}x \right.\\ &\qquad \left. + \int_{B_{1/\sigma}} (|Dv_\varepsilon|^2 - |Dv_\sigma|^2 + W(v_\varepsilon) - W(v_\sigma))\,\mathrm{d}x \right| \quad \le\end{aligned}$$

$$\leq \int_{B_{1/\varepsilon}\setminus B_{1/\sqrt{\varepsilon}}} \Big(\big| |Dv_\varepsilon|^2 - |D\varphi|^2 \big| + \big| W(v_\varepsilon) - W(\varphi) \big| \Big)\, \mathrm{d}x$$
$$+ \int_{B_{1/\sqrt{\varepsilon}}\setminus B_{1/\sigma}} \Big(\big| |Dv_\varepsilon|^2 - |Dv_{\sqrt{\varepsilon}}|^2 \big| + \big| W(v_\varepsilon) - W(v_{\sqrt{\varepsilon}}) \big| \Big)\, \mathrm{d}x$$
$$+ \int_{B_{1/\sqrt{\varepsilon}}\setminus B_{1/\sigma}} \Big(\big| |Dv_{\sqrt{\varepsilon}}|^2 - |D\varphi|^2 \big| + \big| W(v_{\sqrt{\varepsilon}}) - W(\varphi) \big| \Big)\, \mathrm{d}x$$
$$+ \int_{B_{1/\sigma}} \Big(\big| |Dv_\varepsilon|^2 - |Dv_\sigma|^2 \big| + \big| W(v_\varepsilon) - W(v_\sigma) \big| \Big)\, \mathrm{d}x.$$

Let $u^\varepsilon_\sigma(x) = v_\sigma(x/\varepsilon)$. Changing variables we have

$$|\mathcal{G}_\varepsilon(v_\varepsilon) - \mathcal{G}_\sigma(v_\sigma)|$$
$$\leq \int_{B_1\setminus B_{\sqrt{\varepsilon}}} \left(\big| |Du_\varepsilon|^2 - |D\varphi_\varepsilon|^2 \big| + \left| \frac{W(u_\varepsilon) - W(\varphi_\varepsilon)}{\varepsilon^2} \right| \right) \mathrm{d}x$$
$$+ \int_{B_{\sqrt{\varepsilon}}\setminus B_{\varepsilon/\sigma}} \left(\big| |Du_\varepsilon|^2 - |Du^\varepsilon_{\sqrt{\varepsilon}}|^2 \big| + \left| \frac{W(u_\varepsilon) - W(u^\varepsilon_{\sqrt{\varepsilon}})}{\varepsilon^2} \right| \right) \mathrm{d}x$$
$$+ \int_{B_1\setminus B_{\sqrt{\varepsilon}/\sigma}} \left(\big| |Du_{\sqrt{\varepsilon}}|^2 - |D\varphi_{\sqrt{\varepsilon}}|^2 \big| + \left| \frac{W(u_{\sqrt{\varepsilon}}) - W(\varphi_{\sqrt{\varepsilon}})}{\varepsilon} \right| \right) dx$$
$$+ \int_{B_{\varepsilon/\sigma}} \left(\big| |Du_\varepsilon|^2 - |Du^\varepsilon_\sigma|^2 \big| + \left| \frac{W(u_\varepsilon) - W(u_\sigma)}{\varepsilon^2} \right| \right) \mathrm{d}x.$$

Notice that since $\sigma \geq \sqrt{\varepsilon}$, we have that $\varepsilon/\sigma \leq \sqrt{\varepsilon}$. Then using Theorem 4.1 and Corollaries 4.11 and 4.12 we have that for every m there is a constant, that depends on m, such that

$$|\mathcal{G}_\varepsilon(v_\varepsilon) - \mathcal{G}_\sigma(v_\sigma)| \leq C\varepsilon^m. \tag{5.21}$$

- $\sigma \leq \sqrt{\varepsilon}$:

$$|\mathcal{G}_\varepsilon(v_\varepsilon) - \mathcal{G}_\sigma(v_\sigma)|$$
$$\leq \int_{B_{1/\varepsilon}\setminus B_{1/\sigma}} \big(\big| |Dv_\varepsilon|^2 - |D\varphi|^2 \big| + |W(v_\varepsilon) - W(\varphi)| \big)\, \mathrm{d}x$$
$$+ \int_{B_{1/\sigma}} \big(\big| |Dv_\varepsilon|^2 - |Dv_\sigma|^2 \big| + |W(v_\varepsilon) - W(v_\sigma)| \big)\, \mathrm{d}x \quad \leq$$

$$
\begin{aligned}
&\le \int_{B_{1/\varepsilon}\setminus B_{1/\sigma}} \left(\left||Dv_\varepsilon|^2 - |D\varphi|^2\right| + |W(v_\varepsilon) - W(\varphi)|\right) \mathrm{d}x \\
&\quad + \int_{B_{1/\sigma}\setminus B_{1/\sqrt{\varepsilon}}} \left(\left||Dv_\varepsilon|^2 - |D\varphi|^2\right| + |W(v_\varepsilon) - W(\varphi)|\right) \mathrm{d}x \\
&\quad + \int_{B_{1/\sigma}\setminus B_{1/\sqrt{\varepsilon}}} \left(\left||D\varphi|^2 - |Dv_\sigma|^2\right| + |W(v_\sigma) - W(\varphi)|\right) \mathrm{d}x \\
&\quad + \int_{B_{1/\sqrt{\varepsilon}}} \left(\left||Dv_\varepsilon|^2 - |Dv_\sigma|^2\right| + |W(v_\varepsilon) - W(v_\sigma)|\right) \mathrm{d}x.
\end{aligned}
$$

Let $u_\sigma^\varepsilon(x) = v_\sigma(x/\varepsilon)$. Changing variables we have

$$
\begin{aligned}
&|\mathcal{G}_\varepsilon(v_\varepsilon) - \mathcal{G}_\sigma(v_\sigma)| \\
&\quad \le \int_{B_1\setminus B_{\sqrt{\varepsilon}}} \left(\left||Du_\varepsilon|^2 - |D\varphi_\varepsilon|^2\right| + \left|\frac{W(u_\varepsilon) - W(\varphi_\varepsilon)}{\varepsilon^2}\right|\right) \mathrm{d}x \\
&\qquad + \int_{B_1\setminus B_{\sigma/\sqrt{\varepsilon}}} \left(\left||Du_\sigma|^2 - |D\varphi_\sigma|^2\right| + \left|\frac{W(u_\sigma) - W(\varphi_\sigma)}{\sigma^2}\right|\right) \mathrm{d}x \\
&\qquad + \int_{B_{\sqrt{\varepsilon}}} \left(\left||Du_\varepsilon|^2 - |Du_\sigma^\varepsilon|^2\right| + \left|\frac{W(u_\varepsilon) - W(u_\sigma^\varepsilon)}{\varepsilon^2}\right|\right) \mathrm{d}x.
\end{aligned}
$$

Since $\sigma > \varepsilon$, we have that $\sigma/\sqrt{\varepsilon} \ge \sqrt{\sigma}$. Then, Theorem 4.1 and Corollaries 4.11 and 4.12 imply that for every m there is a constant, that depend on m, such that

$$
|\mathcal{G}_\varepsilon(v_\varepsilon) - \mathcal{G}_\sigma(v_\sigma)| \le C(\varepsilon^m + \sigma^m). \tag{5.22}
$$

We conclude from (5.21) and (5.22) that for every $m > 0$ there is a constant C such that

$$
|\mathcal{G}_\varepsilon(v_\varepsilon) - \mathcal{G}_\sigma(v_\sigma)| \le C(\varepsilon^m + \sigma^m).
$$

Therefore $\mathcal{G}_\varepsilon(v_\varepsilon)$ is a Cauchy sequence of real numbers, thus convergent.

Proof of Step (iii). Following the same method of the previous step we can prove that the sequences $\int_{B_{1/\varepsilon}} |Dv_\varepsilon - D\varphi|$ and $\int_{B_{1/\varepsilon}} |v_\varepsilon - \varphi|$ are Cauchy sequences and therefore uniformly bounded. Fatou's lemma implies that

$$
\begin{aligned}
\int_{\mathbb{R}^2} |Dv - D\varphi|\,\mathrm{d}x &\le \int_{B_{1/\varepsilon}} |Dv_\varepsilon - D\varphi|\,\mathrm{d}x < \infty, \\
\int_{\mathbb{R}^2} |v - \varphi| &\le \int_{B_{1/\varepsilon}} |v_\varepsilon - \varphi|\,\mathrm{d}x < \infty.
\end{aligned}
$$

That is, $v \in \mathcal{V}$.

Proof of Step (iv). Consider a smooth function η satisfying $\eta(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\eta(x) = 0$ for $|x| \geq 1$. Define

$$w_\varepsilon(x) = \eta(\varepsilon x)w(x) + (1 - \eta(\varepsilon x))\varphi.$$

Then

$$\begin{aligned}
&|\mathcal{G}_\varepsilon(w_\varepsilon) - \mathcal{G}(w)| \\
&\quad = \Bigg| \int_{\mathbb{R}^2 \setminus B_{1/(2\varepsilon)}} (|Dw|^2 - |D\varphi|^2 + W(w) - W(\varphi))\,\mathrm{d}x \\
&\qquad - \int_{B_{1/\varepsilon} \setminus B_{1/(2\varepsilon)}} (|\eta Dw + (1-\eta)D\varphi + D\eta(w-\varphi)|^2 - |D\varphi|^2)\,\mathrm{d}x \\
&\qquad\qquad + W\big(\eta(\varepsilon x)w(x) + (1-\eta(\varepsilon x))\varphi\big) - W(\varphi)\,\mathrm{d}x \Bigg| \\
&\quad \leq C \left| \int_{\mathbb{R}^2 \setminus B_{1/(2\varepsilon)}} (|Dw - D\varphi| + |w - \varphi|)\,\mathrm{d}x \right| \\
&\qquad + \Bigg| \int_{B_{1/\varepsilon} \setminus B_{1/(2\varepsilon)}} \Big(|\eta Dw + (1-\eta)D\varphi + D\eta(w-\varphi) - D\varphi| \\
&\qquad\qquad + C|\eta(\varepsilon x)||w - \varphi|\Big)\,\mathrm{d}x \Bigg| \\
&\quad \leq C \left| \int_{\mathbb{R}^2 \setminus B_{1/(2\varepsilon)}} (|Dw - D\varphi| + |w - \varphi|)\,\mathrm{d}x \right|.
\end{aligned}$$

Since $w \in \mathcal{V}$ we have

$$\lim_{\varepsilon \to 0} |\mathcal{G}_\varepsilon(w_\varepsilon) - \mathcal{G}(w)| = 0.$$

Proof of Step (v). The previous step implies there is a $\tilde{v}_\varepsilon$ such that

$$\mathcal{G}_\varepsilon(\tilde{v}_\varepsilon) \to \mathcal{G}(v).$$

Since v_ε is a minimizer of $\mathcal{G}_\varepsilon$ we have that

$$\mathcal{G}_\varepsilon(v_\varepsilon) \leq \mathcal{G}_\varepsilon(\tilde{v}_\varepsilon).$$

Taking limits when $\varepsilon \to 0$, we have

$$\lim_{\varepsilon \to 0} \mathcal{G}(v_\varepsilon) \leq \mathcal{G}(v).$$

In particular, $\mathcal{G}(v)$ is bounded below. Fatou's lemma allows us to conclude the other inequality:

$$\mathcal{G}(v) \leq \lim_{\varepsilon \to 0} \mathcal{G}(v_\varepsilon).$$

Proof of Step (vi). Consider $w \in \mathcal{V}$, then take w_ε as in Step (iv). Then the minimality of v_ε implies

$$\mathcal{G}_\varepsilon(v_\varepsilon) \le \mathcal{G}_\varepsilon(w_\varepsilon).$$

Taking limits as $\varepsilon \to 0$ we conclude that

$$\mathcal{G}(v) \le \mathcal{G}(w),$$

which finishes the proof. □

APPENDIX A.

In this appendix we present a collection of technical results used in the previous sections.

We start by stating some results about the heat kernel, used mainly in Section 4. Consider a ball $B \subset \mathbb{R}^2$. Then $\mathcal{H}_B$ can be described as follows:

$$\left(\frac{d}{dt} - \Delta_x\right)\mathcal{H}_B(x,y,t) = 0 \quad \text{for } x,\ y \in B \text{ and } t > 0, \tag{A.1}$$

$$\mathcal{H}_B(x,y,t) = 0 \quad \text{whenever } x \in \partial B, \tag{A.2}$$

$$\lim_{t\to 0^+} \mathcal{H}_B(x,y,t) = \delta_y(x) \quad \text{for } x \in B. \tag{A.3}$$

Hence, the solution to the equation

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_x\right)u(x,t) &= f(x,t) && \text{for } x \in B \text{ and } t > 0,\\ u(x,t) &= 0 && \text{whenever } x \in \partial B,\\ u(x,0) &= g(x) && \text{for } x \in B, \end{aligned}$$

can be represented as

$$u(x,t) = \int_0^t \int_B \mathcal{H}_B(x,y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s + \int_B \mathcal{H}_B(x,y,t)g(y)\,\mathrm{d}y. \tag{A.4}$$

We will use this representation to prove the following lemmas. Let us define P to be the heat operator, that is

$$Pu = \frac{d}{dt}u - \Delta u. \tag{A.5}$$

First we prove some bounds over $\mathcal{H}_B$:

Lemma A.1. *It holds that*

$$0 \le \int_B \mathcal{H}_B(x,y,t-s)\,dy\,ds \le 1, \tag{A.6}$$

$$0 \le \int_s^t \int_B \mathcal{H}_B(x,y,t-s)\,dy\,ds \le (t-s). \tag{A.7}$$

Proof. The proof follows by maximum principle. Notice that the single-valued function

$$v(x,t) = \int_B \mathcal{H}_B(x,y,t-s)\,dy\,ds$$

satisfies the equation

$$Pv(x,t) = 0 \quad \text{for } x \in B, \tag{A.8}$$

$$v(x,t) = 0 \quad \text{whenever } x \in \partial B, \tag{A.9}$$

$$v(x,s) = 1 \quad \text{for } x \in B. \tag{A.10}$$

Since the function 0 is a sub-solution to (A.8)–(A.9)–(A.10) we have that

$$0 \le v(x,t).$$

Similarly, the function 1 is a super-solution. Hence,

$$v(x,t) \le 1,$$

which proves (A.6). Equation (A.7) follows by integrating inequality (A.6).

We also include without proof the following theorem (see [10], [14] for example).

Theorem A.2 (Theorem 3.1 in [10])**.** *Let $\mathcal{M}$ be an n dimensional compact Riemannian manifold with boundary. Then there is a Dirichlet heat kernel, that is a function*

$$\mathcal{H} \in C^\infty(\mathcal{M} \times \mathcal{M} \times (0,\infty)).$$

satisfying (A.1)–(A.3)–(A.2).

The smoothness of $\mathcal{H}(x,x,t)$ may be described as follows

$$\mathcal{H}(x,x,t) = t^{-n/2}(A(x,t) + B(x,t))$$

with $A \in C^\infty(\mathcal{M} \times [0,\infty))$ and B is supported near the boundary, where in local coordinates $(x', x_n) \in U' \times [0,\bar{\delta}) \subset \mathcal{M}$, $U' \subset \mathbb{R}^{n-1}$ open, one has

$$B(x,t) = b\left(x', \frac{x_n}{\sqrt{t}}, t\right), \quad b \in C^\infty(u' \times \mathbb{R}_+ \times [0,\infty)_{\sqrt{t}})$$

with $b(x', \psi_n, t)$ rapidly decaying as $\psi_n \to \infty$.

Now we devote ourselves to prove Lemma 4.2. We start with the following a priori bound:

Theorem A.3. *Let $\tilde{h}_\varepsilon(x,t) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy*

$$P\tilde{h}_\varepsilon + \frac{\nabla_v W(\tilde{h}_\varepsilon)}{2} = 0 \quad \text{for } x \in B_{1/\varepsilon}, \tag{A.11}$$

$$\tilde{h}_\varepsilon(x,t) = \varphi(x) \quad \text{for } x \in \partial B_{1/\varepsilon}, \tag{A.12}$$

$$\tilde{h}_\varepsilon(x,0) = \psi_\varepsilon(x) \quad \text{for } x \in B_{1/\varepsilon}, \tag{A.13}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a proper C^2 function, bounded below, with a finite number of critical points (denoted by $\{c_i\}_{i=1}^m$), and such that the Hessian of $W(u)$ is positive semidefinite for $|u| \geq K$, where K is a fixed real number. Then if $\tilde{h}_\varepsilon(x,0) = \psi_\varepsilon(x)$ is bounded, there is a constant C that depends only on W, φ and ψ_ε such that $|\tilde{h}_\varepsilon(x,t)| \leq C$.

Proof. Consider $\ell_\varepsilon(x,t) = W(\tilde{h}_\varepsilon)(x,t)$; then

$$\begin{aligned}(\ell_\varepsilon)_t - \Delta\ell_\varepsilon &= \nabla_v W(\tilde{h}_\varepsilon) \cdot (\tilde{h}_\varepsilon)_t - \sum_i (\nabla_v W(\tilde{h}_\varepsilon) \cdot (\tilde{h}_\varepsilon)_{x_i})_{x_i} \\ &= \nabla_v W(\tilde{h}_\varepsilon) \cdot (\tilde{h}_\varepsilon)_t - (W''(\tilde{h}_\varepsilon)\nabla\tilde{h}_\varepsilon) \cdot \nabla\tilde{h}_\varepsilon - \nabla_v W(\tilde{h}_\varepsilon) \cdot \Delta\tilde{h}_\varepsilon\end{aligned}$$

where W'' denotes the Hessian matrix of W. Since $\tilde{h}_\varepsilon$ satisfies (A.11), this becomes

$$(\ell_\varepsilon)_t - \Delta\ell_\varepsilon + \frac{|W'(\tilde{h}_\varepsilon)|^2}{2} + (W''(\tilde{h}_\varepsilon)\nabla u) \cdot \nabla\tilde{h}_\varepsilon = 0. \tag{A.14}$$

We are going to find bounds over ℓ_ε at the boundary of $B_{1/\varepsilon}$ and over its possible interior maxima in terms of $\max\varphi$, K, $W(c_i)$, and $\max W(\psi_\varepsilon(x))$.

Since for every $|x| = 1$ it holds that $\tilde{h}_\varepsilon(x,t) = \varphi(x)$ and φ is uniformly bounded, we have that

$$\ell_\varepsilon(x) \leq \max W(\varphi(x)) \quad \text{for every } x \in \partial B_{1/\varepsilon}.$$

Suppose that ℓ_ε has an interior maximum at (x_0,t_0) and $|\tilde{h}_\varepsilon(x_0,t_0)| \geq K$. Since (x_0,t_0) is a maximum for ℓ_ε, it holds that $(\ell_\varepsilon)_t(x_0,t_0) \geq 0$ and $\Delta\ell_\varepsilon(x_0,t_0) \leq 0$. We also have by hypothesis that $W''(u)$ is positive semidefinite for $|u| \geq K$, hence

$$(\ell_\varepsilon)_t - \Delta\ell_\varepsilon + \frac{|\nabla_u W(\tilde{h}_\varepsilon)|^2}{2} + (W''(\tilde{h}_\varepsilon)\nabla\tilde{h}_\varepsilon) \cdot \nabla\tilde{h}_\varepsilon \geq 0.$$

The inequality is strict (which contradicts (A.14)), unless $|\nabla_u W(\tilde{h}_\varepsilon)|^2/\varepsilon^2 = (W''(\tilde{h}_\varepsilon)\nabla\tilde{h}_\varepsilon)\cdot\nabla\tilde{h}_\varepsilon = 0$. If $\nabla_v W(\tilde{h}_\varepsilon(x_0,t_0)) = 0$, we would have $\tilde{h}_\varepsilon(x_0,t_0) = c_i$ for some i, therefore $W(\tilde{h}_\varepsilon(x,t)) \le W(c_i)$. From this and the previous computations we conclude that ℓ_ε is uniformly bounded.

Since W is a proper function, we have that there is a constant C such that

$$|\tilde{h}_\varepsilon(x)| \le C \quad \text{for } x \in \bar{B}_{1/\varepsilon},$$

which finishes the proof. □

By observing that solutions to (4.3)–(4.4)–(4.5) can be written as $h_\varepsilon(x,t) = \tilde{h}_\varepsilon(x,t) - V_{\vec{q}}(x)$, where $\tilde{h}_\varepsilon$ is a solution to (A.11)–(A.12)–(A.13) we have the following result.

Corollary A.4. *Let $h_{\vec{q}}(x,t) : B_{1/\varepsilon} \to \mathbb{R}^2$ be a solution to (4.3)–(4.4)–(4.5), where $W : \mathbb{R}^2 \to \mathbb{R}$ is a proper C^2 function, bounded below, with a finite number of critical points and such that the Hessian of $W(w)$ is positive semidefinite for $|w| \ge K$, where K is a fixed real number. Then if $h_{\vec{q}}(x,0) = \psi_\varepsilon(x)$ is bounded, there is a constant C that depends only on W, φ, $U_{\vec{q}}$, and ψ_ε such that $|h_{\vec{q}}(x,t)| \le C$.*

Proof of Lemma 4.2. Let

$$C_{[\bar{t}_1,\bar{t}_2]}(B) = \{w : \bar{B} \times [\bar{t}_1,\bar{t}_2] \to \mathbb{R}^2 \mid w \text{ is a uniformly bounded continuous function}\}$$

with the standard sup norm.

Consider some $\tau \ge 0$ and define

$$F^\tau_{\vec{q}}(\cdot,\psi^\tau_{\vec{q}}) : C_{[\tau,\tau+2\beta/M]}(B_{\varepsilon^{-1}}) \to C_{[\tau,\tau+2\beta/M]}(B_{\varepsilon^{-1}})$$

by

$$\begin{aligned} F^\tau_{\vec{q}}(w,\psi^\tau_{\vec{q}})(x,t) &= \int_\tau^t \int_{B_{\varepsilon^{-1}}} \mathcal{H}_{B_{\varepsilon^{-1}}}(x,y,t-s)\left(\frac{-W'(w(y,s)+V_{\vec{q}})}{2} + \Delta V_{\vec{q}}(y)\right) \mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_{B_{\varepsilon^{-1}}} \mathcal{H}_{B_{\varepsilon^{-1}}}(x,y,t)\psi^\tau_{\vec{q}}(y)\,\mathrm{d}y\,. \end{aligned} \tag{A.15}$$

Notice that Duhamel's formula implies that fixed points of the function $F^\tau_{\vec{q}}(\cdot,\psi^\tau_{\vec{q}})$ are solutions to (4.3) in $[\tau,\tau+2\beta/M]$. Hence, in order to prove Lemma 4.2 we will use the following strategy: For every τ, $\psi^\tau_{\vec{q}}$ and appropriate constants β, M we find a fixed point of $F^\tau_{\vec{q}}(\cdot,\psi^\tau_{\vec{q}})$ in some appropriate space; then we choose $\psi^\tau_{\vec{q}}$ appropriately so the fixed points (that were found in the previous step) "glue" together appropriately; we finish by showing that in fact (4.3) holds in the whole domain, as well as (4.4) and (4.5).

Claim A.5. *If there is a constant M such that $|W''| \le M$ and $\psi_{\vec q}^{\tau}$ is uniformly bounded, then $F_{\vec q}^{\tau}(\cdot, \psi_{\vec q}^{\tau}) : C_{[\tau,\tau+2\beta/M]}(B_{\varepsilon^{-1}}) \to C_{[\tau,\tau+2\beta/M]}(B_{\varepsilon^{-1}})$ is well defined for each $\vec q \in \mathcal{Q}$, where $\mathcal{Q}$ is given by (4.2). If additionally for any given τ and $\beta \in (0,1)$ we have that $\bar t$ satisfies $|\bar t - \tau| \le 2\beta/M$, then $F_{\vec q}^{\tau}(\cdot, \psi_{\vec q}^{\tau})$ is a contraction mapping with constant β in $C_{[\tau,\tau+2\beta/M]}(B_{\varepsilon^{-1}})$.*

To prove that the function

$$F_{\vec q}^{\tau}(\cdot, \psi_{\vec q}^{\tau}) : C_{[\tau,\tau+2\beta/M]}(B_{\varepsilon^{-1}}) \to C_{[\tau,\tau+2\beta/M]}(B_{\varepsilon^{-1}})$$

is well defined, we need to show that $F_{\vec q}^{\tau}(\cdot, \psi_{\vec q}^{\tau})$ maps any uniformly bounded function into a uniformly bounded function, that is for any function w that satisfies $|w(x,t)| \le C$ for all $(x,t) \in B_{\varepsilon^{-1}} \times [\tau, \bar t\,]$ it holds that $|F_{\vec q}^{\tau}(w, \psi_{\vec q}^{\tau})(x,t)| \le \bar C$ for all $(x,t) \in B_{\varepsilon^{-1}} \times [\tau, \bar t\,]$.

By continuity of W' we have that if $\sup_{B_{\varepsilon^{-1}} \times [\tau,\bar t\,]} |w(x,t)| \le C$; then there is a constant C_1 such that $\sup_{(x,t)\in B_{\varepsilon^{-1}} \times [\tau,\bar t\,]} |W'(w)(x,t)| \le C_1$. Using the definition of $V_{\vec q}$, we can also find constants C_2 and C_3 that $|\Delta V_{\vec q}| \le C_2$ and $|V_{\vec q}| \le C_3$. This implies

$$\begin{aligned}
|F_{\vec q}^{\tau}(w, \psi_{\vec q}^{\tau})|(x,t) \le (C_1 + C_2) &\int_{\tau}^{\bar t} \int_{B_{\varepsilon^{-1}}} \mathcal{H}_{B_{\varepsilon^{-1}}}(x, y,\, t-\tau-s)\, \mathrm{d}y\, \mathrm{d}s \\
&+ \sup_{x \in B_{\varepsilon^{-1}}} |\psi_{\varepsilon}^{\tau}(x)| \int_{B_{\varepsilon^{-1}}} \mathcal{H}_{B_{\varepsilon^{-1}}}(x, y, t-\tau)\, \mathrm{d}y + C_3 \\
&\le (C_1 + C_2)(\bar t - \tau) + \sup_{x \in B_{\varepsilon^{-1}}} |\psi_{\varepsilon}^{\tau}|(x) + C_3 = \bar C < \infty,
\end{aligned}$$

for all (x,t). Hence $F_{\vec q}^{\tau}(\cdot, \psi_{\vec q}^{\tau})$ is well defined for each $\vec q \in \mathcal{Q}$ (where $\mathcal{Q}$ is given by (4.2)).

Now we show that if $|\bar t - \tau| \le 2\beta/M$, then $F_{\vec q}^{\tau}(\cdot, \psi_{\vec q}^{\tau})$ is a contraction mapping.

Since $|W''| \le M$, we have that

$$|W'(w_1) - W'(w_2)| \le M|w_1 - w_2|.$$

Then for every $x \in B_{\varepsilon^{-1}}$ and $t \in [\tau, \bar t\,]$ it holds that

$$\begin{aligned}
&|F_{\vec q}^{\tau}(w_1, \psi_{\vec q}^{\tau}) - F_{\vec q}^{\tau}(w_2, \psi_{\vec q}^{\tau})|(x,t) \\
&\quad = \left| \int_{\tau}^{t} \int_{B_{\varepsilon^{-1}}} \mathcal{H}_{B_{\varepsilon^{-1}}}(x, y, t-s-\tau) \frac{-W'(w_1(y,s)) + W'(w_2(y,s))}{2}\, \mathrm{d}y\, \mathrm{d}s \right| \\
&\quad \le \frac{M(\bar t - \tau)}{2} \sup_{(x,t)\in B_{\varepsilon^{-1}} \times [\tau,\bar t\,]} |w_1 - w_2|(x,t).
\end{aligned}$$

Then for $|\bar{t} - \tau| \le 2\beta/M$ there holds

$$\sup_{(x,t)\in B_{\varepsilon^{-1}}\times[\tau,\bar{t}\,]} |F^{\tau}_{\vec{q}}(w_1, \psi^{\tau}_{\vec{q}}) - F^{\tau}_{\vec{q}}(w_2, \psi^{\tau}_{\vec{q}})|(x,t) \le \beta \sup_{(x,t)\in B_{\varepsilon^{-1}}\times[\tau,\bar{t}\,]} |w_1 - w_2|(x,t)$$

and $F^{\tau}_{\vec{q}}(\cdot\,, \psi^{\tau}_{\vec{q}}) : B_{\varepsilon^{-1}} \times [\tau,\ \tau+2\beta/M] \to B_{\varepsilon^{-1}} \times [\tau,\ \tau+2\beta/M]$ is a contraction with constant β.

We will assume that $|W''| \le M$ and at the end of the proof we will point out the necessary modifications in the general case. Fix $\beta < 1$ and let

$$\tau_i = i\frac{2\beta}{M}, \tag{A.16}$$

$$\bar{t}_i = \tau_{i+1}, \tag{A.17}$$

$$F_{\vec{q},i} = F^{\tau}_{\vec{q}}(\cdot\,, \psi^{\tau}_{\vec{q}}), \tag{A.18}$$

with $i = 0, \dots, I_\beta$, where the constants β, $I_\beta \in \mathbb{N}$ satisfy $TM/(2\beta) \le I_\beta \le 2\bar{t}M/(2\beta)$. By the definition of τ_i, $\bar{t}_i$ we have that $\bar{t}_{I_\beta} \ge \bar{t}$. We will redefine $\bar{t}_{I_\beta} = \bar{t}$.

By the previous claim $F_{\vec{q},i}$ is a contraction, hence it has a unique fixed point: $h^i_{\vec{q}}$. That is

$$F_{\vec{q},i}(h^i_{\vec{q}}(x,t)) = h^i_{\vec{q}}(x,t). \tag{A.19}$$

Moreover, since this fixed point is bounded, we have that

$$F^{\tau}_{\vec{q}}(h^i_{\vec{q}}, \psi^{\tau}_{\vec{q}}) \in C^{1,1/2}(B_{\varepsilon^{-1}} \times (\tau_i, \tau_{i+1}]).$$

Recursively, $h^i_{\vec{q}} \in C^\infty$. From (A.19) and Duhamel's formula we can conclude that (4.3) and (4.4) hold for $t \in [\tau_i, \bar{t}_i]$. We also have

$$h^i_{\vec{q}}(x, \tau_i) = \psi^{\tau_i}_{\vec{q}}(x) \tag{A.20}$$

for $(x,t) \in B_{\varepsilon^{-1}} \times (\tau_i, \bar{t}_i)$.

Now define recursively $\psi^{\tau_i}_{\vec{q}}(x)$:

$$\psi^{\tau_0}_{\vec{q}}(x) = \psi_{\vec{q}}(x), \tag{A.21}$$

$$\psi^{\tau_i}_{\vec{q}}(x) = h^{i-1}_{\vec{q}}(x, \tau_i). \tag{A.22}$$

Then $h_{\vec{q}}(x,t)$ defined by

$$h_{\vec{q}}(x,t) = h^i_{\vec{q}}(x,t) \quad \text{for } t \in [\tau_i, \bar{t}_i] \tag{A.23}$$

satisfies (4.3) for $t \neq \tau_i$. Moreover, by writing

$$h_{\vec{q}}^{i+1}(x,t) = \int_{\bar{t}_i}^{t} \int_{B_{\varepsilon^{-1}}} \mathcal{H}_{B_{\varepsilon^{-1}}}(x,y,t-\bar{t}_i-s) \frac{-W'(h_{\vec{q}}^{i+1}+V_{\vec{q}})(y,s)}{2} \, dy \, ds + \int_{B_{\varepsilon^{-1}}} \mathcal{H}_{B_{\varepsilon^{-1}}}(x,y,t-\bar{t}_i) h_{\vec{q}}^{i}(y,\bar{t}_i) \, dy,$$

standard computations show that $h_{\vec{q}}$ satisfies (4.3) for every t. Since $h_{\vec{q}}$ also satisfies (4.4)–(4.5), we have that $h_{\vec{q}}$ is the desired solution. In particular, this implies that $h_{\vec{q}}$ is the fixed point of $F_{\vec{q}}$. Uniqueness follows from the fact that fixed points of contraction mappings are unique.

In order to prove Equation (4.6) we observe that since $h_{\vec{q}}$ is a fixed point of $F^{\tau}_{\vec{q}}$, standard computations imply for any function $w_{\vec{q}}$

$$|h_{\vec{q}} - w_{\vec{q}}| \leq \frac{1}{1-\beta} \sup_{B_{\varepsilon^{-1}} \times [\tau, \tau+2\beta/M]} |F^{\tau}_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}| \tag{A.24}$$

$$\leq \frac{1}{1-\beta}\Big(\sup_{B_{\varepsilon^{-1}} \times [\tau, \tau+2\beta/M]} |F^{\tau}_{\vec{q}}(w_{\vec{q}}) - F_{\vec{q}}(w_{\vec{q}})| + \sup_{B_{\varepsilon^{-1}} \times [\tau, \tau+2\beta/M]} |F_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}|\Big).$$

The definitions of $F^{\tau}_{\vec{q}}$ and $F_{\vec{q}}$ imply that

$$P(F^{\tau}_{\vec{q}}(w_{\vec{q}}) - F_{\vec{q}}(w_{\vec{q}})) = \frac{\nabla_v W(w_{\vec{q}})}{2} - \frac{\nabla_v W(w_{\vec{q}})}{2} = 0,$$

and

$$F^{\tau}_{\vec{q}}(w_{\vec{q}})(x,\tau) - F_{\vec{q}}(w_{\vec{q}})(x,\tau) = h_{\vec{q}}(x,\tau) - F_{\vec{q}}(w_{\vec{q}})(x,\tau).$$

Using Duhamel's formula we have

$$F^{\tau}_{\vec{q}}(w_{\vec{q}}) - F_{\vec{q}}(w_{\vec{q}}) = \int_{B_{\varepsilon^{-1}}} \mathcal{H}_{B_{\varepsilon^{-1}}}(x,y,t-\tau)(h_{\vec{q}}(y,\tau) - F_{\vec{q}}(w_{\vec{q}})(y,\tau)) \, dy.$$

Together with Lemma A.1, this implies

$$\sup_{B_{\varepsilon^{-1}} \times [\tau, \tau+2\beta/M]} |F^{\tau}_{\vec{q}}(w_{\vec{q}}) - F_{\vec{q}}(w_{\vec{q}})| \leq \sup_{B_{\varepsilon^{-1}}} |h_{\vec{q}}(x,\tau) - F_{\vec{q}}(w_{\vec{q}})|(x,\tau).$$

Using (A.24) we conclude inequality (4.6)

$$|h_{\vec{q}} - w_{\vec{q}}| \leq \frac{1}{1-\beta}\Big(2 \sup_{B_{\varepsilon^{-1}} \times [\tau, \tau+2\beta/M]} |F_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}| + \sup_{B_{\varepsilon^{-1}}} |h_{\vec{q}} - w_{\vec{q}}|(x,\tau)\Big).$$

For the general case (that is when there is no constant M such that $|W''| \leq M$) we fix $K > 0$ large enough. Then we replace W for a function $\tilde{W}$ that satisfies:

- there is an M such that $|\tilde{W}''| \le M$,
- $\tilde{W}(u) = W(u)$ for $u \le \max\{2C, K\}$, where C is the constant given by Theorem A.3,
- $\tilde{W}$ has the same critical points as W.

Then, our previous computations imply that there is a unique solution $h_{\vec{q}}$ to

$$Ph_{\vec{q}} + \frac{\nabla_v \tilde{W}(h_{\vec{q}} + V_{\vec{q}})}{2} + \Delta V_{\vec{q}} = 0 \quad \text{for } x \in B_{1/\varepsilon}, \tag{A.25}$$

$$h_{\vec{q}}(x) = 0 \quad \text{for every } x \in \partial B_{1/\varepsilon}, \tag{A.26}$$

$$h_{\vec{q}}(x, 0) = \psi_\varepsilon(x) \quad \text{for } x \in B_{1/\varepsilon}. \tag{A.27}$$

Moreover, for $w_{\vec{q}}$ as in the hypothesis there holds

$$|h_{\vec{q}} - w_{\vec{q}}| \le \frac{1}{1-\beta}\Big(2 \sup_{B_{\varepsilon^{-1}} \times [\tau, \tau + 2\beta/M]} |\tilde{F}_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}| + \sup_{B_{\varepsilon^{-1}}} |h_{\vec{q}}(x,\tau) - w_{\vec{q}}(x,\tau)|\Big),$$

where $\tilde{F}_{\vec{q}}$ is analogous to $F_{\vec{q}}$ substituting W for $\tilde{W}$.

However, following the proof Theorem A.3 we also have that $|h_{\vec{q}}|(x,t) \le C$, where C is the constant given by Theorem A.3. This fact and the construction of $\tilde{W}$ imply that h_ε is not only a solution to (A.25)–(A.26)–(A.27), but also to (4.3)–(4.4)–(4.5) (since within this range $W = \tilde{W}$). Moreover, for $w_{\vec{q}}$ satisfying $|w_{\vec{q}}| \le K$ we will have $\tilde{F}_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}} = F_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}$, concluding that (4.6) holds and finishing the proof of the theorem. ◻

Theorem A.6. *Let $h_{\vec{q}}$ be a solution to* (4.3)–(4.4)–(4.5)*, with $\psi = 0$; then there is a constant K, independent of $\vec{q}$, such that for every $x \in B_{1/\varepsilon}$*

$$|Dh_{\vec{q}}| \le K. \tag{A.28}$$

Proof. Recall that $h_{\vec{q}}$ is vector-valued. We will denote the i-th coordinate of the vector $h_{\vec{q}}$ by $h^i_{\vec{q}}$ and, similarly, $(\nabla W(h_{\vec{q}}))^i$ is the the i-th coordinate of $\nabla W(h_{\vec{q}})$. We are going to prove separately for each coordinate, that there is a constant C_i such that $|\nabla h^i_{\vec{q}}| \le C_i$.

Let $f : \{(x,y) : y \ge 0\} \to B_{1/\varepsilon}$ be defined by

$$f(x,y) = \frac{1}{\varepsilon}\left(\frac{x^2+y^2-1}{x^2+(y+1)^2}, \frac{-2x}{x^2+(y+1)^2}\right). \tag{A.29}$$

In complex number notation, we can write for $z = x + iy$

$$f(z) = \frac{z-i}{z+i}.$$

Define

$$s^i_{\vec q}(x,y,t) = h^i_{\vec q}(f(x,y),t)). \tag{A.30}$$

It satisfies

$$\frac{8}{\varepsilon(x^2+(y+1)^2)}\frac{ds^i_{\vec q}}{dt} - \Delta s^i_{\vec q} = -\frac{8}{\varepsilon(x^2+(y+1)^2)}(\nabla W(h_{\vec q}))^i + \Delta v^i \qquad \text{for } x\in\mathbb{R},\ y>0,$$

$$\begin{aligned} s^i_{\vec q}(x,y,t) &= 0 \quad \text{for } y=0 \text{ or } |(x,y)|\to\infty,\\ s_{\vec q}(x,y,0) &= 0 \quad \text{for } y\ge 0. \end{aligned}$$

Let $\tilde P$ be the operator defined by

$$\tilde P u = \frac{8}{\varepsilon(x^2+(y+1)^2)}\frac{du}{dt} - \Delta u.$$

Theorem A.3 and the definition of $s^i_{\vec q}$ implies that there is a constant C independent of ε such that

$$\tilde P s^i_{\vec q} \le \frac{C}{\varepsilon}.$$

Moreover,

$$\frac{\partial s^i_{\vec q}}{\partial y}(x,y,0) = 0.$$

Now define

$$w^i_{\vec q}(x,y,t) = s^i_{\vec q}(x,y,t) - \frac{C}{\varepsilon}(y^2+y).$$

Then

$$\begin{aligned} \tilde P w^i_{\vec q} = \tilde P s^i_{\vec q} - 2\frac{C}{\varepsilon} &\le 0,\\ w^i_{\vec q}(x,0,t) = 0 &\quad \text{for every } x\in\mathbb{R}^2 \text{ and } t>0,\\ w^i_{\vec q}(x,y,0) < 0 &\quad \text{for } |(x,y)|\to\infty \text{ and } y>0. \end{aligned}$$

Also,

$$\frac{\partial w^i_{\vec q}}{\partial y}(x,y,0) = -\frac{C}{\varepsilon}(2y+1) \le 0.$$

Claim A.7. *The maximum of $w^i_{\vec q}$ cannot be attained in the interior.*

If the max is attained at some point in the interior, there must hold that $\Delta w^i_{\vec{q}} < 0$ and $dw^i_{\vec{q}}/dt \geq 0$. Hence $\tilde{P}w^i_{\vec{q}} \geq 0$, which is a contradiction and finishes the proof of the claim.

Since the maximum is attained on the boundary, it must be attained at $y = 0$. Therefore

$$\frac{\partial w^i_{\vec{q}}}{\partial y}(x, y, t) \leq 0 \quad \text{for every } t.$$

This implies that

$$\frac{\partial s^i_{\vec{q}}}{\partial y}(x, y, t) \leq \frac{C}{\varepsilon}(2y + 1).$$

This procedure can be repeated for $-s^i_{\vec{q}}$, concluding that

$$\left|\frac{\partial s^i_{\vec{q}}}{\partial y}(x, y, t)\right| \leq \frac{C}{\varepsilon}(2y + 1). \tag{A.31}$$

Since the inverse function of f is

$$f^{-1}(w) = \frac{1 + \varepsilon w}{1 - \varepsilon w},$$

using (A.30), (A.29) and (A.31) we have (in complex number notation) for any $w \in B_{1/\varepsilon}$ that

$$|\nabla h^i_{\vec{q}}(w, t) \cdot (1 - \varepsilon w)^2| \leq 2C\left(\frac{1 - \varepsilon^2|w|^2}{1 + \varepsilon^2|w|^2 - \varepsilon(w + \bar{w})} + \varepsilon\right), \tag{A.32}$$

where $\bar{w}$ is the conjugate of w.

Similarly, if we define (by performing a rotation of f):

$$g(z) = \frac{i}{\varepsilon}\frac{z - i}{z + i} \tag{A.33}$$

and

$$r(x, y, t) = h_{\vec{q}}(g(x, y), t), \tag{A.34}$$

following the same method we obtain

$$|\nabla h^i_{\vec{q}}(w, t) \cdot i(1 + i\varepsilon w)^2| \leq 2C\left(\frac{1 - \varepsilon^2|w|^2}{1 + \varepsilon^2|w|^2 + i\varepsilon(w - \bar{w})} + \varepsilon\right). \tag{A.35}$$

Notice that for w away from $1/\varepsilon$ and i/ε it holds that $i(1+i\varepsilon w)^2$ and $(1-\varepsilon w)^2$ are linearly independent as vectors in $\mathbb{R}^2$. Fixing some δ small enough and considering w such that $|w - i/\varepsilon| \geq \delta$ and $|w - 1/\varepsilon| \geq \delta$, we have that

$$\frac{1-\varepsilon^2|w|^2}{1+\varepsilon^2|w|^2-\varepsilon(w+\bar{w})}+\varepsilon \quad \text{and} \quad \frac{1-\varepsilon^2|w|^2}{1+\varepsilon^2|w|^2+i\varepsilon(w-\bar{w})}+\varepsilon$$

are bounded above and below independent of ε. Hence

$$|\nabla h^i_{\vec{q}}(w,t)| \leq C \quad \text{for every } \left|w-\frac{i}{\varepsilon}\right| \geq \delta, \ \left|w-\frac{1}{\varepsilon}\right| \geq \delta. \tag{A.36}$$

Now considering rotation of f of π and $\frac{3}{2}\pi$ radians, that is,

$$\tilde{f}(z) = -\frac{1}{\varepsilon}\frac{z-i}{z+i} \quad \text{and} \quad \tilde{g}(z) = -\frac{i}{\varepsilon}\frac{z-i}{z+i},$$

and following the same procedure, we find bounds for $|\nabla h^i_{\vec{q}}(w,t)|$ near $1/\varepsilon$ and i/ε, concluding the proof. ◻

Similarly it follows the result below.

Corollary A.8. *Let $k_{\vec{q}}$ be defined by (4.13). Then there is a constant K, independent of $\vec{q}$, such that for every $x \in B_1$*

$$|Dk_{\vec{q}}| \leq \frac{K}{\varepsilon}. \tag{A.37}$$

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