

Stroboscopic prethermalization in weakly interacting periodically driven systems

Elena Canovi,¹ Marcus Kollar,² and Martin Eckstein¹

¹*Max Planck Research Department for Structural Dynamics,
University of Hamburg-CFEL, Hamburg, Germany*

²*Institut für Physik, Theoretische Physik III Center for Electronic
Correlations and Magnetism, University of Augsburg, Augsburg, Germany*
(July 3, 2015)

Time-periodic driving provides a promising route to engineer non-trivial states in quantum many-body systems. However, while it has been shown that the dynamics of *integrable* systems can synchronize with the driving into a non-trivial periodic motion, generic *non-integrable* systems are expected to heat up until they display a trivial infinite-temperature behavior. In this paper we show that a quasi-periodic time evolution over many periods can also emerge in systems with weak integrability breaking, with a clear separation of the timescales for synchronization and the eventual approach of the infinite-temperature state. This behavior is the analogue of prethermalization in quenched systems. The synchronized state can be described using a macroscopic number of approximate constants of motion. We corroborate these findings with numerical simulations for the driven Hubbard model.

PACS numbers: 05.30.Rt, 64.70.Tg, 03.67.Mn, 05.70.Jk

In the last decade, the dynamics of quantum many-particle systems out of equilibrium has become experimentally accessible in a variety of contexts, ranging from ultra-cold atomic gases in optical lattices to the manipulation of solid state systems with femtosecond time-resolved spectroscopy. A particularly important role in this context is played by periodically driven systems. Periodic driving can stabilize novel states both in cold atoms and in condensed matter, including topologically nontrivial states [1, 2, 3, 4, 5, 6] or complex phases such as superconductivity [7, 8]. It can be used to engineer artificial gauge fields in cold atoms [9] and emergent many-body interactions such as magnetic exchange interactions in solids [10, 11, 12], or to transiently modify lattice structures through anharmonic couplings [13].

An important question is thus the theoretical understanding of the long-time dynamics of periodically driven systems. The approach to a steady state has been investigated intensively for the relaxation of isolated systems after a sudden perturbation, both experimentally and theoretically [14, 15, 16]. When a generic non-integrable many-body system is left to evolve with a time-independent Hamiltonian, it is believed to eventually relax to a thermal equilibrium state. If the system is integrable, on the other hand, the steady state is often described by a generalized Gibbs ensemble (GGE) [17, 18], which keeps track of a macroscopic number of constants of motion. When integrability is only slightly broken, the system can display dynamics on separate timescales, such that observables rapidly *prethermalize* to a quasi-steady nonequilibrium state which can be understood by a GGE based on perturbatively constructed constants of motion [19], before thermalizing on much longer time scales [20, 21, 22].

Integrability turns out to be a crucial factor also for periodically driven systems. Their dynamics can syn-

chronize with the driving [23] and display a non-trivial periodic time evolution at long times. A way to understand this is to show that the time evolution over one period T commutes with an infinite number of operators \mathcal{I}_λ , which are thus conserved at stroboscopic times (i.e. integer multiples of the period). Having a fixed expectation value of all \mathcal{I}_λ at stroboscopic times, one can construct a statistical ensemble to describe the long-time behavior of the system (the periodic Gibbs ensemble), which has been analytically and numerically shown to give correct predictions for hard-core bosons [24].

In contrast to integrable systems (and many-body localized states [25, 26, 27, 28]), it has been proposed that generic non-integrable systems “heat up” under the effect of driving and display rather trivial infinite temperature properties as soon as they settle into a periodic motion [29, 30, 31]. One can formulate this statement in terms of the Floquet eigenstates (the exact solutions of the Schrödinger equation with a periodic evolution of all observables [32, 33]), stating that each individual Floquet state displays infinite temperature properties. This conjecture relies on a breakdown of the perturbative expansion of Floquet eigenstates at some order because of unavoidable resonances between transitions in the many-body spectrum with multiples of the driving frequency. A common approach to avoid this problem is to construct effective Floquet Hamiltonians from a high-frequency expansion [25, 9]. In this work we show that quasi-periodic state can also emerge as a consequence of a system being close to integrability, provided that linear absorption can be avoided: the stroboscopic time evolution is constrained by approximately conserved constants of motion $\tilde{\mathcal{I}}_\lambda$. Analogous to prethermalization in weakly interacting systems after a sudden perturbation [20, 21, 19], the system rapidly synchronizes with the driving and remains periodic over a large number of periods m such that

$g^{-1} \gg mT \gg J^{-1}$, where g controls the strength of the integrability-breaking terms in the Hamiltonian, T is the period of the driving and J is coupling of the integrable Hamiltonian, e.g., the bandwidth of the kinetic energy term; we set $\hbar = 1$ throughout. The quasi-periodic state can be described as a periodic Gibbs ensemble based on the $\tilde{\mathcal{I}}_\lambda$, i.e., stroboscopic prethermalization gives access to quasi-periodic states which are entirely different from the infinite temperature final states.

General formalism.— In the following we consider an integrable (or noninteracting) system perturbed by an integrability-breaking periodic driving. The general Hamiltonian is given by

$$H(t) = H_0 + gH_{\text{int}}(t), \quad (1)$$

where the integrable part

$$H_0 = \sum_{\lambda} \epsilon_{\lambda} \hat{\mathcal{I}}_{\lambda}, \quad (2)$$

can be written as a sum of constants of motion (e.g., momentum occupations for independent particles on a lattice), and the small parameter g controls the strength of the interaction $H_{\text{int}}(t)$ which is periodic with period T and frequency $\Omega = 2\pi/T$. To study the time evolution at stroboscopic times $t_m = mT$ (m integer), we extend the approach of Refs. 19, 21 to periodically driven systems, and determine a time-periodic unitary transformation $R(t)$ such that the Hamiltonian $H_{\text{eff}}(t)$ in the rotated frame commutes with the constants of motion $\hat{\mathcal{I}}_{\lambda}$ at any time up to corrections of order $\mathcal{O}(g^3)$. If $|\tilde{\psi}(t)\rangle = R(t)|\psi(t)\rangle$ is the transformed wave function, the Hamiltonian H_{eff} which dictates the evolution in the rotated frame via $i\partial_t|\tilde{\psi}(t)\rangle = H_{\text{eff}}(t)|\tilde{\psi}(t)\rangle$ is given by

$$H_{\text{eff}}(t) = R(t)H(t)R(t)^\dagger - iR(t)\dot{R}(t)^\dagger. \quad (3)$$

We make the ansatz $R(t) \equiv e^{S(t)}$, with an anti-hermitian operator S , and expand all periodic operators $H_{\text{eff}}(t)$, $S(t)$, $H_{\text{int}}(t)$ in a Fourier series $X(t) = \sum_n X_n e^{-in\Omega t}$. One can now construct $S(t) = gS^{(1)}(t) + \frac{1}{2}g^2S^{(2)}(t) + \mathcal{O}(g^3)$ order by order in g , such that $H_{\text{eff}}(t)$ is diagonal in the eigenbasis $\{|\alpha\rangle\}$ of H_0 (see Supplement),

$$H_{\text{eff}}(t) = H_0 + \sum_{\alpha} |\alpha\rangle E_{\text{diag},\alpha}(t) \langle\alpha| + \mathcal{O}(g^3), \quad (4)$$

and thus commutes with all $\hat{\mathcal{I}}_{\lambda}$. In particular, to first order the perturbative corrections to the diagonal entries are given by $E_{\text{diag},\alpha}(t) = g\langle\alpha|H_{\text{int}}(t)|\alpha\rangle + \mathcal{O}(g^2)$. Denoting with $\{E_{\alpha}\}$ the eigenvalues of H_0 , we have

$$\langle\beta|S_n^{(1)}|\alpha\rangle = \langle\beta|H_{\text{int},n}|\alpha\rangle / (E_{\beta} - E_{\alpha} - n\Omega), \quad (5)$$

for $\alpha \neq \beta$, and $\langle\alpha|S_n^{(1)}|\alpha\rangle = 0$. (The role of resonances $E_{\beta} - E_{\alpha} = n\Omega$ will be discussed below.)

Under a general unitary transformation, the time propagator $U(t, 0) = \mathcal{T}e^{-i\int_0^t dt' H(t')}$ is transformed into

$$U(t, 0) = e^{-S(t)} \tilde{U}(t, 0) e^{S(0)}, \quad (6)$$

with $\tilde{U}(t, 0) = \mathcal{T}e^{-i\int_0^t dt' H_{\text{eff}}(t')}$. Because $S(t)$ is periodic, the time evolution at stroboscopic times is thus unitarily equivalent to the time evolution with the diagonal Hamiltonian (4), $U(t_m, 0) = e^{-S(0)} e^{-i\int_0^{t_m} dt H_{\text{eff}}(t)} e^{S(0)} + t_m \mathcal{O}(g^3)$. This implies that the quantities

$$\tilde{\mathcal{I}}_{\lambda} = e^{-S(0)} \hat{\mathcal{I}}_{\lambda} e^{S(0)} \quad (7)$$

are approximately conserved under the evolution over multiple periods T , i.e., $\langle\tilde{\mathcal{I}}_{\lambda}(t_m)\rangle = \langle\tilde{\mathcal{I}}_{\lambda}(0)\rangle + t_m \mathcal{O}(g^3)$. For the example of a weakly interacting Hubbard model studied below, the original constants of motion are momentum occupations $n_{\mathbf{k}}$ of independent particles, while the constants of motion of the stroboscopic time evolution correspond to quasiparticle modes.

We examine the synchronization of these modes in terms of the time evolution

$$\langle A \rangle_t \equiv \langle \psi(0) | U^\dagger(t, 0) A U(t, 0) | \psi(0) \rangle \quad (8)$$

of an observable \hat{A} which is a function of the original constants of motion \mathcal{I}_{λ} (having in mind, e.g., a measurement of momentum occupations $n_{\mathbf{k}}$ or higher-order momentum correlation functions $n_{\mathbf{k}} n_{\mathbf{k}'}$), assuming that the system is in an eigenstate $|\psi(0)\rangle \equiv |\alpha\rangle$ of H_0 before the driving is switched on. Inserting Eq. (6) into (8), expanding the operators $e^{S(0)}$ and $e^{S(t)}$ in powers of g , and using the fact that $[A, \tilde{U}(t, 0)] = t\mathcal{O}(g^3)$ (because A commutes with all \mathcal{I}_{λ}), we obtain

$$\langle A \rangle_t = -2\text{Re}\langle\alpha|S(0)A[S(0) - \bar{S}(t)]|\alpha\rangle + t\mathcal{O}(g^3), \quad (9)$$

with $\bar{S}(t) \equiv \tilde{U}^\dagger(t, 0)S(t)\tilde{U}(t, 0)$. For stroboscopic times, with $S(t_m) = S(0)$ determined by Eq. (5), one finds the final result for the perturbative time evolution

$$\langle A \rangle_{t_m} = \sum_{n,p} \int_{-\infty}^{\infty} d\omega \frac{4g^2 \sin^2(\frac{\omega t_m}{2}) y_{np}(\omega)}{(\omega - n\Omega)(\omega - p\Omega)} + t_m \mathcal{O}(g^3), \quad (10)$$

where $y_{np}(\omega)$ denotes the spectral density

$$y_{np}(\omega) = \langle\alpha|H_{\text{int},-n}A\delta(\omega - H_0 + E_{\alpha})H_{\text{int},-p}|\alpha\rangle. \quad (11)$$

The integral in Eq. (10) gives an accurate description of $\langle A \rangle_{t_m}$ for times $t_m \ll g^{-1}$, where relative corrections $t_m \mathcal{O}(g^3)$ are small. (Note that for finite m the term $\sin(t_m \omega/2)$ regularizes the singularities at $n\omega$.) For $g \rightarrow 0$ there is thus a large time window $g^{-1} \gg t_m \gg T$ in which the dynamics is governed by the long time asymptotics of the integral. To analyze this, we distinguish two different behaviors depending on the frequency Ω :

(i) *Fermi golden rule regime:* If there is nonzero spectral density $y_{nn}(n\Omega) > 0$ at an even pole $1/(\omega - n\Omega)^2$,

the stroboscopic evolution for $m \gg 1$ develops a linear asymptotics $\langle A \rangle_{t_m} \sim g^2 t_m \sum_{\beta \neq \alpha} \langle \beta | A | \beta \rangle \Gamma_{\alpha \rightarrow \beta}$, where $\Gamma_{\alpha \rightarrow \beta}$ is the Fermi golden rule excitation rate

$$\Gamma_{\alpha \rightarrow \beta} = 2\pi \sum_n |\langle \beta | H_{\text{int},n} | \alpha \rangle|^2 \delta(n\Omega - E_\beta + E_\alpha). \quad (12)$$

To see this fact one can consider the contribution to the integral (10) from a small interval $|\omega - n\Omega| \leq \epsilon$ around the pole, in which $y_{nm}(\omega)$ can be approximated by a constant $y_{nn}(n\Omega)$. With a substitution $x = t_m(\omega - n\Omega)$, the remaining integral is $t_m \int_{-\epsilon t_m}^{\epsilon t_m} dx \sin^2(x/2)/x^2 \sim t_m \pi/2$. From a similar consideration for $n \neq p$ one can obtain the subleading terms.

(ii) *Stroboscopic prethermalization*: Assuming that the perturbation involves only a limited number of Fourier components, such as for a harmonic perturbation with $H_{\text{int},n} = 0$ for $|n| > 1$, then the spectral density $y_{nm}(\omega)$ is restricted to a finite band $[-W, W]$, depending on the type of excitation, the bandwidth of the noninteracting single-particle spectrum, and phase space restrictions. If all poles $\omega = n\Omega$ lie outside this band, the limit $m \rightarrow \infty$ integral of (10) is simply obtained by replacing $\sin^2(t_m \omega/2)$ by its average $1/2$, which corresponds to the first term in Eq. (9),

$$\begin{aligned} \langle A \rangle_{\text{pre}} &= -2\text{Re}\langle \alpha | S(0) A S(0) | \alpha \rangle \\ &= 2g^2 \sum_{n,p} \int_{-\infty}^{\infty} d\omega \frac{y_{np}(\omega)}{(\omega - n\Omega)(\omega - p\Omega)}. \end{aligned} \quad (13)$$

In this case the system synchronizes for $t_m \gg T$ (and $t_m \ll g^{-1}$) into a periodic evolution with values $\langle A \rangle_{t_m} = \langle A \rangle_{\text{pre}}$, before further heating takes place on longer timescales. This is the analogue of prethermalization in a quenched system.

Statistical description of the prethermalized state.— The condition $y_{nm}(n\Omega) = 0$ for the absence of linear absorption is equivalent to the absence of resonances in Eq. (5). Outside the Fermi golden rule regime, the constants of motion (7) are thus well-defined, and one can ask whether the prethermalized state can be described by a Gibbs ensemble $\rho_{\tilde{G}} = \sum_{\lambda} e^{-\mu_{\lambda} \tilde{I}_{\lambda}} / Z_{\tilde{G}}$ [34], where the Lagrange multipliers μ_{λ} are determined by the constraint from the initial state, $\langle \tilde{I}_{\lambda} \rangle_0 = \text{tr}[\rho_{\tilde{G}} \tilde{I}_{\lambda}]$,

$$\langle A \rangle_{\text{pre}} = \text{tr}[\rho_{\tilde{G}} A]. \quad (14)$$

Using Eqs. (13) and (7), the proof for this statement only relies on the time-independent matrix $S(0)$ being antihermitian and appearing only to order g in $\text{tr}[\rho_{\tilde{G}} A]$, and thus proceeds analogously to the argument showing that prethermalized states for a sudden quench can be described by a GGE [19].

Relation to the Floquet picture.— Finally, we explain how the prethermalized state Eq. (13) can be related to the Floquet spectrum of the Hamiltonian. According to the Floquet theorem, the exact solution of the

Schrödinger equation with a time-periodic Hamiltonian (1) is given in the form $|\psi_{F,\alpha}(t)\rangle = e^{-iE_{F,\alpha}t} |\psi_{\alpha}(t)\rangle$, where $|\psi_{\alpha}(t)\rangle$ is periodic in time. If a system is in a Floquet state, the time evolution of observables is periodic. By expanding $|\psi_{\alpha}(t)\rangle$ in a Fourier series $|\psi_{\alpha}(t)\rangle = \sum_m e^{-i\Omega m t} |\psi_{\alpha,m}\rangle$, the Floquet quasi-energy spectrum can be obtained by diagonalizing the time-independent block-matrix,

$$(E_{F,\alpha} + m\Omega - H_0) |\psi_{\alpha,m}\rangle = g \sum_l H_{\text{int},l} |\psi_{\alpha,m+l}\rangle. \quad (15)$$

In principle, one can now use standard first-order perturbation theory to construct perturbative Floquet states $|\psi_{\alpha,n}\rangle = |\psi_{\alpha,n}^{(0)}\rangle + g |\psi_{\alpha,n}^{(1)}\rangle + \dots$, where the zeroth order is given by the unperturbed eigenstates $|\psi_{\alpha,m}^{(0)}\rangle = \delta_{m,0} |\alpha\rangle$ ($E_{F,\alpha}^{(0)} = E_{\alpha}$). The perturbative expansion does not converge to the true Floquet eigenstate if there are resonances $E_{\alpha} - E_{\beta} = n\Omega$ in the many-body spectrum, but low orders nevertheless can exist: in particular, the first order is given by $|\psi_{\alpha,m}^{(1)}\rangle = S_m^{(1)} |\alpha\rangle$, and it is well-defined outside the Fermi golden rule regime. This shows that the prethermalized state Eq. (13) is related to the perturbative Floquet state by

$$\langle A \rangle_{\text{pre}} = 2 \langle \psi_{F,\alpha}^{(1)} | A | \psi_{F,\alpha}^{(1)} \rangle. \quad (16)$$

Here the appearance of a factor of two is reminiscent to a similar relation between the prethermalized and ground state expectation values in the quench case.

Application to the Hubbard model.— In order to illustrate the general results above, we now choose as specific example the Hubbard model

$$H(t) = -J \sum_{\langle ij \rangle \sigma} c_{i\sigma}^{\dagger} c_{j\sigma} + U(t) \sum_i (n_{i\uparrow} - \frac{1}{2})(n_{i\downarrow} - \frac{1}{2}), \quad (17)$$

with nearest neighbor hopping J and periodically modulated interaction $U(t) = U(1 - \cos(\Omega t))$. With these choices, the first and the second term of Eq. (17) represent the integrable part H_0 and the periodic integrability-breaking perturbation with $g = U$, respectively. Energy and time are measured in units of J and J^{-1} , respectively. The constants of motion of H_0 are momentum occupation numbers $\hat{n}_{\mathbf{k}\sigma} = c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma}$. To allow for a comparison of the analytical results derived above and a numerical solution, we consider the model in the limit of infinite spatial dimensions with a semi-elliptic density of states $\rho(\epsilon) = \sqrt{4 - \epsilon^2}/(2\pi)$ at half-filling (density $n = 1$). In this limit, the dynamics can be computed using nonequilibrium dynamical-mean-field theory [35], and iterative perturbation theory [36, 37] as impurity solver. The system is assumed to be in an equilibrium state at $U = 0$ and temperature $1/\beta$ before the driving is turned on. Note that it is straightforward to extend our above discussion from pure states to thermal states (see also the Supplement).

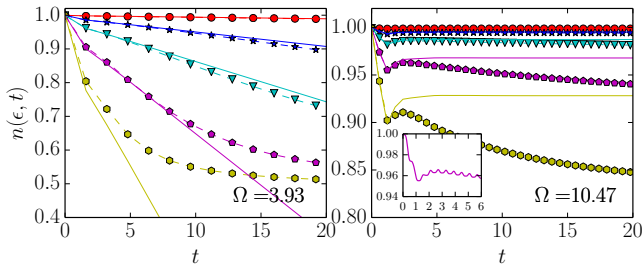


FIG. 1. Momentum occupation $n(\epsilon_{\mathbf{k}})$ for energy $\epsilon_{\mathbf{k}} = -0.4$ at stroboscopic times $2\pi m/\Omega$ for $U = 0.1, 0.3, 0.5, 0.8, 1.2$ from top to bottom curves in each panel. Symbols with dashed lines: DMFT data, continuous lines: perturbative predictions from Eqs. (10) and (18). Inset of right panel: same as main panel for $U = 0.8$, but showing complete time evolution (i.e. also non stroboscopic times).

To investigate the prethermalization dynamics we use the momentum occupations as observables, $A \equiv n_{\mathbf{k}} - n_{\mathbf{k}}^0$, where $n_{\mathbf{k}}^0 = \Theta(-\epsilon_{\mathbf{k}})$ is the initial Fermi sea of the ground state for a symmetric density of states. For harmonic driving $U(t) = U(1 - \cos(\Omega t))$ we have $H_{\text{int},n} = h_n H_{\text{int}}$ with $H_{\text{int}} = \sum_i (n_{i\uparrow} - \frac{1}{2})(n_{i\downarrow} - \frac{1}{2})$, and $h_0 = 1$, $h_{\pm 1} = -\frac{1}{2}$. For the spectral density (11) we obtain $y_{\mathbf{k},np}(\omega) = -\hbar_n h_p J_{\mathbf{k}}(\omega)$, where $J_{\mathbf{k}}(\omega)$ is given by

$$J_{\mathbf{k}}(\omega) = \int d\epsilon_1 d\epsilon_2 d\epsilon_3 \rho(\epsilon_1) \rho(\epsilon_2) \rho(\epsilon_3) [n(\epsilon_3) n(\epsilon) \bar{n}(\epsilon_1) \bar{n}(\epsilon_2) - n(\epsilon_1) n(\epsilon_2) \bar{n}(\epsilon_3) \bar{n}(\epsilon)] \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \omega - \epsilon_{\mathbf{k}}), \quad (18)$$

with Fermi function $n(\epsilon)$ (and $\bar{n}(\epsilon) \equiv (1 - n(\epsilon))$), here given by $n(\epsilon_{\mathbf{k}}) = n_{\mathbf{k}}^0$. Note that the result depends on \mathbf{k} only via the band energy $\epsilon_{\mathbf{k}} \in [-2, 2]$ as a consequence of the large-coordination limit. The function $J_{\mathbf{k}}(\omega)$ has already been obtained for the investigation of the sudden quench [19] (which is contained in our results by setting $h_{\pm 1} = 0$). We thus shift further details of the calculation to the Supplement. From Eq. (18), one can read off the phase space condition for the Fermi golden rule: at zero temperature, $n(\epsilon) = \Theta(-\epsilon)$ and $\rho(\epsilon) = 0$ for $|\epsilon| > 2$, hence linear absorption ($J_{\mathbf{k}}(\pm\Omega) \neq 0$) should occur for $|\epsilon_{\mathbf{k}}| < \Omega < 6 + |\epsilon_{\mathbf{k}}|$.

In Fig. 1 we show the single-particle occupation $n(\epsilon_{\mathbf{k}})$ at stroboscopic times for a specific value of ϵ for $\Omega = 3.93$ and $\Omega = 10.47$, which lie in the Fermi golden-rule regime and in the prethermalization regime, respectively. We find that the perturbative predictions from Eqs. (10) and (18) capture well the initial slope of the occupation in the linear absorption regime, as well as the prethermalization plateau for $\Omega = 10.47$. For later times the numerical results approach the infinite-temperature value $n_{\beta \rightarrow 0}(\epsilon) = 0.5$. As expected, the agreement between the DMFT results and the perturbative predictions improves with decreasing U , where the prethermalization plateau extends to longer times. In the inset of Fig. 1 we show

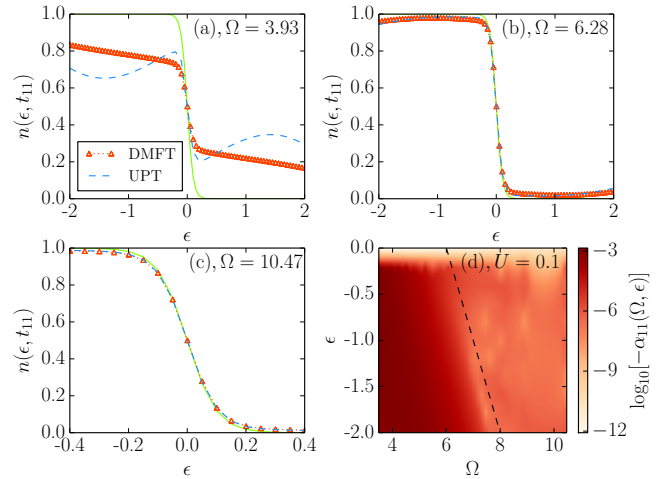


FIG. 2. (a), (b), (c): single particle occupations for a driving term with $U = 0.5$, computed with DMFT (red triangles, with initial inverse temperature $\beta = 20$) and from perturbation theory (blue dashed lines, initially in the ground state) at an intermediate time, $t_{11} = 11T$, for $\Omega = 2\pi/T = 3.93, 6.28$, and 10.47 , respectively; the initial state is prepared with $U = 0$. The green continuous line shows the free initial thermal state at $\beta = 20$. (d): Excitation over one period, measured as $\alpha_m(\epsilon, \Omega)$ (see main text) for $U = 0.1$ using DMFT. The black dashed line corresponds to $\epsilon = -\Omega + 6$, which together with $\epsilon = -\Omega$ delimits the Fermi-golden-rule regime.

the time evolution of the occupation $n(\epsilon)$ at $U = 0.8$. At $t \sim 2-4$ the quasi-periodic prethermalization regime begins where $n(\epsilon)$ is constant at stroboscopic times. In Fig. 2 we plot $n(\epsilon, t_m)$ as a function of ϵ after a given number of periods ($m = 11$). Panel (a) corresponds to a frequency such that every value of ϵ gives rise to linear terms, which are on the contrary absent for $\Omega = 10.47$, see panel (c). Panel (b) refers to an intermediate case ($T = 1.0$, $\Omega = 2\pi$), where only the boundary values of ϵ (i.e. $\epsilon \gtrsim -2$ and $\epsilon \lesssim 2$) give linear contributions and thus at $t_m \gg T$ differ from the DMFT data. Finally, panel (d) shows the absorption of energy, measured by the slope $\alpha_m(\epsilon, \Omega) \equiv n(\epsilon, m2\pi/\Omega) - n(\epsilon, (m-1)2\pi/\Omega)$, which becomes small for $\Omega > 6 - \epsilon$, as predicted by the perturbative calculation (shown with a dashed line for $m = 11$). We point out that the regime of validity of the DMFT calculation with iterative perturbation theory does not allow to explore small values of the frequency ($\Omega \lesssim 1$) where the other boundary ($\epsilon = -\Omega$) lies.

Conclusions.— In conclusion, we discussed the analogue of prethermalization in periodically driven systems. A weakly interacting system can synchronize into a quasi-steady state with nontrivial properties, before reaching the infinite temperature state generic for the long-time behavior of driven non-integrable systems. This stroboscopic prethermalization is a consequence of the existence of a macroscopic set of operators which are almost conserved by the time evolution over one period.

Stroboscopic prethermalization thus provides a way to engineer quantum states with a nontrivial effective dynamics, alternative to the a high frequency expansion. These states reflect the properties of perturbative Floquet states, which can be very different in nature from the exact Floquet states.

The authors would like to acknowledge fruitful discussions with Markus Heyl and Hugo U. R. Strand. M.K. was supported in part by Transregio 80 of Deutsche Forschungsgemeinschaft.

-
- [1] T. Oka and H. Aoki, Phys. Rev. B **79**, 081406 (2009).
- [2] T. Kitagawa, T. Oka, A. Brataas, L. Fu, and E. Demler, Phys. Rev. B **84**, 235108 (2011).
- [3] T. Iadecola, D. Campbell, C. Chamon, C.-Y. Hou, R. Jackiw, S.-Y. Pi, and S. V. Kusminskiy, Phys. Rev. Lett. **110**, 176603 (2013).
- [4] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B **82**, 235114 (2010).
- [5] N. Lindner, G. Refael, and V. Galitski, Nature Physics **7**, 490 (2011).
- [6] Y. H. Wang, H. Steinberg, P. Jarillo-Herrero, and N. Gedik, Science **342**, 453 (2013).
- [7] D. Fausti, R. I. Tobey, N. Dean, S. Kaiser, A. Dienst, M. C. Hoffmann, S. Pyon, T. Takayama, H. Takagi, and A. Cavalleri, Science **331**, 189 (2011).
- [8] W. Hu, S. Kaiser, D. Nicoletti, C. R. Hunt, I. Gierz, M. C. Hoffmann, M. Le Tacon, T. Loew, B. Keimer, and A. Cavalleri, Nat. Mater. **13**, 705 (2014).
- [9] N. Goldman and J. Dalibard, Phys. Rev. X **4**, 031027 (2014).
- [10] J. H. Mentink, K. Balzer, and M. Eckstein, Nat. Commun. **6**, 6708 (2015).
- [11] R. V. Mikhaylovskiy, E. Hendry, A. Secchi, J. H. Mentink, M. Eckstein, A. Wu, R. V. Pisarev, V. V. Kruglyak, M. I. Katsnelson, T. Rasing, and A. V. Kimel, arXiv:1412.7094.
- [12] A. P. Itin and M. I. Katsnelson, arXiv:1401.0402.
- [13] M. Först, C. Manzoni, S. Kaiser, Y. Tomioka, Y. Tokura, R. Merlin, and A. Cavalleri, Nature Physics **7**, 854 (2011).
- [14] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
- [15] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).
- [16] S. Trotzky, Y.-a. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, Nature Physics **8**, 325 (2012).
- [17] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. **98**, 050405 (2007).
- [18] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweigler, M. Kuhnert, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, Science **348**, 207 (2015).
- [19] M. Kollar, F. A. Wolf, and M. Eckstein, Phys. Rev. B **84**, 054304 (2011).
- [20] J. Berges, S. Borsányi, and C. Wetterich, Phys. Rev. Lett. **93**, 142002 (2004).
- [21] M. Moeckel and S. Kehrein, Phys. Rev. Lett. **100**, 175702 (2008).
- [22] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, Science **337**, 1318 (2012).
- [23] A. Russomanno, A. Silva, and G. E. Santoro, Phys. Rev. Lett. **109**, 257201 (2012).
- [24] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. Lett. **112**, 150401 (2014).
- [25] L. D'Alessio and A. Polkovnikov, Annals of Physics **333**, 19 (2013).
- [26] P. Ponte, A. Chandran, Z. Papi, and D. A. Abanin, Annals of Physics **353**, 196 (2015).
- [27] P. Ponte, Z. Papić, F. Huveneers, and D. A. Abanin, Phys. Rev. Lett. **114**, 140401 (2015).
- [28] A. Roy and A. Das, Phys. Rev. B **91**, 121106 (2015).
- [29] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. E **90**, 012110 (2014).
- [30] L. D'Alessio and M. Rigol, Phys. Rev. X **4**, 041048 (2014).
- [31] A. Eckardt and E. Anisimovas, arXiv:1502.06477.
- [32] T. Dittrich, P. Hänggi, G. L. Ingold, B. Kramer, G. Schön, and W. Zwerger, *Quantum Transport and Dissipation* (Wiley-VCH, Weinheim, 1998).
- [33] M. Grifoni and P. Hänggi, Phys. Rep. **304**, 229 (1998).
- [34] The statistical ensemble $\rho_{\tilde{C}}$ is the periodic Gibbs ensemble of Ref. 24 evaluated at stroboscopic times.
- [35] H. Aoki, N. Tsuji, M. Eckstein, M. Kollar, T. Oka, and P. Werner, Rev. Mod. Phys. **86**, 779 (2014).
- [36] M. Eckstein, M. Kollar, and P. Werner, Phys. Rev. B **81**, 115131 (2010).
- [37] N. Tsuji and P. Werner, Phys. Rev. B **88**, 165115 (2013).

Stroboscopic prethermalization in weakly interacting periodically driven systems
— Supplementary Material —

Elena Canovi, Marcus Kollar, and Martin Eckstein

The transformation $S(t)$

We consider an Hamiltonian of the form:

$$H(t) = H_0 + gH_{\text{int}}(t) \quad (19)$$

where H_0 is constant and integrable and can therefore be written as a sum of the constants of motion:

$$H_0 = \sum_{\lambda} \epsilon_{\lambda} \hat{\mathcal{I}}_{\lambda}, \quad (20)$$

where the parameter g is the strength of the perturbation, while $H_{\text{int}}(t)$ is a periodic driving term with period T and frequency $\Omega = 2\pi/T$ and is assumed to be nonintegrable. We now show in detail how to rotate the Hamiltonian Eq. (19) with a transformation $R(t) = e^{S(t)}$ such that

$$H_{\text{eff}} = R(t)H(t)R(t)^{\dagger} - iR(t)\dot{R}(t)^{\dagger} \quad (21)$$

is (i) periodic and (ii) diagonal in the operators that diagonalize H_0 . To implement condition (i), we first expand to second order in g and then write in Fourier series both the effective Hamiltonian

$$\begin{aligned} H_{\text{eff}}(t) &= H_{\text{eff}}^{(0)}(t) + gH_{\text{eff}}^{(1)}(t) + g^2H_{\text{eff}}^{(2)}(t) \\ &= \sum_n e^{-in\Omega t} \left[H_{\text{eff},n}^{(0)} + gH_{\text{eff},n}^{(1)} + g^2H_{\text{eff},n}^{(2)} \right] \end{aligned} \quad (22)$$

and the anti-hermitian operator

$$\begin{aligned} S(t) &= gS^{(1)}(t) + \frac{g^2}{2}S^{(2)}(t) + \mathcal{O}(g^3) \\ &= \sum_n e^{-in\Omega t} \left[gS_n^{(1)} + \frac{g^2}{2}S_n^{(2)} \right] + \mathcal{O}(g^3), \end{aligned} \quad (23)$$

with $H_{\text{eff},n} = H_{\text{eff},-n}^{\dagger}$ and $S_n = -S_{-n}^{\dagger}$. Combining Eqs. (21) and (22) we find:

$$\begin{aligned} H_{\text{eff}}(t) &= H_0 + g \left(H_{\text{int}}(t) + [S^{(1)}(t), H_0] + i\frac{d}{dt}S^{(1)}(t) \right) \\ &+ g^2 \left(\frac{1}{2}[S^{(2)}(t), H_0] + [S^{(1)}(t), H_{\text{int}}(t)] + \right. \\ &\left. \frac{1}{2}[S^{(1)}(t), [S^{(1)}(t), H_0]] + \frac{i}{2}\frac{d}{dt}S^{(2)}(t) + \right. \\ &\left. - \frac{i}{2}(\dot{S}^{(1)}(t)S^{(1)}(t) - S^{(1)}(t)\dot{S}^{(1)}(t)) \right) + \mathcal{O}(g^3) \end{aligned} \quad (24)$$

To ensure condition (ii), we require that

$$[H_{\text{eff},n}^{(X)}, \hat{\mathcal{I}}_{\lambda}] = 0, \quad (25)$$

for any Fourier component, perturbative order and constant of motion, labeled by n , X and λ respectively. As in Ref. 19, we employ the basis $\hat{\mathcal{I}}_{\lambda}|\alpha\rangle = \alpha_{\lambda}|\alpha\rangle$ and assume that the energies ϵ_{λ} are incommensurate, so that the eigenenergies of H_0 , i.e. $E_{\alpha} = \sum_{\lambda} \epsilon_{\lambda} \alpha_{\lambda}$, are nondegenerate. After some lengthy but otherwise straightforward algebra, we can find $H_{\text{eff}}(t)$ and $S(t)$ by repeatedly applying Eq. (25) to each perturbative order in Eq. (24).

In order g^0 we have

$$H_{\text{eff},n}^{(0)} = \begin{cases} H_0 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}, \quad (26)$$

so that $H_{\text{eff},0}^{(0)} = \sum_{\alpha} |\alpha\rangle\langle\alpha|E_{0,\alpha}^{(0)}$, with $E_{0,\alpha}^{(0)} = E_{\alpha}$.

To first order in g the Fourier components of $S(t)$ read:

$$\langle\beta|S_n^{(1)}|\alpha\rangle = \begin{cases} \frac{\langle\beta|H_{\text{int},n}|\alpha\rangle}{E_{\beta} - E_{\alpha} - n\Omega} & \text{if } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases} \quad (27)$$

The first-order perturbative correction to H_{eff} is:

$$H_{\text{eff}}^{(1)}(t) = \sum_{\alpha} e^{-in\Omega t} |\alpha\rangle E_{n,\alpha}^{(1)} \langle\alpha| \quad (28)$$

where

$$E_{n,\alpha}^{(1)} = \langle\alpha|H_{\text{int},n}|\alpha\rangle. \quad (29)$$

At order g^2 the Fourier components of $S^{(2)}$ are found to be:

$$\langle\beta|S_n^{(2)}|\alpha\rangle = \sum_p \frac{\langle\beta| \left[S_p^{(1)}, H_{\text{int},n-p} + H_{\text{diag},n-p}^{(1)} \right] |\alpha\rangle}{E_{\beta} - E_{\alpha} - n\Omega} \quad (30)$$

if $\alpha \neq \beta$ and, as previously, we choose the diagonal elements to be zero. In Eq. (30) we have defined: $H_{\text{diag},n}^{(1)} = \sum_{\alpha} |\alpha\rangle E_{n,\alpha}^{(1)} \langle\alpha|$. Finally, the second order term of the effective Hamiltonian reads:

$$H_{\text{eff},n}^{(2)} = \sum_{\alpha} |\alpha\rangle E_{n,\alpha}^{(2)} \langle\alpha|, \quad (31)$$

with:

$$\begin{aligned} E_{n,\alpha}^{(2)} &= \frac{1}{2} \sum_{\beta \neq \alpha} \sum_p \left[\frac{\langle\alpha|H_{\text{int},p}|\beta\rangle\langle\beta|H_{\text{int},n-p}|\alpha\rangle}{E_{\alpha} - E_{\beta} - p\Omega} \right. \\ &\left. - \frac{\langle\alpha|H_{\text{int},n-p}|\beta\rangle\langle\beta|H_{\text{int},p}|\alpha\rangle}{E_{\beta} - E_{\alpha} - p\Omega} \right]. \end{aligned} \quad (32)$$

Unitary perturbation theory predictions for the Hubbard model in infinite dimensions

To evaluate the momentum occupation for a periodically driven Hubbard model in infinite dimensions we proceed as in Refs. 36 and 19, noting that in that in those derivations also initial free thermal states are allowed by virtue of the finite-temperature version of Wick's theorem. Using the results in the main text the time-dependent occupation of a state with single-particle energy ϵ at time $t_m = mT$ is given by:

$$n_{\text{pert}}(\epsilon, t_m) = n(\epsilon) - 4U^2 F(\epsilon, t_m), \quad (33)$$

where $n(\epsilon_{\mathbf{k}}) = \langle c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} \rangle_{t=0}$ is the momentum distribution in the initial free thermal state, and

$$\begin{aligned} F(\epsilon, t_m) &\equiv \sum_{n,p} \int_{-\infty}^{\infty} d\omega \frac{\sin^2(\omega t_m/2)}{(\omega - n\Omega)(\omega - p\Omega)} h_n h_{-p} J_\epsilon(\omega) \\ &\equiv \sum_{n,p} h_n h_{-p} F_{n,p}(\epsilon, t_m), \end{aligned} \quad (34)$$

where $\bar{n}(\epsilon) \equiv 1 - n(\epsilon)$ (which equals $n(-\epsilon)$ in the case of particle-hole symmetry, which we consider here). Because momentum conservation can be omitted in the limit of infinite dimensions, one has $J_{\mathbf{k}}(\omega) = J_{\epsilon_{\mathbf{k}}}(\omega)$,

$$\begin{aligned} J_\epsilon(\omega) &= \int d\epsilon_1 d\epsilon_2 d\epsilon_3 \rho(\epsilon_1) \rho(\epsilon_2) \rho(\epsilon_3) [n(\epsilon_3) n(\epsilon) \bar{n}(\epsilon_1) \bar{n}(\epsilon_2) \\ &\quad - n(\epsilon_1) n(\epsilon_2) \bar{n}(\epsilon_3) \bar{n}(\epsilon)] \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \omega - \epsilon). \end{aligned} \quad (35)$$

For the time-dependent interaction $U(t) = (1 - \cos(\Omega t))U$ we have $h_0 = 1$, $h_1 = h_{-1} = -1/2$.

As discussed in the main text, the single-particle occupations Eq. (33) at long times (i.e. $t_m \gg T$) display two regimes, namely the Fermi-golden rule absorption regime and the stroboscopic prethermalization regime, depending on the value of Ω and ϵ . Here we discuss these regimes for the specific case of the driven Hubbard interaction by rewriting Eq. (34) and applying a phase-space argument.

As a first step, we express $J_\epsilon(\omega)$ in terms of

$$R(s) \equiv \int_{-\infty}^{\infty} d\epsilon n(\epsilon) \rho(\epsilon) e^{is\epsilon}, \quad (36)$$

using a Fourier representation of the delta function:

$$J_\epsilon(\omega) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \left[n(\epsilon) e^{i(\epsilon+\omega)s} - \bar{n}(\epsilon) e^{-i(\epsilon+\omega)s} \right] R(s)^3. \quad (37)$$

We also note that for an initial zero-temperature state (with $n(\epsilon) = \Theta(-\epsilon)$), $J_\epsilon(\omega)$ is zero unless $\epsilon \leq |\omega| \leq 3D + |\epsilon|$, where D is the half-bandwidth.

A partial fraction decomposition of the functions in Eq. (34) and a shift of the integration variable yield

$$\begin{aligned} F_{n,p}(\epsilon, t_m) &= \frac{F^{(1)}(\epsilon, t_m, n\Omega) - F^{(1)}(\epsilon, t_m, p\Omega)}{(n-p)\Omega}, \quad (n \neq p) \\ F_{n,n}(\epsilon, t_m) &= F^{(2)}(\epsilon, t_m, n\Omega), \end{aligned} \quad (38)$$

where we defined

$$F^{(N)}(\epsilon, t_m, E) \equiv \int_{-\infty}^{\infty} d\omega \frac{\sin^2(\omega t_m/2)}{\omega^N} J_\epsilon(\omega + E). \quad (39)$$

Consider first the case of zero (or sufficiently low) temperature of the initial state and $|\epsilon| \leq |\Omega| \leq 3D + |\epsilon|$. Then a term linear in t_m contributes to $F(\epsilon, t_m)$, namely ($E = |n\Omega|$, $N = 1, 2$, $x = (\omega - E)t_m$)

$$\begin{aligned} F^{(N)}(\epsilon, t_m, E) &= t_m^{N-1} \int_{-\infty}^{\infty} dx \frac{\sin^2(x/2)}{x^N} J_\epsilon\left(\frac{x}{t_m} + E\right) \\ &\sim \delta_{N2} \frac{\pi t_m}{2} J_\epsilon(E) \quad (t_m \rightarrow \infty) \end{aligned} \quad (40)$$

This corresponds to the Fermi golden rule regime with a linear-in-time growth of $n(\epsilon, t_m)$. On the other hand, if Ω is outside the indicated interval, the denominators are never zero (for zero temperature) and a stroboscopic prethermalization plateau is attained.

In all cases we can rewrite the integrals more compactly by using the identities

$$\frac{\sin^2(\omega t/2)}{\omega} = \frac{1}{2} \int_0^t du \sin(\omega u), \quad (41)$$

$$\int_0^\infty d\omega \frac{\sin^2(\omega t/2)}{\omega^2} \cos(\omega s) = \frac{\pi}{4} (t-s) \Theta(t-s), \quad (42)$$

and taking the symmetries of the ω and s integrals into account. We obtain

$$\begin{aligned} F^{(1)}(\epsilon, t_m, E) &= -\frac{1}{2} \int_0^{t_m} ds \text{Im} \left[R(s)^3 \times \right. \\ &\quad \left. \left(n(\epsilon) e^{i(\epsilon+E)s} + \bar{n}(\epsilon) e^{-i(\epsilon+E)s} \right) \right], \end{aligned} \quad (43)$$

$$\begin{aligned} F^{(2)}(\epsilon, t_m, E) &= \frac{1}{2} \int_0^{t_m} ds \text{Re} \left[R(s)^3 \times \right. \\ &\quad \left. \left(n(\epsilon) e^{i(\epsilon+E)s} - \bar{n}(\epsilon) e^{-i(\epsilon+E)s} \right) \right]. \end{aligned} \quad (44)$$

These expressions are suitable for numerical evaluation; they can be further simplified for the zero-temperature case.