Sobolev functions on varifolds

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Abstract

This paper introduces first order Sobolev spaces on certain rectifiable varifolds. These complete locally convex spaces are contained in the generally nonlinear class of generalised weakly differentiable functions and share key functional analytic properties with their Euclidean counterparts.

Assuming the varifold to satisfy a uniform lower density bound and a dimensionally critical summability condition on its mean curvature, the following statements hold. Firstly, continuous and compact embeddings of Sobolev spaces into Lebesgue spaces and spaces of continuous functions are available. Secondly, the geodesic distance associated to the varifold is a continuous, not necessarily Hölder continuous Sobolev function with bounded derivative. Thirdly, if the varifold additionally has bounded mean curvature and finite measure, the present Sobolev spaces are isomorphic to those previously available for finite Radon measures yielding many new results for those classes as well.

Suitable versions of the embedding results obtained for Sobolev functions hold in the larger class of generalised weakly differentiable functions.

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Introduction

Overview

The main purpose of this paper is to present a concept of first order Sobolev functions on nonsmooth "surfaces" in Euclidean space with arbitrary dimension and codimension arising in variational problems involving the area functional. The model for such surfaces are certain rectifiable varifolds, see the general hypothesis below. This class is sufficiently broad to include area minimising rectifiable currents, perimeter minimising "Caccioppoli sets", or typical time slices of "singular" mean curvature flow as well as many surfaces occurring in mathematical models for natural sciences, see [Men15, p. 2].

The envisioned concept should satisfy the following two requirements.

(1) Sobolev functions on varifolds should give rise to Banach spaces.

(2) Sobolev functions on varifolds should share as many embedding estimates and structural results as possible with their Euclidean counterparts.

This is accomplished by the present paper which thus provides the basis for the study of divergence form, second order elliptic partial differential equations on varifolds in their natural setting. The new concept is based on generalised weakly differentiable functions on varifolds introduced by the author in [Men15] along with an array of properties in the spirit of (2). However, *generalised weakly differentiable functions do not form a linear space*, hence they violate the requirement (1) which is necessary for the use of almost any standard tool from functional analysis. The Sobolev spaces on varifolds introduced here provide a way to overcome this difficulty. They satisfy (1) and, as subsets of the nonlinear space of generalised weakly differentiable functions, satisfy (2) as well.

Sobolev functions also provide a new toolbox for the study of the delicate local connectedness properties of varifolds satisfying suitable conditions on their first variation. Understanding these properties is a key challenge in any regularity consideration. Local connectedness is analytically measured by the degree to which control of the gradient of a function entails control on its oscillation. Basic estimates in this respect were provided by the Sobolev Poincaré inequalities obtained in [Men15, § 10] and the oscillation estimate of [Men15, § 13].

The present paper contains three main contributions to the study of local connectedness of varifolds satisfying a uniform lower density bound and dimensionally critical summability condition on their mean curvature, see the hypotheses below. Firstly, a Rellich type embedding theorem for Sobolev functions is proven which is related to the local connectedness structure of the varifold through subtle oscillation estimates in its proof. In fact, a Rellich type embedding theorem is established for generalised weakly differentiable functions in a significantly more general setting. Secondly, it is proven that the geodesic distance – even if the space is incomplete – is an example of a continuous Sobolev function with bounded derivative. In [Men15, § 14] it had only been proven that the geodesic distance is a real valued function. Thirdly, an example is constructed showing that the geodesic distance may fail to be locally Hölder continuous with respect to any exponent. In particular, the embedding of Euclidean Sobolev functions with suitably summable derivative into Hölder continuous functions does not extend to the varifold case.

A distinctive feature of the presently developed theory of first order Sobolev spaces is the key role played by the first variation of the varifold. The latter carries information of the extrinsic geometry of the varifold, considered as generalised submanifold, namely its generalised mean curvature and its "boundary". This is in line with potential use of the present theory in the study of regularity properties of varifolds and it distinguishes the present approach from the completely intrinsic viewpoint of metric measure spaces. The latter perspective is described in the books of Heinonen, see [Hei01], Björn and Björn, see [BB11], and Heinonen, Koskela, Shanmugalingam, and Tyson, see [HKST15].

Hypotheses

The notation is mainly that of Federer [Fed69] and Allard [All72], see Section 1. Throughout the paper footnotes recall some parts of this notation or point to related terminology whenever that appeared to be desirable.

Firstly, a list of hypotheses relevant for the present theory will be drawn up.

General hypothesis. *Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, V is an m dimensional rectifiable varifold in U whose first variation δV is representable by integration, and Y is a finite dimensional normed vectorspace.*[1]

Aspects of the theory involving the isoperimetric inequality are most conveniently developed under the following density hypothesis.

Density hypothesis. *Suppose m, n, U, and V are as in the general hypothesis and satisfy*[2]

$$\boldsymbol{\Theta}^m(\|V\|, x) \geq 1 \quad \textit{for } \|V\| \textit{ almost all } x.$$

Several theorems will also make use of the following additional hypothesis.

Mean curvature hypothesis. *Suppose m, n, U, and V are as in the general hypothesis and satisfies the following condition.*

If $m > 1$ then $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, the function

[1]To recall the notation concerning varifolds from Allard [All72, 3.1, 3.5, 4.2, 4.3], first recall the following items from [Fed69] and [Alm00].

- The inner product of x and y will be denoted $x \bullet y$, see [Fed69, 1.7.1].
- If X is a locally compact Hausdorff space then $\mathscr{K}(X)$ is the vectorspace of all continuous real valued functions on X with compact support, see [Fed69, 2.5.14].
- If U is an open subset of some finite dimensional normed vectorspace and Z is a Banach space then $\mathscr{D}(U, Z)$ denotes the space of all functions $\theta : U \to Z$ of class ∞, i.e. "smooth" functions, with compact support, see [Fed69, 4.1.1].
- Whenever P is an m dimensional plane in $\mathbf{R}^n$, the orthogonal projection of $\mathbf{R}^n$ onto P will be denoted by $P_\natural$, see Almgren [Alm00, T.1 (9)].

Whenever U is an open subset of $\mathbf{R}^n$ an m dimensional varifold V is a Radon measure over $U \times \mathbf{G}(n, m)$, where $\mathbf{G}(n, m)$ denotes the space of m dimensional subspaces of $\mathbf{R}^n$. An m dimensional varifold V in U is called rectifiable if and only if there exist sequences of compact subsets C_i of m dimensional submanifolds M_i of U of class 1 and $0 < \lambda_i < \infty$ such that

$$V(k) = \sum_{i=1}^{\infty} \lambda_i \textstyle\int_{C_i} k(x, \operatorname{Tan}(M_i, x)) \,\mathrm{d}\mathscr{H}^m x \quad \text{for } k \in \mathscr{K}(U \times \mathbf{G}(n, m)),$$

where $\operatorname{Tan}(M_i, x)$ denotes the tangent space of M_i at x. The first variation $\delta V : \mathscr{D}(U, \mathbf{R}^n) \to \mathbf{R}$ of V is defined by

$$(\delta V)(\theta) = \textstyle\int P_\natural \bullet \mathrm{D}\,\theta(x) \,\mathrm{d}V(x, P) \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n),$$

The total variation $\|\delta V\|$ is largest Borel regular measure over U satisfying

$$\|\delta V\|(G) = \sup\{(\delta V)(\theta) : \theta \in \mathscr{D}(U, \mathbf{R}^n), \operatorname{spt}\theta \subset G \text{ and } |\theta| \leq 1\}$$

whenever G is an open subset of U. The first variation δV is representable by integration if and only if $\|\delta V\|$ is a Radon measure. If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ then the generalised mean curvature vector of V, $\mathbf{h}(V, \cdot)$, is $\|V\|$ almost characterised amongst functions in $\mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ by the condition

$$(\delta V)(\theta) = -\textstyle\int \mathbf{h}(V, x) \bullet \theta(x) \,\mathrm{d}\|V\| x \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

[2]If μ measures a metric space X, $a \in X$, and m is a positive integer then

$$\boldsymbol{\Theta}^m(\mu, a) = \lim_{r \to 0+} \frac{\mu\,\mathbf{B}(a, r)}{\boldsymbol{\alpha}(m) r^m}, \quad \text{where } \boldsymbol{\alpha}(m) = \mathscr{L}^m\,\mathbf{B}(0, 1)$$

and $\mathbf{B}(a, r)$ is the closed ball with centre a and radius r, see [Fed69, 2.7.16, 2.10.19].

$\mathbf{h}(V,\cdot)$ *belongs to* $\mathbf{L}_m^{\mathrm{loc}}(\|V\|,\mathbf{R}^n)$,[3] *and* ψ *is Radon measure over* U *such that*

$$\psi(A) = \textstyle\int_A |\mathbf{h}(V,x)|^m \,\mathrm{d}\|V\|\,x \quad \textit{whenever } A \textit{ is a Borel subset of } U.$$

The density hypothesis and the mean curvature hypothesis will be referred to whenever they shall be in force.

Known results

As the present paper extends the author's paper [Men15], it seems expedient to review those results of that paper most relevant for the present development.

Axiomatic approach to Sobolev spaces

Given an open subset of Euclidean space and a finite dimensional normed vectorspace Y, the class of weakly differentiable Y valued functions is clearly closed under addition and composition with members of $\mathscr{D}(Y,\mathbf{R})$ and any Y valued function constant on connected components belongs to that class. Replacing the decomposition into connected components by the decomposition of a varifold in the sense of [Men15, 6.9], one may formulate the following list of desirable properties for a concept of weakly differentiable functions or Sobolev functions on a varifold.

(I) The class is closed under addition.

(II) The class is closed under composition with members of $\mathscr{D}(Y,\mathbf{R})$.

(III) Each appropriately summable function which is constant on the components of *some* decomposition of the varifold belongs to the class.

Owing to the fact that decompositions of varifolds are nonunique, see [Men15, 6.13], one can show that it is impossible to realise all three properties in a single satisfactory concept, see [Men15, 8.28]. Accordingly, three concepts have been developed, two in [Men15] and one in the present paper, each missing precisely one distinct one of the above three properties.

Integration by parts identity

The seemingly most natural way to define a concept of weak differentiability is to employ the fact that the first variation δV is representable by integration to formulate an integration by parts identity.

Definition (see [Men15, 8.27])**.** Suppose m, n, U, V, and Y are as in the general hypothesis.

Then the class $\mathbf{W}(V,Y)$ is defined to consist of all $f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y)$ such that for some $F \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$ there holds

$$(\delta V)((\alpha \circ f)\theta) = \alpha\big(\textstyle\int (P_\natural \bullet \mathrm{D}\,\theta(x)) f(x) + F(x)(\theta(x))\,\mathrm{d}V\,(x,P)\big)$$

whenever $\theta \in \mathscr{D}(U,\mathbf{R}^n)$ and $\alpha \in \mathrm{Hom}(Y,\mathbf{R})$.

[3] The spaces $\mathbf{L}_p(\mu,Y)$ and $\mathbf{L}_p^{\mathrm{loc}}(\mu,Y)$ contain functions rather than equivalence classes of functions.

The function F is $\|V\|$ almost unique and could act as weak derivative of f with respect to V. One readily verifies that this class satisfies (I) and (III). However, it fails to satisfy (II), see [Men15, 8.27]. Moreover, it may happen that $f \in \mathbf{W}(V, \mathbf{R})$ has zero weak derivative but the distributional V boundaries $V\,\partial E(y) : \mathscr{D}(U, \mathbf{R}^n) \to \mathbf{R}$ of the superlevel sets $E(y) = \{x : f(x) > y\}$ satisfy

$$\mathscr{L}^1\big(\mathbf{R} \cap \{y : V\,\partial E(y) \neq 0\}\big) > 0,$$

see [Men15, 8.32]. Consequently, no coarea formula analogous to that for weakly differentiable functions in Euclidean space, see [Fed69, 4.5.9 (13)], may be formulated in this class. These two facts seem to pose a serious obstacle to the development of a satisfactory theory for the class $\mathbf{W}(V, Y)$.

Generalised weakly differentiable functions

To overcome this difficulty, one may modify the integration by parts identity by requiring it to hold also for compositions with a class of nonlinear functions.

Definition (see [Men15, 8.3])**.** Suppose m, n, U, V, and Y are as in the general hypothesis.

Then a Y valued $\|V\| + \|\delta V\|$ measurable[4] function f with domain contained in U is called *generalised V weakly differentiable* if and only if for some $\|V\|$ measurable $\operatorname{Hom}(\mathbf{R}^n, Y)$ valued function F the following two conditions hold:

(1) If K is a compact subset of U and $0 \leq s < \infty$, then

$$\textstyle\int_{K \cap \{x : |f(x)| \leq s\}} \|F\| \,\mathrm{d}\|V\| < \infty.$$

(2) If $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, $\gamma \in \mathscr{E}(Y, \mathbf{R})$ and $\operatorname{spt} \mathrm{D}\,\gamma$ is compact, then[5]

$$\begin{aligned}&(\delta V)((\gamma \circ f)\theta)\\ &\quad = \textstyle\int \gamma(f(x)) P_\natural \bullet \mathrm{D}\,\theta(x) \,\mathrm{d}V\,(x, P) + \int (\mathrm{D}\,\gamma(f(x)) \circ F(x))(\theta(x)) \,\mathrm{d}\|V\|\, x.\end{aligned}$$

The set of all Y valued generalised V weakly differentiable functions will be denoted by $\mathbf{T}(V, Y)$.

The function F is $\|V\|$ almost unique. Accordingly, the *generalised V weak derivative of f*, denoted by $V\,\mathbf{D}\,f$, may be defined to equal a particular such F characterised by an approximate continuity condition, see [Men15, 8.3].

This class has a favourable behaviour under truncation and composition as well as decomposition of the underlying varifold, see [Men15, 8.12, 8.13, 8.15, 8.16, 8.18, 8.24]. In particular, it satisfies properties (II) and (III). In case $m = n$, $U = \mathbf{R}^n$, and $\|V\| = \mathscr{L}^n$, a function f belongs to $\mathbf{T}(V, \mathbf{R})$ if and only if the truncated functions $f_s : \operatorname{dmn} f \to Y$ defined by

$$f_s(x) = f(x) \quad \text{if } |f(x)| \leq s, \qquad f_s(x) = (\operatorname{sign} f(x))s \quad \text{if } |f(x)| > s$$

[4] If μ measures X and f maps a subset of X into a topological space Y, then f is μ measurable if and only if $\mu(X \sim \operatorname{dmn} f) = 0$ and the preimage of every open subset of Y under f is μ measurable, see [Fed69, 2.3.2].

[5] If U is an open subset of some finite dimensional normed vectorspace and Z is Banach space then $\mathscr{E}(U, Z)$ denotes the vectorspace of functions $\theta : U \to Z$ of class ∞.

for $x \in \operatorname{dmn} f$ and $0 < s < \infty$, are weakly differentiable in the classical sense, see [Men15, 8.19], and the subclass

$$\mathbf{T}(V,Y) \cap \mathbf{L}_1^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y) \cap \big\{ f : V\,\mathbf{D}\,f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y) \big\}$$

equals the usual space of weakly differentiable functions, see [Men15, 8.18]. However, considering the varifold associated to three lines in $\mathbf{R}^2$ meeting at a common point at equal angles shows that the indicated subclass need not to be closed with respect to addition, see [Men15, 8.25]. In particular, it does not have property (I). Of course, the class $\mathbf{T}(V,\mathbf{R})$ itself need not to be closed under addition even in case of Lebesgue measure, see Bénilan, Boccardo, Gallouët, Gariepy, Pierre, and Vazquez [BBG+95, p. 245]. This drawback is partially compensated by the fact that the class $\mathbf{T}(V,Y)$ is closed under addition of a locally Lipschitzian function, see [Men15, 8.20 (3)].

Whenever G is a relatively open subset of $\operatorname{Bdry} U$, one may also realise the concept of "zero boundary values on G" for nonnegative functions f in $\mathbf{T}(V,\mathbf{R})$ by means of the class $\mathbf{T}_G(V)$. The resulting class $\mathbf{T}_G(V)$ has good properties under composition and convergence of the functions in measure with appropriate bounds on the derivatives, see [Men15, 9.9, 9.13, 9.14]. Of course, instead of restricting to nonnegative functions, one could also consider the class

$$\mathbf{T}(V,Y) \cap \{ f : |f| \in \mathbf{T}_G(V) \}$$

but stability under compositions would fail in this case, see [Men15, 9.10, 9.11].

The more elaborate properties of $\mathbf{T}(V,Y)$ build on the isoperimetric inequality which works most effectively under the density hypothesis. This hypothesis allows for the formulation of various Sobolev Poincaré type inequalities with and without boundary condition, see [Men15, 10.1, 10.7, 10.9]. Furthermore, pointwise differentiability results both of approximate and integral nature then hold for generalised V weakly differentiable functions, see [Men15, 11.2, 11.4].

Turning to some relevant results concerning the local connectedness structure of varifolds, the mean curvature hypothesis becomes more relevant. Under the density hypothesis and the mean curvature hypothesis, the connected components of $\operatorname{spt} \|V\|$ are relatively open and any two points belonging to the same connected component may be connected by a path of finite length whose image lies in that component, see [Men15, 6.14 (3), 14.2]. At the heart of the proof of the second part of this assertion lies an oscillation estimate for an a priori continuous generalised weakly differentiable function whose derivative satisfies a q-th power summability hypothesis with $q > m$, see [Men15, 13.1]. This estimate differs in two points from the well known oscillation estimate for weakly differentiable functions in Euclidean space. Firstly, the function needs to be continuous a priori; otherwise – in view of property (III) – a counterexample is immediate from considering two crossing lines. Secondly, the estimate does not yield Hölder continuity; in fact, an example showing that Hölder continuity is not implied by those hypotheses will be constructed in the present paper, see Theorems C and D below.

Results of the present paper

The main contributions, apart from introducing the concept of Sobolev functions on varifolds, are Theorems A–E and its Corollaries A and B which will be described below. In order to complete the picture, further theorems are included which essentially follow from combining the present theory with that of [Men15].

Approximation of Lipschitzian functions, see Section 3

The Sobolev spaces on varifold will be defined by a completion procedure starting from locally Lipschitzian functions. As a consequence of the next result, "smooth" functions are dense in these spaces for finite exponents, see Corollary A.

Theorem A, see 3.6 and 3.7. *Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, K is a compact subset of U, Y is a finite dimensional normed vectorspace, $f : U \to Y$ is a Lipschitzian function,* $\operatorname{spt} f \subset \operatorname{Int} K$*, and V is an m dimensional rectifiable varifold in U.*

Then there exists a sequence $f_i \in \mathscr{D}(U, Y)$ satisfying

$$\begin{gathered} f_i(x) \to f(x) \quad \text{uniformly for } x \in \operatorname{spt} \|V\| \text{ as } i \to \infty, \\ \|(\|V\|, m) \operatorname{ap} \mathrm{D}(f_i - f)\| \to 0 \quad \text{in } \|V\| \text{ measure as } i \to \infty, \\ \operatorname{spt} f_i \subset K \quad \text{for each } i, \qquad \limsup_{i \to \infty} \operatorname{Lip} f_i \leq \Gamma \operatorname{Lip} f, \end{gathered}$$

where Γ is a positive, finite number depending only on Y.[6] *Moreover, if $Y = \mathbf{R}^l$, then one may take $\Gamma = 1$.*

As there is no hypothesis on δV, the only available notion of derivative is that of approximate derivative. If V satisfies the general hypothesis, then $(\|V\|, m) \operatorname{ap} \mathrm{D}(f_i - f)$ can be replaced by $V\, \mathbf{D}\,(f - f_i)$, see [Men15, 8.7].

It is instructive to compare Theorem A with the familiar fact that there exists a sequence of continuously differentiable functions $g_i : U \to Y$ such that

$$\|V\|(U \sim \{x : f(x) = g_i(x) \text{ and } \operatorname{ap} \mathrm{D} f(x) = \operatorname{ap} \mathrm{D} g_i(x)\}) \to 0 \quad \text{as } i \to \infty,$$

where "ap" refers to $(\|V\|, m)$ approximate differentiation. The difficulty to construct the asserted functions f_i from the functions g_i is that agreement of the approximate derivatives of f and g_i at x only implies

$$\big\| \mathrm{D}\, g_i(x) | \operatorname{Tan}^m(\|V\|, x) \big\| \leq \operatorname{Lip} f$$

but $\| \mathrm{D}\, g_i(x) \|$ may be much larger than $\operatorname{Lip} f$. This is resolved by employing a special retraction onto continuously differentiable submanifolds of $\mathbf{R}^n$, see 3.2.

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[6] Suppose μ measures an open subset U of a normed vectorspace X, $a \in U$, and m is a positive integer. Then $\operatorname{Tan}^m(\mu, a)$ denotes the closed cone of (μ, m) *approximate tangent vectors* at a consisting of all $u \in X$ such that

$$\begin{gathered} \boldsymbol{\Theta}^{*m}(\mu \llcorner \mathbf{E}(a, u, \varepsilon), a) > 0 \quad \text{for every } \varepsilon > 0, \\ \text{where } \mathbf{E}(a, u, \varepsilon) = X \cap \{x : |r(x - a) - u| < \varepsilon \text{ for some } r > 0\}. \end{gathered}$$

Moreover, if f maps a subset of X into another normed vectorspace Y, then f is called (μ, m) *approximately differentiable* at a if and only if there exist $b \in Y$ and a continuous linear map $L : X \to Y$ such that

$$\boldsymbol{\Theta}^m(\mu \llcorner X \sim \{x : |f(x) - b - L(x - a)| \leq \varepsilon |x - a|\}, a) = 0 \quad \text{for every } \varepsilon > 0.$$

In this case $L | \operatorname{Tan}^m(\mu, a)$ is unique and it is called the (μ, m) *approximate differential* of f at a, denoted $(\mu, m) \operatorname{ap} \mathrm{D} f(a)$, see [Fed69, 3.2.16].

A Rellich type embedding theorem, see Section 4

The Rellich type embedding theorem for Sobolev functions on varifolds, see Corollary B, will be derived as a consequence of the following significantly more general theorem for generalised weakly differentiable functions on varifolds.

Theorem B, see 4.8. *Suppose m, n, U, V, and Y are as in the general hypothesis and the density hypothesis, and $f_i \in \mathbf{T}(V, Y)$ is a sequence satisfying*

$$\lim_{t \to \infty} \sup \big\{ \|V\|(K \cap \{x : |f_i(x)| > t\}) : i = 1, 2, 3, \ldots \big\} = 0,$$
$$\sup \big\{ \textstyle\int_{K \cap \{x : |f_i(x)| \leq t\}} \|V \, \mathbf{D} \, f_i\| \, \mathrm{d}\|V\| : i = 1, 2, 3, \ldots \big\} < \infty \quad \textit{for } 0 \leq t < \infty$$

whenever K is a compact subset of U.

Then there exist a $\|V\|$ measurable Y valued function f and a subsequence of f_i which, whenever K is a compact subset of U, converges to f in $\|V\| \llcorner K$ measure.

As a consequence of Theorem B one obtains sequential closedness results under weak convergence of both functions and derivative for $\mathbf{T}(V, Y)$ and $\mathbf{T}_G(V)$, see 4.9 and 4.10. Such results are of natural importance in considering variational problems in these nonlinear spaces.

Definitions of Sobolev spaces, see Section 5

Next, the definitions of the Sobolev spaces on varifolds and some basic properties that are direct consequences of the theory of [Men15] shall be presented. Firstly, the largest classes, that is the local Sobolev spaces, will be defined.

Definition (local Sobolev space, see 5.1 and 5.2)**.** Suppose m, n, U, V, and Y are as in the general hypothesis and $1 \leq q \leq \infty$.

Then the *local Sobolev space with respect to V and exponent* q, denoted by $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$, is defined to be the class of all $f \in \mathbf{T}(V, Y)$ such that

$$(f, V \, \mathbf{D} \, f) \in \operatorname{Clos} \big\{ (g, V \, \mathbf{D} \, g) : g \in Y^U \text{ and } g \text{ is locally Lipschitzian} \big\},$$

where the closure is taken in $\mathbf{L}_q^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y) \times \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$.[7]

The letter "**H**" is chosen to emphasise the fact that the space is defined by a closure operation, see [AF03, 3.2], and the placement of q as subscript is in line with the symbol $\mathbf{L}_q(\mu, Y)$ employed for Lebesgue spaces, see [Fed69, 2.4.12].

It is important that $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is a subset of the possibly nonlinear space $\mathbf{T}(V, Y)$ but itself is a vectorspace; in fact, it is a complete locally convex space when endowed with its natural topology resulting from its inclusion into

$$\mathbf{L}_q^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y) \times \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y)),$$

see 5.7 and 5.8. Moreover, it has good stability properties under composition, see 5.6 (2) (3). Therefore $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ has properties (I) and (II) but it does not have property (III) as may be seen from considering two crossing lines, see 7.14.

At first sight, one might consider to replace $\mathbf{L}_q^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y)$ in the above definition by $\mathbf{L}_q^{\mathrm{loc}}(\|V\|, Y)$. Indeed, assuming the density hypothesis, the mean

[7]For any Y and U, the expression Y^U denotes the class of all functions $g : U \to Y$.

curvature hypothesis, and $m > 1$, the same definition would have resulted, see 7.7 and 7.8. But in the general case it appears natural (and indispensable) to require control with respect to the measure $\|\delta V\|$ since both the integration by parts identities and the isoperimetric inequalities involve control of the function with respect to $\|\delta V\|$.

Definition (Sobolev space, see 5.11 and 5.14)**.** Suppose m, n, U, V, and Y are as in the general hypothesis and $1 \leq q \leq \infty$.

Then define the *Sobolev space with respect to V and exponent q* by[8]

$$\begin{aligned}&\mathbf{H}_q(V,Y) = \mathbf{H}_q^{\mathrm{loc}}(V,Y) \cap \{f : \mathbf{H}_q(V,f) < \infty\},\\ &\qquad \text{where } \mathbf{H}_q(V,f) = (\|V\| + \|\delta V\|)_{(q)}(f) + \|V\|_{(q)}(V\,\mathbf{D}\,f) \text{ for } f \in \mathbf{T}(V,Y).\end{aligned}$$

The usage of the letter "**H**" in the name of the seminorm $\mathbf{H}_q(V,\cdot)|\mathbf{H}_q^{\mathrm{loc}}(V,Y)$ is modelled on the usage of the letter "**F**" in the name of the seminorm $\mathbf{F}_K$ related to the space of flat chains $\mathbf{F}_{m,K}(U)$, see [Fed69, 4.1.12].

The space $\mathbf{H}_q(V,Y)$ is $\mathbf{H}_q(V,\cdot)|\mathbf{H}_q(V,Y)$ complete, see 5.15.

Definition (Sobolev space with "zero boundary values", see 5.18)**.** Suppose m, n, U, V, and Y are as in the general hypothesis and $1 \leq q \leq \infty$.

Then define $\mathbf{H}_q^\diamond(V,Y)$ to be the $\mathbf{H}_q(V,\cdot)|\mathbf{H}_q(V,Y)$ closure of

$$Y^U \cap \{g : \operatorname{Lip} g < \infty,\ \operatorname{spt} g \text{ is compact}\}$$

in $\mathbf{H}_q(V,Y)$.

The space $\mathbf{H}_q^\diamond(V,Y)$ is $\mathbf{H}_q(V,\cdot)|\mathbf{H}_q^\diamond(V,Y)$ complete, see 5.19. If $U = \mathbf{R}^n$ and $q < \infty$, then $\mathbf{H}_q^\diamond(V,Y) = \mathbf{H}_q(V,Y)$, see 5.23.

The inclusions amongst the various local spaces are given by

$$\begin{aligned}&Y^U \cap \{g : g \text{ locally Lipschitzian}\} \subset \mathbf{H}_q^{\mathrm{loc}}(V,Y)\\ &\qquad \subset \mathbf{T}(V,Y) \cap \mathbf{L}_1^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y) \cap \big\{f : V\,\mathbf{D}\,f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)\big\}\\ &\qquad \subset \mathbf{W}(V,Y),\end{aligned}$$

see 5.2 and [Men15, 8.27]. Finally, one may also consider the quotient of $\mathbf{H}_q^{\mathrm{loc}}(V,Y)$ or $\mathbf{H}_q^{\mathrm{loc}}(V,Y)$ or $\mathbf{H}_q^\diamond(V,Y)$ by

$$\begin{aligned}&\mathbf{H}_q^{\mathrm{loc}}(V,Y) \cap \{f : f(x) = 0 \text{ for } \|V\| \text{ almost all } x\}\\ &\qquad = \mathbf{H}_q^{\mathrm{loc}}(V,Y) \cap \big\{f : f(x) = 0 \text{ for } \|V\| + \|\delta V\| \text{ almost all } x,\\ &\qquad\qquad\qquad V\,\mathbf{D}\,f(x) = 0 \text{ for } \|V\| \text{ almost all } x\big\},\end{aligned}$$

see 5.5, 5.13, 5.17, and 5.26, which allows to conveniently apply certain functional analytic results.

[8] If μ measures X, $1 \leq q \leq \infty$, and f is a μ measurable function with values in some Banach space Y, then one defines (see [Fed69, 2.4.12])

$$\begin{aligned}\mu_{(q)}(f) &= \left(\textstyle\int |f|^q \,\mathrm{d}\mu\right)^{1/q} \quad \text{in case } 1 \leq q < \infty,\\ \mu_{(\infty)}(f) &= \inf \big\{s : s \geq 0,\ \mu(\{x : |f(x)| > s\}) = 0\big\}.\end{aligned}$$

Basic theorems for Sobolev spaces, see Section 5

Having Theorem A at one's disposal, the following density result may be deduced analogously to the Euclidean case.

Corollary A, see 5.9 (2), 5.16, and 5.22. *Suppose m, n, U, V, and Y are as in the general hypothesis, and $1 \leq q < \infty$.*

Then the following three statements hold.

(1) *The set $\mathscr{D}(U, Y)$ is dense in $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$.*

(2) *The set $\mathbf{H}_q(V, Y) \cap \mathscr{E}(U, Y)$ is $\mathbf{H}_q(V, \cdot)|\mathbf{H}_q(V, Y)$ dense in $\mathbf{H}_q(V, Y)$.*

(3) *The set $\mathscr{D}(U, Y)$ is $\mathbf{H}_q(V, \cdot)|\mathbf{H}_q^\diamond(V, Y)$ dense in $\mathbf{H}_q^\diamond(V, Y)$.*

Evidently, the Sobolev space $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is contained in $\mathbf{T}(V, Y)$. The analogous statement involving "zero boundary values" is less obvious.

Theorem, see 5.27. *Suppose m, n, U, V, and Y are as in the general hypothesis, $1 \leq q \leq \infty$, and $f \in \mathbf{H}_q^\diamond(V, Y)$.*

Then $|f| \in \mathbf{T}_{\mathrm{Bdry}\, U}(V)$.

Consequently, the Sobolev inequalities of [Men15, 10.1 (2)] apply in the case of Sobolev functions as well, see 5.28 and 7.18.

Geodesic distance, see Section 6

Apart of the envisioned use of Sobolev functions for certain elliptic partial differential equations on varifolds, Sobolev functions also occur naturally in the study of the geodesic distance on the support of the weight measure of a varifold.

Theorem C, see 6.8. *Suppose m, n, U, and V are as in the density hypothesis and the mean curvature hypothesis, $X = \operatorname{spt} \|V\|$, X is connected, ϱ is the geodesic distance on X, see 6.6, and $W \in \mathbf{V}_{2m}(U \times U)$ satisfies*

$$W(k) = \textstyle\int k((x_1, x_2), P_1 \times P_2) \,\mathrm{d}(V \times V)\,((x_1, P_1), (x_2, P_2))$$

whenever $k \in \mathscr{K}(U \times U, \mathbf{G}(\mathbf{R}^n \times \mathbf{R}^n, 2m))$.

Then the following two statements hold.

(1) *The function ϱ is continuous, a metric on X, and belongs to $\mathbf{H}_q^{\mathrm{loc}}(W, \mathbf{R})$ for $1 \leq q < \infty$ with*

$$|\langle (u_1, u_2), W\, \mathbf{D}\, \varrho(x_1, x_2)\rangle| \leq |u_1| + |u_2| \quad \textit{whenever } u_1, u_2 \in \mathbf{R}^n$$

for $\|W\|$ almost all (x_1, x_2).

(2) *If $a \in X$, then $\varrho(a, \cdot) \in \mathbf{H}_q^{\mathrm{loc}}(V, \mathbf{R})$ for $1 \leq q < \infty$ and*

$$|V\, \mathbf{D}\, (\varrho(a, \cdot))(x)| = 1 \quad \textit{for } \|V\| \textit{ almost all } x.$$

The previous result from [Men15, 14.2] only showed that the function ϱ is real valued. The sharpness of the preceding theorem is illustrated by an example whose properties are summarised in the next theorem.

Theorem D, see 6.11. *There exist m, n, U, and V satisfying the density hypothesis and the mean curvature hypothesis and $a \in \operatorname{spt} \|V\|$ such that the geodesic distance ϱ on $\operatorname{spt} \|V\|$, see 6.6, has the following two properties.*

(1) *The function $\varrho(a, \cdot)$ does not belong to $\mathbf{H}_\infty^{\mathrm{loc}}(V, \mathbf{R})$.*

(2) *The function $\varrho(a, \cdot)$ is not Hölder continuous with respect to any exponent.*

Further theorems on Sobolev spaces, see Section 7

All further results focus on the case when the density hypothesis and the mean curvature hypothesis are satisfied.

Corollary B, see 7.21 (2) and 7.22. *Suppose m, n, U, V, and ψ are as in the density hypothesis and the mean curvature hypothesis, $\|V\|(U) < \infty$, $\Lambda = \Gamma_{[\text{Men15, 10.1}]}(n)$, $\psi(U) \leq \Lambda^{-1}$, $1 \leq q < m$, $1 \leq \alpha < mq/(m-q)$, and Y is a finite dimensional normed vectorspace.*

Then any sequence $f_i \in \mathbf{H}_q^\diamond(V, Y)$ with

$$\sup\left\{\|V\|_{(q)}(V\,\mathbf{D}\,f_i) : i = 1, 2, 3, \ldots\right\} < \infty$$

admits a subsequence converging in $\mathbf{L}_\alpha(\|V\|, Y)$.

The smallness condition on $\psi(U)$ ensures that $\mathbf{H}_q^\diamond(V, Y)$ does not contain nontrivial functions with vanishing derivative as would be the case if V corresponded to a sphere for example. Moreover, a similar result holds for $m = 1$, see 7.21 (1) and 7.22. Both results have an analogous formulation in the space $\mathbf{H}_q^{\text{loc}}(V, Y)$, see 7.11 and 7.15.

Next, an embedding theorem into the space $\mathscr{C}(\operatorname{spt}\|V\|, Y)$ of continuous functions from $\operatorname{spt}\|V\|$ into Y endowed with the topology of locally uniform convergence, see 2.13, is stated.

Theorem, see 7.12 and 7.15. *Suppose m, n, U, V, and ψ are as in the density hypothesis and the mean curvature hypothesis, $1 < m < q$, and Y is a finite dimensional normed vectorspace.*

Then there exists a continuous linear map $L : \mathbf{H}_q^{\text{loc}}(V, Y) \to \mathscr{C}(\operatorname{spt}\|V\|, Y)$ uniquely characterised by

$$L(f)(x) = f(x) \quad \text{for } \|V\| \text{ almost all } x.$$

Moreover, if $f_i \in \mathbf{H}_q^{\text{loc}}(V, Y)$ form a sequence satisfying

$$\sup\left\{(\|V\| \llcorner K)_{(q)}(f_i) + (\|V\| \llcorner K)_{(q)}(V\,\mathbf{D}\,f_i) : i = 1, 2, 3, \ldots\right\} < \infty$$

whenever K is a compact subset of U, then the sequence $L(f_i)$ admits a subsequence converging in $\mathscr{C}(\operatorname{spt}\|V\|, Y)$.

In view of Theorems C and D concerning the geodesic distance to a point, it follows that, in contrast to the Euclidean case, $L(f)$ need not to be Hölder continuous with respect to any exponent, see 7.13.

Finally, a subspace of $\mathbf{H}_q^{\text{loc}}(V, Y)$ is considered which is defined by a seminorm not involving $\|\delta V\|$ and hence more closely resembles the Euclidean case.

Theorem E, see 7.16 (1) (3) (5). *Suppose m, n, U, V, and ψ are as in the density hypothesis and the mean curvature hypothesis, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace,*

$$\sigma(f) = \|V\|_{(q)}(f) + \|V\|_{(q)}(V\,\mathbf{D}\,f) \quad \text{for } f \in \mathbf{H}_q^{\text{loc}}(V, Y),$$

and $E = \mathbf{H}_q^{\text{loc}}(V, Y) \cap \{f : \sigma(f) < \infty\}$.

Then the following three statements hold.

(1) The vectorspace E is σ complete.

(2) *If $q < \infty$, then $\mathscr{E}(U,Y) \cap \{f : \sigma(f) < \infty\}$ is σ dense in E, and if additionally $U = \mathbf{R}^n$, then $\mathscr{D}(\mathbf{R}^n, Y)$ is σ dense in E.*

(3) *If $U = \mathbf{R}^n$ and $\psi(\mathbf{R}^n) < \infty$, then $E = \mathbf{H}_q(V, Y)$.*

(1) is simple unless $m = 1$ in which case the non absolutely continuous part of $\|\delta V\|$ with respect to $\|V\|$ requires additional care, see 7.1. (2) is a corollary to Theorem A. Finally, (3) relies on an estimate of $\|\delta V\|_{(q)}(f)$ for generalised weakly differentiable functions f, see 7.6.

In view of Theorem E, depending on the intended usage, both

$$\mathbf{H}_q(V, Y) \quad \text{and} \quad \mathbf{H}_q^{\mathrm{loc}}(V, Y) \cap \{f : \sigma(f) < \infty\}$$

could act as substitute for the Euclidean Sobolev space.

Comparison to other Sobolev spaces, Section 8

Finally, the presently introduced notion of Sobolev space shall be compared to notions of Sobolev space for finite Radon measures μ over $\mathbf{R}^n$ defined in Bouchitté, Buttazzo and Fragalà, see [BBF01]; see also Bouchitté, Buttazzo and Seppecher in [BBS97]. To describe this approach suppose $1 \leq q \leq \infty$, $1 \leq r \leq \infty$, and $1/q + 1/r = 1$. Firstly, one defines vectorspaces $T_\mu^q(x)$ for $x \in \mathbf{R}^n$, acting as tangent space, by means of r-th power μ summable vectorfields such that a suitable distributional divergence involving μ is also r-th power summable, see 8.2.[9] Accordingly, the gradient $\nabla_\mu^q f$ of $f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ is given by[10]

$$\nabla_\mu^q f(x) = T_\mu^q(x)_\natural(\operatorname{grad} f(x)) \quad \text{for } \mu \text{ almost all } x.$$

Then one obtains both the strong Sobolev space $H_\mu^{1,q}(\mathbf{R}^n)$ and the associated weak derivative by taking a suitable closure of the afore-mentioned gradient operator ∇_μ^q.

Examples of Di Marino and Speight [DMS15, Theorem 1] show that the vectorspace $T_\mu^q(x)$ and hence also the weak derivative depend on q, see 8.2. This is in some sense analogous to the dependency of the (μ, m) approximate derivative on the dimension m. It appears to be largely unknown what is the weakest condition on the first variation, for instance amongst those considered in 6.1, to ensure that the two concepts agree for an m dimensional rectifiable varifold. By results of Fragalà and Mantegazza in [FM99], one is at least assured that they agree provided m, n, U, and V are as in the mean curvature hypothesis, $\|V\|(\mathbf{R}^n) < \infty$, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, and $\mathbf{h}(V, \cdot) \in \mathbf{L}_\infty(\|V\|, \mathbf{R}^n)$, see 8.3. In case the tangent spaces agree, taking σ as in the preceding theorem, one may isometrically identify the strong Sobolev space $H_{\|V\|}^{1,q}(\mathbf{R}^n)$ with the quotient space

$$\big(\mathbf{H}_q^{\mathrm{loc}}(V, Y) \cap \{f : \sigma(f) < \infty\}\big) \Big/ \big(\mathbf{H}_q^{\mathrm{loc}}(V, Y) \cap \{f : \sigma(f) = 0\}\big)$$

provided m, n, U, and V are as in the density hypothesis and the mean curvature hypothesis, $U = \mathbf{R}^n$, $1 \leq q < \infty$, and $\|V\|(\mathbf{R}^n) < \infty$, see 8.4. Under the

[9] In [BBF01, p. 403] Bouchitté, Buttazzo and Fragalà define T_μ^q to be an equivalence class of functions agreeing μ almost everywhere; see 8.2 for a canonical representative.

[10] In [BBF01, p. 403] Bouchitté, Buttazzo and Fragalà suppress the dependency on q, and ∇_μ is considered as a linear map of a subset of $L_q(\mu, \mathbf{R})$ into $(L_q(\mu, \mathbf{R}))^n$, see 2.6.

same hypotheses, one may similarly identify the weak Sobolev space $W^{1,q}_{\|V\|}(\mathbf{R}^n)$ introduced by Bouchitté, Buttazzo, and Fragalà in [BBF01, p. 403] with a quotient space based on $\mathbf{W}(V, Y)$, see 8.5.

Summarising, the approach initiated by Bouchitté, Buttazzo and Seppecher in [BBS97] allows to treat geometric objects consisting of pieces of different dimensions. However, it seems not to be tailored for the study of varifolds as is exemplified by the behaviour of the different notions of tangent planes. Moreover, apart from a general coarea formula, see Bellettini, Bouchitté and Fragalà [BBF99, §4], very few structural results and no embedding estimates appear to have been known for those spaces even if the Radon measure is the weight of a suitable varifold. Now, in the cases where the above-mentioned isometry to the spaces developed here is valid, much of the theory of the present paper and its predecessor, [Men15], applies to Sobolev spaces in the sense of Bouchitté, Buttazzo and Seppecher as well.

Possible lines of further study

Second order elliptic partial differential equations in divergence form

The primary motivation for this paper was to provide a natural framework for the study of divergence form, second order elliptic partial differential equations. More concretely, the author's motivation stems from two results announced already in [Men12]. Firstly, a local maximum estimate for subsolutions generalising those of Allard [All72, 7.6 (5)] or Michael and Simon [MS73, 3.4] could be used to deduce a strong second order differentiability of the support of *integral*[11] varifolds satisfying the mean curvature hypothesis from the author's second order rectifiability result [Men13, 4.8], see [Men12, Corollary 2 (1)]. Secondly, a suitable version of a weak Harnack estimate could be employed to prove an area formula for a suitable defined substitute of the Gauss map of at least two dimensional such varifolds in codimension one, see [Men12, Theorem 3]. For these two specific applications, it would suffice to formulate the estimates for Lipschitzian subsolutions respectively Lipschitzian solutions. However, as these estimates are of independent significance they shall be formulated in their – yet to be determined – natural generality using the presently introduced Sobolev spaces. Finally, the author hopes that dispensing with ad hoc formulations in favour of using Sobolev spaces will also facilitate the exchange of ideas between varifold theory and other areas of geometric analysis.

Minimisation of integral functionals

For integral functionals based on the presently introduced Sobolev spaces certain minimisation problems possess a solution. A simple example is given by the Rayleigh quotient where one may check the conditions of the abstract framework of Arnlind, Björn and Björn, see [ABB15, 5.3], in the situation of 7.21 if $1 < q < \infty$ using 5.26 and 7.22. A more comprehensive study of the problem would include investigation of lower semicontinuity of integral functionals defined on the presently introduced Sobolev spaces in the spirit of quasiconvexity. In

[11] An m dimensional rectifiable varifold V in an open subset of $\mathbf{R}^n$ is integral if and only if $\mathbf{\Theta}^m(\|V\|, x)$ is an integer for $\|V\|$ almost all x, see Allard [All72, 3.5 (1c)].

the context of Sobolev spaces over compactly supported Radon measures in $\mathbf{R}^n$ this topic was considered by Fragalà, see [Fra03].

In view of the closedness results 4.9 and 4.10, similar questions might be considered in the framework of $\mathbf{T}(V, Y)$ and $\mathbf{T}_G(V)$ spaces.

Logical prerequisites

The present paper is a continuation of the author's paper [Men15]. Concerning newer results on varifolds, additionally only some auxiliary results from [Men09, § 1] and Kolasiński and the author [KM15, § 3] are employed. Concerning Sobolev spaces over finite Radon measures, certain items from Fragalà and Mantegazza [FM99] and Bouchitté, Buttazzo, and Fragalà [BBF01] are used.

Additionally, a number of classical results are employed. For those items, as a service to the reader, detailed references to Whitney [Whi57], Dunford and Schwartz [DS58], Federer [Fed69], Allard [All72], Kelley [Kel75], Castaing and Valadier [CV77], Bourbaki [Bou87, Bou89a, Bou89b], do Carmo [dC92], and Adams and Fournier [AF03] are given.

Throughout the paper comments in small font are included. These are not part of the logical line of arguments but are rather offered to the reader as a guide through the formal presentation of the material.

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1 Notation

The notation of [Men15, § 1] will be employed which follows with some additions and modifications Federer [Fed69] and Allard [All72].

Modifications If X is a metric space and M is the class of Borel regular measures ψ over X such that $\psi(U_i) < \infty$ for $i \in \mathscr{P}$ for some sequence of open sets $U_1, U_2, U_3, \ldots$ covering X, measures $\psi_\phi \in M$ will be defined by

$$\psi_\phi(A) = \inf\{\psi(B) : B \text{ is a Borel set and } \phi(A \sim B) = 0\} \quad \text{for } A \subset X$$

whenever $\phi, \psi \in M$. This extends [Fed69, 2.9.1, 2.9.2, 2.9.7] to certain measures failing to be finite on bounded sets.

Definitions in the text The notion of *pseudometric* is introduced in 2.2. The local Lebesgue space $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ and the space of continuous functions $\mathscr{C}(X, Y)$ are defined in 2.7 and 2.13 respectively. The local Sobolev space $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ and its topology is defined in 5.1 and 5.7. The quantity $\mathbf{H}_q(V, f)$ for certain functions f is defined in 5.11. Finally, the Sobolev space $\mathbf{H}_q(V, Y)$ and its subspace $\mathbf{H}_q^\diamond(V, Y)$ are defined in 5.14 and 5.18.

2 Locally convex spaces

The purpose of this section is to summarise properties of locally convex spaces, including definitions of some particular spaces, for convenient reference.

Firstly, some properties of Lebesgue spaces are given.

2.1. Suppose $1 \leq p < \infty$, μ is a Radon measure over an open subset of U of $\mathbf{R}^n$, and Y is a separable Banach space. Then $\mathscr{D}(U, Y)$ is $\mu_{(p)}$ dense in $\mathbf{L}_p(\mu, Y)$ and $\mathbf{L}_p(\mu, Y)$ is $\mu_{(p)}$ separable; in fact, whenever G is an open subset of U and K is a compact subset of G, there exists $\zeta \in \mathscr{D}(U, \mathbf{R})$ such that $0 \leq \zeta(x) \leq 1$ for $x \in U$, $\zeta(x) = 1$ for $x \in K$, and $\operatorname{spt} \zeta \subset G$, hence [Fed69, 2.2.5, 2.4.12] implies the denseness and the separability follows from [Men15, 2.2, 2.15, 2.24].

Next, the meaning of the term pseudometric is specified.

2.2 Definition (see [Bou89a, II, §1.2, def. 3], [Bou89b, IX, §1.1, def. 1; IX, §1.2, def. 2])**.** Suppose X is a set.

Then $\varrho : X \times X \to \{t : 0 \leq t \leq \infty\}$ will be called a *pseudometric on* X if and only if the following three conditions are satisfied.

(1) If $x \in X$, then $\varrho(x, x) = 0$.

(2) If $a, x \in X$, then $\varrho(a, x) = \varrho(x, a)$.

(3) If $a, x, \chi \in X$, then $\varrho(a, \chi) \leq \varrho(a, x) + \varrho(x, \chi)$.

The sets $X \cap \{x : \varrho(a, x) < r\}$ corresponding to $a \in X$ and $0 < r < \infty$ form a base of a topology on X, called the *topology induced by* ϱ.

2.3 Remark. Notice that ∞ may occur amongst the values of ϱ.[12]

The next two items provide basic properties of locally convex spaces whose topology is defined by a set of real valued seminorms. All of these properties are readily verified and most of them can also be found in Bourbaki [Bou87].

2.4. Suppose E is a vectorspace endowed with the topology induced by a nonempty family Σ of real valued seminorms on E, see [Bou87, II, p. 3].[13] Then E is a locally convex space and the family of sets

$$E \cap \{x : \sigma(x - a) < r\}$$

corresponding to $a \in E$, $0 < r < \infty$, and $\sigma \in \Sigma$ form a subbase of the topology of E, see [Bou87, II, p. 24, cor.]. A subset B of E is bounded if and only if

$$\sup \sigma[B] < \infty \quad \text{whenever } \sigma \in \Sigma,$$

see [Bou87, III, p. 2]. In case for $\sigma_1, \sigma_2 \in \Sigma$ there exists $\sigma_3 \in \Sigma$ with $\sup\{\sigma_1, \sigma_2\} \leq \sigma_3$ the above-mentioned family forms a base of the topology of E. If $\sigma_1, \sigma_2, \sigma_3, \ldots$ form an enumeration of Σ, then the topology of E is induced by the translation invariant real valued pseudometric with value

$$\sum_{i=1}^{\infty} 2^{-i} \inf\{1, \sigma_i(x - a)\} \quad \text{at } (a, x) \in E \times E$$

which is a metric if and only if E is Hausdorff.

[12] This is in contrast with the definition of "pseudo-metric" in [Kel75, p. 119].

[13] A real valued seminorm is precisely a "semi-norm" in the sense of [Bou87, II, p. 1, def. 1].

2.5. Suppose E and Σ are as in 2.4 and V is the closure of $\{0\}$ in E. Then $V = \bigcap\{\sigma^{-1}[\{0\}] : \sigma \in \Sigma\}$ is a vector subspace and E is Hausdorff if and only if $V = \{0\}$. Moreover, denoting the canonical projection of E onto E/V by π and endowing E/V with the topology induced by the family $\{\sigma \circ \pi^{-1} : \sigma \in \Sigma\}$ of real valued seminorms on E/V, one obtains the quotient locally convex space, see [Bou87, II, p. 5; II, p. 29, Example I].[14] Clearly, E/V is Hausdorff and, if E is complete, so is E/V.

Often, it is more convenient to consider functions – as contained in $\mathbf{L}_p(\mu, Y)$ – instead of equivalence classes of functions. However, to access certain functional analytic results, the quotient space $L_p(\mu, Y)$ will be introduced as well.

2.6 Example. Occasionally, the Banach spaces

$$L_q(\mu, Y) = \mathbf{L}_q(\mu, Y) \Big/ \big(\mathbf{L}_q(\mu, Y) \cap \{f : \mu_{(q)}(f) = 0\}\big)$$

corresponding to $1 \leq q \leq \infty$, measures μ, and Banach spaces Y will be employed. If $1 < q < \infty$ and $\dim Y < \infty$, then $L_q(\mu, Y)$ is reflexive, see [Fed69, 2.5.7 (i)]; in fact, a basis of Y induces an isomorphism $L_q(\mu, Y) \simeq L_q(\mu, \mathbf{R})^{\dim Y}$.

Next, the local Lebesgue spaces for Radon measures over locally compact Hausdorff spaces X are introduced; in fact, X will always be an open subset of some Euclidean space in the later sections.

2.7 Definition. Suppose $1 \leq p \leq \infty$, μ is a Radon measure over a locally compact Hausdorff space X, and Y is a Banach space.

Then $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ is defined to be the vectorspace consisting of all functions f mapping a subset of X into Y such that $f \in \mathbf{L}_p(\mu \llcorner K, Y)$ whenever K is a compact subset of X. Moreover, $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ is endowed with the topology induced by the family of seminorms mapping $f \in \mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ onto $(\mu \llcorner K)_{(p)}(f)$ corresponding to all compact subsets K of X, see 2.4. Let $\mathbf{L}_p^{\mathrm{loc}}(\mu) = \mathbf{L}_p^{\mathrm{loc}}(\mu, \mathbf{R})$.

2.8 Remark. This definition is in accordance with [Men15, p. 16].

2.9 Remark. If $K(i)$ is a sequence of compact subsets of X with $K(i) \subset \operatorname{Int} K(i+1)$ for $i \in \mathscr{P}$ and $X = \bigcup_{i=1}^{\infty} K(i)$, then the topology on $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ is induced by the seminorms $(\mu \llcorner K(i))_{(p)}$ corresponding to $i \in \mathscr{P}$ by 2.4.

2.10 Remark. The topological vector space $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ is a complete locally convex space and

(1) $\mu(T \sim \operatorname{dmn} f) = 0$ whenever $\mu(T) < \infty$,

(2) $f^{-1}[B]$ is μ measurable whenever B is a Borel subset of Y,

whenever $f \in \mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$; this is evident if X is countably μ measurable[15] and may be verified using the family G constructed in [Fed69, 2.5.10] in the general case.

[14] Whenever f and g are relations the inverse and composition satisfy

$$f^{-1} = \{(y, x) : (x, y) \in f\}, \quad g \circ f = \{(x, z) : (x, y) \in f \text{ and } (y, z) \in g \text{ for some } y\}.$$

[15] A set is called countably μ measurable if and only if it equals the union of a countable family of μ measurable sets with finite μ measure, see [Fed69, 2.3.4].

2.11 Remark. If $p < \infty$, X is an open subset of $\mathbf{R}^n$, and Y is separable, then $\mathscr{D}(U, Y)$ is dense in $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ and $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ is separable; in fact, the inclusion map of $\mathbf{L}_p(\mu, Y)$ topologised by $\mu_{(p)}$ into $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ is continuous with dense image by 2.4, hence 2.1 implies the conclusion.

2.12 Remark. The quotient locally convex space $Q = \mathbf{L}_q^{\mathrm{loc}}(\mu, Y)\big/V$, where

$$V = \mathbf{L}_q^{\mathrm{loc}}(\mu, Y) \cap \big\{f : (\mu \llcorner K)_{(q)}(f) = 0 \text{ whenever } K \text{ is a compact subset of } U\big\},$$

see 2.5, is Hausdorff and complete by 2.10. Under the conditions of 2.9, the topology of Q is induced by a translation invariant metric by 2.4 and Q is an "F-space" in the terminology of [DS58, II.1.10].

Finally, the locally convex space of continuous functions defined on some locally compact Hausdorff space with values in some Banach space is introduced.

2.13 Definition. Suppose X is a locally compact Hausdorff space and Y is a Banach space.

Then $\mathscr{C}(X, Y)$ denotes the vectorspace of all continuous functions mapping X into Y. Its topology is induced by the seminorms ν_K defined by

$$\nu_K(f) = \sup(\{0\} \cup \{|f(x)| : x \in K\}) \quad \text{for } f \in \mathscr{C}(X, Y)$$

corresponding to all compact subsets K of X, see 2.4.[16] Let $\mathscr{C}(X) = \mathscr{C}(X, \mathbf{R})$.

2.14 Remark. The topological vector space $\mathscr{C}(X, Y)$ is a Hausdorff complete locally convex space. If $K(i)$ is a sequence of compact subsets of X with $K(i) \subset \operatorname{Int} K(i+1)$ for $i \in \mathscr{P}$ and $X = \bigcup_{i=1}^\infty K(i)$, then the topology on $\mathscr{C}(X, Y)$ is induced by the seminorms $\nu_{K(i)}$ corresponding to $i \in \mathscr{P}$ by 2.4.

2.15 Remark. The inclusion map of $\mathscr{K}(X)$ into $\mathscr{C}(X)$ is continuous. Moreover, $\mathscr{K}(X)$ is dense in $\mathscr{C}(X)$ since for every compact subset K of X there exists $\zeta \in \mathscr{K}(X)$ with $\zeta(x) = 1$ for $x \in K$ by [Kel75, 5.17, 5.18].

3 Locally Lipschitzian functions

In this section an approximation result for Lipschitzian functions by functions of class 1 over rectifiable varifolds is proven, see 3.5. Its main additional feature is that agreement outside a set of small weight measure may be achieved while essentially maintaining the Lipschitz constant of the original function. This rests on the observation that every submanifold of class 1 of $\mathbf{R}^n$ may be expressed as the image of a retraction of class 1 whose differential at each point of the submanifold equals the orthogonal projection of $\mathbf{R}^n$ onto the tangent space at that point, see 3.2.

The approximation result of this section will be used to prove various density results of function of class ∞ in Sobolev spaces with exponent $q < \infty$, see 5.9 (2), 5.16, and 5.22.

The following theorem is a direct consequence of the definition of submanifolds and Whitney's extension theorem, see [Fed69, 3.1.14].

[16] The topological space $\mathscr{C}(X, Y)$ is denoted $\mathscr{C}_c(X; Y)$ in [Bou89b, X, §1.6, p. 280], where the letter "c" indicates the topology of compact convergence.

3.1 Theorem. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, M is an m dimensional submanifold of class 1 of $\mathbf{R}^n$, Y is a normed vectorspace, and $f : M \to Y$ is of class 1 relative to M.*

Then the following two statements hold:

(1) If $\varrho(C, \delta)$ denotes the supremum of all numbers

$$|f(x) - f(a) - \langle \operatorname{Tan}(M, a)_\natural (x - a), \operatorname{D} f(a) \rangle| / |x - a|$$

corresponding to $\{x, a\} \subset C$ with $0 < |x - a| \leq \delta$ whenever $C \subset M$ and $\delta > 0$, then $\varrho(C, \delta) \to 0$ as $\delta \to 0+$ whenever C is a compact subset of M.

(2) There exist an open subset U of $\mathbf{R}^n$ with $M \subset U$ and a function $g : U \to Y$ of class 1 with $g|M = f$ and

$$\operatorname{D} g(a) = \operatorname{D} f(a) \circ \operatorname{Tan}(M, a)_\natural \quad \textit{for } a \in M.$$

Proof. (1) is readily verified by use of [Fed69, 3.1.19 (1), 3.1.11].

Define $P_a : \mathbf{R}^n \to Y$ by $P_a(x) = f(a) + \langle \operatorname{Tan}(M, a)_\natural (x - a), \operatorname{D} f(a) \rangle$ for $a \in M$ and $x \in \mathbf{R}^n$. Noting (1) and

$$\operatorname{D} P_a(x) - \operatorname{D} P_x(x) = \operatorname{D} f(a) \circ \operatorname{Tan}(M, a)_\natural - \operatorname{D} f(x) \circ \operatorname{Tan}(M, x)_\natural$$

for $a, x \in M$, one applies [Fed69, 3.1.14] to construct for each closed subset A of $\mathbf{R}^n$ with $A \subset M$, a function $g_A : \mathbf{R}^n \to Y$ of class 1 with $g_A|A = f|A$ and

$$\operatorname{D} g_A(a) = \operatorname{D} f(a) \circ \operatorname{Tan}(M, a)_\natural \quad \text{for } a \in A.$$

Therefore g is constructable by use of a partition of unity. □

The existence of a retraction with the desired additional property now follows from the known existence result of retractions, see [Whi57, p. 121].

3.2 Corollary. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, and M is an m dimensional submanifold of class 1 of $\mathbf{R}^n$.*

Then there exists a function r of class 1 retracting some open subset of $\mathbf{R}^n$ onto M and satisfying

$$\operatorname{D} r(a) = \operatorname{Tan}(M, a)_\natural \quad \textit{whenever } a \in M.$$

Proof. Obtaining from [Whi57, p. 121] a map h of class 1 retracting some open subset of $\mathbf{R}^n$ onto M and from 3.1 (2) with $Y = \mathbf{R}^n$ and $f = \mathbf{1}_M$ a function g, one may take $r = h \circ g$. □

Two more preparatory lemmata for the approximation result are needed. The first one is fairly elementary.

3.3 Lemma. *Suppose U is an open subset of $\mathbf{R}^n$, μ is a Radon measure over U, $h : U \to \mathbf{R}$ is of class 1, $A = \{x : h(x) \geq 0\}$, and $\varepsilon > 0$.*

Then there exists a nonnegative function $g : U \to \mathbf{R}$ of class 1 such that

$$\mu(A \sim \{x : h(x) = g(x)\}) \leq \varepsilon.$$

Proof. Applying [Fed69, 3.1.13] with $\Phi = \{U\}$ one obtains $\zeta_1, \zeta_2, \zeta_3, \ldots$ forming a partition of unity on U associated to $\{U\}$. Abbreviating $K_i = \operatorname{spt} \zeta_i$, choose $\delta_i > 0$ and nonnegative functions $f_i : \mathbf{R} \to \mathbf{R}$ of class 1 with

$$\mu(K_i \cap \{x : 0 < h(x) < \delta_i\}) \leq 2^{-i}\varepsilon,$$
$$f_i(t) = \sup\{t, 0\} \quad \text{if either } t \leq 0 \text{ or } t \geq \delta_i$$

whenever $i \in \mathscr{P}$. Since

$$A \sim \left\{ x : h(x) = \sum_{i=1}^{\infty} \zeta_i(x)(f_i \circ h)(x) \right\} \subset \bigcup_{i=1}^{\infty} K_i \cap \{x : 0 < h(x) < \delta_i\},$$

one may take $g = \sum_{i=1}^{\infty} \zeta_i (f_i \circ h)$. $\square$

The second one is slightly more elaborate and relies, among other things, on Kirszbraun's extension theorem, see [Fed69, 2.10.43], and a partition of unity, see [Fed69, 3.1.13].

3.4 Lemma. *Suppose $l, n \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, $A \subset U$, $f : U \to \mathbf{R}^l$ is of class 1, and $\varepsilon > 0$.*

Then there exist an open subset X of U and a function $g : \mathbf{R}^n \to \mathbf{R}^l$ of class 1 such that $A \subset X$, $f|X = g|X$, and

$$\operatorname{Lip} g \leq \varepsilon + \sup\{\operatorname{Lip}(f|A), \sup \|\operatorname{D} f\|[A]\}.$$

Moreover, if $l = 1$ and $f \geq 0$ then one may require $g \geq 0$.

Proof. Assume $\kappa = \sup\{\operatorname{Lip}(f|A), \sup \|\operatorname{D} f\|[A]\} < \infty$ and that A is relatively closed in U. Firstly, it will be shown that there exists an open subset G of U with

$$A \subset G, \quad \operatorname{Lip}(f|G) \leq \varepsilon/2 + \kappa.$$

Define $\eta = 2^{-4}\varepsilon(\varepsilon + \kappa)^{-1}$, note $\eta \leq 1/2$, choose $\delta : A \to \{r : r > 0\}$ such that

$$\mathbf{U}(a, \delta(a)) \subset U \quad \text{and} \quad \operatorname{Lip}(f|\mathbf{U}(a, \delta(a))) \leq \varepsilon/2 + \kappa \quad \text{whenever } a \in A,$$

and let $G = \bigcup\{\mathbf{U}(a, \eta\delta(a)) : a \in A\}$. Suppose $a, x \in A$, $\alpha, \chi \in \mathbf{R}^n$, $|a - \alpha| < \eta\delta(a)$ and $|x - \chi| < \eta\delta(x)$. In case $\delta(a) + \delta(x) \leq 4|\chi - \alpha|$, one estimates

$$\begin{aligned} &|a - \alpha| + |x - \chi| < 4\eta|\chi - \alpha|, \quad |x - a| \leq (1 + 4\eta)|\chi - \alpha|, \\ &|f(\chi) - f(\alpha)| \leq (\varepsilon/2 + \kappa)(|a - \alpha| + |x - \chi|) + \kappa|x - a| \\ &\qquad\qquad \leq (8\eta(\varepsilon + \kappa) + \kappa)|\chi - \alpha| = (\varepsilon/2 + \kappa)|\chi - \alpha| \end{aligned}$$

and, in case $\delta(a) + \delta(x) > 4|\chi - \alpha|$ and $\delta(a) \geq \delta(x)$ one estimates

$$|\chi - a| \leq |\chi - \alpha| + |\alpha - a| < (1/2 + \eta)\delta(a) \leq \delta(a), \quad |\alpha - a| < \delta(a),$$

hence always $|f(\chi) - f(\alpha)| \leq (\varepsilon/2 + \kappa)|\chi - \alpha|$.

Next, choose $g_0 : \mathbf{R}^n \to \mathbf{R}^l$ with $g_0|G = f|G$ and $\operatorname{Lip} g_0 = \operatorname{Lip}(f|G)$ such that $g_0 \geq 0$ if $l = 1$ and $f \geq 0$, see [Fed69, 2.10.43, 4.1.16]. Using [Fed69, 3.1.13] with

Φ replaced by $\{G, U \sim A\}$, one constructs nonnegative functions $\phi_0 \in \mathscr{E}(U, \mathbf{R})$ and $\phi_i \in \mathscr{D}(U, \mathbf{R})$ for $i \in \mathscr{P}$ such that

$$\operatorname{card}(\mathscr{P} \cap \{i : K \cap \operatorname{spt} \phi_i \neq \varnothing\}) < \infty \quad \text{whenever } K \text{ is compact subset of } U,$$
$$A \subset \operatorname{Int}\{x : \phi_0(x) = 1\}, \qquad \operatorname{spt} \phi_0 \subset G, \qquad \operatorname{spt} \phi_i \subset U \sim A \quad \text{for } i \in \mathscr{P},$$
$$\textstyle\sum_{j=0}^\infty \phi_j(x) = 1 \quad \text{for } x \in U.$$

Employing convolution, one obtains functions $g_i : \mathbf{R}^n \to \mathbf{R}^l$ of class 1 satisfying

$$\operatorname{Lip} g_i \leq \operatorname{Lip} g_0, \quad (\operatorname{Lip} \phi_i) \sup \operatorname{im} |g_i - g_0| \leq 2^{-i-1}\varepsilon,$$
$$\text{if } l = 1 \text{ and } f \geq 0 \text{ then } g_i \geq 0$$

for $i \in \mathscr{P}$. Let $g = \sum_{j=0}^\infty \phi_j g_j$ and observe that g is of class 1. Also for $x, \chi \in U$

$$g(x) - g(\chi) = \textstyle\sum_{j=0}^\infty \big(\phi_j(x)(g_j(x) - g_j(\chi)) + (\phi_j(x) - \phi_j(\chi))(g_j(\chi) - g_0(\chi))\big),$$
$$\operatorname{Lip} g \leq \varepsilon/2 + \operatorname{Lip} g_0 = \varepsilon/2 + \operatorname{Lip}(f|G) \leq \varepsilon + \kappa.$$

Therefore one may take $X = \operatorname{Int}\{x : \phi_0(x) = 1\}$. $\square$

In combination with basic properties of rectifiable varifolds the approximation result for locally Lipschitzian functions now readily follows.

3.5 Theorem. *Suppose $l, m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, C is a relatively closed subset of U, $V \in \mathbf{RV}_m(U)$, $f : U \to \mathbf{R}^l$ is locally Lipschitzian,* $\operatorname{spt} f \subset \operatorname{Int} C$, *and $\varepsilon > 0$.*[17]

Then there exists $g : U \to \mathbf{R}^l$ of class 1 satisfying

$$\operatorname{spt} g \subset C, \quad \operatorname{Lip} g \leq \varepsilon + \operatorname{Lip} f, \quad \|V\|(U \sim \{x : f(x) = g(x)\}) \leq \varepsilon.$$

Moreover, if $l = 1$ and $f \geq 0$ then one may require $g \geq 0$.

Proof. Let $X = \operatorname{Int} C$. Employing for instance [Men15, 11.1 (2) (3)] and [KM15, 3.6 (1)], one may construct a function $h : X \to \mathbf{R}^l$ of class 1 and an m dimensional submanifold M of class 1 of $\mathbf{R}^n$ such that

$$M \subset X, \quad \|V\|(X \sim (M \cap \{x : f(x) = h(x)\})) < \varepsilon.$$

In view of 3.3, one may require $h \geq 0$ if $l = 1$ and $f \geq 0$. Since

$$\mathrm{D}(f|M)(x) = \mathrm{D}(h|M)(x) \quad \text{for } \mathscr{H}^m \text{ almost all } x \in M \text{ with } f(x) = h(x),$$

by [Fed69, 2.8.18, 2.9.11, 3.1.5, 3.1.22], the set

$$B = M \cap \{x : f(x) = h(x) \text{ and } \mathrm{D}(f|M)(x) = \mathrm{D}(h|M)(x)\}$$

satisfies $\|V\|(X \sim B) < \varepsilon$. From 3.2 one obtains a function r of class 1 retracting some open subset G of X onto M such that

$$\mathrm{D}\, r(a) = \operatorname{Tan}(M, a)_\natural \quad \text{whenever } a \in M,$$

hence $\sup \|\mathrm{D}(h \circ r)\|[B] \leq \operatorname{Lip} f$. Applying 3.4 with U, A, and f replaced by $(U \sim C) \cup G$, $(U \sim C) \cup B$, and $((U \sim C) \times \{0\}) \cup (h \circ r)$, one obtains a function $g : U \to \mathbf{R}^l$ of class 1 such that

$$g|U \sim X = 0, \quad g|B = f|B, \quad \operatorname{Lip} g \leq \varepsilon + \operatorname{Lip} f$$

and $g \geq 0$ if $l = 1$ and $f \geq 0$. $\square$

[17] The symbol $\mathbf{RV}_m(U)$ denotes the set of m dimensional rectifiable varifolds in U, see Allard [All72, 3.5].

For functions with compact support the result takes the following form.

3.6 Corollary. *Suppose $l, m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, K is a compact subset of U, $f : U \to \mathbf{R}^l$ is a Lipschitzian function,* $\operatorname{spt} f \subset \operatorname{Int} K$, *and $V \in \mathbf{RV}_m(U)$.*

Then there exists a sequence $f_i \in \mathscr{D}(U, \mathbf{R}^l)$ satisfying

$$f_i(x) \to f(x) \quad \text{uniformly for } x \in \operatorname{spt} \|V\| \text{ as } i \to \infty,$$
$$\|(\|V\|, m) \operatorname{ap} \mathrm{D}(f_i - f)\| \to 0 \quad \text{in } \|V\| \text{ measure as } i \to \infty,$$
$$\operatorname{spt} f_i \subset K \quad \text{for } i \in \mathscr{P}, \qquad \limsup_{i \to \infty} \operatorname{Lip} f_i \leq \operatorname{Lip} f.$$

Moreover, if $l = 1$ and $f \geq 0$ one may require $f_i \geq 0$ for $i \in \mathscr{P}$.

Proof. The problem may be reduced firstly to the construction of functions f_i of class 1 by means of convolution and secondly to establishing convergence of f_i to f in $\|V\|$ measure as $i \to \infty$ by [Fed69, 2.10.21]. In view of [Men15, 11.1 (4)], the conclusion now follows from 3.5. □

3.7 Remark. In the preceding statement $\mathbf{R}^l$ may be replaced by a finite dimensional normed vectorspace Y provided "$\operatorname{Lip} f$" is replaced by "$\Gamma \operatorname{Lip} f$" in the conclusion, where Γ is a positive, finite number depending only on Y.

4 Rellich type embeddings

In the present section a Rellich type compactness result for generalised weakly differentiable functions will be established in 4.8. As a consequence one obtains a sequential closedness result under weak convergence for the nonlinear space of generalised weakly differentiable functions with and without "boundary conditions", see 4.9 and 4.10.

The key is to quantify the approximability on large sets by Lipschitzian functions obtained in [Men15, 11.1, 11.2] by means of maximal function techniques. This procedure is complicated by the fact that Sobolev Poincaré inequalities in their usual form with one median are only available near almost every point centred at that point on all scales below a certain threshold which depends on the point and the varifold considered in a rather nonuniform way. An instructive example of qualitative nature is given by Brakke in [Bra78, 6.1], two quantified forms of which were presented by Kolasiński and the author in [KM15, 10.3, 10.8].

Firstly, a simple closedness result assuming convergence locally in measure of the functions and local weak convergence of the generalised derivative is recorded. Several proofs in this section employ the duality of Lebesgue spaces, see [Fed69, 2.5.7], and properties of weak convergence in general, see [DS58, II.3.27], and in $L_1(\mu, Y)$ in particular, see 2.6 and [DS58, IV.8.9–IV.8.12].

4.1 Lemma. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, Y is a finite dimensional normed vectorspace,*[18]

$$f \in \mathbf{A}(\|V\| + \|\delta V\|, Y), \quad F \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)),$$

[18]Whenever μ is a measure and Y is a separable Banach space, $\mathbf{A}(\mu, Y)$ equals the vectorspace of μ measurable functions with values in Y, see [Fed69, 2.3.8].

and $f_i \in \mathbf{T}(V, Y)$ is a sequence with $V \mathbf{D} f_i \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$ satisfying

$$f_i \to f \quad \textit{in } (\|V\| + \|\delta V\|) \llcorner K \textit{ measure as } i \to \infty,$$
$$\lim_{i\to\infty} \textstyle\int_K \langle V \mathbf{D} f_i, G\rangle \,\mathrm{d}\|V\| = \int_K \langle F, G\rangle \,\mathrm{d}\|V\| \quad \textit{for } G \in \mathbf{L}_\infty\big(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y)^*\big)$$

whenever K is a compact subset of U.

Then $f \in \mathbf{T}(V, Y)$ and

$$F(x) = V \mathbf{D} f(x) \quad \textit{for } \|V\| \textit{ almost all } x.$$

Proof. Recall [Fed69, 2.5.7 (ii)] and [DS58, II.3.27]. Since

$$\lim_{i\to\infty} \textstyle\int \langle \theta(x), \mathrm{D}\,\gamma(f_i(x)) \circ V \mathbf{D} f_i(x)\rangle \,\mathrm{d}\|V\|\, x = \int \langle \theta(x), \mathrm{D}\,\gamma(f(x)) \circ F(x)\rangle \,\mathrm{d}\|V\|\, x$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, $\gamma \in \mathscr{E}(Y, \mathbf{R})$ and $\operatorname{spt} \mathrm{D}\,\gamma$ is compact by [DS58, IV.8.10, IV.8.11], the conclusion is readily verified by means of [Men15, 8.3]. $\square$

4.2 Remark. If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, one may replace "$\|V\| + \|\delta V\|$" by "$\|V\|$" in the preceding lemma; in fact, [Fed69, 2.4.11, 2.8.18, 2.9.2, 2.9.7] implies that f is $\|\delta V\|$ measurable and that f_i converges to f in $\|\delta V\| \llcorner K$ measure as $i \to \infty$ whenever K is a compact subset of U.

4.3. The following proposition is an elementary fact about pseudometric spaces. *If Z is the space of a complete pseudometric ϱ, f_i is a sequence in Z and for every $\varepsilon > 0$ there exists a sequence g_i in Z with $\varrho(f_i, g_i) \le \varepsilon$ for $i \in \mathscr{P}$ such that each subsequence of g_i admits a convergent subsequence, then f_i possesses a convergent subsequence.*

Next, an elementary but useful criterion for sequential compactness for local convergence in measure, ultimately based on the Ascoli theorem for Lipschitzian functions, see [Fed69, 2.10.21], is proven.

4.4 Lemma. *Suppose U is an open subset of $\mathbf{R}^n$, μ is a Radon measure over U, Y is a finite dimensional normed vectorspace, f_i is a sequence in $\mathbf{A}(\mu, Y)$ such that whenever X is a μ measurable set with $\mu(X) < \infty$ and $\varepsilon > 0$ there exists $\kappa < \infty$ such that for each $i \in \mathscr{P}$ there exists a subset A of X with $\mu(X \sim A) \le \varepsilon$ and $\sup |f_i|[A] + \mathrm{Lip}(f_i|A) \le \kappa$.*

Then there exist $f \in \mathbf{A}(\mu, Y)$ and a subsequence of f_i which, whenever K is a compact subset of U, converges to f in $\mu \llcorner K$ measure.

Proof. One may assume $\mu(U) < \infty$ and $Y = \mathbf{R}$. Suppose $\varepsilon > 0$. Taking κ as in the hypotheses for $X = U$, one constructs functions $g_i : U \to \mathbf{R}$ with $\sup \operatorname{im} |g_i| + \operatorname{Lip} g_i \le \kappa$ and $\mu(U \sim \{x : f_i(x) = g_i(x)\}) \le \varepsilon$ for $i \in \mathscr{P}$ by [Fed69, 2.10.44, 4.1.16], in particular $|f_i - g_i|_\mu \le \varepsilon$.[19] Therefore, in view of [Fed69, 2.3.8, 2.3.10, 2.10.21], one may apply 4.3 with Z and $\varrho(f, g)$ replaced by $\mathbf{A}(\mu, \mathbf{R})$ and $|f - g|_\mu$ to obtain the conclusion. $\square$

For the use in the present and the next section, a set of hypotheses is collected.

[19] Whenever μ is a measure and Y is a separable Banach

$$|f|_\mu = \inf \big\{ t : \mu(\{x : |f(x)| > t\}) \le t \big\} \quad \text{for } f \in \mathbf{A}(\mu, Y),$$

see [Fed69, 2.3.8].

4.5. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x. In particular, V is rectifiable by Allard [All72, 5.5 (1)].

The key estimate for the Rellich type embedding result will now be formulated. It is based on the Sobolev Poincaré type inequalities obtained in [Men15, 10.1].

4.6 Lemma. *Suppose m, n, U, and V are as in 4.5, $n \leq M < \infty$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $1 < \lambda \leq 2$, $U = \mathbf{U}(a, \lambda r)$, $1 \leq Q \leq M$, $0 \leq \kappa < \infty$, $f \in \mathbf{T}(V)$,*

$$\beta = \infty \quad \text{if } m = 1, \qquad \beta = m/(m-1) \quad \text{if } m > 1,$$
$$\Lambda = 2\Gamma_{[\text{Men15, 10.1}]}(M)\big(1 + 16(\lambda - 1)^{1-m} M^{1/\beta}\big),$$
$$C = \{(x, \mathbf{B}(x, s)) : x \in \mathbf{U}(a, \lambda r) \text{ and } 0 < s + |x - a| < \lambda r\},$$
$$f \text{ is } (\|V\|, C) \text{ approximately continuous at } a,$$

and, for $0 < s \leq r$,

$$\|V\| \, \mathbf{B}(a, s) \geq (1/2)\boldsymbol{\alpha}(m) s^m, \quad \|V\| \, \mathbf{U}(a, \lambda s) \leq 2(Q - M^{-1})\boldsymbol{\alpha}(m) s^m,$$
$$\|V\|(\mathbf{U}(a, \lambda s) \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Gamma_{[\text{Men15, 10.1}]}(M)^{-1} s^m,$$
$$\textstyle\int_{\mathbf{U}(a,\lambda s)} |V \, \mathbf{D} \, f| \, \mathrm{d}\|V\| + (\|V\| \llcorner \mathbf{U}(a, \lambda s))_{(\infty)}(f) \|\delta V\| \, \mathbf{U}(a, \lambda s) \leq \kappa s^m;$$

here $0 \cdot \infty = \infty \cdot 0 = 0$.

Then there holds

$$(\|V\| \llcorner \mathbf{B}(a, (\lambda - 1)s))_{(\beta)}(f(\cdot) - f(a)) \leq \Lambda \kappa s^m \quad \text{for } 0 < s \leq r.$$

Proof. Abbreviate $\Delta = \Gamma_{[\text{Men15, 10.1}]}(M)$. Choose $y(s) \in \mathbf{R}$ such that

$$\|V\|(\mathbf{U}(a, \lambda s) \cap \{x : f(x) < y(s)\}) \leq (1/2)\|V\| \, \mathbf{U}(a, \lambda s),$$
$$\|V\|(\mathbf{U}(a, \lambda s) \cap \{x : f(x) > y(s)\}) \leq (1/2)\|V\| \, \mathbf{U}(a, \lambda s)$$

for $0 < s \leq r$, in particular

$$|y(s)| \leq (\|V\| \llcorner \mathbf{U}(a, \lambda s))_{(\infty)}(f) \quad \text{and} \quad f(a) = \lim_{s \to 0+} y(s).$$

Define $g_s(x) = f(x) - y(s)$ whenever $0 < s \leq r$ and $x \in \operatorname{dmn} f$, hence

$$(\|\delta V\| \llcorner \mathbf{U}(a, \lambda s))_{(\infty)}(g_s) \leq 2(\|V\| \llcorner \mathbf{U}(a, \lambda s))_{(\infty)}(f)$$

by [Men15, 8.33]. Recalling [Men15, 8.12, 8.13 (4), 9.2], one applies [Men15, 10.1 (1a)] with U, G, f, and r replaced by $\mathbf{U}(a, \lambda s)$, $\varnothing$, $g_s^+ | \mathbf{U}(a, \lambda s)$ respectively $g_s^- | \mathbf{U}(a, \lambda s)$, and s to infer

$$\begin{aligned}(\|V\| \llcorner \mathbf{B}(a, (\lambda - 1)s))_{(\beta)}(g_s) &\leq \Delta\big(\textstyle\int_{\mathbf{U}(a,\lambda s)} |V \, \mathbf{D} \, f| \, \mathrm{d}\|V\| + \int_{\mathbf{U}(a,\lambda s)} |g_s| \, \mathrm{d}\|\delta V\|\big) \\ &\leq 2\Delta\kappa s^m\end{aligned}$$

for $0 < s \leq r$. Finally, one estimates

$$\begin{aligned}&|y(s) - y(s/2)| \cdot \|V\|(\mathbf{B}(a, (\lambda - 1)s/2))^{1/\beta} \\ &\quad \leq (\|V\| \llcorner \mathbf{B}(a, (\lambda - 1)s/2))_{(\beta)}(g_{s/2}) + (\|V\| \llcorner \mathbf{B}(a, (\lambda - 1)s))_{(\beta)}(g_s) \leq 4\Delta\kappa s^m,\end{aligned}$$
$$|y(s) - f(a)| \leq 16\Delta\boldsymbol{\alpha}(m)^{-1/\beta}(\lambda - 1)^{1-m}\kappa s,$$
$$(\|V\| \llcorner \mathbf{B}(a, (\lambda - 1)s))_{(\beta)}(f(\cdot) - f(a)) \leq \Lambda\kappa s^m$$

for $0 < s \leq r$. □

4.7 Remark. The preceding lemma is extracted from the proof of [Men15, 11.2].

Proving the following Rellich type embedding result now mainly amounts to applying the preceding two lemmata, defining the necessary parameters in the appropriate order in this process, and using Egoroff's theorem, see [Fed69, 2.3.7], to construct large sets on which certain conditions are satisfied uniformly.

4.8 Theorem. *Suppose m, n, U, and V are as in 4.5, Y is a finite dimensional normed vectorspace, and $f_i \in \mathbf{T}(V,Y)$ is a sequence satisfying*

$$\lim_{t\to\infty} \sup \big\{ \|V\|(K \cap \{x : |f_i(x)| > t\}) : i \in \mathscr{P} \big\} = 0,$$
$$\sup \big\{ \textstyle\int_{K\cap\{x : |f_i(x)| \leq t\}} \|V \, \mathbf{D} \, f_i\| \, \mathrm{d}\|V\| : i \in \mathscr{P} \big\} < \infty \quad \textit{for } 0 \leq t < \infty$$

whenever K is a compact subset of U.

Then there exist $f \in \mathbf{A}(\|V\|, Y)$ and a subsequence of f_i which, whenever K is a compact subset of U, converges to f in $\|V\| \llcorner K$ measure.

Proof. The proof will be conducted by verifying the hypotheses of 4.4 with μ replaced by $\|V\|$. For this purpose suppose X is a $\|V\|$ measurable set with $\|V\|(X) < \infty$. Defining $C = \{(a, \mathbf{B}(a,r)) : \mathbf{B}(a,r) \subset U\}$, one may assume that for some M with $\sup\{4, n\} \leq M < \infty$ there holds

$$1 \leq \boldsymbol{\Theta}^m(\|V\|, x) \leq M,$$
$$\boldsymbol{\Theta}^m(\|V\|, \cdot) \text{ and } f_i \text{ are } (\|V\|, C) \text{ approximately continuous at } x$$

whenever $x \in X$ and $i \in \mathscr{P}$ by [Fed69, 2.8.18, 2.9.13] and that X is compact.

Choose a compact subset K of U with $X \subset \operatorname{Int} K$ and let $\delta = \operatorname{dist}(X, \mathbf{R}^n \sim K)$. Define $\lambda = (1.1)^{1/m}$, hence $1 < \lambda \leq 2$, let

$$Q(x) = \sup\{1, (5/6)\boldsymbol{\Theta}^m(\|V\|, x)\} \quad \text{whenever } x \in X$$

and notice that

$$\boldsymbol{\Theta}^m(\|V\|, x) < 2\lambda^{-m}(Q(x) - M^{-1}) \quad \text{for } x \in X.$$

Abbreviate $\Delta_1 = \Gamma_{[\text{Men15, 10.1}]}(M)^{-1}$.

Suppose $\varepsilon > 0$.

In order to define κ, first observe that one may construct, by means of [Fed69, 2.3.7, 2.6.2], a $\|V\|$ measurable subset X' of X with $\|V\|(X \sim X') \leq \varepsilon/3$ and $0 < r \leq \delta/2$ satisfying

$$\|V\| \, \mathbf{B}(x,s) \geq (1/2)\boldsymbol{\alpha}(m)s^m, \quad \|V\| \, \mathbf{U}(x, \lambda s) \leq 2(Q(x) - M^{-1})\boldsymbol{\alpha}(m)s^m,$$
$$\|V\|(\mathbf{U}(x,\lambda s) \cap \{\chi : \boldsymbol{\Theta}^m(\|V\|, \chi) < Q(x)\}) \leq \Delta_1 s^m$$

for $x \in X'$ and $0 < s \leq r$. Choose $0 \leq \Delta_2 < \infty$ such that

$$\|V\|(X \cap \{x : |f_i(x)| > \Delta_2\}) \leq \varepsilon/3 \quad \text{for } i \in \mathscr{P}$$

and $g \in \mathscr{D}(Y,Y)$ with $g(y) = y$ whenever $y \in \mathbf{B}(0, \Delta_2)$ and $\sup \operatorname{im} |g| \leq 2\Delta_2$. Define $h_i = g \circ f_i$ and notice that [Men15, 8.12] implies that $h_i \in \mathbf{T}(V,Y)$ and

$$\Delta_3 = \sup \big\{ \textstyle\int_K \|V \, \mathbf{D} \, h_i\| \, \mathrm{d}\|V\| + \|\delta V\|(K) : i \in \mathscr{P} \big\} < \infty.$$

Define β and Λ to be related to m, M, and λ as in 4.6, and let

$$\Delta_4 = 8M(1+2\Delta_2)\Delta_3\boldsymbol{\alpha}(m)\boldsymbol{\beta}(n)\varepsilon^{-1}\Lambda,$$
$$\kappa = \sup\big\{2\Delta_2, 2^{m+3}(\lambda-1)^{-m}\boldsymbol{\alpha}(m)^{-1/\beta}\Delta_4, 8\Delta_2(\lambda-1)^{-1}r^{-1}\big\}.$$

Suppose $i \in \mathscr{P}$.

Defining $F : X \to \overline{\mathbf{R}}$ by

$$F(x) = \sup\left\{\frac{\int_{\mathbf{B}(x,s)} \|V\,\mathbf{D}\,h_i\|\,\mathrm{d}\|V\| + \|\delta V\|\,\mathbf{B}(x,s)}{\|V\|\,\mathbf{B}(x,s)} : 0 < s \le \delta\right\}$$

for $x \in X$, let

$$A = X' \cap \big\{x : |f_i(x)| \le \Delta_2 \text{ and } |F(x)| \le 3\Delta_3\boldsymbol{\beta}(n)\varepsilon^{-1}\big\}$$

and observe $\|V\|(X \sim A) \le \varepsilon$. Noting $f_i|A = h_i|A$ and $\Delta_2 \le \kappa/2$, it is sufficient (by [DS58, II.3.15]) to prove that $\operatorname{Lip}(\alpha \circ h_i|A) \le \kappa/2$ whenever $\alpha \in \operatorname{Hom}(Y, \mathbf{R})$ and $\|\alpha\| \le 1$. Noting [Men15, 8.18], one applies 4.6 with Q, κ, and f replaced by $Q(a)$, $2M(1+2\Delta_2)\boldsymbol{\alpha}(m)F(a)$, and $\alpha \circ h_i|\mathbf{U}(a, \lambda s)$ to infer

$$(\|V\| \llcorner \mathbf{B}(a, (\lambda-1)s))_{(\beta)}((\alpha \circ h_i)(\cdot) - (\alpha \circ h_i)(a))) \le \Delta_4 s^m$$

for $a \in A$ and $0 < s \le r$. Therefore, if $a, x \in A$ and $|x-a| \le (\lambda-1)r/2$, then taking $s = 2|x-a|/(\lambda-1)$ yields

$$\begin{aligned}
&|(\alpha \circ h_i)(x) - (\alpha \circ h_i)(a)| \\
&\qquad \le 2^{m+1}(\lambda-1)^{-m}\Delta_4\|V\|(\mathbf{B}(a, 2|x-a|) \cap \mathbf{B}(x, 2|x-a|))^{-1/\beta}|x-a|^m \\
&\qquad \le 2^{m+2}(\lambda-1)^{-m}\boldsymbol{\alpha}(m)^{-1/\beta}\Delta_4|x-a| \le (\kappa/2)|x-a|.
\end{aligned}$$

Finally, notice that $|(\alpha \circ h_i)(x) - (\alpha \circ h_i)(a)| \le 2\Delta_2 \le (\kappa/2)|x-a|$ whenever $a, x \in A$ and $|x-a| > (\lambda-1)r/2$. $\square$

Combining the theorem with basic properties of weak convergence, the first corollary concerning the closedness of the space of generalised weakly differentiable functions is readily verified.

4.9 Corollary. *Suppose m, n, U, and V are as in 4.5, Y is a finite dimensional normed vectorspace,*

$$f \in \mathbf{A}(\|\delta V\|, Y) \cap \mathbf{L}_1^{\mathrm{loc}}(\|V\|, Y), \quad F \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)),$$

and $f_i \in \mathbf{T}(V, Y) \cap \mathbf{L}_1^{\mathrm{loc}}(\|V\|, Y)$ is a sequence satisfying

$$V\,\mathbf{D}\,f_i \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)) \quad \text{for } i \in \mathscr{P},$$
$$\lim_{i\to\infty} \textstyle\int_K \langle f_i, g\rangle\,\mathrm{d}\|V\| = \int_K \langle f, g\rangle\,\mathrm{d}\|V\| \quad \text{for } g \in \mathbf{L}_\infty(\|V\|, Y^*),$$
$$\lim_{i\to\infty} \textstyle\int_K \langle V\,\mathbf{D}\,f_i, G\rangle\,\mathrm{d}\|V\| = \int_K \langle F, G\rangle\,\mathrm{d}\|V\| \quad \text{for } G \in \mathbf{L}_\infty\big(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)^*\big),$$
$$f_i \to f \quad \text{in } (\|\delta V\| - \|\delta V\|_{\|V\|}) \llcorner K \text{ measure as } i \to \infty$$

whenever K is a compact subset of U, see page 14.

Then $f \in \mathbf{T}(V, Y)$ and

$$F(x) = V\,\mathbf{D}\,f(x) \quad \text{for } \|V\| \text{ almost all } x,$$
$$\lim_{i\to\infty} (\|V\| \llcorner K)_{(1)}(f_i - f) = 0 \quad \text{whenever } K \text{ is a compact subset of } U.$$

Proof. Recall [Fed69, 2.5.7 (ii)] and [DS58, II.3.27]. Applying 4.8 and [DS58, IV.8.12] yields

$$\lim_{i\to\infty} (\|V\| \llcorner K)_{(1)}(f_i - f) = 0 \quad \text{whenever } K \text{ is a compact subset of } U,$$

hence [Fed69, 2.4.11, 2.8.18, 2.9.7] implies

$$f_i \to f \quad \text{in } (\|V\| + \|\delta V\|) \llcorner K \text{ measure as } i \to \infty$$

whenever K is a compact subset of U and the conclusion follows from 4.1. □

Taking the more basic closedness result obtained in [Men15, 9.13] into account, the second corollary involving a "boundary condition" follows similarly.

4.10 Corollary. *Suppose m, n, U, and V are as in 4.5, G is a relatively open subset of* $\operatorname{Bdry} U$*, $B = (\operatorname{Bdry} U) \sim G$,*

$$0 \leq f \in \mathbf{A}(\|\delta V\|, \mathbf{R}) \cap \mathbf{A}(\|V\|, \mathbf{R}), \quad F \in \mathbf{A}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R})),$$
$$\|\delta V\|(U \cap K) < \infty, \quad \textstyle\int_K f + |F| \,\mathrm{d}\|V\| < \infty$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$, and $f_i \in \mathbf{T}_G(V)$ is a sequence satisfying

$$\textstyle\int_{U\cap K} f_i + |V \,\mathbf{D}\, f_i| \,\mathrm{d}\|V\| < \infty \quad \textit{for } i \in \mathscr{P},$$
$$\lim_{i\to\infty} \textstyle\int_{U\cap K} f_i g \,\mathrm{d}\|V\| = \int_{U\cap K} f g \,\mathrm{d}\|V\| \quad \textit{for } g \in \mathbf{L}_\infty(\|V\|),$$
$$\lim_{i\to\infty} \textstyle\int_{U\cap K} \langle \theta, F\rangle \,\mathrm{d}\|V\| = \int_{U\cap K} \langle \theta, V \,\mathbf{D}\, f_i\rangle \,\mathrm{d}\|V\| \quad \textit{for } \theta \in \mathbf{L}_\infty(\|V\|, \mathbf{R}^n),$$
$$f_i \to f \quad \textit{in } (\|\delta V\| - \|\delta V\|_{\|V\|}) \llcorner U \cap K \textit{ measure as } i \to \infty$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$, see page 14.

Then $f \in \mathbf{T}_G(V)$ and

$$F(x) = V \,\mathbf{D}\, f(x) \quad \textit{for } \|V\| \textit{ almost all } x,$$
$$\lim_{i\to\infty} (\|V\| \llcorner U \cap K)_{(1)}(f_i - f) = 0$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$.

Proof. Recall [Fed69, 2.5.7 (ii)] and [DS58, II.3.27].

Applying 4.9 with $Y = \mathbf{R}$ yields $f \in \mathbf{T}(V)$ with F and $V \,\mathbf{D}\, f$ being $\|V\|$ almost equal and, in combination with [DS58, IV.8.9, IV.8.10], also

$$\lim_{i\to\infty} (\|V\| \llcorner U \cap K)_{(1)}(f_i - f) = 0$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$. By [Fed69, 2.4.11, 2.8.18, 2.9.7] this implies

$$f_i \to f \quad \text{in } (\|V\| + \|\delta V\|) \llcorner U \cap K \text{ measure as } i \to \infty$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$, in particular for such K

$$\limsup_{i\to\infty} \|V\|(K \cap A \cap \{x : f_i(x) > y\}) \leq \|V\|(K \cap A \cap \{x : f(x) > b\})$$

whenever A is $\|V\|$ measurable and $0 < b < y < \infty$. Noting [DS58, IV.8.10, IV.8.11], the conclusion now follows from [Men15, 9.13]. □

4.11 Remark. If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, then the hypotheses of $\|\delta V\|$ measurability of f and convergence in $(\|\delta V\| - \|\delta V\|_{\|V\|}) \llcorner K$ measure in 4.9 and 4.10 are evidently redundant by [Fed69, 2.8.18, 2.9.2, 2.9.7].

5 Sobolev spaces

In this section, mainly definitions and basic properties of Sobolev spaces with respect to certain rectifiable varifolds for functions with values in a finite dimensional normed space are provided, see 5.1–5.26. In the proof of the deeper properties of these spaces their link to the spaces of generalised weakly differentiable functions will be used heavily. The relation to generalised weakly differentiable functions is immediate in case of local Sobolev spaces, see 5.2, and takes the form of a theorem for Sobolev functions with "zero boundary values", see 5.27. A first example of the utility of this link is provided by the Sobolev inequality for Sobolev functions with "zero boundary values" in 5.28.

Firstly, the local Sobolev space is defined as a vectorspace; its topology will be defined only after some basic properties are established.

5.1 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, Y is a finite dimensional normed vectorspace, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, and $1 \leq q \leq \infty$.

Then the *local Sobolev space with respect to V and exponent q*, denoted by $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$, is defined to be the vectorspace consisting of all $f \in \mathbf{L}_q^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y)$ such that there exists $F \in \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$ with the following property. If K is a compact subset of U and $\varepsilon > 0$ then

$$((\|V\| + \|\delta V\|) \llcorner K)_{(q)}(f - g) + (\|V\| \llcorner K)_{(q)}(F - V\,\mathbf{D}\,g) \leq \varepsilon$$

for some locally Lipschitzian function $g : U \to Y$. Abbreviate $\mathbf{H}_q^{\mathrm{loc}}(V, \mathbf{R}) = \mathbf{H}_q^{\mathrm{loc}}(V)$.

5.2 Remark. Notice that [Men15, 8.7] and 4.1 imply

$$Y^U \cap \{f : f \text{ is locally Lipschitzian}\} \subset \mathbf{H}_q^{\mathrm{loc}}(V, Y) \subset \mathbf{T}(V, Y)$$

and F is $\|V\|$ almost equal to $V\,\mathbf{D}\,f$.

5.3 Remark. In some cases the definition may be reformulated.

(1) If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ and $q = \infty$, then "$\|V\| + \|\delta V\|$" may be replaced by "$\|V\|$".

(2) One may require g to have compact support.

(3) If $q < \infty$, then one may require $g \in \mathscr{D}(U, Y)$. If additionally $Y = \mathbf{R}$ and $f \geq 0$, then one may in turn also require $g \geq 0$.

(4) If $q = \infty$, $Y = \mathbf{R}$, and $f \geq 0$ then one may require $g \geq 0$.

(5) The family of all compacts subsets of U may be replaced by a family of compact subsets of U whose interiors cover U.

(1), (2), and (4) are evident. (3) follows from (2), 3.6, 3.7, and [Men15, 8.7]. (5) may be verified by means of a partition of unity.

5.4 Remark. If $f \in \mathbf{H}_\infty^{\mathrm{loc}}(V, Y)$ then there exists a continuous function $g : \operatorname{spt} \|V\| \to Y$ such that $f(x) = g(x)$ for $\|V\| + \|\delta V\|$ almost all x. However, modifying [Men09, 1.2 (v)] shows that g may fail to be locally Lipschitzian.

5.5 Remark. If $f \in \mathbf{T}(V, Y)$ and $f(x) = 0$ for $\|V\|$ almost all x, then $f(x) = 0$ for $\|\delta V\|$ almost all x by [Men15, 8.33], hence $V\,\mathbf{D}\,f(x) = 0$ for $\|V\|$ almost all x; in particular, this applies to $f \in \mathbf{H}_q^{\mathrm{loc}}(V, Y)$ by 5.2.

5.6 Remark. The following four basic statements hold.

(1) If $1 \leq r \leq \infty$, $1 \leq s \leq \infty$, $1/r + 1/s = 1/q$, $f \in \mathbf{H}_r^{\mathrm{loc}}(V, Y)$, and $g \in \mathbf{H}_s^{\mathrm{loc}}(V)$, then $gf \in \mathbf{H}_q^{\mathrm{loc}}(V, Y)$ and

$$V\,\mathbf{D}\,(gf)(x) = V\,\mathbf{D}\,g(x)\,f(x) + g(x) V\,\mathbf{D}\,f(x) \quad \text{for } \|V\| \text{ almost all } x.$$

(2) If $f \in \mathbf{H}_q^{\mathrm{loc}}(V, Y)$, Z is a finite dimensional normed vectorspace, and $g : Y \to Z$ is of class 1 with $\operatorname{Lip} g < \infty$, then $g \circ f \in \mathbf{H}_q^{\mathrm{loc}}(V, Z)$ and

$$V\,\mathbf{D}\,(g \circ f)(x) = \mathrm{D}\,g(f(x)) \circ V\,\mathbf{D}\,f(x) \quad \text{for } \|V\| \text{ almost all } x.$$

(3) If $f \in \mathbf{H}_q^{\mathrm{loc}}(V)$ and $q < \infty$, then $\{f^+, f^-, |f|\} \subset \mathbf{H}_q^{\mathrm{loc}}(V)$.

(4) If $f \in \mathbf{H}_1^{\mathrm{loc}}(V, Y)$, $g \in \mathbf{T}(V, Y)$, and $V\,\mathbf{D}\,g \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$, then $f + g \in \mathbf{T}(V, Y)$ and

$$V\,\mathbf{D}\,(f + g)(x) = V\,\mathbf{D}\,f(x) + V\,\mathbf{D}\,g(x) \quad \text{for } \|V\| \text{ almost all } x.$$

(1) and (2) are direct consequences of the definition and [Men15, 8.7]. (3) follows from (2) and the approximation technique employed in [Men15, 8.13 (4)]. (4) follows from [Men15, 8.20 (3)] in conjunction with 4.1.

Now, the locally convex topologies on the local Sobolev spaces can be defined without referring to approximating functions.

5.7 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, Y is a finite dimensional normed vectorspace, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, and $1 \leq q \leq \infty$.

Then $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is endowed with the topology induced by the family of seminorms mapping $f \in \mathbf{H}_q^{\mathrm{loc}}(V, Y)$ onto

$$((\|V\| + \|\delta V\|) \llcorner K)_{(q)}(f) + (\|V\| \llcorner K)_{(q)}(V\,\mathbf{D}\,f)$$

corresponding to all compact subsets K of U, see 2.4.

5.8 Remark. Clearly, $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is a locally convex space and whenever $K(i)$ is a sequence of compact subsets of U with $K(i) \subset \operatorname{Int} K(i+1)$ for $i \in \mathscr{P}$ and $U = \bigcup_{i=1}^{\infty} K(i)$ the topology of $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is induced by the seminorms corresponding to $K(i)$ for $i \in \mathscr{P}$, see 2.4, hence the topology of $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is induced by a real valued translation invariant pseudometric. Moreover, $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is complete since (see 5.2 and 5.3 (2))

$$\begin{aligned} &\{(f, F) : f \in \mathbf{H}_q^{\mathrm{loc}}(V, Y) \text{ and } F(x) = V\,\mathbf{D}\,f(x) \text{ for } \|V\| \text{ almost all } x\} \\ &\quad = \operatorname{Clos}\big\{(g, V\,\mathbf{D}\,g) : g \in Y^U,\ \operatorname{Lip} g < \infty, \text{ and } \operatorname{spt} g \text{ is compact}\big\}, \end{aligned}$$

where the closure is taken in $\mathbf{L}_q^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y) \times \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$, is complete by 2.10 and [Kel75, 6.22, 6.25].

5.9 Remark. The following three basic statements will be verified.

(1) The subspace $Y^U \cap \{g : \operatorname{Lip} g < \infty, \operatorname{spt} g \text{ is compact}\}$ is dense in $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$.

(2) If $q < \infty$, then $\mathscr{D}(U, Y)$ is dense in $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ and $\mathscr{D}(U, \mathbf{R}) \cap \{g : g \geq 0\}$ is dense in $\mathbf{H}_q^{\mathrm{loc}}(V) \cap \{f : f \geq 0\}$.

(3) If $q < \infty$, then $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is separable.

(1) is a consequence of 5.3 (2). 5.3 (3) implies (2). 2.4, 2.11, and 5.8 yield (3).

5.10 Remark. If $q = \infty$ then the topology of $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is induced by the family of seminorms mapping $f \in \mathbf{H}_q^{\mathrm{loc}}(V, Y)$ onto

$$(\|V\| \llcorner K)_{(q)}(f) + (\|V\| \llcorner K)_{(q)}(V \mathbf{D} f)$$

corresponding to all compact subsets K of U by 5.4.

Next, in order to conveniently formulate the quotient local Sobolev space and to prepare for the definition of the Sobolev space, the following quantity is defined.

5.11 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, Y is a finite dimensional normed vectorspace, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, and $1 \leq q \leq \infty$.

Then define

$$\mathbf{H}_q(V, f) = (\|V\| + \|\delta V\|)_{(q)}(f) + \|V\|_{(q)}(V \mathbf{D} f) \quad \text{for } f \in \mathbf{T}(V, Y).$$

5.12 Remark. The function $\mathbf{H}_q(V, \cdot)|E$ is a seminorm whenever E is a vectorspace contained in $\mathbf{T}(V, Y)$. However, the function $\mathbf{H}_q(V, \cdot)$ may not be a seminorm as its domain may fail to be a vectorspace, see [Men15, 8.25].

5.13 Remark. The quotient locally convex space

$$Q = \mathbf{H}_q^{\mathrm{loc}}(V, Y) \Big/ \big(\mathbf{H}_q^{\mathrm{loc}}(V, Y) \cap \{f : \mathbf{H}_q(V, f) = 0\}\big),$$

see 2.5, is Hausdorff and complete (by 5.8) and the topology of Q is induced by a translation invariant metric by 2.4. In particular, Q is an "F-space" in the terminology of [DS58, II.1.10].

The definition of Sobolev space is now obvious.

5.14 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, Y is a finite dimensional normed vectorspace, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, and $1 \leq q \leq \infty$.

Then define the *Sobolev space with respect to V and exponent q* by

$$\mathbf{H}_q(V, Y) = \mathbf{H}_q^{\mathrm{loc}}(V, Y) \cap \{f : \mathbf{H}_q(V, f) < \infty\}.$$

Abbreviate $\mathbf{H}_q(V, \mathbf{R}) = \mathbf{H}_q(V)$.

5.15 Remark. Notice that $\mathbf{H}_q(V, Y)$ *is a* $\mathbf{H}_q(V, \cdot)|\mathbf{H}_q(V, Y)$ *complete topological vector space* by 5.8; in particular the set

$$\{(f, F) : f \in \mathbf{H}_q(V, Y) \text{ and } F(x) = V \mathbf{D} f(x) \text{ for } \|V\| \text{ almost all } x\}$$

is closed in $\mathbf{L}_q(\|V\| + \|\delta V\|, Y) \times \mathbf{L}_q(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$.

5.16 Remark. The vector subspaces

$$\mathbf{H}_q(V,Y) \cap \mathscr{E}(U,Y) \text{ if } q < \infty,$$
$$\text{and } \mathbf{H}_\infty(V,Y) \cap \{g : g \text{ is locally Lipschitzian}\} \text{ if } q = \infty$$

are $\mathbf{H}_q(V,\cdot)|\mathbf{H}_q(V,Y)$ *dense in* $\mathbf{H}_q(V,Y)$; in fact, suppose $\varepsilon > 0$ and $f \in \mathbf{H}_q(V,Y)$, choose a sequence ζ_i forming a partition of unity on U associated with $\{U\}$ as in [Fed69, 3.1.13], abbreviate $\kappa_i = \sup \operatorname{im} |\mathrm{D}\,\zeta_i|$ and $K_i = \operatorname{spt} \zeta_i$, select $g_i \in \mathscr{E}(U,Y)$ if $q < \infty$ by 5.3 (3), respectively locally Lipschitzian functions $g_i : U \to Y$ if $q = \infty$, with

$$(1+\kappa_i)((\|V\| + \|\delta V\|) \llcorner K_i)_{(q)}(f - g_i) + (\|V\| \llcorner K_i)_{(q)}(V\,\mathbf{D}\,(f-g_i)) \leq 2^{-i}\varepsilon$$

for $i \in \mathscr{P}$, and define $g = \sum_{i=1}^\infty \zeta_i g_i$, hence one verifies $\mathbf{H}_q(V, f-g) \leq \varepsilon$ by means of 5.6 (1). Observe that *if* $q < \infty$ *then* $\mathscr{E}(U,Y)$ *may be replaced by* $\mathscr{E}(U,Y) \cap \{g : \operatorname{spt} g$ *is bounded*$\}$ *in the preceding statement.* Finally, notice that in case $Y = \mathbf{R}$ similar results for the corresponding cones of nonnegative functions may be formulated.

The next remark employs the fact that closed subspaces of reflexive Banach spaces are reflexive, see [DS58, II.3.23].

5.17 Remark. In view of 2.5, the quotient space

$$Q = \mathbf{H}_q(V,Y) \Big/ \big(\mathbf{H}_q(V,Y) \cap \{f : \mathbf{H}_q(V,f) = 0\}\big)$$

is a Banach space normed by $\mathbf{H}_q(V,\cdot) \circ \pi^{-1}$, where $\pi : \mathbf{H}_q(V,Y) \to Q$ denotes the canonical projection. If $1 < q < \infty$, then Q is reflexive by 2.6, 5.15, and [DS58, II.3.23].

Also, the definition of the subspace of Sobolev functions with "zero boundary values" now follows the usual pattern.

5.18 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, Y is a finite dimensional normed vectorspace, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, and $1 \leq q \leq \infty$.

Then define $\mathbf{H}^\diamond_q(V,Y)$ to be the $\mathbf{H}_q(V,\cdot)|\mathbf{H}_q(V,Y)$ closure of

$$Y^U \cap \{g : \operatorname{Lip} g < \infty, \operatorname{spt} g \text{ is compact}\}$$

in $\mathbf{H}_q(V,Y)$. Abbreviate $\mathbf{H}^\diamond_q(V,\mathbf{R}) = \mathbf{H}^\diamond_q(V)$.

5.19 Remark. Notice that $\mathbf{H}^\diamond_q(V,Y)$ is $\mathbf{H}_q(V,\cdot)|\mathbf{H}^\diamond_q(V,Y)$ complete by 5.15; in particular the set

$$\{(f,F) : f \in \mathbf{H}^\diamond_q(V,Y) \text{ and } F(x) = V\,\mathbf{D}\,f(x) \text{ for } \|V\| \text{ almost all } x\}$$

is closed in $\mathbf{L}_q(\|V\| + \|\delta V\|, Y) \times \mathbf{L}_q(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$.

5.20 Remark. If K is a compact subset of U, $f \in \mathbf{H}_q(V,Y)$, and $f(x) = 0$ for $\|V\| + \|\delta V\|$ almost all $x \in U \sim K$, then $f \in \mathbf{H}^\diamond_q(V,Y)$.

5.21 Remark. Similarly to 5.6, one obtains the following three basic properties.

(1) If $1 \leq r \leq \infty$, $1 \leq s \leq \infty$, $1/r + 1/s = 1/q$, $f \in \mathbf{H}_r(V,Y)$, and $g \in \mathbf{H}^\diamond_s(V)$, then $gf \in \mathbf{H}^\diamond_q(V,Y)$.

(2) If $f \in \mathbf{H}_q^\diamond(V, Y)$, Z is a finite dimensional normed vectorspace, and $g : Y \to Z$ is of class 1 with $\operatorname{Lip} g < \infty$ and $g(0) = 0$, then $g \circ f \in \mathbf{H}_q^\diamond(V, Z)$.

(3) If $f \in \mathbf{H}_q^\diamond(V)$ and $q < \infty$, then $\{f^+, f^-, |f|\} \subset \mathbf{H}_q^\diamond(V)$.

5.22 Remark. If $q < \infty$, then $\mathscr{D}(U, Y)$ is $\mathbf{H}_q(V, \cdot)|\mathbf{H}_q^\diamond(V, Y)$ dense in $\mathbf{H}_q^\diamond(V, Y)$ and $\mathscr{D}(U, \mathbf{R}) \cap \{g : g \geq 0\}$ is $\mathbf{H}_q(V, \cdot)|\mathbf{H}_q^\diamond(V, \mathbf{R})$ dense in $\mathbf{H}_q^\diamond(V) \cap \{f : f \geq 0\}$ by 3.6, 3.7, and [Men15, 8.7].

5.23 Remark. If $U = \mathbf{R}^n$ and $q < \infty$, then $\mathbf{H}_q^\diamond(V, Y) = \mathbf{H}_q(V, Y)$ by 5.16.

5.24 Remark. If $f : \operatorname{spt} \|V\| \to Y$ is continuous and $f \in \mathbf{H}_\infty^\diamond(V, Y)$, then

$$\{x : |f(x)| \geq t\} \text{ is compact whenever } 0 < t < \infty;$$

in fact, this is trivial if f has compact support and the asserted condition is closed under uniform convergence.

5.25 Remark. If $f \in \mathbf{H}_q^\diamond(V, Y)$, then

$$(\|V\| + \|\delta V\|)(\{x : |f(x)| \geq t\}) < \infty \quad \text{whenever } 0 < t < \infty$$

by 5.4 and 5.24 if $q = \infty$ and trivially else.

5.26 Remark. In view of 2.5, the quotient space

$$Q = \mathbf{H}_q^\diamond(V, Y) \Big/ \big(\mathbf{H}_q^\diamond(V, Y) \cap \{f : \mathbf{H}_q(V, f) = 0\}\big)$$

is a Banach space normed by $\mathbf{H}_q(V, \cdot) \circ \pi^{-1}$, where $\pi : \mathbf{H}_q^\diamond(V, Y) \to Q$ denotes the canonical projection. If $1 < q < \infty$, then Q is reflexive by 2.6, 5.19, and [DS58, II.3.23].

Next, the link between the two realisations, for Sobolev functions and generalised weakly differentiable functions, of the concept of "zero boundary values" will be established. The proof uses an approximation procedure and relies on basic properties of generalised weakly differentiable functions and the corresponding concept of "zero boundary values".

5.27 Theorem. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace, and $f \in \mathbf{H}_q^\diamond(V, Y)$.*

Then $|f| \in \mathbf{T}_{\operatorname{Bdry} U}(V)$.

Proof. Firstly, it will be proven that *if $g : Y \to \mathbf{R}$ is a nonnegative, proper Lipschitzian function of class 1 and $g(0) = 0$, then $g \circ f \in \mathbf{T}_{\operatorname{Bdry} U}(V)$ and*

$$V\,\mathbf{D}\,(g \circ f)(x) = \mathrm{D}\,g(f(x)) \circ V\,\mathbf{D}\,f(x) \quad \textit{for } \|V\| \textit{ almost all } x.$$

By [Men15, 8.12], $g \circ f \in \mathbf{T}(V)$ and the asserted formula holds. Observe that

$$(\|V\| + \|\delta V\|)(\{x : g(f(x)) \geq z\}) < \infty \quad \text{for } 0 < z < \infty$$

by 5.25. Choose Lipschitzian functions $f_i : U \to Y$ with compact support and

$$\mathbf{H}_q(V, f - f_i) \to 0 \quad \text{as } i \to \infty.$$

Noting $g \circ f_i \in \mathbf{T}_{\operatorname{Bdry} U}(V)$ by [Men15, 9.2, 9.4] and, if $q = 1$, then

$$\textstyle\int |V\,\mathbf{D}\,(g \circ f)|\,\mathrm{d}\|V\| < \infty, \quad \lim_{i \to \infty} \|V\|_{(1)}(V\,\mathbf{D}\,(g \circ f) - V\,\mathbf{D}\,(g \circ f_i)) = 0.$$

Now, applying [Men15, 9.13, 9.14] with f and f_i replaced by $g \circ f$ and $g \circ f_i$ yields the assertion.

Secondly, construct functions $g_i : Y \to \mathbf{R}$ of class 1 with

$$g_i \geq 0, \quad \operatorname{Lip} g_i \leq 1, \quad \delta_i = \sup\{|g_i(y) - |y|| : y \in Y\} < \infty$$

for $i \in \mathscr{P}$ and $\delta_i \to 0$ as $i \to \infty$. In particular, the maps g_i are proper. Observe that one may require $g_i(0) = 0$ for $i \in \mathscr{P}$. Notice that $|f| \in \mathbf{T}(V)$ by 5.2 and [Men15, 8.16] and

$$\begin{aligned} \limsup_{i \to \infty} \textstyle\int_{A \cap \{x : |g_i(f(x))| \geq z\}} &|V \mathbf{D} (g_i \circ f)| \, \mathrm{d}\|V\| \\ &\leq \textstyle\int_{A \cap \{x : |f(x)| \geq c\}} \|V \mathbf{D} f\| \, \mathrm{d}\|V\| < \infty \end{aligned}$$

whenever $0 < c < z < \infty$ and A is $\|V\|$ measurable by 5.25 and Hölder's inequality. In view 5.25 and [Fed69, 2.4.11], the assertion of the preceding paragraph allows to apply [Men15, 9.13] with G, f, and f_i replaced by $\operatorname{Bdry} U$, $|f|$, and $g_i \circ f$ to obtain the conclusion. □

The Sobolev inequality now follows immediately.

5.28 Corollary. *Suppose m, n, U, and V are as in 4.5, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace, $f \in \mathbf{H}_q^\diamond(V, Y)$, and*

$$\beta = \infty \quad \textit{if } m = 1, \qquad \beta = m/(m-1) \quad \textit{if } m > 1.$$

Then there holds

$$\|V\|_{(\beta)}(f) \leq \Gamma_{[\text{Men15, 10.1}]}(n) \big(\|V\|_{(1)}(V \mathbf{D} f) + \|\delta V\|(f) \big).$$

Proof. In view of 5.25 and 5.27, the conclusion is a consequence of [Men15, 8.16, 10.1 (2a)]. □

6 Geodesic distance

In this section and in the following section, varifolds satisfying a dimensionally critical summability condition on the mean curvature and a lower bound on their densities as described in 6.1 with $p = m$ will be investigated. Here, the properties of the geodesic distance in the support of the weight measure of such varifolds are investigated. Since connected components of this support are relatively open by [Men15, 6.14], one may assume for this purpose that the support of the weight measure is connected, see 6.9. Moreover, it is known from [Men15, 14.2] that in this case the geodesic distance between any two points is finite.

In the present section, it is established that – under the previously described hypotheses – the geodesic distance is a continuous function and gives rise to a local Sobolev function with bounded generalised weak derivative, see 6.8. Moreover, an example is constructed that shows that this function need not be locally Hölder continuous with respect to any exponent, see 6.11. This example also yields that the embedding result of local Sobolev functions into continuous functions which will be obtained in 7.12 is sharp, see 7.13.

The proof of the properties of the geodesic distance consists of a refinement of the techniques used in [Men15, 14.2]. In particular, it relies as well on the oscillation estimates for continuous generalised weakly differentiable functions obtained in [Men15, 13.1].

Firstly, the condition on the varifolds will be formulated in which, usually, the case $p = m$ will be considered.

6.1. Suppose $m, n \in \mathscr{P}$, $m \leq n$, $1 \leq p \leq \infty$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x. If $p > 1$, then suppose additionally that $\mathbf{h}(V, \cdot) \in \mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ and

$$(\delta V)(\theta) = -\textstyle\int \mathbf{h}(V, x) \bullet \theta(x) \,\mathrm{d}\|V\| x \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

In particular, V is rectifiable by Allard [All72, 5.5 (1)]. If $p = 1$ let $\psi = \|\delta V\|$. If $1 < p < \infty$ define a Radon measure ψ over U by

$$\psi(A) = \textstyle\int_A^* |\mathbf{h}(V, x)|^p \,\mathrm{d}\|V\| x \quad \text{for } A \subset U.$$

Secondly, an observation concerning the differential of a function relative to a set will be made.

6.2. Suppose X and Y are a normed vectorspaces, $a \in A \subset X$, and $f : A \to Y$ is differentiable relative to A at a.[20] Then

$$\sup\{|\operatorname{D} f(a)(u)| : u \in \operatorname{Tan}(A, a) \text{ and } |u| = 1\} \leq \limsup_{x \to a} \frac{|f(x) - f(a)|}{|x - a|}$$

and equality holds if $\dim X < \infty$; in fact, by [Fed69, 3.1.21, 3.1.22] it is sufficient to note that if x_i is a sequence in $A \sim \{a\}$, $u \in X$, $x_i \to a$ as $i \to \infty$, and $|x_i - a|^{-1}(x_i - a) \to u$ as $i \to \infty$ then

$$|\operatorname{D} f(a)(u)| = \lim_{i \to \infty} \frac{|f(x_i) - f(a)|}{|x_i - a|}.$$

Notice that if U is a proper subset of $\mathbf{R}^n$ in 6.1 then $\operatorname{spt} \|V\|$, which by definition is a subset of U, may be incomplete. This makes the study of the geodesic distance on $\operatorname{spt} \|V\|$ more delicate. Therefore, initially, the geodesic distance on the closure of $\operatorname{spt} \|V\|$ in $\mathbf{R}^n$ will investigated before treating the general case by means of an exhaustion procedure.

Some well known facts concerning geodesic distances are summarised below.

6.3. Suppose Y is a boundedly compact metric space metrised by τ and X is a dense subset of Y. Whenever $0 < \delta \leq \infty$ one may define a pseudometric $\sigma_\delta : X \times X \to \overline{\mathbf{R}}$ by letting $\sigma_\delta(a, x)$ denote the infimum of the set of numbers

$$\sum_{i=1}^{j} \tau(x_i, x_{i-1})$$

[20] Suppose X and Y are normed vectorspaces, $A \subset X$, $a \in \operatorname{Clos} A$, and $f : A \to Y$. Then the *tangent cone* of A at a, denoted $\operatorname{Tan}(A, a)$, is the set of all $u \in X$ such that for every $\varepsilon > 0$ there exist $x \in A$ and $0 < r \in \mathbf{R}$ with

$$|x - a| < \varepsilon \quad \text{and} \quad |r(x - a) - u| < \varepsilon,$$

see [Fed69, 3.1.21]. Moreover, f is called *differentiable relative to A at a* if and only if there exist a neighbourhood U of a in X and function $g : U \to Y$ such that

$$g|A \cap U = f|A \cap U, \quad g \text{ is differentiable at } a.$$

In this case $\operatorname{D} g(a)| \operatorname{Tan}(A, a)$ is determined by f and a and denoted $\operatorname{D} f(a)$, see [Fed69, 3.1.22].

corresponding to all finite sequences $x_0, x_1, \dots, x_j$ in X with $x_0 = a$, $x_j = x$ and $|x_i - x_{i-1}| \leq \delta$ for $i = 1, \dots, j$ and $j \in \mathscr{P}$. Clearly, $\sigma_\infty = \tau | X \times X$ and $\sigma_\delta \geq \sigma_\varepsilon$ whenever $0 < \delta \leq \varepsilon$. Defining $\sigma : X \times X \to \overline{\mathbf{R}}$ by

$$\sigma(a,x) = \lim_{\delta \to 0+} \sigma_\delta(a,x) \quad \text{for } a, x \in X,$$

one obtains a pseudometric over X such that $\sigma(a,x)$ equals the infimum of the set of numbers $\mathbf{V}_{\inf I}^{\sup I} g$ corresponding to continuous maps $g : \mathbf{R} \to Y$ such that $g(\inf I) = a$ and $g(\sup I) = x$ for some compact subinterval I of $\mathbf{R}$, where the length of g from $\inf I$ to $\sup I$ is computed with respect to τ; in fact, if $\sigma(a,x) < \infty$, then there exists g mapping $\mathbf{R}$ into Y satisfying

$$g(0) = a, \quad g(\sigma(a,x)) = x, \quad \operatorname{Lip} g \leq 1.$$

These classical facts may be verified by means of [Fed69, 2.5.16, 2.10.21].

For the auxiliary pseudometric σ the desired result now follows by approximating by the pseudometrics σ_δ and passing to the limit with the help of the oscillation estimate [Men15, 13.1] and Ascoli's theorem.

6.4 Lemma. *Suppose m, n, p, U, and V are as in 6.1, $p = m$, $X = \operatorname{spt} \|V\|$, X is connected, σ is associated to X as in 6.3 with $Y = \operatorname{Clos} X$, and $W \in \mathbf{V}_{2m}(U \times U)$ satisfies*

$$W(k) = \textstyle\int k((x_1,x_2), P_1 \times P_2) \, \mathrm{d}(V \times V)\, ((x_1,P_1),(x_2,P_2))$$

whenever $k \in \mathscr{K}(U \times U, \mathbf{G}(\mathbf{R}^n \times \mathbf{R}^n, 2m))$.

Then the following two statements hold.

(1) *The function σ is continuous, a metric on X, and belongs to $\mathbf{H}_q^{\mathrm{loc}}(W)$ for $1 \leq q < \infty$ with*

$$|\langle (u_1,u_2), W \, \mathbf{D}\, \sigma(x_1,x_2) \rangle| \leq |u_1| + |u_2| \quad \textit{whenever } u_1, u_2 \in \mathbf{R}^n$$

for $\|W\|$ almost all (x_1,x_2).

(2) *If $a \in X$, then $\sigma(a,\cdot) \in \mathbf{H}_q^{\mathrm{loc}}(V)$ for $1 \leq q < \infty$ and*

$$|V \, \mathbf{D}\, (\sigma(a,\cdot))(x)| = 1 \quad \textit{for } \|V\| \textit{ almost all } x.$$

Proof. Define a norm ν over $\mathbf{R}^n \times \mathbf{R}^n$ by $\nu(x_1,x_2) = |x_1| + |x_2|$ for $x_1, x_2 \in \mathbf{R}^n$. Quantities derived with ν replacing the norm associated to the inner product on $\mathbf{R}^n \times \mathbf{R}^n$ will be distinguished by the subscript ν. Notice that W satisfies the conditions of 6.1 with m, n, p, and U replaced by $2m$, $2n$, m, and $U \times U$ and

$$\|P_\natural\|_\nu \leq 1 \quad \text{for } W \text{ almost all } (z,P)$$

by [KM15, 3.7 (1) (2) (4) (5) (6)]. Let σ_δ for $0 < \delta \leq \infty$ be defined as in 6.3.

Notice that

$$\sigma_\delta(a,x) \leq \sigma_\delta(\alpha,\chi) + \nu((a,x) - (\alpha,\chi))$$

whenever $a, x, \alpha, \chi \in X$ and $\sup\{|a - \alpha|, |x - \chi|\} \leq \delta$. Since X is connected and $\sigma_\delta(a,a) = 0$ for $a \in X$, it follows that σ_δ is a locally Lipschitzian real valued function with

$$\operatorname{Lip}_\nu(\sigma_\delta | A) \leq 1 \quad \text{whenever } A \subset X \times X \text{ and } \operatorname{diam} A \leq \delta,$$

in particular $\sigma_\delta \in \mathbf{H}^{\mathrm{loc}}_\infty(W)$ and $\sigma_\delta(a,\cdot) \in \mathbf{H}^{\mathrm{loc}}_\infty(V)$ with

$$\|W\,\mathbf{D}\,\sigma_\delta(z)\|_\nu = \|(\|W\|,2m)\,\mathrm{ap}\,\mathrm{D}\,\sigma_\delta(z)\|_\nu \le 1 \quad \text{for } \|W\| \text{ almost all } z,$$
$$|V\,\mathbf{D}\,(\sigma_\delta(a,\cdot))(x)| \le 1 \quad \text{for } \|V\| \text{ almost all } x$$

for $a \in X$ by [Men15, 8.7] in conjunction with [Fed69, 3.2.16] and 6.2. Since $\{\sigma_\delta(a,\cdot)|K : a \in X, \delta > 0\}$ is an equicontinuous family of functions whenever K is a compact subset of X by 5.2 and [Men15, 4.8 (1), 13.1] and σ is real valued by [Men15, 14.2], one obtains that

$$\sigma_\delta(a,\cdot) \uparrow \sigma(a,\cdot) \quad \text{locally uniformly as } \delta \to 0+ \text{ for } a \in X.$$

by the Ascoli theorem, see [Kel75, 7.14, 7.18]. Therefore $\sigma(a,\cdot)$ is continuous for $a \in X$, hence σ is continuous as σ is a metric and

$$\sigma_\delta \uparrow \sigma \quad \text{locally uniformly as } \delta \to 0+$$

by Dini's theorem, see [Kel75, Problem 7.E]. Consequently, 4.1 and [Men15, 8.14] yield $\sigma \in \mathbf{T}(W)$ and $\sigma(a,\cdot) \in \mathbf{T}(V)$ with

$$\|W\,\mathbf{D}\,\sigma(z)\rangle\|_\nu \le 1 \quad \text{for } \|W\| \text{ almost all } z,$$
$$|V\,\mathbf{D}\,(\sigma(a,\cdot))(x)| \le 1 \quad \text{for } \|V\| \text{ almost all } x$$

for $a \in X$. From 2.1, 4.1, [Fed69, 2.5.7 (ii)], and Alaoglu's theorem, see [DS58, V.4.2, V.5.1], one obtains

$$\textstyle\int \langle \theta, W\,\mathbf{D}\,\sigma_\delta\rangle \,\mathrm{d}\|W\| \to \int \langle \theta, W\,\mathbf{D}\,\sigma\rangle \,\mathrm{d}\|W\| \quad \text{for } \theta \in \mathbf{L}_1(\|W\|, \mathbf{R}^n \times \mathbf{R}^n),$$
$$\textstyle\int \langle \theta, V\,\mathbf{D}\,(\sigma_\delta(a,\cdot))\rangle \,\mathrm{d}\|V\| \to \int \langle \theta, V\,\mathbf{D}\,(\sigma(a,\cdot))\rangle \,\mathrm{d}\|V\| \quad \text{for } \theta \in \mathbf{L}_1(\|V\|, \mathbf{R}^n)$$

as $\delta \to 0+$ for $a \in X$, so that Mazur's lemma, see [DS58, V.3.14], and [Fed69, 2.5.7 (i)] in fact yield $\sigma \in \mathbf{H}^{\mathrm{loc}}_q(W)$ and $\sigma(a,\cdot) \in \mathbf{H}^{\mathrm{loc}}_q(V)$ for $a \in X$ and $1 \le q < \infty$.

Suppose $a \in X$. Then, by [Men15, 11.2, 11.4 (4)], $\|V\|$ almost all x satisfy $x \in X$, $x \neq a$, and $\sigma(a,\cdot)$ is differentiable relative to X at x and $|\,\mathrm{D}(\sigma(a,\cdot))(x)| = |V\,\mathbf{D}\,(\sigma(a,\cdot))(x)|$. Consider such x, abbreviate $b = \sigma(a,x)$, and choose g as in 6.3. Then $\Upsilon = g^{-1}[X]$ is neighbourhood of b and one observes that

$$\sigma(a, g(\upsilon)) = \upsilon \quad \text{whenever } \upsilon \in \Upsilon \text{ and } 0 \le \upsilon \le b,$$
$$1 \le \big(\limsup_{\chi \to x} |\sigma(a,\chi) - \sigma(a,x)|/|\chi - x|\big)\big(\limsup_{\upsilon \to b-} |g(\upsilon) - g(b)|/|\upsilon - b|\big).$$

Consequently, one infers $1 \le |\,\mathrm{D}(\sigma(a,\cdot))(x)|$ by 6.2. □

Next, the exhaustion procedure is prepared to treat the general case.

6.5. Suppose X is a connected, locally connected, locally compact, separable metric space. Then there exists a sequence of connected, open subsets A_i of X with compact closure and $\mathrm{Clos}\, A_i \subset A_{i+1}$ for $i \in \mathscr{P}$ and $X = \bigcup_{i=1}^\infty A_i$; in fact, if Φ is a nonempty countable base of the topology of X consisting of nonempty connected open subsets of X with compact closure, then one observes that there exists an enumeration $B_1, B_2, B_3, \ldots$ of Φ such that $B_{i+1} \cap \bigcup_{j=1}^i B_j \neq \varnothing$ for $i \in \mathscr{P}$, hence one may inductively select a strictly increasing sequence of positive integers $j(i)$ such that

$$A_i = \bigcup_{k=1}^{j(i)} B_k \quad \text{satisfies} \quad \mathrm{Clos}\, A_i \subset \bigcup_{k=1}^{j(i+1)} B_k \quad \text{for } i \in \mathscr{P}.$$

6.6. Suppose X is a metric space and ϱ is the pseudometric on X defined by letting $\varrho(a, x)$ for $a, x \in X$ denote the infimum of the set of numbers

$$\mathbf{V}_{\inf I}^{\sup I} g$$

corresponding to continuous maps $g : \mathbf{R} \to X$ and compact subintervals I of $\mathbf{R}$ with $g(\inf I) = a$ and $g(\sup I) = x$. If A_i form a sequence of open subsets of X with compact closure and $\operatorname{Clos} A_i \subset A_{i+1}$ for $i \in \mathscr{P}$ and $X = \bigcup_{i=1}^{\infty} A_i$, and ϱ_i are the pseudometrics on A_i such that $\varrho_i(a, x)$ for $a, x \in X$ equals the infimum of the set of numbers $\mathbf{V}_{\inf I}^{\sup I} g$ corresponding to continuous maps $g : \mathbf{R} \to \operatorname{Clos} A_i$ and compact subintervals I of $\mathbf{R}$ such that $g(\inf I) = a$ and $g(\sup I) = x$, then ϱ_i equals the metric constructed in 6.3 under the name "σ" with X and Y replaced by A_i and $\operatorname{Clos} A_i$, $\varrho_{i+1} | A_i \times A_i \leq \varrho_i$ for $i \in \mathscr{P}$ and

$$\varrho_i(a, x) \to \varrho(a, x) \quad \text{as } i \to \infty \text{ for } a, x \in X.$$

Evidently, if X is a dense subset of some boundedly compact metric space Y, then the pseudometric σ constructed in 6.3 satisfies $\sigma \leq \varrho$.

If X is incomplete the infimum occurring in the definition of $\varrho(a, x)$ need not to be attained even if $\varrho(a, x) < \infty$. The following lemma serves as a substitute.

6.7 Lemma. *Suppose Y is a boundedly compact metric space, $X \subset Y$, ϱ is associated to X as in 6.6, ϱ is continuous, $a, x \in X$, and $b = \varrho(a, x) < \infty$.*

Then there exists a map $g : \mathbf{R} \to Y$ satisfying

$$g(0) = a, \quad g(b) = x, \quad \operatorname{Lip} g \leq 1,$$
$$\varrho(a, g(\upsilon)) = \upsilon \quad \textit{whenever } 0 \leq \upsilon \leq b \textit{ and } g(\upsilon) \in X.$$

Proof. For $i \in \mathscr{P}$ choose continuous maps $g_i : \mathbf{R} \to X$ and $0 \leq b_i < \infty$ such that $g_i(0) = a$, $g_i(b_i) = x$, and $\mathbf{V}_0^{b_i} g_i \to b$ as $i \to \infty$. In view of [Fed69, 2.5.16] one may require additionally $\operatorname{Lip} g_i \leq 1$ and $b_i = \mathbf{V}_0^{b_i} g_i$, hence $\mathbf{V}_y^{\upsilon} g_i = \upsilon - y$ for $0 \leq y \leq \upsilon \leq b_i$. Possibly passing to a subsequence, one constructs a map $g : \mathbf{R} \to Y$ as the locally uniform limit of g_i as $i \to \infty$ with

$$g(0) = a, \quad g(b) = x, \quad \operatorname{Lip} g \leq 1,$$

see [Fed69, 2.10.21]. If $0 \leq \upsilon \leq b$ and $g(\upsilon) \in X$, then

$$\varrho(a, g(\upsilon)) = \lim_{i \to \infty} \varrho(a, g_i(\upsilon)) \leq \liminf_{i \to \infty} \mathbf{V}_0^{\upsilon} g_i = \upsilon,$$
$$\varrho(g(\upsilon), x) = \lim_{i \to \infty} \varrho(g_i(\upsilon), g_i(b)) \leq \liminf_{i \to \infty} \mathbf{V}_{\upsilon}^{b} g_i = b - \upsilon,$$

hence $\varrho(a, g(\upsilon)) = \upsilon$. □

Now, the general case may be treated using the same pattern of proof as in 6.4.

6.8 Theorem. *Suppose m, n, p, U, and V are as in 6.1, $p = m$, $X = \operatorname{spt} \|V\|$, X is connected, ϱ is associated to X as in 6.6, and $W \in \mathbf{V}_{2m}(U \times U)$ satisfies*

$$W(k) = \textstyle\int k((x_1, x_2), P_1 \times P_2) \, \mathrm{d}(V \times V) \, ((x_1, P_1), (x_2, P_2))$$

whenever $k \in \mathscr{K}(U \times U, \mathbf{G}(\mathbf{R}^n \times \mathbf{R}^n, 2m))$.

Then the following two statements hold.

(1) *The function ϱ is continuous, a metric on X, and belongs to $\mathbf{H}_q^{\mathrm{loc}}(W)$ for $1 \leq q < \infty$ with*

$$|\langle (u_1, u_2), W\,\mathbf{D}\,\varrho(x_1, x_2)\rangle| \leq |u_1| + |u_2| \quad \textit{whenever } u_1, u_2 \in \mathbf{R}^n$$

for $\|W\|$ almost all (x_1, x_2).

(2) *If $a \in X$, then $\varrho(a, \cdot) \in \mathbf{H}_q^{\mathrm{loc}}(V)$ for $1 \leq q < \infty$ and*

$$|V\,\mathbf{D}\,(\varrho(a, \cdot))(x)| = 1 \quad \textit{for } \|V\| \textit{ almost all } x.$$

Proof. Firstly, define a norm ν over $\mathbf{R}^n \times \mathbf{R}^n$ by $\nu(x_1, x_2) = |x_1| + |x_2|$ for $x_1, x_2 \in \mathbf{R}^n$. Quantities derived with ν replacing the norm associated to the inner product on $\mathbf{R}^n \times \mathbf{R}^n$ will be distinguished by the subscript ν.

Secondly, observe that [Men15, 6.14 (3)] implies that X is locally connected, hence one may choose subsets A_i of X as in 6.5 and apply 6.6 to obtain ϱ_i. Defining $U_i = U \sim (X \sim A_i)$, one obtains an increasing sequence of open subsets U_i of U such that $A_i = U_i \cap X$ for $i \in \mathscr{P}$ and $U = \bigcup_{i=1}^\infty U_i$. Let

$$V_i = V|\mathbf{2}^{U_i \times \mathbf{G}(n,m)}, \quad W_i = W|\mathbf{2}^{(U_i \times U_i) \times \mathbf{G}(\mathbf{R}^n \times \mathbf{R}^n, 2m)}$$

for $i \in \mathscr{P}$. Applying 6.4 with U, V, X, and W replaced by U_i, V_i, A_i, and W_i yields that ϱ_i are continuous metrics on A_i such that

$$\varrho_i \in \mathbf{H}_q^{\mathrm{loc}}(W_i) \quad \text{with} \quad \|W_i\,\mathbf{D}\,\varrho_i(z)\|_\nu \leq 1 \quad \text{for } \|W_i\| \text{ almost all } z,$$
$$\varrho_i(a, \cdot) \in \mathbf{H}_q^{\mathrm{loc}}(V_i) \quad \text{with} \quad |V_i\,\mathbf{D}\,(\varrho_i(a, \cdot))(x)| \leq 1 \quad \text{for } \|V_i\| \text{ almost all } x$$

whenever $1 \leq q < \infty$, $i \in \mathscr{P}$, and $a \in A_i$.

Thirdly, suppose $j \in \mathscr{P}$. Then $\{\varrho_i(a, \cdot)|K : a \in A_j, j \leq i \in \mathscr{P}\}$ is an equicontinuous family of functions whenever K is a compact subset of A_j by 5.2 and [Men15, 4.8 (1), 13.1], hence one obtains that

$$\varrho_i(a, \cdot)|A_j \downarrow \varrho(a, \cdot)|A_j \quad \text{locally uniformly as } i \to \infty \text{ for } a \in A_j$$

by the Ascoli theorem, see [Kel75, 7.14, 7.18]. Therefore $\varrho(a, \cdot)|A_j$ is continuous for $a \in A_j$, hence $\varrho|A_j \times A_j$ is continuous as $\varrho|A_j \times A_j$ is a metric and

$$\varrho_i|A_j \times A_j \downarrow \varrho|A_j \times A_j \quad \text{locally uniformly as } i \to \infty$$

by Dini's theorem, see [Kel75, Problem 7.E]. Consequently, 4.1, 5.2, and [Men15, 8.14] yield $\varrho|A_j \times A_j \in \mathbf{T}(W_j)$ and $\varrho(a, \cdot)|A_j \in \mathbf{T}(V_j)$ with

$$\|W_j\,\mathbf{D}\,(\varrho|A_j \times A_j)(z)\rangle\|_\nu \leq 1 \quad \text{for } \|W_j\| \text{ almost all } z,$$
$$|V_j\,\mathbf{D}\,(\varrho(a, \cdot)|A_j)(x)| \leq 1 \quad \text{for } \|V_j\| \text{ almost all } x$$

for $a \in A_j$. From 2.1, 4.1, [Fed69, 2.5.7 (ii)] and Alaoglu's theorem, see [DS58, V.4.2, V.5.1], one obtains

$$\lim_{i \to \infty} \textstyle\int \langle \theta, W_j\,\mathbf{D}\,(\varrho_i|A_j \times A_j)\rangle \,\mathrm{d}\|W_j\| = \textstyle\int \langle \theta, W_j\,\mathbf{D}\,(\varrho|A_j \times A_j)\rangle \,\mathrm{d}\|W_j\|$$

for $\theta \in \mathbf{L}_1(\|W_j\|, \mathbf{R}^n \times \mathbf{R}^n)$ and

$$\lim_{i \to \infty} \textstyle\int \langle \theta, V_j\,\mathbf{D}\,(\varrho_i(a, \cdot)|A_j)\rangle \,\mathrm{d}\|V_j\| = \textstyle\int \langle \theta, V_j\,\mathbf{D}\,(\varrho(a, \cdot)|A_j)\rangle \,\mathrm{d}\|V_j\|$$

for $\theta \in \mathbf{L}_1(\|V_j\|, \mathbf{R}^n)$, so that 5.8, Mazur's lemma, see [DS58, V.3.14], and [Fed69, 2.5.7 (i)] in fact yield $\varrho | A_j \times A_j \in \mathbf{H}_q^{\mathrm{loc}}(W_j)$ and $\varrho(a, \cdot)|A_j \in \mathbf{H}_q^{\mathrm{loc}}(V_j)$ for $a \in A_j$ and $1 \leq q < \infty$.

Finally, suppose $a \in X$. Then, by [Men15, 11.2, 11.4 (4)], $\|V\|$ almost all x satisfy $x \in X$, $x \neq a$, and $\varrho(a, \cdot)$ is differentiable relative to X at x and $|\operatorname{D}(\varrho(a,\cdot))(x)| = |V \mathbf{D}\,(\varrho(a,\cdot))(x)|$. Consider such x, abbreviate $b = \varrho(a,x)$, and choose g as in 6.7 with $Y = \operatorname{Clos} X$. Therefore one obtains, as $b \in \operatorname{Int} g^{-1}[X]$,

$$1 \leq \big(\limsup_{\chi \to x} |\varrho(a,\chi) - \varrho(a,x)|/|\chi - x| \big) \big(\limsup_{\upsilon \to b-} |g(\upsilon) - g(b)|/|\upsilon - b| \big).$$

and infers $1 \leq |\operatorname{D}(\varrho(a,\cdot))(x)|$ by 6.2. □

6.9 Remark. The connectedness hypothesis on X is not as restrictive as it may seem since the theorem may otherwise be applied separately to $V \llcorner C \times \mathbf{G}(n,m)$ whenever C is a connected component of $\operatorname{spt} \|V\|$ by [Men15, 6.14].

Finally, an example concerning the possible behaviour of the geodesic distance to a point is constructed which uses the following observation.

6.10. The following fact is of elementary geometric nature. *If $0 < \delta_i < r_i < \infty$ for $i \in \{1,2\}$, $2 \leq n \in \mathscr{P}$, $u \in \mathbf{S}^{n-1}$, $\delta_2 < \delta_1$, and $r_2^2 - (r_2 - \delta_2)^2 < r_1^2 - (r_1 - \delta_1)^2$, then $\mathbf{B}((\delta_2 - r_2)u, r_2) \cap \{x : x \bullet u \geq 0\} \subset \mathbf{U}((\delta_1 - r_1)u, r_1)$.*

6.11 Example. Suppose $2 \leq m \in \mathscr{P}$, $n = m+1$, $p = m$, $U = \mathbf{R}^n$, $\varepsilon > 0$, and $\omega : \{t : 0 < t \leq 1\} \to \{t : 0 < t < \infty\}$ satisfies $\lim_{t \to 0+} \omega(t) = 0$.

Then there exist $T \in \mathbf{G}(n,m)$ and V related to m, n, p, and U as in 6.1 with

$$\begin{gathered}
\|V\| \llcorner \mathbf{R}^n \sim \mathbf{B}(0,1) = \mathscr{H}^m \llcorner T \sim \mathbf{B}(0,1), \quad \|V\| \, \mathbf{B}(0,1) \leq 7\boldsymbol{\alpha}(m), \\
\operatorname{dist}(x,T) \leq \varepsilon \quad \text{for } x \in \operatorname{spt}\|V\|, \qquad \textstyle\int |\mathbf{h}(V,x)|^m \, \mathrm{d}\|V\| \, x \leq \varepsilon, \\
\|\operatorname{Tan}^m(\|V\|,x)_\natural - T_\natural\| \leq \varepsilon \quad \text{and} \quad \boldsymbol{\Theta}^m(\|V\|,x) \leq 3 \qquad \text{for } \|V\| \text{ almost all } x, \\
T \subset \operatorname{spt}\|V\|, \quad \operatorname{spt}\|V\| \text{ is connected}, \\
\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner \operatorname{spt}\|V\|, 0) = \infty, \quad \operatorname{Tan}(\operatorname{spt}\|V\|, 0) = \mathbf{R}^n
\end{gathered}$$

such that the metric ϱ on $X = \operatorname{spt}\|V\|$, see 6.6 and 6.8 (1), satisfies

$$\limsup_{x \to 0} \varrho(0,x)/\omega(|x|) = \infty \quad \text{and} \quad \varrho(0,\cdot) \notin \mathbf{H}_\infty^{\mathrm{loc}}(V).$$

Construction. Assume $\omega(t) \geq t$ for $0 < t \leq 1$. The projections $\mathbf{p} : \mathbf{R}^n \to \mathbf{R}^m$ and $\mathbf{q} : \mathbf{R}^n \to \mathbf{R}$ defined by

$$\mathbf{p}(x) = (x_1, \ldots, x_m), \quad \mathbf{q}(x) = x_n$$

for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ will be employed, see [Fed69, 5.1.9]. Let $T = \operatorname{im} \mathbf{p}^*$. Choose a nonincreasing, locally Lipschitzian function $\gamma : \{t : 0 < t \leq 1\} \to \mathbf{R}$ such that $\lim_{t \to 0+} \gamma(t) = \infty$, $\gamma(1) = 0$, and $\zeta : T \cap \{x : 0 < |x| < 1\} \to \mathbf{R}$ defined by $\zeta(x) = \gamma(|x|)$ whenever $x \in T$ and $0 < |x| < 1$ satisfies

$$\textstyle\int_{T \cap \mathbf{U}(0,1)} |\operatorname{D}\zeta|^m \, \mathrm{d}\mathscr{H}^m \leq 2^{-m}\varepsilon,$$

see for instance [AF03, 4.43]. Let $s_0 = 1$ and choose a strictly decreasing sequence s_i of positive numbers such that

$$s_i^m \leq 2^{-i} \quad \text{and} \quad \gamma(s_i) \geq \gamma(s_{i-1}) + 2 \quad \text{for } i \in \mathscr{P}.$$

Abbreviating $\pi = \boldsymbol{\Gamma}(1/2)^2$ (≈ 3.14), see [Fed69, 3.2.13], next choose sequences δ_i, α_i and r_i such that for $i \in \mathscr{P}$ the following eight conditions are satisfied:

$$\begin{gathered} 0 < \delta_{i+1} < \delta_i \le \varepsilon, \quad 0 \le \alpha_i \le \pi/4, \quad 0 < r_i < \infty, \\ r_i^2 = (r_i - \delta_i)^2 + s_i^2, \quad \sin \alpha_i = s_i / r_i \le \varepsilon, \\ \delta_i \le s_{2i+2}, \quad i\,\omega(\delta_i) \le 2 s_i, \quad 4\boldsymbol{\alpha}(m) m^m r_i^{-m} \le 2^{-i-m}\varepsilon; \end{gathered}$$

in fact the first equation is equivalent to $r_i = (\delta_i^2 + s_i^2)/(2\delta_i)$, hence it is sufficient to inductively choose δ_i small enough. Notice that

$$r_i > \delta_i \text{ and } \cos \alpha_i = (r_i - \delta_i)/r_i \text{ for } i \in \mathscr{P}, \quad \lim_{i \to \infty} s_i = 0, \quad \lim_{i \to \infty} \delta_i = 0.$$

Let $S_i = T \cap \{x : |x| = s_i\}$ for $i \in \mathscr{P} \cup \{0\}$.

Let $V_0 \in \mathbf{RV}_m(\mathbf{R}^n)$ by defined by

$$V_0(k) = \textstyle\int_{T \cap \{x : |x| > 1\}} k(x, T) \, \mathrm{d}\mathscr{H}^m x \quad \text{for } k \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n, m)),$$

hence one obtains for $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$ that

$$(\delta V_0)(\theta) = -\textstyle\int_{S_0} x \bullet \theta(x) \, \mathrm{d}\mathscr{H}^{m-1} x.$$

For $i \in \mathscr{P}$ define $V_i \in \mathbf{RV}_m(\mathbf{R}^n)$ by

$$V_i(k) = \textstyle\int_{T \cap \mathbf{U}(0, s_{i-1}) \sim \mathbf{U}(0, s_i)} k(x, T)(1 + \inf\{2 \cos \alpha_i, \zeta(x) - \gamma(s_{i-1})\}) \, \mathrm{d}\mathscr{H}^m x$$

for $k \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n, m))$ and compute for $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$ that

$$\begin{aligned} (\delta V_i)(\theta) = {} & \textstyle\int_{S_{i-1}} |x|^{-1} x \bullet \theta(x) \, \mathrm{d}\mathscr{H}^{m-1} x - \int_{A_i} \langle T_\natural(\theta(x)), \mathrm{D}\,\zeta(x) \rangle \, \mathrm{d}\mathscr{H}^m x \\ & - (1 + 2 \cos \alpha_i) \textstyle\int_{S_i} |x|^{-1} x \bullet \theta(x) \, \mathrm{d}\mathscr{H}^{m-1} x, \end{aligned}$$

where $A_i = T \cap \mathbf{U}(0, s_{i-1}) \cap \{x : \gamma(|x|) < \gamma(s_{i-1}) + 2 \cos \alpha_i\}$. Define $A = \bigcup_{i=1}^\infty A_i$.

Let $u = \mathbf{q}^*(1)$. Define the sets

$$B_i = \{x : x \bullet u > 0, |x - (\delta_i - r_i) u| = r_i\} \cup \{x : x \bullet u < 0, |x - (r_i - \delta_i) u| = r_i\}.$$

for $i \in \mathscr{P}$. Let C_i for $i \in \mathscr{P}$ denote the closed convex hull of B_i and verify

$$C_i = \mathbf{B}((\delta_i - r_i) u, r_i) \cap \mathbf{B}((r_i - \delta_i) u, r_i).$$

Notice that $0 \in C_i$, $\operatorname{Clos} B_i = \operatorname{Bdry} C_i$, and by 6.10, also

$$C_{i+1} \subset \operatorname{Int} C_i \subset \mathbf{B}(0, 1) \quad \text{for } i \in \mathscr{P},$$

in particular $B = \bigcup_{i=1}^\infty B_i$ is an m dimensional submanifold of class ∞ of $\mathbf{R}^n$. The condition $\sin \alpha_i \le \varepsilon$ for $i \in \mathscr{P}$ implies

$$\| \operatorname{Tan}(B, x)_\natural - T_\natural \| \le \varepsilon \quad \text{for } x \in B,$$

see for instance [KM15, 5.1 (2)]. For $i \in \mathscr{P}$ define $W_i \in \mathbf{RV}_m(\mathbf{R}^n)$ by

$$W_i(k) = \textstyle\int_{B_i} k(x, \operatorname{Tan}(B_i, x)) \, \mathrm{d}\mathscr{H}^m x \quad \text{for } k \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n, m))$$

and compute for $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$ that

$$(\delta W_i)(\theta) = -\textstyle\int_{B_i} \mathbf{h}(B_i, x) \bullet \theta(x) \,\mathrm{d}\mathscr{H}^m x + 2\cos\alpha_i \textstyle\int_{S_i} |x|^{-1} x \bullet \theta(x) \,\mathrm{d}\mathscr{H}^{m-1} x.$$

Since $\sum_{i=1}^\infty \|V_i + W_i\|(\mathbf{R}^n) \leq 7\boldsymbol{\alpha}(m)$, one may define $V \in \mathbf{RV}_m(\mathbf{R}^n)$ by $V = V_0 + \sum_{i=1}^\infty (V_i + W_i)$. One computes and estimates

$$(\delta V)(\theta) = -\textstyle\int_B \mathbf{h}(B, x) \bullet \theta(x) \,\mathrm{d}\mathscr{H}^m x - \textstyle\int_A \langle T_\natural(\theta(x)), \mathrm{D}\,\zeta(x)\rangle \,\mathrm{d}\mathscr{H}^m x,$$

$$|(\delta V)(\theta)| \leq \big(\textstyle\int_B |\mathbf{h}(B, \cdot)|^m \,\mathrm{d}\mathscr{H}^m\big)^{1/m} + \textstyle\int_{T \cap \mathbf{U}(0,1)} |\mathrm{D}\,\zeta|^m \,\mathrm{d}\mathscr{L}^m\big)^{1/m} \leq \varepsilon^{1/m}$$

whenever $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$ and $\|V\|_{(m/(m-1))}(\theta) \leq 1$. Using 2.1 and [Fed69, 2.4.16, 2.5.7 (i)], one deduces that V is related to m, n, p, and U as in 6.1 with

$$\textstyle\int |\mathbf{h}(V, x)|^m \,\mathrm{d}\|V\| x \leq \varepsilon, \quad \|V\| \llcorner \mathbf{R}^n \sim \mathbf{B}(0,1) = \mathscr{H}^m \llcorner T \sim \mathbf{B}(0,1),$$

$$\operatorname{spt}\|V\| \text{ is connected}, \qquad \boldsymbol{\Theta}^m(\|V\|, x) \leq 3 \quad \text{for } \|V\| \text{ almost all } x.$$

Moreover, defining $X = \operatorname{spt}\|V\|$ and noting $X = T \cup B$, one infers

$$\operatorname{Tan}(X, 0) = \mathbf{R}^n, \quad \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner X, 0) = \infty;$$

in fact, to prove the second equation notice that

$$\operatorname{card}(X \cap \{x : \mathbf{p}(x) = \mathbf{p}(a) \text{ and } |\mathbf{q}(x)| \leq t\}) \geq 2i + 3$$

whenever $a \in T \cap \mathbf{B}(0, t)$, $s_{2i+1} \leq t < s_{2i}$, and $i \in \mathscr{P}$.

The well known structure of length minimising geodesics on spheres, see for instance [dC92, Chap. 3, Example 2.11; Chap. 3, Proposition 3.6], implies the following statement. *If $i \in \mathscr{P}$, then*

$$\varrho(0, \delta_i u) = s_i + \alpha_i r_i,$$

and if $i \in \mathscr{P}$, $v \in T$, $|v| = 1$, and $g : \{t : 0 \leq t \leq s_i + \alpha_i r_i\} \to X$ satisfies

$$g(t) = tv \quad \text{if } t \leq s_i,$$

$$g(t) = \sin(\alpha_i + (s_i - t)/r_i) r_i v + \big(\delta_i - r_i + \cos(\alpha_i + (s_i - t)/r_i) r_i\big) u \quad \text{if } s_i < t$$

whenever $0 \leq t \leq s_i + \alpha_i r_i$, then $g(0) = 0$, $g(s_i + \alpha_i r_i) = \delta_i u$, $\operatorname{Lip} g = 1$ *and $|g'(t)| = 1$ for $\mathscr{L}^1$ almost all $t \in \operatorname{dmn} g$.* Consequently, $\varrho(0, \delta_i u) \geq i\,\omega(|\delta_i u|)$ for $i \in \mathscr{P}$ and

$$\limsup_{x \to 0} \varrho(0, x)/\omega(|x|) = \infty.$$

Suppose $f : \mathbf{R}^n \to \mathbf{R}$ were a Lipschitzian function satisfying $f(0) = 0$ and

$$|V\,\mathbf{D}\,(\varrho(0, \cdot))(x) - V\,\mathbf{D}\,f(x)| \leq 1/2 \quad \text{for } \|V\| \text{ almost all } x \in \mathbf{B}(0,1).$$

Then 6.8 (2) in conjunction with 5.2 and [Men15, 11.4 (4)] would imply

$$|\mathrm{D}(\varrho(0, \cdot))(x) - \mathrm{D}(f|X)(x)| \leq 1/2 \quad \text{for } \|V\| \text{ almost all } x \in \mathbf{B}(0,1).$$

Whenever $i \in \mathscr{P}$ one could select $v \in T$ with $|v| = 1$ such that the function g associated to i and v in the statement of the preceding paragraph would satisfy

$$|\mathrm{D}(\varrho(0, \cdot))(g(t)) - \mathrm{D}(f|X)(g(t))| \leq 1/2 \quad \text{for } \mathscr{L}^1 \text{ almost all } t \in \operatorname{dmn} g,$$

hence, noting $\varrho(0,\cdot)\circ g = \mathbf{1}_{\operatorname{dmn} g}$ and $\operatorname{Lip}(f\circ g) < \infty$, integration would yield

$$\begin{aligned} |\varrho(0,\delta_i u) - f(\delta_i u)| &\leq \textstyle\int_0^{s_i+\alpha_i r_i} |\langle g'(t), \mathrm{D}(\varrho(0,\cdot) - f|X)(g(t))\rangle| \,\mathrm{d}\mathscr{L}^1 t \\ &\leq \varrho(0,\delta_i u)/2. \end{aligned}$$

Consequently, one would obtain

$$f(\delta_i u) \geq \varrho(0,\delta_i u)/2 \quad \text{for } i \in \mathscr{P}$$

in contradiction to $\limsup_{x\to 0} |f(x)|/|x| < \infty$. Therefore $\varrho(0,\cdot) \notin \mathbf{H}_\infty^{\mathrm{loc}}(V)$. $\square$

6.12 Remark. Considering large balls centred at 0, the preceding example also shows that the Reifenberg type flatness result of Allard in [All72, 8.8] does not extend to the case of dimensionally critical mean curvature, $p = m$ in 6.1. To which extent the behaviour of integral varifolds satisfying $p = m$ in 6.1 is more regular is only partially understood. Properties of the density ratio specific to the integral case were obtained by Kuwert and Schätzle in [KS04, Appendix A] and [Men10, 3.9], see also [Men15, 7.6]. On the other hand nonuniqueness of tangent cones occurs naturally for $p = m$ even for integral varifolds associated to Lipschitzian functions, see Hutchinson and Meier [HM86].

7 Further implications of critical mean curvature

In this section the study of varifolds satisfying a dimensionally critical summability condition on the mean curvature and a lower bound on their densities as described in 6.1 with $p = m$ will be continued. Initially, estimates for generalised weakly differentiable functions are derived, see 7.1–7.6. Subsequently, continuous and compact embeddings of Sobolev spaces into Lebesgue spaces and spaces of continuous functions along with topological properties of the various Sobolev spaces are compiled. These results follow readily from the corresponding results on generalised weakly differentiable functions obtained in [Men15, §§ 8–10, § 13], Section 4, and 7.1–7.6. The treatment includes local Sobolev spaces in 7.7–7.15, an intermediate space between $\mathbf{H}_q(V,Y)$ and $\mathbf{H}_q^{\mathrm{loc}}(V,Y)$ in 7.16–7.17, and Sobolev spaces with "zero boundary values" in 7.18–7.22.

In implementing this study, the case of one dimensional varifolds needs particular care since the present hypotheses permit that the variation measure of the first variation is not absolutely continuous with respect to the weight measure of the varifold.

The first statement concerns a local Sobolev estimate near a single "boundary point" for one dimensional varifolds.

Its proof is based on the Sobolev Poincaré inequality with several medians, see [Men15, 10.9], and involves the concept of distributional boundary of a set with respect to a varifold as defined in [Men15, 5.1].

7.1 Theorem. *Suppose m, n, p, U, and V are as in 6.1, $p = m = 1$, $n \leq M < \infty$, $\Lambda = \Gamma_{[\text{Men15, 10.1}]}(1+M)$, $0 < r < \infty$, $A = \{x : \mathbf{U}(x,r) \subset U\}$, $a \in A$,*

$$\|V\|(U) \leq M\boldsymbol{\alpha}(1)r, \quad \|\delta V\|(U \sim \{a\}) \leq (2+\Lambda)^{-1},$$

Y is a finite dimensional normed vectorspace, and $f \in \mathbf{T}(V,Y)$.

Then there holds

$$(\|V\| \llcorner A)_{(\infty)}(f) \leq \Gamma\big(r^{-1}\|V\|_{(1)}(f) + \|V\|_{(1)}(V\,\mathbf{D}\,f)\big),$$

where $\Gamma = 2^4 M(1+\Lambda)$.

Proof. Assume $r = 1$ and $\|V\|_{(1)}(f) + \|V\|_{(1)}(V\,\mathbf{D}\,f) < \infty$. In view of [Men15, 8.16] one may also assume $Y = \mathbf{R}$. Choose $N \in \mathscr{P}$ such that $N \leq 4M \leq N+1$ and abbreviate $X = \{x : \mathbf{B}(x, 1/2) \subset U\}$. Applying [Men15, 10.9 (2)] with M, Q, r, and X replaced by $1+M$, 1, $1/2$, and $\{a\}$, one obtains a subset Υ of $\mathbf{R}$ such that $1 \leq \operatorname{card} \Upsilon \leq N+1$ and

$$(\|V\| \llcorner X)_{(\infty)}(g) \leq \Lambda \|V\|_{(1)}(V\,\mathbf{D}\,f), \quad \text{where } g = \operatorname{dist}(\cdot, \Upsilon) \circ f.$$

Let $s = \Lambda \|V\|_{(1)}(V\,\mathbf{D}\,f)$ and $B = \bigcup\{\mathbf{B}(\upsilon, s) : \upsilon \in \Upsilon\}$. Employing [Men15, 8.15, 8.33], one infers

$$f(x) \in B \quad \text{for } \|V\| + \|\delta V\| \text{ almost all } x \in X.$$

Suppose E is a connected component of B and let $F = f^{-1}[E]$. The proof will be concluded by showing that

$$(\|V\| \llcorner A \cap F)_{(\infty)}(f) \leq \Gamma\big(\|V\|_{(1)}(f) + \|V\|_{(1)}(V\,\mathbf{D}\,f)\big).$$

Hence one may assume $A \cap \operatorname{spt}(\|V\| \llcorner F) \neq \varnothing$. Observe that in view of [Men15, 8.15] one may use [Men15, 8.30] with f replaced by $-\operatorname{dist}(\cdot, E) \circ f$ to deduce $\|V\,\partial F\|(X) = 0$. Abbreviating $W = V \llcorner F \times \mathbf{G}(n, 1)$, this implies

$$\|\delta W\|(X) < \infty \quad \text{and} \quad \|\delta W\|(X \sim \{a\}) \leq 1/2.$$

Choose $\chi \in A \cap \operatorname{spt}\|W\|$ such that $\chi = a$ if $a \in \operatorname{spt}\|W\|$. By [Men15, 4.8 (1) (4)] with U, V, and a replaced by X, $W | \mathbf{2}^{X \times \mathbf{G}(n,1)}$, and χ one obtains that

$$\|W\|(X) \geq \|W\|\,\mathbf{U}(\chi, 1/2) \geq 1/4.$$

Since $\operatorname{diam} E \leq \mathscr{L}^1(E) \leq 2^4 M s$, integrating the inequality

$$(\|W\| \llcorner X)_{(\infty)}(f) \leq |f(x)| + \operatorname{diam} E \quad \text{for } \|W\| \text{ almost all } x \in X$$

over X now yields the conclusion. □

In the "interior" the corresponding estimate is much simpler and obtainable in all dimensions by localising the Sobolev inequality [Men15, 10.1 (2)].

7.2 Theorem. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$, $\Lambda = \Gamma_{\text{[Men15, 10.1]}}(n)$, $\psi(U) \leq \Lambda^{-1}$, $0 < r < \infty$, $A = \{x : \mathbf{U}(x, r) \subset U\}$,*

(1) *either $m = q = 1$, $\alpha = \infty$, and $\kappa = \Lambda$,*

(2) *or $1 \leq q < m$, $\alpha = mq/(m-q)$, and $\kappa = \Lambda(m-q)^{-1}$,*

(3) *or $1 < m < q$, $\alpha = \infty$, and $\kappa = \Lambda^{1/(1/m-1/q)}\|V\|(U)^{1/m-1/q} < \infty$,*

and Y is a finite dimensional normed vectorspace.

Then there holds

$$(\|V\| \llcorner A)_{(\alpha)}(f) \leq \kappa\big(\|V\|_{(q)}(V\,\mathbf{D}\,f) + r^{-1}\|V\|_{(q)}(f)\big) \quad \textit{for } f \in \mathbf{T}(V, Y).$$

Proof. Assume $Y = \mathbf{R}$ and $f \geq 0$ by [Men15, 8.16] and $\gamma = r^{-1}\|V\|_{(q)}(f) + \|V\|_{(q)}(V\,\mathbf{D}\,f) < \infty$. Whenever $g : U \to \mathbf{R}$ is a nonnegative Lipschitzian function with compact support, $\sup \operatorname{im} g \leq 1$ and $\operatorname{Lip} g \leq r^{-1}$, one infers from [Men15, 8.20 (4), 9.2, 9.4] that

$$gf \in \mathbf{T}_{\operatorname{Bdry} U}(V) \quad \text{and} \quad \|V\|_{(q)}(V\,\mathbf{D}\,(gf)) \leq \gamma,$$

hence [Men15, 10.1 (2b) (2c) (2d)] implies $\|V\|_{(\alpha)}(gf) \leq \kappa\gamma$. $\square$

Next, estimates of the Lebesgue seminorm of a generalised weakly differentiable function with respect to the variation measure of the first variation of the varifold will be studied.

7.3 Lemma. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$, $\Lambda = \Gamma_{[\text{Men15, 10.1}]}(n)$, $1 \leq q \leq \infty$, $f \in \mathbf{T}_{\operatorname{Bdry} U}(V)$, and*

$$\|V\|(\{x : f(x) > y\}) < \infty \quad \text{for } 0 < y < \infty, \qquad \psi(\{x : f(x) > 0\}) \leq \Lambda^{-1},$$
$$\|V\|_{(q)}(f) + \|V\|_{(q)}(V\,\mathbf{D}\,f) < \infty.$$

Then there holds

$$\|\delta V\|_{(q)}(f) \leq \Gamma \|V\|_{(q)}(f)^{1-1/q} \|V\|_{(q)}(V\,\mathbf{D}\,f)^{1/q},$$

where $\Gamma = 2(1+\Lambda)$ and $0^0 = 1$.

Proof. One may assume $q < \infty$ by [Men15, 8.33] and that f is bounded by [Men15, 8.12, 8.13 (4), 9.9]. Define $\alpha = \infty$ if $m = 1$ and $\alpha = qm/(m-1)$ if $m > 1$. Then Hölder's inequality, in case $m = 1$ in conjunction with [Men15, 8.33], implies

$$\|\delta V\|_{(q)}(f) \leq \psi(\{x : f(x) > 0\})^{1/(mq)} \|V\|_{(\alpha)}(f) \leq \Lambda^{-1/(mq)} \|V\|_{(\alpha)}(f)$$

and applying [Men15, 10.1 (2b) (2c)] with f and q replaced by f^q and 1 in conjunction with [Men15, 8.6, 9.9] yields

$$\|V\|_{(\alpha)}(f) \leq (\Lambda q)^{1/q} \|V\|_{(1)}(f^{q-1} V\,\mathbf{D}\,f)^{1/q}$$

and the conclusion follows from Hölder's inequality and the fact $q^{1/q} \leq 2$. $\square$

7.4 Remark. In particular, one infers

$$\|\delta V\|_{(q)}(f) \leq \Gamma\big(\|V\|_{(q)}(f) + \|V\|_{(q)}(V\,\mathbf{D}\,f)\big).$$

Localising the preceding estimate, one obtains the following theorem.

7.5 Theorem. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$, $\Lambda = \Gamma_{[\text{Men15, 10.1}]}(n)$, $\psi(U) \leq \Lambda^{-1}$, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace, $f \in \mathbf{T}(V, Y)$, $0 < r < \infty$, and $A = \{x : \mathbf{U}(x, r) \subset U\}$.*

Then there holds

$$(\|\delta V\| \llcorner A)_{(q)}(f) \leq \Gamma\big(r^{-1/q}\|V\|_{(q)}(f) + r^{1-1/q}\|V\|_{(q)}(V\,\mathbf{D}\,f)\big),$$

where $\Gamma = 4(1+\Lambda)$.

Proof. Assume firstly $r = 1$, secondly $Y = \mathbf{R}$ and $f \geq 0$ by [Men15, 8.16] and thirdly $\gamma = \|V\|_{(q)}(f) + \|V\|_{(q)}(V \mathbf{D} f) < \infty$. Whenever $g : U \to \mathbf{R}$ is a nonnegative Lipschitzian function with compact support, $\sup \operatorname{im} g \leq 1$ and $\operatorname{Lip} g \leq 1$, one infers from [Men15, 8.20 (4), 9.2, 9.4] that

$$gf \in \mathbf{T}_{\operatorname{Bdry} U}(V) \quad \text{and} \quad \|V\|_{(q)}(gf) + \|V\|_{(q)}(V \mathbf{D}(gf)) \leq 2\gamma,$$

hence 7.3 and 7.4 imply $\|V\|_{(\alpha)}(gf) \leq \Gamma\gamma$. □

If $U = \mathbf{R}^n$, a global estimate under a weaker condition on ψ will be proven.

7.6 Theorem. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$, $U = \mathbf{R}^n$, and $\psi(\mathbf{R}^n) < \infty$.*

Then there exists a positive, finite number Γ with the following property.

If $1 \leq q \leq \infty$ and Y is a finite dimensional normed vectorspace, then

$$\|\delta V\|_{(q)}(f) \leq \Gamma\big(\|V\|_{(q)}(f) + \|V\|_{(q)}(V \mathbf{D} f)\big) \quad \textit{for } f \in \mathbf{T}(V, Y).$$

Proof. Define $\Delta_1 = \Gamma_{[\text{Men15, 10.1}]}(n)$. Choose a compact subset K of $\mathbf{R}^n$ such that $\psi(\mathbf{R}^n \sim K) \leq \Delta_1^{-1}$. If $m = 1$ then let

$$M = \sup\big(\{n\} \cup \{r^{-1}\|V\| \mathbf{U}(a, 2r) : a \in K, 0 < r \leq 1\}\big),$$

hence $M < \infty$ by [Men15, 4.8 (1)], and define $\Delta_2 = 2 + \Gamma_{[\text{Men15, 10.1}]}(1 + M)$. If $m > 1$ then let $\Delta_2 = \Delta_1$. Next, choose $k \in \mathscr{P}$ and $a_j \in K$, $0 < r_j \leq 1$ for $j = 1, \ldots, k$ such that

$$\|V\| \mathbf{U}(a_j, 2r_j) \leq M\boldsymbol{\alpha}(1) r_j \quad \text{if } m = 1,$$
$$\psi(\mathbf{U}(a_j, 2r_j) \sim \{a_j\}) \leq \inf\{\Delta_1^{-1}, \Delta_2^{-1}\}, \quad K \subset \bigcup\{\mathbf{U}(a_j, r_j) : j = 1, \ldots, k\},$$

and a closed subset A of $\mathbf{R}^n \sim K$ such that $\mathbf{R}^n = A \cup \bigcup_{j=1}^k \mathbf{U}(a_j, r_j)$. Abbreviating

$$\begin{aligned} \Delta_3 &= \sup\big\{2^5 M^2 \Delta_2(1 + \psi(\mathbf{R}^n)), 4(1 + \Delta_1)\big\}, && \text{if } m = 1, \\ \Delta_3 &= 4(1 + \Delta_1), && \text{if } m > 1, \\ r &= \inf\big(\{\operatorname{dist}(A, K)\} \cup \{r_j : j = 1, \ldots, k\}\big), \end{aligned}$$

define $\Gamma = (k+1)\Delta_3 r^{-1}$.

Suppose $1 \leq q \leq \infty$ and Y is a finite dimensional normed vectorspace.

Then one obtains

$$(\|\delta V\| \llcorner \mathbf{U}(a_j, r_j))_{(q)}(f) \leq \Delta_3\big(r^{-1}\|V\|_{(q)}(f) + \|V\|_{(q)}(V \mathbf{D} f)\big)$$

for $j = 1, \ldots, k$ from 7.1, [Men15, 8.33] and Hölder's inequality if $m = 1$ and from 7.5 if $m > 1$. Moreover, 7.5 yields

$$(\|V\| \llcorner A)_{(q)}(f) \leq \Delta_3\big(r^{-1}\|V\|_{(q)}(f) + \|V\|_{(q)}(V \mathbf{D} f)\big).$$

Summing the preceding inequalities yields the conclusion. □

Turning to local Sobolev spaces, firstly two properties of their topologies implied by the preceding estimates are gathered.

7.7 Theorem. *Suppose m, n, p, U, and V are as in 6.1, $p = m$, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace, and $\sigma_K : \mathbf{H}_q^{\mathrm{loc}}(V, Y) \to \mathbf{R}$ satisfy*

$$\sigma_K(f) = (\|V\| \llcorner K)_{(q)}(f) + (\|V\| \llcorner K)_{(q)}(V \, \mathbf{D} \, f) \quad \text{for } f \in \mathbf{H}_q^{\mathrm{loc}}(V, Y)$$

whenever K is a compact subset of U.

Then the following two statements hold.

(1) The topology of $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ is induced by the seminorms σ_K corresponding to all compact subsets K of U.

(2) The set of

$$(f, F) \in \mathbf{L}_q^{\mathrm{loc}}(\|V\|, Y) \times \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$$

such that some $g \in \mathbf{H}_q^{\mathrm{loc}}(V, Y)$ satisfies

$$f(x) = g(x) \quad \text{and} \quad F(x) = V \, \mathbf{D} \, g(x) \quad \text{for } \|V\| \text{ almost all } x$$

is closed in $\mathbf{L}_q^{\mathrm{loc}}(\|V\|, Y) \times \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$.

Proof. Noting 2.4 and 5.2, and in case $m = 1$ also [Men15, 4.8 (2)], one may employ 7.1, [Men15, 8.33], and Hölder's inequality if $m = 1$ and 7.5 if $m > 1$ to verify (1). Moreover, 5.8 and (1) yield (2). ☐

7.8 Remark. If V satisfies the hypotheses of 7.7, $\|\delta V\|$ is not absolutely continuous with respect to $\|V\|$ and $\|\delta V\|(\{x\}) = 0$ for $x \in U$, then the set occurring in (2) contains pairs (f, F) with f being $\|\delta V\|$ nonmeasurable by [Fed69, 2.2.4, 2.9.2]. Such V do exist if and only if $m = 1$, see [Men15, 12.3].

Continuous embeddings into local Lebesgue spaces closely resemble the behaviour of Sobolev spaces in the Euclidean case.

7.9 Theorem. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$,*

(1) either $m = q = 1$ and $\alpha = \infty$,

(2) or $1 \leq q < m$ and $\alpha = mq/(m - q)$,

(3) or $1 < m < q$ and $\alpha = \infty$,

and Y is a finite dimensional normed vectorspace.

Then $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ embeds continuously into $\mathbf{L}_\alpha^{\mathrm{loc}}(\|V\|, Y)$.

Proof. In view of 2.4, 2.5, 2.12, 5.13, and [DS58, II.1.14], it is enough to show that bounded sets in $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ are bounded in $\mathbf{L}_\alpha^{\mathrm{loc}}(\|V\|, Y)$. Noting 5.2 and in case $m = 1$ also [Men15, 4.8 (2)], this is a consequence of 7.1 and Hölder's inequality if $m = 1$ and of 7.2 (2) (3) if $m > 1$. ☐

7.10 Remark. In case of (1) or (3) a continuous embedding of $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ into $\mathscr{C}(\operatorname{spt} \|V\|, Y)$ will be constructed in 7.12.

In combination with the Rellich type embedding result for generalised weakly differentiable functions, see 4.8, one directly infers a Rellich type embedding for local Sobolev functions.

7.11 Theorem. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$,*

(1) *either* $m = 1$, $1 \leq q \leq \infty$, *and* $1 \leq \alpha < \infty$,

(2) *or* $1 \leq q < m$ *and* $1 \leq \alpha < mq/(m-q)$,

and Y *is a finite dimensional normed vectorspace.*

Then bounded subsets of $\mathbf{H}_q^{\mathrm{loc}}(V,Y)$ *have compact closure in* $\mathbf{L}_\alpha^{\mathrm{loc}}(\|V\|,Y)$.

Proof. Noting 5.2 and 7.9 (1) (2), the conclusion is a consequence of 4.8 and Hölder's inequality. □

The following embedding theorem into continuous functions rests on the oscillation estimate [Men15, 13.1]; the absence of an embedding into locally Hölder continuous functions shows a notable difference to the Euclidean case, see 7.13.

7.12 Theorem. *Suppose* m, n, p, U, *and* V *are as in 6.1,* $p = m$, *either* $1 = m \leq q$ *or* $1 < m < q$, *and* Y *is a finite dimensional normed vectorspace.*

Then for every $f \in \mathbf{H}_q^{\mathrm{loc}}(V,Y)$ *there exists* $L(f) \in \mathscr{C}(\operatorname{spt}\|V\|,Y)$ *uniquely characterised by*

$$L(f)(x) = f(x) \quad \textit{for } \|V\| + \|\delta V\| \textit{ almost all } x$$

and L *is a continuous linear map. Moreover, if additionally* $q > 1$, *then* L *maps bounded subsets of* $\mathbf{H}_q^{\mathrm{loc}}(V,Y)$ *onto sets with compact closure in* $\mathscr{C}(\operatorname{spt}\|V\|,Y)$.

Proof. To prove the existence of $L(f)$, suppose $f \in \mathbf{H}_q^{\mathrm{loc}}(V,Y)$ and K is a compact subset of U. Choose a compact subset C of U with $K \subset \operatorname{Int} C$ and locally Lipschitzian functions $f_i : U \to Y$ with

$$((\|V\| + \|\delta V\|) \llcorner C)_{(q)}(f_i - f) + (\|V\| \llcorner C)_{(q)}(V\,\mathbf{D}\,(f_i - f)) \to 0 \quad \text{as } i \to \infty.$$

By [Fed69, 2.3.10] one may assume

$$(\|V\| + \|\delta V\|)(K \sim A) = 0, \quad \text{where } A = K \cap \Big\{x : \lim_{i\to\infty} f_i(x) = f(x)\Big\}.$$

The family $\{f_i | K \cap \operatorname{spt}\|V\| : i \in \mathscr{P}\}$ is equicontinuous by [Men15, 4.8 (1), 13.1], hence $f | A \cap \operatorname{spt}\|V\|$ is uniformly continuous.

The continuity of L follows from 7.9 (1) (3) as $\mathscr{C}(\operatorname{spt}\|V\|,Y)$ is homeomorphically included in $\mathbf{L}_\infty^{\mathrm{loc}}(\|V\|,Y)$.

To prove the postscript, suppose $q > 1$ and B is a bounded subset of $\mathbf{H}_q^{\mathrm{loc}}(V,Y)$, hence $L[B]$ is bounded in $\mathscr{C}(\operatorname{spt}\|V\|,Y)$. Moreover, 5.2 and [Men15, 4.8 (1), 13.1] yield that the family $\{L(f)|K : f \in B\}$ is equicontinuous whenever K is a compact subset of $\operatorname{spt}\|V\|$. Consequently, the conclusion follows from the Ascoli theorem, see [Kel75, 7.14, 7.18]. □

7.13 Remark. Notice that some functions $L(f)$ may not be locally Hölder continuous with any exponent by 6.8 (2) and 6.11.

7.14 Remark. Simple examples, see [Men15, 5.5, 5.6], show that not all members of $f \in \mathbf{T}(V,Y)$ with

$$((\|V\| + \|\delta V\|) \llcorner K)_{(q)}(f) + (\|V\| \llcorner K)_{(q)}(V\,\mathbf{D}\,f) < \infty$$

whenever K is a compact subset of U admit a continuous function $g : \operatorname{spt}\|V\| \to Y$ which is $\|V\| + \|\delta V\|$ almost equal to f.

7.15 Remark. 7.9, 7.11, and 7.12 are most useful in conjunction with the alternate description of the topology of $\mathbf{H}_q^{\mathrm{loc}}(V, Y)$ obtained in 7.7 (1).

Sometimes, the seminormed space E defined in the following theorem may function as alternate substitute to $\mathbf{H}_q(V, Y)$, for the Euclidean Sobolev space.

7.16 Theorem. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace, $\sigma : \mathbf{H}_q^{\mathrm{loc}}(V, Y) \to \overline{\mathbf{R}}$ satisfies*

$$\sigma(f) = \|V\|_{(q)}(f) + \|V\|_{(q)}(V\,\mathbf{D}\,f) \quad \textit{for } f \in \mathbf{H}_q^{\mathrm{loc}}(V, Y),$$

and $E = \{f : \sigma(f) < \infty\}$.

Then the following six statements hold.

1. *The vectorspace E is σ complete.*
2. *The subspace $Y^U \cap \{g : g$ locally Lipschitzian, $\sigma(g) < \infty\}$ is σ dense in E.*
3. *If $q < \infty$, then $\mathscr{E}(U, Y) \cap \{g : \sigma(g) < \infty$, $\operatorname{spt} g$ bounded$\}$ is σ dense in E.*
4. *If $q < \infty$, then E is σ separable.*
5. *If $U = \mathbf{R}^n$ and $\psi(\mathbf{R}^n) < \infty$, then $E = \mathbf{H}_q(V, Y)$.*
6. *The set of*

$$(f, F) \in \mathbf{L}_q(\|V\|, Y) \times \mathbf{L}_q(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$$

such that there exists $g \in E$ satisfying

$$f(x) = g(x) \quad \textit{and} \quad F(x) = V\,\mathbf{D}\,g(x) \quad \textit{for } \|V\| \textit{ almost all } x$$

is closed in $\mathbf{L}_q(\|V\|, Y) \times \mathbf{L}_q(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$.

Proof. (1) follows from 5.8 and 7.7 (1). Replacing $\mathbf{H}_q(V, Y)$ and $\mathbf{H}_q(V, \cdot)$ by E and σ in the argument of 5.16, a proof of (2) and (3) results. 2.1 implies (4). 5.2 and 7.6 yield (5). (6) follows from 7.7 (2). $\square$

7.17 Remark. In view of 2.5 and 7.16 (1), the quotient space

$$Q = E \big/ \{f : \sigma(f) = 0\}$$

is a Banach space normed by $\sigma \circ \pi^{-1}$, where $\pi : E \to Q$ denotes the canonical projection. If $1 < q < \infty$, then Q is reflexive by 2.6, 7.16 (6) and [DS58, II.3.23].

Turning to Sobolev spaces with "zero boundary values", firstly the corresponding Sobolev inequalities are stated.

7.18 Theorem. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$, $\Lambda = \Gamma_{[\text{Men15, 10.1}]}(n)$, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace, $f \in \mathbf{H}_q^\diamond(V, Y)$, and $E = U \sim \{x : f(x) = 0\}$.*

Then the following three statements hold.

1. *If $m = 1$ and $\psi(E) \leq \Lambda^{-1}$, then*

$$\|V\|_{(\infty)}(f) \leq \Lambda\, \|V\|_{(1)}(V\,\mathbf{D}\,f).$$

(2) *If* $1 \le \alpha < m$ *and* $\psi(E) \le \Lambda^{-1}$, *then*

$$\|V\|_{(m\alpha/(m-\alpha))}(f) \le \Lambda (m-\alpha)^{-1} \|V\|_{(\alpha)}(V \mathbf{D} f).$$

(3) *If* $1 < m < \alpha \le \infty$ *and* $\psi(E) \le \Lambda^{-1}$, *then*

$$\|V\|_{(\infty)}(f) \le \Lambda^{1/(1/m-1/\alpha)} \|V\|(E)^{1/m-1/\alpha} \|V\|_{(\alpha)}(V \mathbf{D} f),$$

here $0 \cdot \infty = \infty \cdot 0 = 0$.

Proof. In view of 5.25, 5.27 and [Men15, 8.16], the conclusion is a consequence of [Men15, 10.1 (2b) (2c) (2d)] with q replaced by α. ☐

In relation to the topology of $\mathbf{H}^\diamond_q(V,Y)$, also the following consequence of the Sobolev inequalities becomes relevant.

7.19 Theorem. *Suppose* m, n, p, U, V, *and* ψ *are as in 6.1,* $p = m$, $\Lambda = \Gamma_{[\text{Men15, 10.1}]}(n)$, $1 \le q \le \infty$, Y *is a finite dimensional normed vectorspace,* $f \in \mathbf{H}^\diamond_q(V,Y)$, *and* $\psi(U \sim \{x : f(x) = 0\}) \le \Lambda^{-1}$.

Then there holds

$$\|\delta V\|_{(q)}(f) \le \Gamma\big(\|V\|_{(q)}(f) + \|V\|_{(q)}(V \mathbf{D} f)\big),$$

where $\Gamma = 2(1+\Lambda)$.

Proof. In view of 5.25, 5.27, and [Men15, 8.16], the conclusion is a consequence 7.3 and 7.4. ☐

In the Euclidean case the following theorem is a direct consequence of a suitable Poincaré inequality; here also 7.19 enters.

7.20 Theorem. *Suppose* m, n, p, U, V, *and* ψ *are as in 6.1,* $p = m$, $\|V\|(U) < \infty$, $\Lambda = \Gamma_{[\text{Men15, 10.1}]}(n)$, $\psi(U) \le \Lambda^{-1}$, $1 \le q \le \infty$, Y *is a finite dimensional normed vectorspace, and* $\tau : \mathbf{H}^\diamond_q(V,Y) \to \mathbf{R}$ *satisfies*

$$\tau(f) = \|V\|_{(q)}(V \mathbf{D} f) \quad \text{for } f \in \mathbf{H}^\diamond_q(V,Y).$$

Then the topology of $\mathbf{H}^\diamond_q(V,Y)$ *is induced by* τ.

Proof. This is a consequence of 7.18, 7.19, and Hölder's inequality. ☐

Finally, the Sobolev inequalities, see 7.18, and the Rellich type embedding for generalised weakly differentiable functions, see 4.8, combine to the following Rellich type embedding for Sobolev functions with "zero boundary values".

7.21 Theorem. *Suppose* m, n, p, U, V, *and* ψ *are as in 6.1,* $p = m$, $\|V\|(U) < \infty$, $\Lambda = \Gamma_{[\text{Men15, 10.1}]}(n)$, $\psi(U) \le \Lambda^{-1}$,

(1) *either* $m = 1$, $1 \le q \le \infty$, *and* $1 \le \alpha < \infty$,

(2) *or* $1 \le q < m$ *and* $1 \le \alpha < mq/(m-q)$,

and Y *is a finite dimensional normed vectorspace.*

Then $\mathbf{H}^\diamond_q(V,Y)$ *embeds compactly into* $\mathbf{L}_\alpha(\|V\|,Y)$.

Proof. Bounded subsets of $\mathbf{H}^\diamond_q(V,Y)$ are bounded in $\mathbf{L}_\infty(\|V\|,Y)$ if $m = 1$ by 7.18 (1) and bounded in $\mathbf{L}_{mq/(m-q)}(\|V\|,Y)$ if $m > 1$ by 7.18 (2). Therefore 4.8 and Hölder's inequality imply the conclusion. ☐

7.22 Remark. The preceding corollary is most useful in conjunction with the alternate description of the topology of $\mathbf{H}^\diamond_q(V,Y)$ obtained in 7.20.

8 Comparison to other Sobolev spaces

In this section, the notion of Sobolev space developed in the present paper will be compared to the notion of strong Sobolev space for finite Radon measures from Bouchitté, Buttazzo and Fragalà in [BBF01], see 8.2–8.4, and the space $\mathbf{W}(V, \mathbf{R})$ defined in [Men15, 8.28] will be compared to the weak Sobolev space for finite Radon measures introduced in Bouchitté, Buttazzo and Fragalà in [BBF01], see 8.5. In both cases, this mainly amounts to relating the two notions of tangent spaces involved by means of the results of Fragalà and Mantegazza in [FM99].

The following space occurs in the pointwise variant of the definition of the tangent space of Bouchitté, Buttazzo and Fragalà in [BBF01] given below.

8.1. Suppose X is a locally compact, separable metric space and

$$E = \mathscr{C}(X) \cap \{f : \operatorname{Lip} f \leq 1\}.$$

Then $\mathscr{C}(X)$ and $\mathbf{R}^X$ endowed with the Cartesian product topology induce the same metrisable topology on E, see 2.4, 2.14, and [Fed69, 2.10.21]. Also, E is separable by 2.15 and [Men15, 2.2, 2.23]. Consequently, the Borel family of E is generated by the sets $E \cap \{f : f(x) < t\}$ corresponding to all $x \in X$ and $t \in \mathbf{R}$.

For finite Radon measures the concept of tangent space of Bouchitté, Buttazzo and Fragalà in [BBF01] is implemented as follows.

8.2. Suppose μ is a Radon measure over $\mathbf{R}^n$ with $\mu(\mathbf{R}^n) < \infty$, $1 \leq q \leq \infty$, $1 \leq r \leq \infty$, $1/q + 1/r = 1$, and $C = \{(a, \mathbf{B}(a, r)) : a \in \mathbf{R}^n,\ 0 < r < \infty\}$. Define Z to be the vector subspace of $\mathbf{L}_r(\mu, \mathbf{R}^n)$ consisting of those $\theta \in \mathbf{L}_r(\mu, \mathbf{R}^n)$ such that there exists $0 \leq \kappa < \infty$ satisfying

$$\textstyle\int \langle \theta, \mathrm{D}\, \zeta \rangle \,\mathrm{d}\mu \leq \kappa\, \mu_{(q)}(\zeta) \quad \text{for } \zeta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}).$$

Define vector subspaces of $\mathbf{R}^n$ by

$$P(x) = \mathbf{R}^n \cap \left\{ u : u = (\mu, C) \operatorname*{ap\,lim}_{\chi \to x} \theta(\chi) \text{ for some } \theta \in Z \right\} \quad \text{for } x \in \mathbf{R}^n.$$

Whenever D is a nonempty countable $|\cdot|_\mu$ dense subset of Z there holds

$$P(x) = \operatorname{Clos}\{\theta(x) : \theta \in D\} \quad \text{for } \mu \text{ almost all } x;$$

in fact, defining E as in 8.1 with $X = \mathbf{R}^n$ and $g : \mathbf{R}^n \to E$ by

$$g(x)(u) = \inf\{|u - \theta(x)| : \theta \in D\} \quad \text{for } x, u \in \mathbf{R}^n,$$

g is μ measurable and one observes that the equation holds at x whenever g and all members of D are (μ, C) approximately continuous at x which is the case for μ almost all x by [Fed69, 2.8.18, 2.9.13]. Consequently, in view of [CV77, p. 59], the function P is a representative of the equivalence class introduced under the name "T^q_μ" by Bouchitté, Buttazzo and Seppecher in [BBS97, p. 38] in case $1 < q < \infty$ and by Bouchitté, Buttazzo and Fragalà in [BBF01, p. 403] for all q. There exists a nonzero Radon measure μ such that the vector subspace $P(x)$

depends at μ almost all points x on the parameter q involved in its definition; in fact, let $q < s < \infty$ and consider a Radon measure μ over $\mathbf{R}$ with

$$\mu \leq \mathscr{L}^1 \llcorner \mathbf{U}(0,1), \quad \textstyle\int \mathbf{\Theta}^1(\mu, x)^{1/(1-s)} \,\mathrm{d}\mathscr{L}^1 x < \infty,$$
$$\textstyle\int_a^b \mathbf{\Theta}^1(\mu, x)^{1/(1-q)} \,\mathrm{d}\mathscr{L}^1 x = \infty \quad \text{whenever } -1 \leq a < b \leq 1.$$

The latter argument is a variant of Di Marino and Speight [DMS15, Theorem 1]. (The reader interested in relating the approach of Bouchitté, Buttazzo and Seppecher [BBS97, Theorem 3] to concepts of Sobolev spaces on metric measure spaces should consult Ambrosio, Gigli, and Savaré [AGS13] where the equivalence of several of the latter concepts is established.)

8.3. Continuing 8.2, results concerning the question which conditions on $V \in \mathbf{RV}_m(\mathbf{R}^n)$ guarantee

$$P(x) = \operatorname{Tan}^m(\|V\|, x) \quad \text{for } \|V\| \text{ almost all } x,$$

where P is defined with reference to $\mu = \|V\|$, will be briefly summarised. Firstly, in view of Allard [All72, 3.5 (1)], Fragalà and Mantegazza [FM99, 2.4] implies

$$P(x) \subset \operatorname{Tan}^m(\|V\|, x) \quad \text{for } \|V\| \text{ almost all } x$$

whenever $\mu = \|V\|$ for some $n \geq m \in \mathscr{P}$, $V \in \mathbf{RV}_m(\mathbf{R}^n)$ with $\|V\|(\mathbf{R}^n) < \infty$. If additionally $\|\delta V\|$ is a Radon measure and $\mathbf{h}(V, \cdot) \in \mathbf{L}_r^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ with

$$(\delta V)(\theta) = -\textstyle\int \mathbf{h}(V, x) \bullet \theta(x) \,\mathrm{d}\|V\| x \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n),$$

then equality holds; in fact, whenever $u \in \mathbf{R}^n$ and $\varrho \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ the function mapping $x \in \mathbf{R}^n$ with $\operatorname{Tan}^m(\|V\|, x) \in \mathbf{G}(n, m)$ onto $\varrho(x) \operatorname{Tan}^m(\|V\|, x)_\natural(u)$ belongs to Z. The latter argument is a variant of Fragalà and Mantegazza [FM99, 3.8].

Now, under suitable conditions, the presently introduced Sobolev space may be identified with the strong Sobolev space of Bouchitté, Buttazzo, and Fragalà [BBF01].

8.4. If m, n, U, V, and ψ are as in 6.1, $p = m$, $U = \mathbf{R}^n$, $\|V\|(\mathbf{R}^n) < \infty$, $1 \leq q < \infty$,

$$P(x) = \operatorname{Tan}^m(\|V\|, x) \quad \text{for } \|V\| \text{ almost all } x,$$

where $P(x)$ is related to $\mu = \|V\|$ and q as in 8.2, and σ, Q, and π are as in 7.16 and 7.17, then $H^{1,q}_{\|V\|}(\mathbf{R}^n)$ with notion of derivative $\nabla_{\|V\|}$ and norm $\|\cdot\|_{1,q,\|V\|}$ defined by Bouchitté, Buttazzo, and Fragalà in [BBF01, p. 403] is isometrically isomorphic to Q with notion of derivative induced by $V\,\mathbf{D}$ and norm $\sigma \circ \pi^{-1}$ by 5.5 and 7.16 (1) (3). It appears to be unknown whether the condition on P is redundant, see 8.3.

Finally, turning to the weak Sobolev space of Bouchitté, Buttazzo, and Fragalà [BBF01], a comparison may be given as follows.

8.5. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_m(U)$, $\|V\|(\mathbf{R}^n) < \infty$, $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$, and $\mathbf{h}(V, \cdot) \in \mathbf{L}_\infty(\|V\|, \mathbf{R}^n)$. In particular, 8.3 implies

$$P(x) = \operatorname{Tan}^m(\|V\|, x) \quad \text{for } \|V\| \text{ almost all } x,$$

where $P(x)$ is related to $\mu = \|V\|$ and q as in 8.2. Recalling 2.6, 2.12, and [Men15, 8.27] and denoting by $D : \mathbf{W}(V, \mathbf{R}) \to L_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}))$ the notion of derivative implicit there, define the quotient space

$$\begin{gathered} Q = \big(\mathbf{W}(V, \mathbf{R}) \cap \mathbf{L}_q(\|V\|) \cap \{f : D(f) \in L_q(\|V\|, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}))\}\big)/W, \\ \text{where } W = \mathbf{W}(V, \mathbf{R}) \cap \{f : f(x) = 0 \text{ for } \|V\| \text{ almost all } x\}, \end{gathered}$$

with associated canonical projection π, and define the value of a norm on Q at f to be the sum of the $L_q(\|V\|, \mathbf{R})$ norm of f and the $L_q(\|V\|, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}))$ norm of $(D \circ \pi^{-1})(f)$. Then Q with notion of derivative $D \circ \pi^{-1}$ is isometrically isomorphic to the space $W^{1,q}_{\|V\|}(U)$ with notion of derivative $D_{\|V\|}$ defined by Bouchitté, Buttazzo, and Fragalà in [BBF01, p. 403].

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