On the Stability of Helically Symmetric $\ell = 2$ and $\ell = 3$ Equilibria

by
E. Rebhan
Max-Planck-Institut für Plasma-Physik, Garching b.München (Westgermany)

Abstract: A sufficient stability criterion is applied to helical $\ell = 2$ and $\ell = 3$ equilibria in the neighbourhood of the magnetic axis. While $\ell = 2$ configurations violate this criterion, $\ell = 3$ configurations are stable if a properly defined local ß is small enough. Rough estimates for a nonlocal critical ß yield ≈ 10 %.

We consider helically symmetric magnetohydrodynamic equilibria. All equilibrium quantities depend on s and $u = \varphi + kz$ only, where s, φ, z are cylindrical coordinates. The magnetic surfaces are F (s,u) = const., the pressure is p (F) and the magnetic field is

$$\vec{B} = \vec{w} \times \nabla F + f(F)\vec{w} , \quad \vec{w} = \frac{s}{1+k^2s^2} \nabla s \times \nabla u \quad . \tag{1}$$

The following stability criterion, derived in [1], is applied

$$\langle A \rangle \leq \max A - \frac{2}{X}\sqrt{\max A} + g\frac{X}{2}\sqrt{\max A} \tag{2}$$

where

$$A = 2q\left[f'f k^2 q s F_s - f'kq|\nabla F|^2 + (p'/q + f'f)(kqf - |\nabla F|\kappa)\right]/|\nabla F|^4 \tag{3}$$

$$q = (1+k^2s^2)^{-1}$$

$$\kappa = \left[F_u^2 F_{ss} - 2F_uF_sF_{us} + F_s^2F_{uu}\right]/(s^2q|\nabla F|^3) + \left[2F_u^2F_s/s^3 + qF_s^3/s\right]/|\nabla F|^3$$

$$X = \int_{period} du\, |\vec{w}\times\nabla F|^{-1}|\nabla F|^2 , \qquad \langle A \rangle = X^{-1}\int_{period} du\, |\vec{w}\times\nabla F|^{-1}|\nabla F|^2 A$$

A necessary condition for (2) to be satisfied is jz = 0 on the magnetic axis s = 0 [1]. With allowance for this restriction the usual expansion technique is employed:

$$f = f_0 + f_1 F + f_2\frac{F^2}{2} + \dots , \quad p = p_0 + p_1 F + p_2\frac{F^2}{2} + \dots \tag{4}$$

$$F = F_2(u)s^2 + F_3(u)s^3 + \dots$$

To lowest order one obtains for $\langle A \rangle$

$$\langle A \rangle = \frac{p_0 - p}{f_0^2 F}\,\frac{\varepsilon^2}{1-\varepsilon^2} \qquad (\varepsilon < 1) \tag{5}$$

where ε determines the ellipticity of the surfaces:

$$F_2 = \frac{kf_0}{2}s^2(1 + \varepsilon\cos 2u) \tag{6}$$

We are interested only in decaying pressure $p < p_0$, hence it follows from the equality (5) that $\langle A \rangle \geq 0$. Since the right-hand side of (2) is ≤ 0, the criterion can only be fulfilled with $\varepsilon = 0$. This means that $\ell = 2$ configurations do not satisfy the criterion.

With $\varepsilon = 0$ the equilibrium equations yield up to third order

$$F = \frac{kf_0}{2}s^2 + (a\cos 3u)s^3 \tag{7}$$

With this one obtains

$$\langle A \rangle = \frac{1}{8}\mathfrak{F}_2^2/k^2f_0^2 - 6\mathfrak{F}_2 a^2/(kf_0)^4 + 3\varepsilon k f_1 a^2/(kf_0)^4 \tag{8}$$

$$\mathfrak{F}_2 = p_2 + f_0 f_2 + f_1^2$$

The right-hand side of (2) turns out to be of the order $\sqrt{F}$ and thus $\langle A \rangle < 0$ is already sufficient for stability. Minimizing $\langle A \rangle$ with respect to $\mathfrak{F}_2$ yields the criterion

$$-p_1 < 2a^2/k^3 f_0 \tag{9}$$

Defining a local β by $\beta_\ell = (p_0 - p)/(B_0^2/2)$ gives

$$\beta_\ell < a^2s^2/(kf_0)^2 \tag{10}$$

Here a measures the derivation from the circular shape and f_0 is the field stength on the magnetic axis.

A rough extrapolation of formula (10) to plasmas with finite extent from the magnetic axis yields for the nonlocal ß values of the order of magnitude of 10 %.

[1] D.Lortz, E.Rebhan, G.Spies, Nucl.Fusion 11, (1971) 583