Mode Coupling in Dynamic Stabilization of MHD Modes

by

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Abstract: The $m = 1$ mode of the screw pinch has been stabilized by HF B_z -fields. Existing linearized theories fail to explain the measured dynamic force ($\vec{F}$), since mode coupling has to be taken into account. $\vec{F}$ is calculated on the basis of $(\vec{B}\vec{\nabla})\vec{B}$ -coupling between $m = 0$ and $m = 1$.

1. Introduction: In the Isar III linear screw pinch the $m = 1$ mode has been dynamically stabilized by HF B_z -fields ($\vec{B}_z$) [1]. A theoretical explanation of the mechanism is needed in order to minimize the HF power required.

2. Theory and experiment: The theoretical papers on the subject [2,3] and the linearized $m = 1$ equation of motion with $\vec{B}_z$, that analyze normal modes, fail to explain the experimental dynamic force ($\vec{F}_{exp}$). From the experiment a stabilizing condition $\epsilon > C \frac{\omega_1}{\omega_s}$ with $C = 2 \pm 1.5$ and $\vec{F}_{exp} = -\frac{1}{2} \rho_l \epsilon \omega_s^2 \cdot \cos \omega_s t \cdot \vec{\xi}_1$ are derived, where $\epsilon = \frac{\vec{B}_z \cdot \vec{\xi}_1}{B_{z0}}$ (experimentally ≤ 0.2), ω_1 is the $m = 1$ growth rate, ω_s the stabilizing frequency, ρ_l the line mass and $\vec{\xi}_1$ the $m = 1$ perturbation. $\vec{F}_{exp}$ results from coupling of the $m = 1$ mode to cylindrical oscillations that are induced by $\vec{B}_z$ and approximately described by $\xi_0 = \xi_{00} \cos \omega_s t$ (experimentally $\xi_{00} = \epsilon r_0$, where r_0 is the radius of an element).

Now the discrepancy between normal mode analysis and experiment shall be explained. Mode coupling in the case of no external dynamic fields can be demonstrated by a model with anharmonic $m = 0$ and $m = 1$ oscillations (other modes are not found in the experiment), which is related to the problem of $m = 1$ stabilization by $\vec{B}_z$. According to [4] the Lagrangian reads

(1) $L = \frac{1}{2} \sum_i (\dot{\xi}_i^2 - \omega_i^2 \xi_i^2) + \frac{1}{2} \sum_{ijk} \lambda_{ijk} \xi_i \xi_j \xi_k - \frac{1}{3} \sum_{ijkl} \mu_{ijkl} \xi_i \xi_j \xi_k \xi_l$
where $i, j, k = 0, 1$. The following equations of motion are obtained for $\frac{\omega_0}{\omega_1} \approx \frac{1}{\epsilon}$, when only important terms are retained

$$(2) \quad \ddot{\xi}_0 + C_1 (\ddot{\xi}_0 \xi_0 + \frac{1}{2} \dot{\xi}_0^2) + C_2 \ddot{\xi}_0 \xi_1 + \omega_0^2 \xi_0 = 0$$

$$(3) \quad \ddot{\xi}_1 + C_3 \ddot{\xi}_0 \xi_1 + \omega_1^2 \xi_1 + C_4 \ddot{\xi}_0 \xi_1 + C_5 \ddot{\xi}_0 \xi_0 + C_6 \xi_0^2 = 0$$

with $C_1 = \lambda_{000}$, $C_2 = \lambda_{001}$, $C_3 = \lambda_{110}$, $C_4 = \frac{1}{2} (\lambda_{011} + \lambda_{101})$,
 $C_5 = \frac{1}{2} (\lambda_{010} + \lambda_{100})$ and $C_6 = C_5 - C_2$.

The first and last term in (2) are of order $\frac{1}{\epsilon}$, the other terms of order $\frac{1}{\epsilon}$. Thus the $m = 0$ mode essentially is a harmonic oscillator. The mode coupling terms in (3) (last three terms) are of order $\frac{1}{\epsilon}$, whereas the terms in a $m = 1$ normal mode equation are only of order 1. Thus, a special case has been used to show, that even for small perturbations the linearized theory is not correct, if the eigenfrequencies of various modes differ by an order of magnitude. MHD analysis has to take into account anharmonic terms in δW in these cases. As far as we know, no such theory is available. One sees, that only the term $\sim \xi_0 \xi_1$ in (3) can be responsible for dynamic stabilization of the $m = 1$ and indeed, $\vec{F}_{exp}$ has this structure.

In the following a model calculation will be presented, where the essential mode coupling terms for $m = 0$ and $m = 1$ result from $(\vec{B}\vec{\nabla})\vec{B}$ -forces within the plasma. It can be deduced from geometry, that the model should yield a transfer of force (S) between the modes, that is not axially symmetric.

3. Calculation of mode coupling and discussion: The induced $m = 0$ mode acts as driving mode with so much reserve energy that the amount fed back by the $m = 1$ instability (driven mode) is negligible. S will be calculated for a sharp pressure profile, constant B_z in the plasma and homogeneous volume z -current. The coupling results from non-linear superposition of a field helix in the unperturbed plasma (radius r_0 , pitch λ , $k = \frac{2\pi}{\lambda}$) and a second field helix (radius ξ_1 , pitch λ_1 , $k_1 = \frac{2\pi}{\lambda_1}$) due to the $m = 1$ instability. The $(\vec{B}\vec{\nabla})\vec{B}$ -force on the ξ_1 -helix (axis in 1, cf. Fig.1) for a cross sectional element dq in 2 is: $F_1 = \frac{1}{4\pi} B_z^2 dq k_1^2 \xi_1 = A k_1^2 \xi_1$ and is directed from 2 to 1, i.e. it points to the position of the B_z -field line in 1 with $\xi_1 = 0$, from which the helix through 2 has been formed. An r_0 -helix through 1 (axis in z -axis) correspondingly experiences a force $F = A k^2 r_0$, which is directed to the z -axis.

Fig.1 Superposition of r_0 - and ξ_1 -helices

results in a curve through 2. The force $\vec{F}_{res}$, resulting from a superposition of the $(\vec{B}\vec{\nabla})\vec{B}$ -forces of the helices, is directed to the z -axis, since it has to point to the position of the straight field line from which the field line in 2 can be formed. F_{res} is equal to the sum of the linear component $F_L = F + F_1$ and the non-linear component F_N . In the case $\omega_s \gg \omega_1$, F_N causes only small corrections of the normal modes. Therefore, it should be permissible to put $F_{res} = F_L$ in zeroth order. The coupling force F_N can be derived from Fig.1

(4) $F_N = (F_1 - F_L) \frac{\sin(\varphi - \theta - \Omega)}{\cos \delta/2}$ where $\delta = \Omega - \epsilon$. For $k \neq k_1$ the $(\vec{B}\vec{\nabla})\vec{B}$ -forces of the helices are non-linearly superposed. $\frac{F_1}{F} \cdot \frac{\xi_1}{r_0}$ represents the fraction of $\vec{F}$ which is transferred to the ξ_1 -direction. Normalization to F yields for $k \geq k_1$

$$(5) \quad S = [1 - (\frac{k_1}{k})^2] \frac{\xi_1}{r_0} \frac{\sin(\varphi - \theta - \Omega)}{\cos \frac{\delta}{2} (\Omega - \epsilon)} \frac{\sin[\varphi - \theta - \frac{1}{2}(\Omega + \epsilon)]}{\cos \frac{\delta}{2} (\Omega - \epsilon)}$$

$$\text{where } \tan \Omega = \frac{\xi_1 \sin(\varphi - \theta)}{r_0 + \xi_1 \cos(\varphi - \theta)} \text{ and } \tan \epsilon = \frac{\xi_1 \sin(\varphi - \theta)}{(k/k_1)^2 r_0 + \xi_1 \cos(\varphi - \theta)}$$

For $k \geq k_1$ and $\frac{\xi_1}{r_0} \ll 1$, which holds experimentally for the bulk of the plasma, $S \sim \sin^2(m - \theta)$, i.e. it is not axially symmetric. Thus, the destabilizing $m = 1$ force can be transferred to the $m = 0$ oscillation due to asymmetric $(\vec{B}\vec{\nabla})\vec{B}$ -coupling and is compensated by the coil. The $\vec{B}_z$ -field produces a force $dm \cdot \vec{\xi}_0$ at a mass element dm and a dynamic force per unit length parallel to $\vec{\xi}_1$

$$(6) \quad \vec{F} = \int dm \ddot{\xi}_0 S = -\frac{1}{2} [1 - (\frac{k_1}{k})^2] \rho_l g \epsilon \omega_s^2 \cos \omega_s t \cdot \vec{\xi}_1$$

where $\frac{\xi_{00}}{r_0} = g \cdot \epsilon$ with a numerical factor g has been used. $\vec{F}$ agrees well with $\vec{F}_{exp}$ in magnitude and structure ($g = 1$, $k \approx 2k_1$). For a given ξ_1 ($\ll r_0$) $S \sim \frac{1}{\epsilon}$ holds, but F is nevertheless independent of r_0 because $\xi_{00} \sim r_0$. According to this model mode coupling cannot be improved. HF power can be saved, however, by working in resonance with the $m = 0$ natural frequency ω_0 , which is rather large for hot plasmas [1] ($g > 1$, limited by $m = 0$ damping).

References

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