Time Evolution and Bifurcation of Temperature Profiles

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Introduction. Multiple stationary solutions of the one-dimensional heat conduction equation and bifurcation phenomena are known to be caused by the non-monotonic dependence of the impurity radiation function on the temperature [1], and the non-linearity of the heat flux in the high-recycling regime in front of the divertor target plates [2]. In this paper we report on bifurcation and time evolution of temperature profiles owing to localized heat sources and energy loss due to impurity radiation which are described by simple Gaussian model functions. The main result is that in dependence on the impurity density there may exist one, two or three stationary profiles. A linear stability analysis of stationary temperature profiles leads to a Schrödinger-type equation for the temperature perturbation. The solution of this equation shows that one of three temperature profiles is unstable. Finaly the fully time-dependent problem is solved.

Problem. Solutions to the following problem for the time dependent one-dimensional diffusion equation for the temperature $T(x,t)$ are considered:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} + s(x) - q(T), \ x \in X := [0,1], t \in [0,\infty) \tag{1}$$

$$T(x,0) = T_1(x) \ (a); \ \ T(0,t) = c_0, \ T(1,t) = c_1 \ (b). \tag{2}$$

T_1 is a given function characterizing the initial condition (2a); (2b) represents a simple Dirichlet boundary (2) condition with the constants $c_{0,1}$. κ is the constant heat conductivity. The localized heat source function $s(x)$ and temperature dependent radiation loss function $q(T)$ are assumed to be given by the simple Gaussian model expressions:

$$s(x) := f_s \exp\left[-\frac{(x-x^*)^2}{\Delta_s}\right], \ q(T) := f_q \exp\left[-\frac{(T-T^*)^2}{\Delta_q}\right] \tag{3}$$

where the $f_{s,q}$ denote the strengths of the sources, x^*, T^* their localization and $\Delta_{s,q}$ their widths. Dimesionless quantities are used in what follows.

Stationary solution and bifurcation. Bifurcation of steady solutions $T_0(x)$ are considered. In order to elucidate its mechanism we analyse the simplified case $\Delta_s >> 1$ of a nearly constant source function ($s \simeq f_s$). Then the steady solution of (1) is implicitly given by

$$\left[\theta\left(x_{max} - x\right) \int_{\tau_0}^{\tau} \ + \ \theta\left(x - x_{max}\right)\left(\int_{\tau_0}^{1} + \int_{\tau}^{1}\right)\right] \ \frac{d\tau'}{\sqrt{1 - \tau' + g(\tau') - g(1)}} = ax, \tag{4}$$

$$g(\tau) := \frac{\pi}{2}\frac{f_q}{f_s}\frac{\sqrt{\Delta_q}}{T_{max}}\mathrm{erf}\left(\frac{\tau T_{max} - T^*}{\sqrt{\Delta_q}}\right), \ \tau := \frac{T_0}{T_{max}}, \ \tau_{0,1} := \frac{c_{0,1}}{T_{max}} \ a := \sqrt{\frac{2 f_s}{\kappa T_{max}}}.$$

$T_{max} \equiv T_0(x_{max})$ follows by inserting $\tau = \tau_1$, $x = 1$. ($\theta(x)$ - step function, erf(x) - error function). Its dependence on the parameters κ, $f_{q,s}$, T^*, x^*, $\Delta_{q,s}$ shows that the solution is *not unique*, in the sense that 1, 2 or 3 values may exist. Assuming $\kappa = 0.5, \Delta_q = 0.1, T^* = 1$, this multiple valuedness is demonstrated in Fig. 1 for different parameters, where the high- and low-temperature bifurcation points $(T_{max}^{(h)}, f_q^{(h)}), (T_{max}^{(l)}, f_q^{(l)})$ are determined by the relation $dT_{max}/df_q|_{f_q^{(h)}, f_q^{(l)}} = \infty$ appearing above a threshold value $f_{s,thr}$ (Fig 1a).
Stationary temperature and radiation profiles are displayed in Fig. 2. The larger the maximum temperature the more the radiation is shifted to the boundary.

Stability analysis. Suppose that the boundary value problem (2b) to eq. (1) has the *steady solution* $T = T_0(x)$; its *stability* will now be investigated. Expressing $T = T_0(x) + \delta T(x,t)$, and linearizing eq. (1) with respect to the small perturbation δT, leads to the Schrödinger-like equation

$$\frac{\partial}{\partial t}\delta T = \kappa \frac{\partial^2}{\partial x^2}\delta T - q'(T_0)\delta T, \; q'(T) := \frac{d}{dT}q(T), \; \delta T = 0 \text{ at } x = 0, 1. \tag{5}$$

where $q'(T_0)$ plays the role of the effective potential. Taking normal modes of the form $\delta T = \tilde{T}(x)e^{-\lambda t}$, the *eigenvalue problem*

$$\kappa \frac{d^2\tilde{T}}{dx^2} + [\lambda - q'(T_0)]\,\tilde{T} = 0, \; \tilde{T}(0) = 0, \tilde{T}(1) = 0 \tag{6}$$

results. Since the operators are Hermitian, the eigenvalue is real and can be found by a minimisation procedure

$$\lambda := \text{Min}\; \frac{\kappa \int_0^1 dx\; \tilde{T}'^2 + \int_0^1 dx\; q'(T_0)\tilde{T}^2}{\int_0^1 dx\; \tilde{T}^2}, \tag{7}$$

where $q' = -2(T_0 - T^*)q(T_0)/\Delta_q$ can change its sign and depends nonlinearly on the radiation factor f_q ($T_0 = T_0(x, f_q)$). Expanding $\tilde{T}$ in harmonics of $\sin \pi x$, $\tilde{T} = \sum_n a_n \sin(n\pi x)$, the dependence of λ on f_q shows that the low- and high-temperature profiles are stable but the middle-temperature branch is unstable. This is demonstrated in Fig. 3 where the effective potential and the eigenvalues λ as a function of f_q are displayed.

Time evolution. Solutions of the fully time-dependent problem (1), (2) with the initial temperature profile

$$T_1(x) = 4x(1-x)[u_0 + u_1 \sin(u_2 \pi x)] \tag{8}$$

show that in dependence of the parameters u_n (n=0,1,2) either the low- or the high-temperature state can be attained. This is demonstrated in Fig. 4.

Summary. In the frame of a simple time dependent one-dimensional model we have shown that (i) bifurcation phenomena strongly depend on the parameters of the system, especially on the impurity radiation factor f_q, (ii) multiple stationary temperature and radiation profiles may exist, (iii) in the case of three stationary states the middle-temperature branch is unstable, (iv) solutions of the fully time-dependent problem in dependence of the initial temperature profiles attain either the stationary low- or the high-temperature state.

References.
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[2] D. Sünder, H. Wobig, 20th EPS, Lisboa 1993, Contrib. Papers II-819

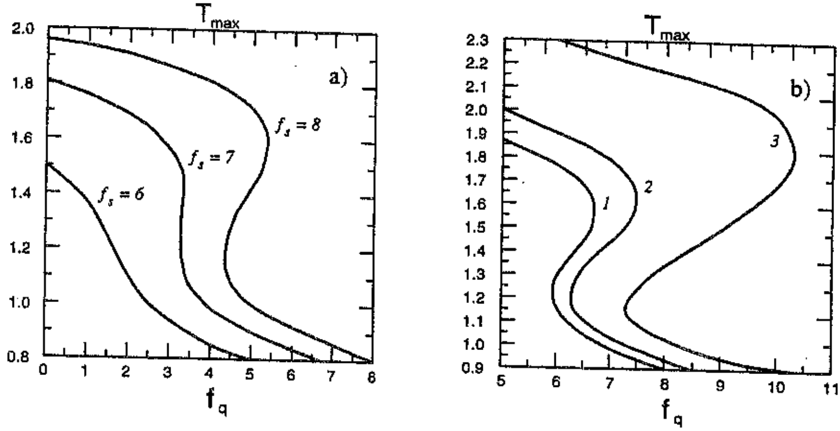

Fig. 1. T_{max} as a function of f_q for (a) $c_{0,1} = 0, x^* = 0.5, \Delta_s = 5$ and (b) 1 - $c_{0,1} = 0, f_s = 15, x^* = 0.4, \Delta_s = 0.05$, 2 - $c_{0,1} = 0, f_s = 15, x^* = 0.5, \Delta_s = 0.05$, 3 - $c_0 = 0, c_1 = 0.5, f_s = 15, x^* = 0.5, \Delta_s = 0.05$.

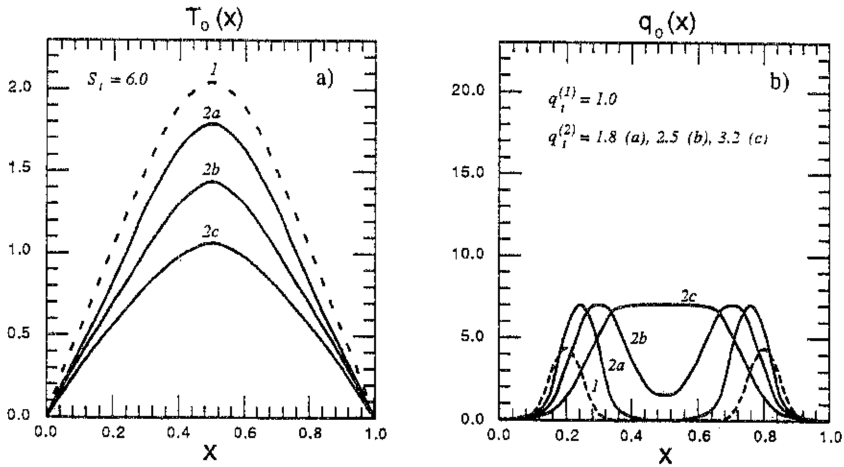

Fig. 2. Stationary temperature (a) and radiation (b) profiles for $f_s = 15, x^* = 0.5, \Delta_s = 0.05$ and $f_q = 4.3$ (1) and $f_q = 7$ ($2a, b, c$) where s_t and q_t are the total energy input and radiation power.

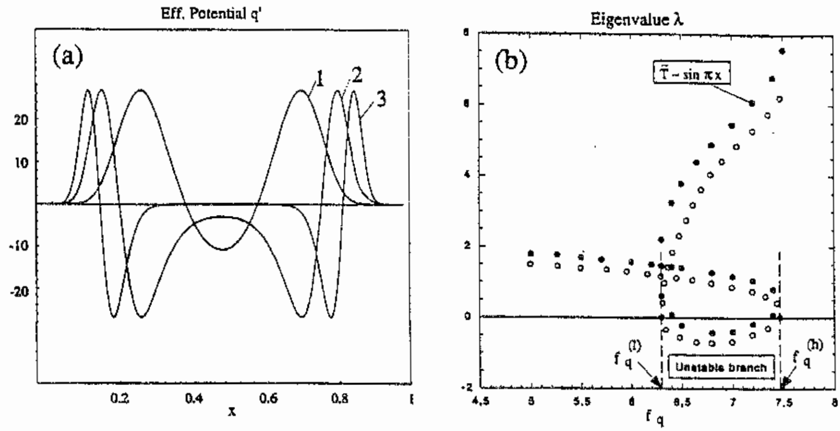

Fig. 3. Effective potentials (a) for (1) low-, (2) middle- and (3) high-temperature profiles and the eigenvalue (b) vs f_q for $f_s = 15, x^* = 0.5, \Delta_s = 0.05$.

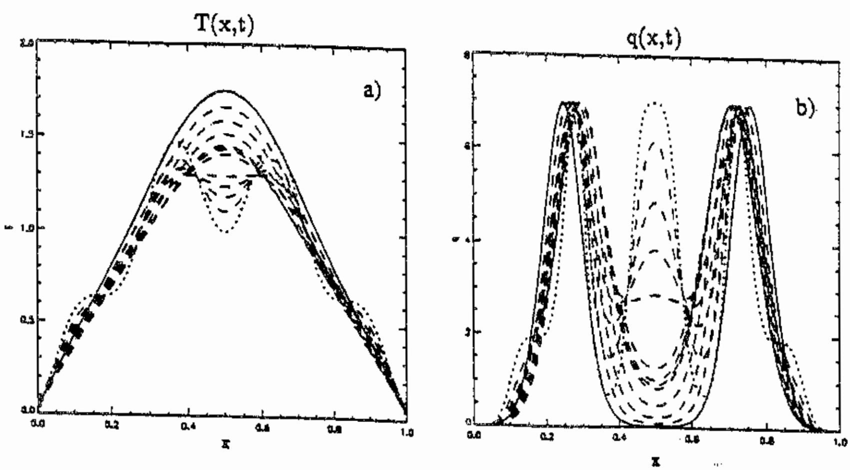

Fig. 4. Time evolution of temperature (a) and radiation (b) profiles for $c_{0,1} = 0, f_s = 15, x^* = 0.5, \Delta_s = 0.05, f_q = 7, u_0 = 1.30, u_1 = 0.2$ and $u_2 = 7$ ($\cdots$ - $t = 0$, — - $t = \infty$).