Transmission of Information in Active Networks

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Shannon's Capacity Theorem is the main concept behind the Theory of Communication. It says that if the amount of information contained in a signal is smaller than the channel capacity of a physical media of communication, it can be transmitted with arbitrarily small probability of error. This theorem is usually applicable to ideal channels of communication in which the information to be transmitted does not alter the passive characteristics of the channel that basically tries to reproduce the source of information. For an *active channel*, a network formed by elements that are dynamical systems (such as neurons, chaotic or periodic oscillators), it is unclear if such theorem is applicable, once an active channel can adapt to the input of a signal, altering its capacity. To shed light into this matter, we show, among other results, how to calculate the information capacity of an active channel of communication. Then, we show that the *channel capacity* depends on whether the active channel is self-excitable or not and that, contrary to a current belief, desynchronization can provide an environment in which large amounts of information can be transmitted in a channel that is self-excitable. An interesting case of a self-excitable active channel is a network of electrically connected Hindmarsh-Rose chaotic neurons.

I. INTRODUCTION

Given an arbitrary time dependent stimulus that externally excites an active channel, a network formed by systems that have some intrinsic dynamics (e.g. oscillators and neurons), how much information from such stimulus can be realized by measuring the time evolution of one of the elements of the channel ? The work in Ref. [1] shows that 50% of the information about light displacements might be lost after being processed by the H1 neuron, sensitive to image motion around a vertical axis, a neuron localized in a small neural network of the Chrysomya magacephala fly , the lobula plate. Does that mean that the H1 neuron has an information capacity lower than the information contained in the light stimulus ? Or does that mean that information is lost due to the presence of internal noise ?

In order to be able to shed light into these questions, we need to know how to calculate the information capacity of an active channel, and, for practical purposes, to understand how an active channel (an active network composed by dynamical systems) behaves if it is set up to operate with its information capacity. As we shall see, there is a parameter route through which the information capacity is reached, and this route can be established in terms of either the coupling strengths or the level of synchronism (behavior) among the elements forming the active channel. While information might not always be easy to be measured or quantified in experiments, one might be able to state about how good is an active channel to transmit information by measuring synchronization, a phenomenom which is often not only possible to observe but also relatively easy to characterize.

Synchronization is vital for modern methods of digital communication that rely on the synchronous operation of many subsystems [2]. Similarly, transport networks depend crucially on the synchronous operation of each subnetwork. If one subnetwork gets out of synchrony, the whole network might failure to function properly. So, it would be intuitive to say that complex systems should have subsystems that operate in synchrony for a proper functioning. In fact, synchronization between neurons in the brain is believed to provide a good environment for information transmission. This comes from a fundamental hypothesis of neurobiology [3–6] that synchronization [7, 8] functionally binds neural networks coding the same feature or objects. This hypothesis raised one of the most important contemporary debates in neurobiology, but is still controversial [9, 10] because desynchronization seems to play an important role in the perception of objects as well.

The analyses are carried out using among others two quantities suitable for the treatment of information transmission in active channels, the *channel capacity* and the *system capacity*. In short, the channel capacity measures the maximum rate with which information is exchanged between two elements of the active channel, a path along which information can flow in the active channel. On the other hand, the system capacity is the maximum of the Kolmogorov-Sinai entropy, the total amount of independent information that can be simultaneously transmitted between all the pairs of elements of the active channel.

Among our main results, we show that the channel capacity of an active channel depends on whether the active channel is *self-excitable* or not (see definition in Sec. III). Active channels composed of non self-excitable systems (such as Rössler-type oscillators) achieve its maximal channel capacity to transmit information whenever

its elements are in complete synchrony. On the other hand, active channels composed of self-excitable systems (such as neurons), achieve its maximal channel capacity when there is still at least one degree of freedom or characteristic oscillation (time-scale) that is out of synchrony. In the case of active channels formed by spiking-bursting neurons (Hindmarsh-Rose electrically coupled), the maximal channel capacity to transmit information is achieved when the neurons phase synchronize in the slow time-scale (bursting) and desynchronize in the fast time-scale (spiking). Thus, synchronization in neural networks might be two folded. Depending on the type of measurement made, one can agree or disagree with the binding hypothesis.

Therefore, in this work, we build a bridge between Shannon's Theory of Communication [11] and the Theory of Information in dynamical systems [12] contributing to the development of a nonlinear Theory of Communication applied to dynamical systems, shading some light in these paradigms of neurobiology. These new ideas, concepts, and theoretical approaches unravel the relation between stimuli [13], information capacity, and synchronization, in a nonlinear media of communication, the *active channel*, a network formed by elements that are dynamical systems.

II. MUTUAL INFORMATION RATE

A good communication system as visualized by Shannon comprises a *transmitter* who transforms a message in a signal suitable to be transmitted through the channel and a *receiver* who recovers the message from the signal. In analogy to this definition and in order to properly deal with information transmission in dynamical systems, we define three subspaces in the active channel. The subspace α that generates a suitable message to be transmitted, regarded as the transmitter (an element of the network), the subspace β where the message can be recovered, regarded as the receiver (another element of the network), and finally the composed subspace (α, β), considered to form one *communication channel*, an union of the subspaces α and β, also a subspace of the active channel. So, both the transmitter and the receiver belong to the communication channel. Herein, the more is the information exchanged between the receiver and the transmitter, the more information about the transmitter's trajectory can be realized in the receiver. The trajectory of the transmitter (receiver) represents the evolution in time of the transmitter's (receiver's) position in the subspaces α (β).

According to Shannon, the amount of information that can be measured in the receiver, $x^{(\beta)}$, about the transmitter of information, $x^{(\alpha)}$, is given by

$$I(\alpha, \beta) = H^{(\alpha)} + H^{(\beta)} - H^{(\alpha,\beta)}, \qquad (1)$$

also known as mutual information between the transmitter and the receiver. $H^{(\alpha)}$ is the information produced by the transmitter, $H^{(\beta)}$, the one produced (or measured in) by the receiver, and $H^{(\alpha,\beta)}$ the one produced in the composed subspace, also known as the joint entropy between the receiver and the transmitter. To calculate the terms in Eq. (1) for systems where events in the future are connected to events in the past (systems with correlation), one usually needs to coarse grain the domain of the subspaces α and β into n equal size-ϵ [14] intervals α_i and β_j, with $i, j = 1, \ldots, n$ ($n = 1/\epsilon$), being that $P_m^{(\alpha)}$ ($P_m^{(\beta)}$) represents the probability of an event, e.g. a trajectory point in the subspace α (β) visiting a sequence of L intervals. The trajectory remains a time τ in an interval ϵ. If one is working with maps, $\tau=1$. The term $P_m^{(\alpha,\beta)}$ represents the probability of a composed event, e.g. a trajectory point visiting an itinerary following a sequence of L areas, each area delimited by the intervals α_i and β_j, as represented in Fig. 1. Then, in Eq. (1), $H^{(\alpha)}=- \sum_m P_m^{(\alpha)} \ln P_m^{(\alpha)}$, $H^{(\beta)}=- \sum_m P_m^{(\beta)} \ln P_m^{(\beta)}$, and $H^{(\alpha,\beta)}=-\sum_m P_m^{(\alpha,\beta)} \ln P_m^{(\alpha,\beta)}$, where we have taken the limit of $(\epsilon, \tau) \to 0, L \to \infty$.

Notice that all the terms in Eq. (1) tend to infinity as $(\epsilon, \tau) \to 0, L \to \infty$. So, it is advantageous to work with terms $\sigma = H/(\tau L)$ that measure the amount of information per time unit, which are finite quantities in the active channel. So, we rewrite Eq. (1) as

$$I(\alpha, \beta) = \lim_{L \to \infty, (\tau, \epsilon) \to 0} \tau L(\sigma^{(\alpha)} + \sigma^{(\beta)} - \sigma^{(\alpha,\beta)}) \qquad (2)$$

$I_C(\alpha, \beta) = I(\alpha, \beta)/(\tau L)$ is the **mutual information rate** (MIR) between the transmitter (α) and the receiver (β). Based on the results of [15], the term $\sigma^{(\alpha,\beta)}$ is the Kolmogorov-Sinai (KS) entropy [16] of the trajectory in the subspace (α, β) [14], regarded as $H_{KS}^{(\alpha,\beta)}$. Imagine that the receiver has a finite physical coupling with the transmitter. From Takens theorem [17], the entropy of a trajectory calculated in a subspace, e.g. (α), should provide the entropy of the trajectory in the whole space (α, β), which leads to that $\sigma^{(\alpha)}=\sigma^{(\beta)}=\sigma^{(\alpha,\beta)}$, and therefore, $I_C(\alpha, \beta)=H_{KS}^{(\alpha,\beta)}$. Independent on the coupling strength and on the synchronization level between the receiver and the transmitter, one arrives that the MIR is constant and given by $H_{KS}^{(\alpha,\beta)}$. Naturally, in order for one to gain such an amount of information rate, one might have to realize an infinite number of observations in the receiver's trajectory and one has to have access to a good trajectory projection (subspace). But, in communication, it is desirable that information about the transmitter can be "instantaneously" realized in the receiver. In addition, measurements performed in the subspace β of the receiver do not necessarily contain all the information content produced by either the whole active channel or a communication channel. For these reasons, we introduce a consistent definition for the MIR between two subspaces (elements) in a network

$$I_C(\alpha, \beta) = D_1^{(\alpha)}|\lambda^{(\alpha)}| + D_1^{(\beta)}|\lambda^{(\beta)}| - H_{KS}^{(\alpha,\beta)} \qquad (3)$$

where $\lambda^{(\alpha)}$ and $\lambda^{(\beta)}$ are the Lyapunov exponents of the trajectories in the subspaces α and β, respectively, which measures how nearby trajectories exponentially diverge in these subspaces, $D_1^{(\alpha)}$ and $D_1^{(\beta)}$ are the information dimension of the trajectory in these subspaces [see Appendix A], and $|.|$ is the modulus operation [18]. By doing that, we assume that the first two terms $(D_1^{(\alpha)}|\lambda^{(\alpha)}| + D_1^{(\beta)}|\lambda^{(\beta)}|)$ in the right side of Eq. (3) measure the information produced by both the receiver and the transmitter as if they were uncoupled, i.e., uncorrelated "random" variables (no phase-space reconstruction [17] is employed in the measurable data from the subspaces α and β), and the last term $(H_{KS}^{(\alpha,\beta)})$ provides the dependence between them. These are actually the basic assumptions behind the definition of mutual information provided by Shannon [11] to random variables being transmitted through a noisy channel. For more details about how to analytically calculate the terms $\lambda^{(\cdot)}$, see Appendixes A and B.

III. INFORMATION CAPACITY, EXCITABILITY, AND SUSCEPTIBILITY

In order to study the way information is transmitted in active networks, we introduce quantities and terminologies that assist us to better present our ideas and approaches.

An **active channel** is an active network constructed using Q elements that have some intrinsic dynamics and can be described by classical dynamical systems, such as chaotic oscillators, neurons, phase oscillators, and so on. Every pair of elements forms a **communication channel** and the rate with which information is exchanged between these elements, a transmitter and a receiver, is given by the mutual information rate (MIR) between them.

The **channel capacity**, $\mathcal{C}_C$, of a communication channel is defined as the maximum of the MIR for this communication channel formed by a pair of elements, the receiver and the transmitter, with respect to many possible coupling strengths among the elements, for a given network topology. Thus, the channel capacity is the maximal possible amount of information that two elements within the network with a given topology can exchange, a local measure that quantifies the point-to-point rate with which information is being transmitted. Notice that a communication channel is a subset of an active network.

The **system capacity**, $\mathcal{C}_S$, of an active network composed by Q elements is defined as the maximum of the Kolmogorov-Sinai (KS) entropy, H_{KS}, of the whole active network ($H_{KS} \geq H_{KS}^{(\alpha,\beta)}$), with respect to many possible coupling strengths among the elements, for a given network topology. The Kolmogorov-Sinai entropy offers an appropriate way of obtaining the entropy production of a dynamical system. Here, it provides a global measure of how much information can be simultaneously transmit-

ted among all communication channels. Therefore, $\mathcal{C}_S$ bounds $\mathcal{C}_C(\alpha,\beta)$ as well as the KS-entropy, H_{KS}, of an active network, calculated for a given coupling strength, bounds the MIR between two elements, $I_C(\alpha,\beta)$, calculated for the same coupling strength. Thus,

$$\begin{aligned} \mathcal{C}_C(\alpha,\beta) &\leq \mathcal{C}_S \\ I_C(\alpha,\beta) &\leq H_{KS} \end{aligned} \tag{4}$$

An active channel is said to be **self-excitable (not self-excitable)** when $\mathcal{C}_C > H_{KS}^{(0)}$ (when $\mathcal{C}_C \leq H_{KS}^{(0)}$), with $H_{KS}^{(0)}$ representing the KS entropy of one of the Q elements forming the active channel, before they are coupled. Analogously, we can also define self-excitability in terms of the channel capacity. For a self-excitable channel, it is true that $\mathcal{C}_S/Q > H_{KS}^{(0)}$. Thus, in a self-excitable active channel H_{KS} increases as the coupling strength among the elements increases.

An active channel is said to be **susceptible** if $\mathcal{C}_C > H_{KS}^{(r)}$ and **not susceptible** if $\mathcal{C}_C \leq H_{KS}^{(r)}$, where $H_{KS}^{(r)}$ represents the KS entropy of the uncoupled receiver. So, a susceptible channel does not resist the action of the stimulus provided by the transmitter or the dynamical alteration caused by the coupling configuration in the active channel. These alterations might produce also a self-excitable channel. It is to be expected that a self-excitable channel is also a susceptible one.

IV. THE CHAOTIC CHANNEL

Here, we analyze how a source of information can be transmitted through a channel that stretches the amplitude of the information signal [19, 20]. The Lyapunov exponent of the receiver, λ^β, is always positive even if there is no coupling between the transmitter and the receiver. Part of the information transmitted might be lost due to the presence of chaos in the channel. We assume that a general source of information can be modeled by a chaotic system.

A model of the chaotic channel is given by two one-dimensional chaotic maps bidirectionally coupled [21],

$$x_{n+1}^{(\beta)} = 2x_n^{(\beta)} + 2c(x_n^{(\alpha)} - x_n^{(\beta)}), mod(1) \tag{5}$$

$$x_{n+1}^{(\alpha)} = 2x_n^{(\alpha)} + 2c(x_n^{(\beta)} - x_n^{(\alpha)}), mod(1) \tag{6}$$

where the subspace of Eq. (5) is regarded as the receiver and Eq. (6), the transmitter. In Fig. 1, we represent the map trajectory for $c = 0.24$, the coupling strength. This map has two Lyapunov exponents: $\lambda_1 = \ln(2)$ and $\lambda_2 = \ln(2 - 4c)$, λ_1 measures the exponential divergence of nearby trajectories in the direction of the synchronization manifold defined as $x^{(\alpha)} - x^{(\beta)} = 0$, and λ_2 the exponential divergence of nearby trajectories in the direction transversal to the synchronization manifold. The exponents $\lambda^{(\alpha)}$ and $\lambda^{(\beta)}$ that measure the exponential divergence of trajectories along the subspaces α and β are
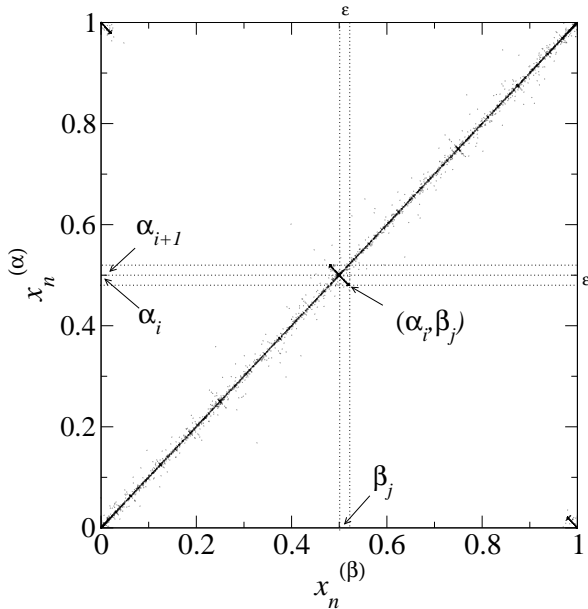
FIG. 1: The subspace α of the transmitter who generates the information, the subspace β, where information about the transmitter can be realized, and the composed subspace (α, β) that represents a two-dimensional active channel of communication.

equal to $\max(\lambda_1, \lambda_2) = \lambda_1$, and therefore, $\lambda^{(\alpha)} = \lambda^{(\beta)} = \lambda_1$, since the maps have equal parameters [see Sec. VII]. One can also arrive to this result by noting that the Lyapunov exponent of a typical 1D projection of a 2D chaotic set (with two positive Lyapunov exponents) is given by the Largest exponent. Since the trajectories in the subspaces α and β have uniform probability distribution and the information dimension of the trajectory in the composed subspace is $D_1=2$, a one dimensional projection of it should provide $D_1^{(\alpha)} = D_1^{(\beta)} = 1$ (see Appendixes A and B). Using a result by Pesin [16], the Kolmogorov-Sinai entropy of a chaotic system is the summation of all the positive Lyapunov exponents, $H_{KS} = \sum^{+} \lambda$. For a two-dimensional channel, $H_{KS}^{(\alpha,\beta)}$ equals the entropy of the whole channel, H_{KS}. So, $H_{KS} = \lambda_1 + \lambda_2$, if $\lambda_2 \geq 0$, or $H_{KS} = \lambda_1$, otherwise. Therefore, we arrive that the rate with which information can be retrieved in the receiver about the stimulus generated in the transmitter is given by $I_C = \lambda_1 - \lambda_2$, if $\lambda_2 \geq 0$, or $I_C = \lambda_1$, otherwise. So, $\mathcal{C}_C = \ln(2)$ and $\mathcal{C}_S = 2\ln(2)$. An increase in the coupling leads to an increase in the MIR and a decrease in H_{KS}.

To relate the MIR with the synchronization level of this chaotic channel, we make a convenient coordinate transformation into new variables $x^{\parallel}$ and $x^{\perp}$ (see Appendix A) such that the exponential divergence on $x^{\perp}$ is minimal ($x^{\perp}$ is oriented along the contracting direction) and $x^{\parallel}$ is orthogonal to $x^{\perp}$ ($x^{\parallel}$ is oriented along the expanding direction). Such a transformation, for the systems to be treated here, is $x_n^{(\alpha\beta)\parallel} = x_n^{(\alpha)} + x_n^{(\beta)}$ and

$x_n^{(\alpha\beta)\perp} = x_n^{(\alpha)} - x_n^{(\beta)}$, with the synchronization manifold given by $x_n^{\perp} = 0$. The mapping in this new coordinate generates the conditional Lyapunov exponents $\lambda^{\parallel} = \lambda_1$ and $\lambda^{\perp} = \lambda_2$. In contrast to the conditional exponents defined in Ref. [22], the obtained exponents produce physically consistent quantities (ergodic invariants) [23] even for situations when complete synchronization is absent. The transformed equations of motion in these new variables provide not only the same Lyapunov exponents but also, for this example, the same KS entropy ($H_{KS} = \lambda^{\parallel} + \lambda^{\perp}$, if $\lambda^{\perp} \geq 0$, or $H_{KS} = \lambda^{\parallel}$, otherwise) of the original equations, and advantageously supply us with a way to understand the synchronization level between the two subsystems. $\lambda^{\parallel} > \lambda^{\perp}$, and we recover the conjecture of Ref. [20], $I_C = \lambda^{\parallel} - \lambda^{\perp}$, if $\lambda^{\perp} \geq 0$ or $I_C = \lambda^{\parallel}$, otherwise. This conjecture provides an easy way of solving Eq. (3) for an active channel linking information to synchronization. The more synchronization, the smaller is $x^{\perp}$, and therefore, nearby initial conditions in this variable will diverge exponentially in a smaller rate, i.e., the conditional exponent associated with this variable, $\lambda^{\perp}$, is smaller. If $\lambda^{\perp}$ is very small, $\lambda^{\parallel}$ can be associated with the amount of information production of the synchronous trajectories (between the transmitter and receiver), otherwise, it is associated with the excitation of the channel. The more excitation in the channel the larger is $\lambda^{\parallel}$. So, to achieve larger amounts of information transmission it is required that either the excitation or the synchronization level are large, or both. In this channel, as we increase the coupling, H_{KS} decreases due to an increase in the synchronization level, which leads to an increase in the MIR. $\mathcal{C}_C = \min(H_{KS}) \neq \mathcal{C}_S$ is achieved for a configuration when the synchronization is maximal. *Therefore, the larger the coupling is, the less information the whole active channel produces (KS-entropy, H_{KS}), but the larger the MIR between a receiver and a transmiter is, which means that the more information about the transmitter can be measured in the receiver. More synchronization implies more information transmission. This channel is not self-excitable and since $\mathcal{C}_C = H_{KS}^{(r)}$, it is not susceptible, because its capacity is limited by the capacity of the receiver to generate information*

In order to calculate the MIR of a communication channel in a large chaotic active channel, we need to use the coordinate transformation $x^{\parallel}$ and $x^{\perp}$. This transformation enables one to calculate the MIR between two subsystems as if they were detached from the active channel. Imagine an active channel formed by Q fully bidirectionally coupled chaotic systems:

$$x_{n+1}^{(j)} = 2x_n^{(j)} + \sum_{i=1}^{Q} 2c(x_n^{(i)} - x_n^{(j)}) mod(1) \qquad (7)$$

with $j = [1, \ldots, Q]$. Now, we can define $[Q \times (Q-1)]/2$ pairs of subspaces. For instance, the pair of subspaces formed by the subsystem $x^{(1)}$ and the subsystem $x^{(2)}$, with $x_n^{(12)\parallel} = x_n^{(1)} + x_n^{(2)}$ and $x_n^{(12)\perp} = x_n^{(1)} - x_n^{(2)}$. Any pair of subspaces produces the same conditional exponents

$\lambda^{\parallel} = \ln(2)$ and $\lambda^{\perp} = \ln|2(1 - Qc)|$ [see Appendix A]. In fact, this system produces one Lyapunov exponent $\lambda = \lambda^{\parallel}$ and $(Q - 1)$ equal others $\lambda = \lambda^{\perp}$, and so, our defined conditional exponents can be related to the Lyapunov exponents even in higher-dimensional systems. So, the MIR [see Eq. (B1)] between any two subsystems $x^{(k)}$ and $x^{(l)}$ is $I_C(x^{(k)}, x^{(l)}) = -\ln(1 - Qc)$ bits per iteration of the mapping, for $c \leq 1/(2Q)$ ($\lambda^{\perp} \geq 0$). If there is no coupling ($c=0$), then $I_C(x^{(k)}, x^{(l)}) = 0$ and no information is exchanged between both subspaces. For $c \leq 1/(2Q)$, the larger is c the more synchronous a transmitter, say $x^{(k)}$, is with a receiver, say $x^{(l)}$, and the more information is exchanged. The channel capacity for all communication channels is achieved for $c \geq 1/(2Q)$, when $I_C(x^{(k)}, x^{(l)}) = \ln(2) = \min(H_{KS})$, and the network completely synchronizes ($\lambda^{\perp} < 0$). *This type of active channel is not self-excitable. Notice that the introduction of one more element into this channel [13] does not alter $\mathcal{C}_C$. It is also not susceptible.*

V. THE PERIODIC CHANNEL

The purpose of the present work is to describe how information is transmitted via an active media, a network formed by dynamical systems. There are three possible asymptotic stable behaviors for an autonomous dynamical system: chaotic, periodic, or quasi-periodic. A quasi-periodic behavior can be usually replaced by either a chaotic or a periodic one, by an arbitrary perturbation. For that reason, we neglect such a state and focus the attention on active channels that are either chaotic or periodic.

The purpose of the present section is dedicated to analyze how a source of information can be transmitted through active channels that are periodic, channels that squeeze the amplitude of the information signal. More specifically, channels whose receiver behaves in a periodic fashion (its Lyapunov exponent, λ^{β}, is negative).

It is to be expected that in the periodic channel a fractal set appears, when $\lambda_1 \leq |\lambda_2|$ (assuming a bidimensional channel). This clearly imposes severe limits for the recovery of information in the receiver. The periodic channel can be imagined as a filter. As shown in [24], chaotic signals being transmitted through filters might produce an output with higher dimension due to the appearance of a fractal set. To see that we study the generalized baker's map [25], shown in Figs. 2(**a-b**) and in Fig. 3. All the information produced in the transmitter $x^{(\alpha)}$ is transfered to the receiver $x^{(\beta)}$, but with a time delay. To make it more clear, note that in Fig. 2**b**, by recognizing if the received signal is either smaller or larger than b at the iteration time $n + 1$, one is able to know if the position of the transmitter was lower or higher than a, at the iteration time n. By looking the received signal at higher resolution one is able to predict with higher resolution the position of the transmitter farer away in the past.
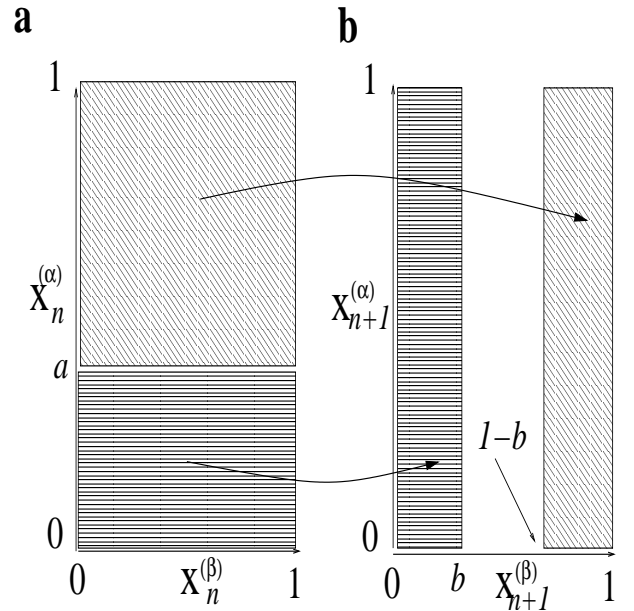

FIG. 2: **a**, The baker's map squeezes the rectangle above a in **a** into the right rectangle in **b** and it squeezes the rectangle bellow a in **a** into the left rectangle in **b**. Both rectangles in **a** are stretched in the horizontal direction. From **a** to **b**, we represent one application of the generalized baker's map.

In order to calculate the MIR between the transmitter and the receiver using Eq. (3) note that $\lambda^{(\alpha)}=a\ln[1/a]+(1-a)\ln[1/(1-a)]$, $D_1^{(\alpha)}=1$, $\lambda^{(\beta)}=a\ln(b)+(1-a)\ln(b)$, and $D_1^{(\beta)}=\lambda^{(\alpha)}/|\lambda^{(\beta)}|$ (with $D_1^{(\beta)} < 1$) and $\lambda^{(\alpha,\beta)} = H_{KS} = \lambda^{(\alpha)} + \lambda^{(\beta)}$. If $D_1^{(\beta)} < 1$, $\lambda^{(\beta)} < 0$, and $\lambda^{(\alpha)} \leq |\lambda^{(\beta)}|$, and a fractal set takes place. As demonstrated in Ref. [26], the information content of this fractal set is $D_1^{(\beta)}|\lambda^{(\beta)}|$ [27]. Thus, $I_C=\lambda^{(\alpha)} + D_1^{(\beta)}|\lambda^{(\beta)}| - H_{KS}$, and we arrive at $I_C=\lambda^{(\alpha)}$ per iteration, since $H_{KS} = \lambda^{(\alpha)}$ and $|\lambda^{(\beta)}|D_1^{(\beta)} = \lambda^{(\alpha)}$. $H_{KS} = \lambda^{(\alpha)}$ because the fractal set does not contribute to the KS entropy of the full chaotic map. *So, $\mathcal{C}_C$ depends on the amount of information produced in the transmitter, a typical characteristic of a susceptible channel. Unlike the chaotic channel that is robust to small noise intensities [28], in the periodic channel $\mathcal{C}_C$ might be extreme sensitive to noise of even arbitrary amplitudes.*

In active channels, the receiver might influence the transmitter behavior. We can imagine a bidirectional coupling scheme for which a periodic uncoupled receiver might make an uncoupled chaotic transmitter to behave also periodically, after the coupling is switched on. This type of periodic channel is thus not susceptible to adapt to stimuli, and $\mathcal{C}_C=0$.

The periodic channel might be relevant to understand the role of subthreshold oscillations in the processing of information. These oscillations are observed in the brain, in particular to regions associated with motor reaction and learning like the Inferior Olive [29]. They appear because groups of neurons interact in such a way that
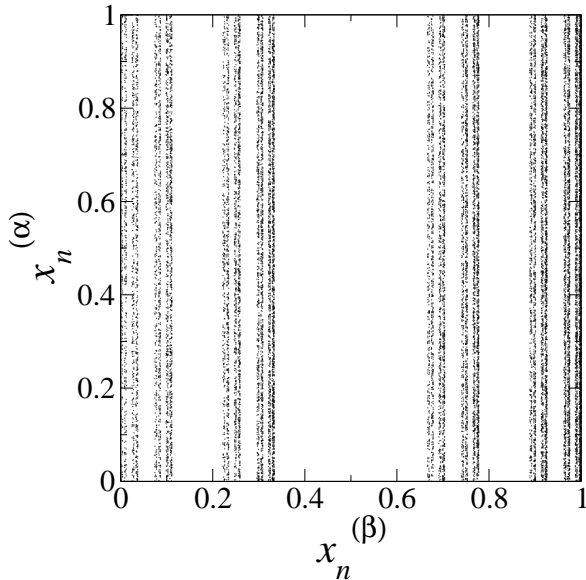
FIG. 3: The asymptotic solution of the map in Fig. 2, formed by a series of vertical strips forming a fractal set in the horizontal direction.

the potential in the neurons membrane is not sufficient to induce a spike. Usually, the oscillations are reduced to a limit cycle, a periodic behavior.

VI. THE NEURON CHANNEL

We illustrate our ideas in a relevant type of excitable active channel, the chaotic neural channel [30–32], formed by a network of electrically connected Hindmarsh-Rose (HR) neuron models [33]. This network possesses both characteristics of the periodic and chaotic channels, since it has both positive and negative Lyapunov exponents. A fractal set that occupies a small portion of the phase space coexists with a chaotic set that occupies most of the phase space. Due to the negative exponents, the dynamics is strongly compressed along the stable directions, the stable manifolds. The result is that the observable dynamics of the neurons lies along the unstable manifolds, and thus, the negative exponents do not contribute to the recovered information. This is a consequence of the Sinai-Ruelle-Bowen (SRB) assumption.

Each pair of neurons can be treated as an active channel of communication, one neuron α_k performing the transmitter task and the other β_l the receiver. In Ref. [20], we have stated that the amount of information in one chaotic channel should be always smaller than the information produced by the network, H_{KS}. Thus, $I_C(\alpha_k, \beta_l) \leq H_{KS}$ [see Eq. (4)]. By working with large networks, composed of many elements, we should expect that the same information travels simultaneously along many different channels. This property, often desired in networks, makes it a reliable medium for informa-

tion transmission because it introduces in the network a large amount of redundancy, which results into a sum of all the MIR in the communication channels larger than H_{KS}. Even if one or many channels are blocked, the information still finds its destination. So, to treat networks composed by Q chaotic systems, we expect that

$$\langle I_C \rangle \leq H_{KS}. \qquad (8)$$

where $\langle I_C \rangle = \sum_{k,l} I_C(\alpha_k, \beta_l)/P$ represents the average amount of MIR of the whole network, P is the number of communication channels given by $P = [Q(Q-1)]/2$, and Q is the number of neurons. For the neuron channel, we consider that $\mathcal{C}_C$ represents the maximal of $\langle I_C \rangle$ for many coupling configurations. The average amount of redundancy in the network is defined to be $\langle R \rangle = \frac{\langle I_C \rangle}{H_{KS}}$, and thus, if the network is completely synchronized $\langle R \rangle = 1$, and if the network is completely uncoupled $\langle R \rangle = 0$.

We consider a network composed of $Q=4$ bidirectionally fully coupled neurons:

$$
\begin{aligned}
\dot{x}_i &= y_i + 3x_i^2 - x_i^3 - z_i + I_i + \sum_j A_{ji}(x_j - x_i) \\
\dot{y}_i &= 1 - 5x_i^2 - y_i \qquad\qquad (9) \\
\dot{z}_i &= -rz_i + 4r(x_i + 1.6)
\end{aligned}
$$

The parameter that modulates the slow dynamics is set to $r=0.005$, such that each neuron is chaotic. i and j, with $j \neq i$ assume values within the set $[1, \ldots, Q]$. α_k represents the subsystem of the variables (x_k, y_k, z_k) and β_l represents the subsystem of the variables (x_l, y_l, z_l), where $k=[1, \ldots, Q-1]$ and $l=[k+1, \ldots, Q]$. $A_{ji} = A_{ij}$ [21] is the strength of the electrical coupling between the neurons represented by α_j and α_i. The external stimulus I_1, in α_1, is set to be equal to $I_1 = 3.25$-δI, and then, $I_2=I_1+\delta I$, $I_3=I_1$-δI, $I_4=I_1+\delta I$, with $\delta I = 0.00001$. Initial conditions are x=-1.3078+η, y=-7.3218+η, and z=3.3530+η, where η is an uniform random number within [0,0.001].

Four synchronization phenomena are relevant to be considered [see Appendixes A and C]. Bursting phase synchronization (BPS), when at least one pair of neurons is phase synchronous in the bursts, partial phase synchronization (PPS) when at least one pair of neurons is phase synchronous in the bursts and in the spikes, phase synchronization (PS), when all the pairs of neurons are phase synchronous in the bursts and in the spikes, and complete synchronization (CS). An evidence for the presence of bursting or spiking phase synchronization is found if the condition $\max(\Delta N^n)/P \leq 1$ is satisfied, where $\Delta N^n = \sum_{k,l} |N_k^n - N_l^n|$ and N_k^n represents the number of spikes/bursts in α_k, at the time the neuron α_1 suffered its n-th spike/burst. This condition is threshold dependent, but it will be employed here for the purposes of illustration.

This example, illustrated by Fig. 4, shows three fundamental characteristics of an active channel:

(i) Excitation enhances H_{KS} and the MIR of the communication channels. With no coupling, the rate of information production in each neuron is approximately

$H_{KS}^{(0)} \cong 0.014$ and $\langle I_C \rangle$ is null. For a coupling strength of $A_{kl,lk} > 0.01$, each pair of neuron exchanges (in average) information with each other in a rate larger than the individual rate with no coupling. So, an increase in the coupling strength is simultaneously followed by an increase in both H_{KS} and the rate of information production of each individual neuron, resulting in an increase of $\langle I_C \rangle$, meaning also an increase in the MIR of the communication channels, a typical characteristic of a self-excitable channel.
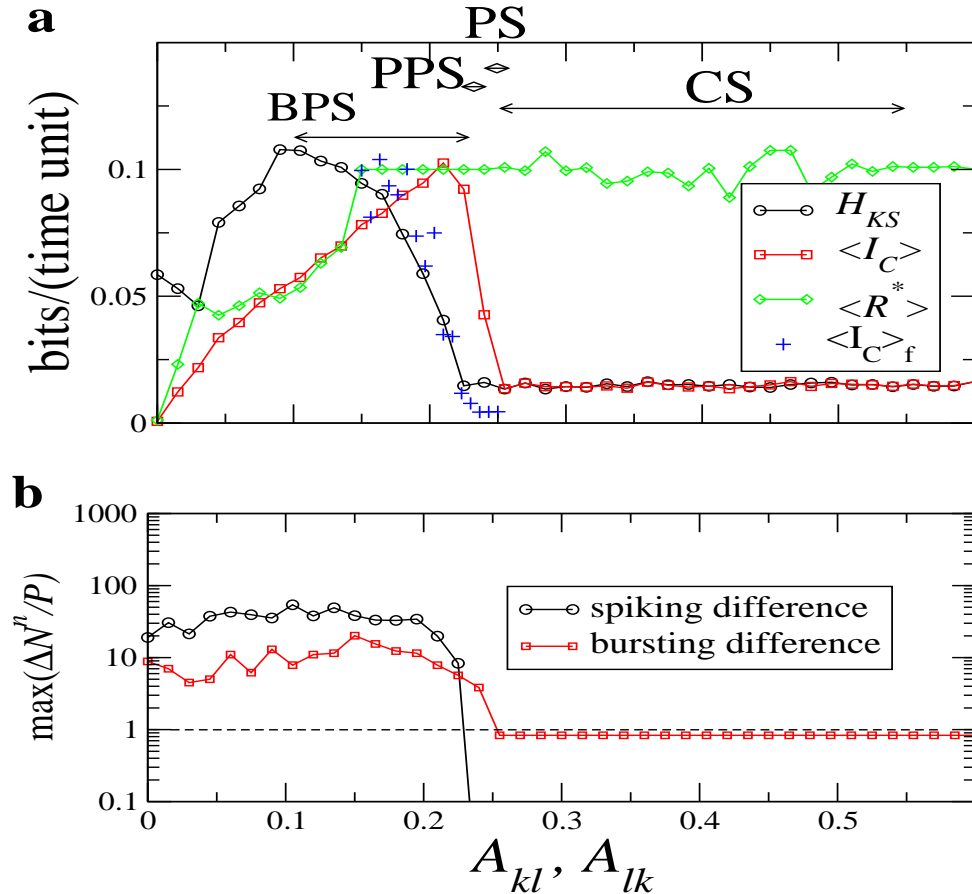

FIG. 4: [Color online] **a**, Circles, squares, and diamonds show H_{KS}, $\langle I_C \rangle$, and $\langle R^* \rangle = \langle R \rangle / 10$ for a network of 4 fully connected neurons. The necessary conditions for the proper calculation of $\langle I_C \rangle$ are not satisfied for coupling strengths that produce stronger types of PS ($A_{kl} = [0.15, 0.25]$), see Appendix A. There, $\langle I_C \rangle$ should be estimated by a finite time MIR, indicated by $\langle I_C \rangle_f$. For this coupling strength interval, we set $\langle R^* \rangle = 1$. **b**, Circles and squares represent the maximal average spiking and bursting difference, $\max(\Delta N^n / P)$, after $n = 200$ bursts, in a log vertical axis. Values of $\max(\Delta N^n / P)$ smaller than the dashed line are an evidence that there is PS. The upper arrows in **a** indicate the coupling strength intervals for which we find BPS ($A_{kl} \cong [0.1, 0.23]$), PPS ($A_{kl} = [0.23, 0.245]$), PS ($A_{kl} = [0.245, 0.25]$), and CS ($A_{kl} \geq 0.25$). To obtain the information in units of bits we divide the related equations by $\ln(2)$. I_C is calculated using Eq. (B1), which produces similar values to the ones obtained by calculating the MIR from the entropy and joint entropy of symbolic sequences generated from the trajectory of pair of neurons.

(ii) Synchronization does not necessarily mean high levels of information transmission. When the network reaches its system capacity ($A_{kl} \approx 0.08$), the spikes and the bursts are highly desynchronous [Fig. 4**b**] by usual definitions of phase synchronization [see Appendix C], but both $\langle I_C \rangle$ and the redundancy $\langle R \rangle$ are high. At this point, we have to remember that MIR means the amount of excitation minus the amount of desynchronization. The amount of excitation is of the order of the maximal Lyapunov exponent of the network, which is large, since the network is excited, much larger than the amount of desynchronization. On the other hand, for $A_{kl} \geq 0.23$, when the neurons phase (PPS or PS) or completely synchronize ($A_{kl} \geq 0.23$), $\langle I_C \rangle$ abruptly drops

approaching the low value of H_{KS}, much lower than $\mathcal{C}_S$, as if the whole network were formed by one single neuron. The redundancy is high but few information can be transmitted in the network.

(iii) BPS provides an ideal environment for information transmission. When BPS is present, $\langle I_C \rangle$ and the redundancy are high. $H_{KS} \cong \langle I_C \rangle$. That suggests that BPS plays an important role in the reliable exchange of information that demands rapid responses and a large amount of information transmission. Each neuron maintains a high level of independent activity (given by the desynchronous spikes) and simultaneously a moderate level of synchrony (synchronism in the bursts) that allows a neuron to "talk" to another. These characteristics are usually desirable in sensory neurons [34] and in the ones responsible for motor reaction processes.

VII.   ACTIVE CHANNELS FORMED BY NON-EQUAL ELEMENTS

Here, we briefly describe the dependence of the MIR on the parameter mismatches between the elements forming an active channel. For such a case, $\lambda^{(\alpha)}$ and $\lambda^{(\beta)}$ typically differ, in Eq. (3). As a consequence, the channel capacity is lower than when the parameters do not mismatch and Eq. (B1) should be considered as an upper bound (it might overestimate the real value) for the MIR between the subspaces α_k and α_l [35]. Also, a parameter mismatch might enhance the value of the MIR if the coupling is kept constant while the parameters are changed. For simplicity, let us represent an active channel formed by two coupled unidimensional maps by $(\vec{x}_{n+1}^{(\alpha)}, \vec{x}_{n+1}^{(\beta)})$. Thus, $\lambda^{(\alpha)}$, which gives how nearby trajectories exponentially diverge along the subspace α, can be calculated from $\frac{\partial \vec{x}_{n+1}^{(\alpha)}}{\partial \vec{x}_n^{(\alpha)}} + \frac{\partial \vec{x}_{n+1}^{(\alpha)}}{\partial \vec{x}_n^{(\beta)}}$, the sum of the terms in the row of the Jacobian with respect to subsystem α, and $\lambda^{(\beta)}$ from $\frac{\partial \vec{x}_{n+1}^{(\beta)}}{\partial \vec{x}_n^{(\alpha)}} + \frac{\partial \vec{x}_{n+1}^{(\beta)}}{\partial \vec{x}_n^{(\beta)}}$, the sum of the terms in the row of the Jacobian with respect to subsystem β.

As an illustration, imagine the following chaotic channel:

$$x_{n+1}^{(\beta)} = (a - \epsilon)x_n^{(\beta)} + 2c(x_n^{(\alpha)} - x_n^{(\beta)}), mod(1)$$
$$x_{n+1}^{(\alpha)} = ax_n^{(\alpha)} + 2c(x_n^{(\beta)} - x_n^{(\alpha)}), mod(1) \quad (10)$$

with $a > 1$ and ϵ is the parameter mismatch. Then, we arrive at $\lambda^{(\alpha)}=\ln(a)$, $\lambda^{(\beta)}=\ln(|a - \epsilon|)$, $\lambda_1=\lambda^{\parallel}=\ln(|\mu_1 + \mu_2|)$, and $\lambda_2=\lambda^{\perp}=\ln(|\mu_1 - \mu_2|)$, with $\mu_1 = a - 2c - \epsilon/2$ and $\mu_2 = 2\sqrt{(c^2 + \epsilon^2/16)}$. If there is no coupling ($c=0$), and even if $\epsilon \neq 0$, $I_C(\alpha, \beta) = 0$. To see that, we assume $\epsilon \rightarrow 0$, which lead us to $\lambda_1 = \ln(|a - \epsilon/2|)$ and $\lambda_2 = \ln(|a - 4c - \epsilon/2|)$, and then we use the expansion $\ln(a - \epsilon)=\ln(a) - \epsilon/2$, in the exponents. The larger is ϵ, the smaller is the channel capacity, $\mathcal{C}_C$, which is reached for parameters that lead to $\lambda_2 = 0$ ($a - 4c - \epsilon/2 = 1$), when the maps, although not completely synchronous,

have close trajectories. If $\lambda_2 > 0$ ($c \leq -1/4 - \epsilon/8 + a/4$), then $I_C(c, \epsilon > 0) \geq I_C(c, \epsilon = 0)$, which means that a parameter mismatch can enhance the MIR of the channel, more precisely by a maximal amount of $\epsilon/4$. That means that parameter mismatches for sufficiently small coupling strengths enhance the synchronization level in an active channel. Notice that Eq. (B1) is indeed an upper bound for the MIR, being that the difference between I_C obtained from Eq. (B1) and the one from Eq. (3) is of the order of ϵ^2, for small ϵ and c, and for larger c it is smaller than $\epsilon/4$. If $\epsilon=0$, then $\lambda^{(\alpha)}=\lambda^{(\beta)}=\ln(a)$, which results in $\mathcal{C}_C=\ln(a)$, the maximal possible value for the MIR. Finally, the system capacity, $\mathcal{C}_S = 2\ln(a)$, is reached when $c=0$ and $\epsilon = 0$ [36]. *This channel is not susceptible for $\epsilon = 0$. If $\epsilon \neq 0$, it becomes susceptible, since $\mathcal{C}_C > H_{KS}^{(r)} = \ln(|a - \epsilon|)$. However, independently on ϵ, this channel is not self-excitable.*

For larger HR neural networks (up to 50 neurons) described by Eq. (9), the system capacity is reached for a network of equal neurons, but with a non-zero coupling strength. We find that $\mathcal{C}_S$ is reached for a Small World network geometry [37], being that $\mathcal{C}_S$ increases linearly with the number of neurons, Q, by $\mathcal{C}_S =13.75Q$ bits/burst. The larger Q is, the smaller the coupling strengths, which are considered to be equal. Also, the average number of connections, γ, that each neuron receives scales linearly with the number of neurons as $\gamma \propto 0.5Q$.

Networks formed by HR neurons connected to their nearest neighbors, forming a ring, can be regarded as models for the small networks of electrically connected neurons found in the Inferior Olive [38] that regulates the transmission of information between the cerebellum and the cortex, and is responsible for motor control and learning. In this type of network, we find that the system capacity increases linearly with the number of neurons by $\mathcal{C}_S =9.5N$ bits/burst, being achieved always for the same small coupling strength (0.08). Therefore, the capacity does not depend on the coupling strength. Network configurations for which the system capacity is reached operate also with a large MIR in each communication channel. This is an optimal configuration for networks found in the Inferior Olive that demand large amounts of information transmission for an efficient cerebellar learning. Naturally, we do not expect that the neural networks found in the Inferior Olive are formed by equal neurons. So, the calculated channel and system capacities should be interpreted as an upper bound for these quantities in realistic models of the networks found in there.

VIII.   THE MUTUAL INFORMATION RATE BETWEEN OTHER SUBSPACES

In this work, we are mainly interested in calculating the MIR between the subspaces α and β [Eqs. (3) and (B1)]. That means that we are mainly interested in knowing the rate of information that can be realized from a transmitter (one element of the active channel) by measuring

the signal of a receiver (another element of the active channel).

However, it is of general interest to learn how to calculate the MIR between groups of elements or between different subspaces of the active channel. For example, in many experimental situations, one cannot obtain the signal of an isolated element but rather an average field, or average quantity, such as the quantity $\vec{x}^{\parallel}$, which can be imagined as an average field ($\vec{x}_{kl}^{\parallel} = \vec{x}_k + \vec{x}_l$) between two elements.

As briefly described here, in fact, we can also calculate the MIR between other subspaces. As an illustration, we consider the calculation of the MIR between the subspaces defined by the coordinate transformations $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$, regarded as $I_C(\vec{x}^{\parallel}, \vec{x}^{\perp})$.

Typically, we should expect that $I_C(\vec{x}^{\parallel}, \vec{x}^{\perp}) \neq I_C(\alpha, \beta)$, and therefore, the MIR is coordinate dependent. While $I_C(\alpha, \beta)$ measures the rate with which information about the transmitter α can be realized by observing the receiver β, $I_C(\vec{x}^{\parallel}, \vec{x}^{\perp})$ measures the rate with which information about $\vec{x}^{\parallel}$ can be realized by observing $\vec{x}^{\perp}$. Naturally, at a situation when CS takes place, nothing about $\vec{x}^{\parallel}$ can be said by observing $\vec{x}^{\perp}$, since $\vec{x}^{\perp}=0$, and therefore, $I_C(\vec{x}^{\parallel}, \vec{x}^{\perp})=0$. So, to know from $I_C(\vec{x}^{\parallel}, \vec{x}^{\perp})$ if one element exchanges high amounts of information with another element, not only $I_C(\vec{x}^{\parallel}, \vec{x}^{\perp})$ should be low, but also $\lambda^{\perp}$.

Notice that if the network is composed by elements with a linear dynamics (or piecewise linear for continuous dynamics), we can always find a linear transformation from which $I_C(\vec{x}^{\parallel}, \vec{x}^{\perp})$ can be calculated from the state variables $\vec{x}$, or $I_C(\alpha, \beta)$ from the transformed variables $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$. Usually, one would calculate $I_C(\vec{x}^{\parallel}, \vec{x}^{\perp})$ using the variables $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$ and $I_C(\alpha, \beta)$ using the variables $\vec{x}^{\alpha}$ and $\vec{x}^{\beta}$.

Therefore, even if a measurement cannot provide the signal of isolated elements in an active network, there might be situations in which we can still calculate the mutual information rate exchanged between two isolate elements from the measurements of averages.

IX. THE TIME DEPENDENT (NOISY) ACTIVE CHANNEL

If all the elements forming the active channel suffer the influence of the same time dependent stimulus (or noise) and both the stimulus is completely uncorrelated with respect to the variables $\vec{x}^{\perp}$ and $\vec{x}^{\parallel}$ and also $\vec{x}^{\perp} \to 0$ [such that the Jacobian in Eq. (A1) is approximately block diagonal], then both exponents $\lambda^{\perp}$ and $\lambda^{\parallel}$, calculated for the autonomous channel, are not modified by the introduction of the stimulus. Also, if the largest Lyapunov exponent of the channel is not affected by the introduction of the stimulus, $\lambda^{\parallel}$ is also not modified. Thus, if the previous conditions are satisfied, Eq. (B1) calculated for the autonomous channel gives the upper bound for the mutual information of the non-autonomous channel.

The periodic channel fully preserves the transmitted information, however it might transform it in a fractal set which is vulnerable to noise. So, the information rate recovered in the receiver can be sensitively dependent on the noise level in the channel. Chaotic channels tend to destroy part of the information transmitted, even without the presence of noise ($\sum \lambda^{\perp} > 0$). However, they might offer a nice way to deal with additive noise. As shown in Ref. [28], Gaussian noise with small variance added to a chaotic trajectory can be completely filtered out.

The action of more general types of time dependent stimulus that alters the dynamics of the active channel still needs better clarification.

X. CONCLUSION

An *active channel* is an active network composed by dynamical systems. Every pair of elements forms a *communication channel* and the rate with which information is exchanged between two elements, a transmitter and a receiver, is given by the mutual information rate (MIR) between them. The maximum rate of information that can be transmitted in a communication channel of an active channel is regarded as the *channel capacity*, $\mathcal{C}_C$, and the maximum rate of information produced in the whole active channel is regarded as *system capacity*, the maximum of the Kolmogorov-Sinai entropy, H_{KS}, of the active channel. All these maximums are calculated with respect to many possible coupling strengths among the elements, for a given network topology.

We can organize the active channels in two main classes. Periodic or chaotic. A chaotic (periodic) channel is composed by a receiver that behaves in a chaotic (periodic) fashion, for long time intervals. Chaotic channels formed by HR neuron networks are self-excitable, which means that the channel capacity, $\mathcal{C}_C$, is larger than H_{KS}^0, the Kolmogorov-Sinai entropy of all the elements forming the active channel when they are uncoupled. In a self-excitable channel, a transmitter and a receiver (together with all the other elements forming the channel) mutually increase their capacity for information production, leading to an increase in their channel capacity. So, the introduction of stimuli in a self-excitable active channel might increase its channel capacity for information transmission. Not all chaotic channels present self-excitability. For example, we have not verified such a property in active channels formed by linearly coupled Rössler oscillators or by linearly coupled Chua's circuit (while in the Double Scroll attractor regime). It is to be expected that a periodic channel is non self-excitable.

More synchronization results in an increase of the MIR between two elements in an active channel, regarded as transmitter and receiver, if as the transmitter becomes more synchronous with the receiver, simultaneously also the Kolmogorov-Sinai entropy decreases, meaning that synchronization is accompanied by a reduction of the ex-

citation in the channel. This situation is to be expected in non self-excitable channels. In self-excitable channels, a large amount of information transmission can be obtained when the bursts are phase synchronous while the spikes are highly desynchronous.

Periodic channels might allow the complete transmission of the information signal provided by the transmitter. On the other hand, the transmitted information in a chaotic channel might be lost due to the presence of non-synchronous chaotic trajectories, if the transmitter is weakly coupled to the receiver. However, while a periodic channel might be very sensible to the presence of even arbitrarily small noise intensities, chaotic channels might be robust.

APPENDIX A: CONDITIONAL EXPONENTS AND COMPLETE SYNCHRONIZATION (CS)

Assume $\vec{x}_k$ to describe the state variables of subsystem k. The parallel and perpendicular subspaces are defined

to be a transformation in the variables of the subsystems that maximize the calculated mutual information. For the cases here studied, the parallel subspace between α_k and α_l is defined as $\vec{x}_{kl}^{\parallel} = \vec{x}_k + \vec{x}_l$, and the transversal subspace is defined as $\vec{x}_{kl}^{\perp} = \vec{x}_k - \vec{x}_l$. For a network of Q elements, $k = [1, Q-1]$ and $l = [k+1, Q]$. Writing the equations of motion in these new variables, one can separate the transformed equations into subsystems that contain only terms of that subsystem. So, a network of Q elements formed by systems of dimension m ($\mathcal{R}^m$), can be broken down in $Q(Q-1)/2$ subsystems of dimensionality $2m$ ($\mathcal{R}^{2m}$). Then, the conditional exponents of the neural network between two subspaces measure the exponential divergence of nearby trajectories of the transformed equations for these two subspaces, which in practice is calculated using the following Jacobian in the method of Ref.[12]

$$
\begin{array}{cc}
\dfrac{\partial \dot{\vec{x}}_{kl}^{\perp}}{\partial \vec{x}_{kl}^{\perp}} & \dfrac{\partial \dot{\vec{x}}_{kl}^{\perp}}{\partial \vec{x}_{kl}^{\parallel}} \\[2ex]
\dfrac{\partial \dot{\vec{x}}_{kl}^{\parallel}}{\partial \vec{x}_{kl}^{\perp}} & \dfrac{\partial \dot{\vec{x}}_{kl}^{\parallel}}{\partial \vec{x}_{kl}^{\parallel}}
\end{array}
\tag{A1}
$$

which is

$$
\begin{array}{cccccc}
3x^{\parallel} - \frac{3(x^{\perp 2} + x^{\parallel 2})}{4} + F & 1 & -1 & 3x^{\perp} - \frac{3x^{\parallel}x^{\perp}}{2} & 0 & 0 \\
-5x^{\parallel} & -1 & 0 & -5x^{\perp} & 0 & 0 \\
4r & 0 & -r & 0 & 0 & 0 \\
3x^{\perp} - \frac{3x^{\perp}x^{\parallel}}{2} + G & 0 & 0 & 3x^{\parallel} - \frac{3(x^{\parallel 2} + x^{\perp 2})}{4} & 1 & -1 \\
-5x^{\perp} & 0 & 0 & -5x^{\parallel} & -1 & 0 \\
0 & 0 & 0 & 4r & 0 & -r
\end{array}
$$

where x stands for x_{kl}. For a network of Q fully connected neurons with equal coupling strengths, $F = -QA_{kl}$. When $\xi = [(Q-2)/2]\vec{x}_{ij}^{\parallel} - \sum_{k=1}^{Q} \vec{x}_k$ (with $k \neq i, j$) is either orders of magnitude smaller than the quadratic terms (no synchronization) or $\xi = 0$ (CS), then $G \cong 0$. For parameter values close to or when PPS or PS is present, the quadratic terms are also small and G cannot be neglected. In Fig. 4, this happens for the parameter region $A_{kl}, A_{lk} \cong [0.15, 0.25]$. There, Eq. (8) seems to be violated. To resolve that, we force $\xi \cong 0$, which leads to $G \cong 0$, condition for which the Jacobian in Eq. (A1) can be used. We set the initial conditions all equal, and

integrate the system for a small time interval (5 bursts which is equivalent to about 50 spikes) to estimate a finite time averaged MIR, indicated in this figure by $\langle I_C \rangle_f$. Within the time scale for which phenomena happen in real biological neural networks, periodic state as well as chaotic state in the asymptotic sense might never be observed, but rather a transient state whose subspaces $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$ possess finite time conditional exponents. Finite time quantities are well defined in dynamical systems.

For $Q=2$, $F = -2A_{kl}$ and $G = 0$. When CS takes place in a network formed by Q neurons, the only term of the Jacobian that changes is F. For this Jacobian,

we can calculate that CS appears if $F(Q) \leq F(Q = 2)$, where $F(Q = 2) = -2A_{kl}(Q = 2)$, with $A_{kl}(Q = 2)=0.5$ being the coupling for which complete synchronization appears for a configuration of two coupled neurons [see also [39]]. So, the coupling to reach CS is $A_{kl} \geq 1/Q$, when the second largest Lyapunov as well as all transversal conditional exponents are negative. At this point, the trajectory distance between any pair of neurons tends to zero.

For the active channels composed by coupled one-dimensional maps

$$x_{n+1}^{(j)} = 2x_n^{(j)} + \sum_{i=1}^{Q} 2c(x_n^{(i)} - x_n^{(j)}), mod(1),$$

we can calculate the mutual information in each communication channel exactly, with no need of any special conditions. The network equations can be broken down in subspaces that depend only on the parallel or transversal variables of that subspace. So, $x_{n+1}^{(kl)\perp}=Fx_n^{(kl)\perp}$, and $x_{n+1}^{(kl)\parallel}=Gx_n^{(kl)\parallel} + H(x_n^{(kl)\perp})$. H does not participate in the calculation of the conditional exponents and can be ignored. For a fully connected network formed by Q maps with equal coupling strengths c, $F=2(1 - Qc)$ and $G = 2$. The conditional exponents are $\lambda^{\perp}=\ln(|F|)$ and $\lambda^{\parallel} = \ln(|G|)$. For a network of $Q=4$ maps bidirectionally connected to their nearest neighbors forming a closed ring, i.e. $x^{(i)}$ is connected to $x^{(i+1)}$ and to $x^{(i-1)}$, and $x^{(Q)}$ is connected to $x^{(Q-1)}$ and to $x^{(1)}$, then $F=2[1-2c]$ ($F=2[1-c]$) and $G=2$ for subspaces whose pair of systems have a direct connection (no connection). This network completely synchronizes when all the $\lambda^{\perp} <0$, and thus, when $c > 1/2$.

APPENDIX B: INFORMATION DIMENSION, LYAPUNOV EXPONENTS AND MIR

We consider an active channel formed by only one communication channel, composed by two coupled unidimensional systems which produce an attractor Γ with at most two positive Lyapunov exponents. Γ is corse-grained with volumes of size ϵ [14], and for $\epsilon \to 0$, we have that $\sum_i P_i' \ln P_i'/ \ln(\epsilon) = D_1$, with D_1 being the information dimension of Γ, a quantity that measures the information content of Γ and P_i' is the probability of finding a trajectory point in one of the i volumes of size ϵ. The average probability $\langle P \rangle$ of finding a trajectory following an itinerary visiting one of the possible m combinations of sequences of L volumes of size ϵ for a time interval $L\tau$ is $\langle P \rangle \propto \exp^{-\tau L \sum_j^+ D_1^{(j)} \bullet \lambda_j}$, and $D_1^{(j)}$ ($\sum_j D_1^{(j)} = D_1$ and $D_1^{(j)} \in [0,1]$) are the partial information dimensions [40], a quantity that measures the information content of Γ along the direction j, either parallel or orthogonal to the trajectory, and λ_j are the Lyapunov exponents in the direction j. $\bullet$ is the inner product. Assuming that the distribution of trajectory points is smooth

along unstable directions (associated with positive exponents) and the system possesses a Sinai-Ruelle-Bowen (SRB) measure [12], $D_1^{(j)} \to 1$ if $\lambda_j > 0$. From [16, 26], $1/(\tau L)\sum_m P_i \ln P_i = H_{KS}$ for SRB systems. To understand how that is derived, we assume uniformity in the probability distribution, $-\sum_m P_i \ln P_i = -\ln \langle P \rangle$. Then, the term $H_{KS}^{(\alpha,\beta)}$ [in Eq. (3)] can be calculated by knowing that $-\ln \langle P \rangle/(\tau L) \sim \sum_j^+ \lambda_j$. Now, we make the intuitive hypothesis that the terms $\sigma^{(\alpha)}$ and $\sigma^{(\beta)}$, in Eq. (2), preserve the physical quantities used to calculate $\sigma^{(\alpha,\beta)}$. So, if $\sigma^{(\alpha,\beta)}$ is a function of the information dimension and the Lyapunov exponents of the trajectory on the subspaces (α,β), similarly, $\sigma^{(\alpha)}$ and $\sigma^{(\beta)}$ should be a function of these quantities. This hypothesis provides the terms $D_1^{(\alpha)}|\lambda^{(\alpha)}| + D_1^{(\beta)}|\lambda^{(\beta)}|$ in Eq. (3). If a projection of Γ onto the lower-dimensional subspaces α and β produces a fractal set, one has to consider the absolute value of the negative Lyapunov exponent [26]. Otherwise, $D_1^{(\alpha)} = D_1^{(\beta)} = \max [D_1^{(j)}]$, assuming that the subspaces α and β contain only the dynamics of the expanding directions, i.e., either they are unidimensional or they can be reduced to a unidimensional subspace by a mapping of the flow [18]. For most of the chaotic channels here studied, $\max [D_1^{(j)}]=1$, for typical projections of Γ. Further, we calculate the Lyapunov exponents of the transmitter and the receiver as if they were detached from each other. If the elements that compose the channel have equal parameters, $\lambda^{(\alpha)} = \lambda^{(\beta)}=\lambda_1=\lambda^{\parallel}$, otherwise, $\lambda^{(\alpha)} \neq \lambda^{(\beta)}$.

For the considered networks, each pair of elements, in the transformed variables of the parallel and transversal subspaces, forms a composed subspace (α_k, β_l) that produces at most two positive conditional exponents. The previous analysis applies for each communication channel of this network, since each pair of neurons has a dynamics equivalent to a bidimensional discrete map with at most two positive conditional Lyapunov exponents, a larger one $\lambda_{kl}^{\parallel}$ and a smaller one $\lambda_{kl}^{\perp}$. Then,

$$I_C(\alpha_k, \beta_l) = \lambda_{kl}^{\parallel} - \lambda_{kl}^{\perp}, \tag{B1}$$

if $\lambda_{kl}^{\perp} > 0$, or $I_C(\alpha_k, \beta_l) = \lambda_{kl}^{\parallel}$, otherwise. To derive Eq. (B1), we have used that $\lambda^{(\alpha_k)}=\lambda^{(\beta_l)}= \lambda_{kl}^{\parallel}$, $D_1^{(\cdot)} = \max(D_1^{(j)})=1$, and $H_{KS}^{(\alpha_k,\beta_l)}$ is the sum of all positive conditional exponents of the composed subspace (α_k, β_l). This can be done whenever the subspace (α_k, β_l) is separable from the whole network. If the elements forming the channel have different parameters and one still wants to use Eq. (B1), be in mind that such an equation might provide an upper bound for the MIR between these two subspaces.

For arbitrary networks, for simplicity let us imagine two coupled oscillators $\vec{x}^{(\alpha)}$ and $\vec{x}^{(\beta)}$, coupled by a term c, the terms $\lambda^{(\alpha)}$ and $\lambda^{(\beta)}$ from Eq. (3) can be analytically or semi-analytically calculated if the coupling is either sufficiently small (such that the elements forming

the network are almost decoupled) or sufficiently large (such that the whole network has a high level of synchrony). At this situation, $\lambda^{(\alpha)}$ ($\lambda^{(\beta)}$) is the Lyapunov exponent in the subspace of the oscillator $\vec{x}^{(\alpha)}$ ($\vec{x}^{(\beta)}$), i.e., assume $\vec{x}^{(\beta)} = 0$ ($\vec{x}^{(\alpha)} = 0$) and $c=0$, and then calculate the Lyapunov exponents by the usual methods. For such coupling strengths, this is an equivalent approach to the one described in Sec. VII. For other coupling strengths, we expect that $D_1^{(\alpha)}|\lambda^{(\alpha)}| + D_1^{(\beta)}|\lambda^{(\beta)}| - H_{KS}^{(\alpha,\beta)} \leq H_{KS}^{(\alpha,\beta)}$. If not, that points to the existence of trajectory foldings in the lower-dimensional subspaces α and β which results in an overestimation of the Lyapunov exponents, $\lambda^{(\alpha)}$ and $\lambda^{(\beta)}$. In such cases, $D_1^{(.)}$ should be underestimated [41], so balancing the action of the trajectory foldings.

If the trajectory is very close to the synchronization manifold and so, $\vec{x}_1 \cong \vec{x}_2 \cong \ldots \cong \vec{x}_Q$, the term $\partial \dot{\vec{x}}_{kl}^{\parallel}/\partial \vec{x}_{kl}^{\parallel}$ in Eq. (A1) gives the largest Lyapunov exponent λ of the network which equals the largest exponent of one neuron, and thus, Eq. (B1) can be estimated by $I_C(\alpha_k, \alpha_l) = \lambda - \lambda_{kl}^{\perp}$. This equation agrees with the intuitive idea that the amount of information exchanged between two systems within a large network is given by the amount of information production of one system (λ) minus the error in the transmission between both systems ($\lambda_{kl}^{\perp}$).

APPENDIX C: PHASE AND PHASE SYNCHRONIZATION

Phase synchronization [7] is a phenomenon defined by

$$|\phi_k - m\phi_l| \leq r, \qquad (C1)$$

where ϕ_k and ϕ_l are the phases of two neurons α_k and α_l, $m = \omega_l/\omega_k$ is a real number [42], and ω_k and ω_l are the average frequencies of oscillation of the neurons α_k and α_l, and r is a finite number [43]. In this work, we have used in Eq. (C1) $m = 1$, which means that we search for $\omega_k : \omega_l$=1:1 (rational) phase synchronization [7]. If another type of $\omega_k : \omega_l$-PS is present, the methods in Refs. [43, 46, 47] can detect.

The phase ϕ is a function constructed on a 2D subspace, whose trajectory projection has proper rotation, i.e, it rotates around a well defined center of rotation. So, the phase is a function of a subspace. Usually, a good 2D subspace of the HR neurons is formed by the variables x and y, and whenever there is proper rotation in this subspace the phase can be calculated by [44]

$$\phi(t) = \int_0^t \frac{\dot{y}x - \dot{x}y}{(x^2 + y^2)}dt. \qquad (C2)$$

If there is no proper rotation in the subspace (x, y) one can still find proper rotation in the velocity subspace $(\dot{x}, \dot{y})$ and a phase can be defined by [44]

$$\phi(t) = \int_0^t \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)}dt. \qquad (C3)$$

If a good 2D subspace can be found, one can also define a phase by means of Hilbert transformation, which basically transforms an oscillatory scalar signal into a two components signal [45]. In the active channel of Eq. (9), for the coupling strength interval $A_{kl} \cong [0, 0.05]$, the subspace (x, y) has proper rotation, and therefore, phase is well defined and can be calculated by Eq. (C2). However, for this coupling interval, Eq. (C1) is not satisfied, and therefore, there is no PS between any pair of neurons in the subspace (x, y).

For the coupling strength interval $A_{kl} \cong [0.05, 0.23]$, the neurons trajectories lose proper rotation both in the subspaces (x, y) and $(\dot{x}, \dot{y})$. The phase cannot be calculated by Eq. (C2) or by Eq. (C3). That is due to the fact that the chaotic trajectory gets arbitrarily close to the neighborhood of the equilibrium point (x, y)=(0, 0), a manifestation that a homoclinic orbit to this point exists.

In fact, the Hilbert transformation also fails to provide the phase from either scalar signals x or y, since these signals do not present any longer an oscillatory behavior close to the equilibrium point. In such cases, even the traditional technique to detect PS by defining the phase as a function that grows 2π, whenever a trajectory component crosses a threshold cannot be used. Since the trajectory comes arbitrarily close to the equilibrium point, no threshold can be defined such that the phase difference between pairs of neurons is bounded. Notice that by this definition the phase difference equals $2\pi\Delta N$. For that reason, Fig. 4**b** would remain roughly as it is even if the thresholds that define a spike and a burst are modified or even if another variable (either y or z) is used. In this figure, a burst (spike) in a neuron α_k is considered to start/end if x_k crosses the threshold defined by x_k=-1.0 (x_k=0.0).

In order to check if PS indeed exists in at least one subspace, alternative methods of detection can be employed as proposed in Refs. [46, 47]. In short, if PS exists in a subspace then by observing one neuron trajectory at the time the other bursts or spikes (or any typical event), there exists at least one special curve, Γ, in this subspace, for which the points obtained from these conditional observations do not visit the neighborhood of Γ. A curve Γ is defined in the following way. Given a point x_0 in the attractor projected on the subspace of one neuron where the phase is defined, Γ is the union of all points for which the phase, calculated from this initial point x_0 reaches $n\langle r \rangle$, with $n = 1, 2, 3, \ldots, \infty$ and $\langle r \rangle$ a constant, usually 2π. So, note that an infinite number of curves Γ can be defined. For coupled systems with sufficiently close parameters that present in some subspace proper rotation, if the points obtained from the conditional observations do not visit the whole attractor projection on this subspace, one can always find a curve Γ that is far away from the conditional observations. Therefore, for such cases, to state the existence of PS one just has to check if the conditional observations are localized with respect to the attractor projection on the subspace where the phase is calculated.
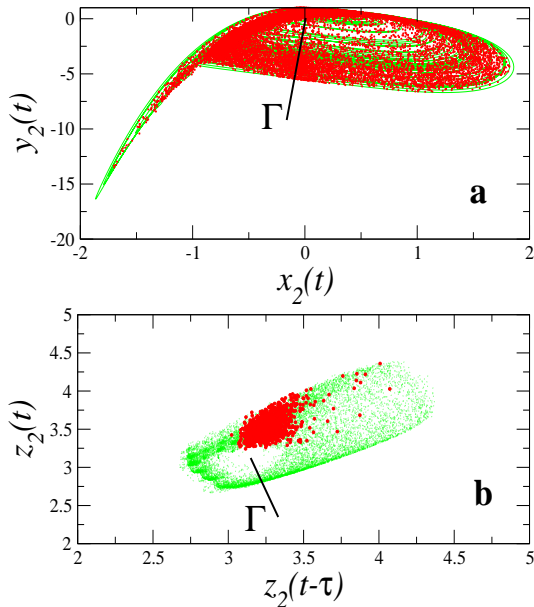
FIG. 5: [Color online] The network of Eqs. (9) for A_{kl}=0.1. The curve Γ, a continuous curve transversal to the trajectory, is pictorially represented by the straight line Γ. **a**, the conditional observations are not localized and thus there is no PS in this subspace. The light gray line (green online) represents the attractor projection on the subspace (x, y) of the neuron α_2, and filled gray circles (red online) represent the points obtained from the conditional observations of the neuron α_2 whenever the neuron α_4 spikes. The point $(x, y) = (0.0)$ does not belong to Γ. **b**, the conditional observations are localized and thus there is PS in this subspace. Light gray dots (green online) represent the reconstructed attractor $z_2(t) \times z_2(t-\tau)$, for τ=30, and filled circles (red online) represent the points obtained from the conditional observation of neuron α_2, whenever the reconstructed trajectory of the neuron α_4 crosses the threshold line $z_4(t-\tau) = 3.25$ and $z_4(t) > 3$.

Conditional observations of the neuron trajectory α_k in the subspace (x, y), whenever another neuron α_l spikes, in the system modeled by Eqs. (9), are not localized with respect to a curve Γ, for the coupling strength $0.05 < A_{kl} < 0.23$. An example can be seen in Fig. 5**a**, for $A_{kl} = 0.1$. The set of points produced by the conditional observations are represented by dark gray circles (red online), and the attractor by the light gray points (green online). Therefore, there is no PS in the subspace (x, y). However, the points obtained from the conditional observations do not visit the whole attractor in the subspace (x, y). This is an evidence that there is PS in some other subspace.

In order to know on which subspace PS occurs, we proceed in the following way. We reconstruct the neuron attractors by means of the time-delay technique, using the variable z. This variable describes the slow timescale, responsible for the occurrence of bursts. The reconstructed attractor $z(t) \times z(t - \tau)$ has proper rotation (see Fig. 5**b**) and the points obtained from the condi-

tional observations do not visit the neighborhood of a curve Γ, then, there is PS in this subspace. Indeed, we find localized sets with respect to a curve Γ in the system of Eqs. (9), in the reconstructed subspace $(z(t) \times z(t-\tau))$, for $A_{kl} \geq 0.1$.

So, for the parameter interval $A_{kl} = [0.1, 23[$, there is no PS in the subspace (x, y) but there is PS in the subspace of the variable z. In this type of synchronous behavior, the bursts are phase synchronized while the spikes are not. This behavior is regarded as bursting phase synchronization (BPS).

For simplicity in the analysis, we say that BPS happens when at least one pair of neurons is phase synchronous in the bursts. Partial phase synchronization (PPS) happens in the network when it is true that for at least one pair of neurons Eq. (C1) is satisfied by the phases as defined by either Eq. (C2) or Eq. (C3). In addition, at the coupling strengths for which PPS appears, one positive Lyapunov and one positive transversal conditional exponent become negative. At the coupling strengths for which Eq. (C1) is satisfied for all pair of neurons (there is PS), the second largest Lyapunov exponent and all the transversal conditional exponents become non positive.

Notice that these phenomena happen in a hierarchical way organized by the "intensity" of synchronization. The presence of a stronger type of synchronization implies in the presence of other softer types of synchronization in the following order: CS $\rightarrow$ PS $\rightarrow$ PPS $\rightarrow$ BPS.

APPENDIX D: RECOVERY OF INFORMATION

Equations (2) and (B1) give the amount of information that can be retrieved in the receiver per time unit. Imagine the neural network. If, in average, a burst happens for a time interval ΔT, one can retrieve in the receiver an amount $I_C \Delta T$ of information about the transmitter *per burst*. It is often desirable to known how much information one single observation with precision ϵ can provide. Assuming that observations are taken over in time intervals not smaller than $\delta t = -\ln(\epsilon)/\lambda^{\parallel}$, the maximal amount of information, I_m, that can be retrieved in the receiver about the transmitter in each observation is estimated by $I_m = (\lambda^{\perp}/\lambda^{\parallel} - 1) \ln(\epsilon)$. We arrive at this result by assuming δt to be the memory time of the channel, the time interval for which observations in the receiver trajectory with precision ϵ, at a time t_0, will provide no information about the transmitter trajectory, at the time $t_0 + \delta t$, and $I_m = I_C \times \delta t$.

APPENDIX E: TRANSIENT DYNAMICS

If an active channel is being externally stimulated or if the initial conditions are far away from the asymptotic (for large time intervals) stable state (periodic or chaotic behavior), the channel will present a transient dynamics. In such a case, Eqs. (3) and (B1) should remain valid by

the use of finite time conditional or Lyapunov exponents (assuming $D_1^j=1$). As an illustrative example, an active channel that has an asymptotic chaotic attractor might have to be treated as a periodic channel (space contracting dynamics), if the initial conditions are far away from the chaotic set and the dynamics is dominated by the stable directions.

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