The self-energy of the uniform electron gas in the second order of exchange

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The on-shell self-energy of the homogeneous electron gas in second order of exchange, $\Sigma_{2x} = \mathrm{Re}\ \Sigma_{2x}(k_F, k_F^2/2)$, is given by a certain integral. This integral is treated here in a similar way as Onsager, Mittag, and Stephen [Ann. Physik (Leipzig) **18**, 71 (1966)] have obtained their famous analytical expression $e_{2x} = \frac{1}{6}\ln 2 - \frac{3}{4}\frac{\zeta(3)}{\pi^2}$ (in atomic units) for the correlation energy in second order of exchange. Here it is shown that the result for the corresponding on-shell self-energy is $\Sigma_{2x} = e_{2x}$. The off-shell self-energy $\Sigma_{2x}(k, \omega)$ correctly yields $2e_{2x}$ (the potential component of e_{2x}) through the Galitskii-Migdal formula. The quantities e_{2x} and Σ_{2x} appear in the high-density limit of the Hugenholtz-van Hove (Luttinger-Ward) theorem.

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I. INTRODUCTION

Although not present in the Periodic Table the homogeneous electron gas (HEG) is still an important model system for electronic structure theory, cf. e.g. [1]. In its spin-unpolarized version the HEG ground state is characterized by only one parameter r_s, such that a sphere with the radius r_s contains *on average* one electron [2]. It determines the Fermi wave number as $k_\mathrm{F} = 1/(\alpha r_s)$ with $\alpha = (4/(9\pi))^{1/3}$ and it measures simultaneously the interaction strength and the density such that high density corresponds to weak interaction and hence weak correlation. For recent papers on this limit cf. [3–5]. Naively one should expect that in this weak-correlation limit the Coulomb repulsion $\alpha r_s/r$ (where lengths and energies are measured in units of k_F^{-1} and k_F^2, respectively) can be treated as perturbation. But in the early theory of the HEG, Heisenberg [6] has shown, that ordinary perturbation theory does not apply. With e_0 being the energy per particle of the ideal Fermi gas and e_x being the exchange energy in lowest (1st) order, the total energy $e = e_0 + e_\mathrm{x} + e_\mathrm{c}$ defines the correlation energy $e_\mathrm{c} = e_2 + e_3 + \cdots$. In 2nd order, there is a direct (d) term $e_{2\mathrm{d}}$ and an exchange (x) term $e_{2\mathrm{x}}$, so that $e_2 = e_{2\mathrm{d}} + e_{2\mathrm{x}}$. Whereas $e_{2\mathrm{x}}/(\alpha r_s)^2$ is a pure finite number b_x, the direct term $e_{2\mathrm{d}}$ logarithmically diverges along the Fermi surface (i.e. for vanishing transition momenta q): $e_{2\mathrm{d}} \to \ln q$ for $q \to 0$. This failure of perturbation theory has been repaired by Macke [7] with an appropriate partial summation of higher-order terms up to infinite order. The result of this (ring-diagram) summation for the correlation energy in its weak-correlation limit is $e_\mathrm{c}/(\alpha r_s)^2 = a \ln r_s + \cdots$ with $a = (1 - \ln 2)/\pi^2 \approx 0.031091$. This has been confirmed later by Gell-Mann and Brueckner [8], who in addition to the logarithmic term numerically calculated contributions to the next (constant, i.e. not depending on r_s) term b, namely b_r and b_d arising from the ring-diagram summation and from $e_{2\mathrm{d}}$, respectively: $b_\mathrm{r} \approx a \ln \frac{\alpha}{\pi} - 0.001656 \approx -0.057514$ and $b_\mathrm{d} \approx -0.013586$. The total constant term is $b = b_\mathrm{r} + b_\mathrm{d} + b_\mathrm{x} \approx -0.046921$, where the exchange term $b_\mathrm{x} = \frac{1}{6}\ln 2 - \frac{3}{4}\frac{\zeta(3)}{\pi^2} \approx +0.0241792$ has been *analytically* calculated by Onsager, Mittag, and Stephen in a very tricky way [9]. Although the integrand is rather simple, the Pauli principle causes complicated boundary conditions, which need a sophisticated treatment through several substitutions of the integral variables.

Here their method is used to calculate the analog term of the self-energy on the energy shell, namely $\Sigma_{2\mathrm{x}} = \mathrm{Re}\ \Sigma_{2\mathrm{x}}(1,1/2)$ [with k measured in units of k_{F} and ω measured in units of k_{F}^2]. The more general quantity $\Sigma_{2\mathrm{x}}(k,\omega)$ appears (i) in a recent study of the spectral moments of the HEG [11], (ii) in the context of self-consistent *GW* calculations [12], and within a project 'spectral functions for a hydrogen plasma' [13]. For $k=1$, $\omega=\mu$ it appears in the Hugenholtz-van Hove (Luttinger-Ward) identity [14], which relates the chemical potential μ to the self-energy $\Sigma(k,\omega)$ according to $\mu-\mu_0=\Sigma(1,\mu)$. The chemical potential follows from the total energy according to $\mu=(\frac{5}{3}-\frac{1}{3}r_s\frac{d}{dr_s})e$. In the weak-correlation limit $r_s\to 0$ the total energy $e=e_0+e_{\mathrm{x}}+e_{\mathrm{c}}$ and the chemical potential $\mu=\mu_0+\mu_{\mathrm{x}}+\mu_{\mathrm{c}}$ are given by

$$\begin{aligned} e_0 &= \frac{3}{10}, \qquad e_{\mathrm{x}} = -\frac{3}{4}\frac{\alpha r_s}{\pi}, \qquad e_{\mathrm{c}} = (\alpha r_s)^2[a\ln r_s + b + O(r_s)], \\ \mu_0 &= \frac{1}{2}, \quad \mu_{\mathrm{x}} = -\frac{\alpha r_s}{\pi}, \quad \mu_{\mathrm{c}} = (\alpha r_s)^2\left[a\ln r_s + \left(-\frac{1}{3}a+b\right) + O(r_s)\right]. \end{aligned} \tag{1.1}$$

Because of the above mentioned identity the self-energy $\Sigma(1,\mu)$ must have a corresponding behavior. The exchange in lowest order yields $\Sigma_{\mathrm{x}}(1) = -\frac{\alpha r_s}{\pi}$ or $\mu_{\mathrm{x}} = \Sigma_{\mathrm{x}}(1)$, exactly in agreement with the mentioned identity. To obtain also the logarithmic term of $\Sigma(1,1/2)$ the ring-diagram summation has to be done for the self-energy [15]. To the next term beyond the logarithmic term contributes the 2nd-order exchange self-energy $\Sigma_{2\mathrm{x}} = \mathrm{Re}\ \Sigma_{2\mathrm{x}}(1,1/2)$. Just this term is calculated here using the tricky method of Onsager, Mittag, and Stephen [9] *mutatis mutandi*. The Feynman diagrams of $e_{2\mathrm{x}}$ and $\Sigma_{2\mathrm{x}}$ are given in Figs. 1 and 2. As shown in the Appendix, from the Feynman diagram rules it follows $\Sigma_{2\mathrm{x}} = -\frac{(\alpha r_s)^2}{4\pi^4}(X_1+X_2)$, where $X_{1,2}$ mean the integrals defined in Eqs. (A.10) and (A.11). They are calculated in the following sections. The final results are $X_1 = -\pi^4\left[\frac{4}{3}\ln 2 - 5\frac{\zeta(3)}{\pi^2}\right]$, $X_2 = \pi^4\left[\frac{2}{3}\ln 2 - 2\frac{\zeta(3)}{\pi^2}\right]$, thus $\Sigma_{2\mathrm{x}} = (\alpha r_s)^2\left[\frac{1}{6}\ln 2 - \frac{3}{4}\frac{\zeta(3)}{\pi^2}\right] = e_{2\mathrm{x}}$, thus $\mu_{2\mathrm{x}} = \Sigma_{2\mathrm{x}}$. This relation appears in the weak-correlation limit of the above mentioned Hugenholtz-van Hove theorem [14, 15]. This paper is a contribution to the mathematics of the weakly-correlated (high-density) HEG.

II. THE INTEGRAL X_1

In Eq. (A.10), new variables $\boldsymbol{q}' = (\boldsymbol{q}_1+\boldsymbol{q}_2)/2$, $\boldsymbol{p}' = (\boldsymbol{q}_1-\boldsymbol{q}_2)/2$, and $\boldsymbol{s}' = \boldsymbol{e}+(\boldsymbol{q}_1+\boldsymbol{q}_2)/2$ lead to

$$\begin{aligned} \boldsymbol{q}_1 = \boldsymbol{q}'+\boldsymbol{p}', \qquad (\boldsymbol{e}+\boldsymbol{q}_1)^2 &= (\boldsymbol{s}'+\boldsymbol{p}')^2, \quad (\boldsymbol{e}+\boldsymbol{q}_2)^2 = (\boldsymbol{s}'-\boldsymbol{p}')^2, \\ \boldsymbol{q}_2 = \boldsymbol{q}'-\boldsymbol{p}', \quad (\boldsymbol{e}+\boldsymbol{q}_1+\boldsymbol{q}_2)^2 &= (\boldsymbol{s}'+\boldsymbol{q}')^2, \qquad \boldsymbol{e}^2 = (\boldsymbol{s}'-\boldsymbol{q}')^2. \end{aligned} \tag{2.1}$$

Therefore (with $\Theta(x)$ = Heaviside step function)

$$X_1 = \int d^3q' d^3p' \; \frac{2^3}{q'^2 - p'^2} \; \frac{1}{(\boldsymbol{p}' + \boldsymbol{q}')^2 (\boldsymbol{p}' - \boldsymbol{q}')^2} \; \times$$

$$\times \int \frac{d^2s'}{\pi} \; \delta((\boldsymbol{s}' - \boldsymbol{q}')^2 - 1) \Theta[1 - (\boldsymbol{s}' + \boldsymbol{q}')^2] \Theta[(\boldsymbol{s}' + \boldsymbol{p}')^2 - 1] \Theta[(\boldsymbol{s}' - \boldsymbol{p}')^2 - 1]. \tag{2.2}$$

The next step scales $\boldsymbol{q}'$, $\boldsymbol{p}'$, $\boldsymbol{s}'$ with $\lambda = 1/\sqrt{1 - s'^2}$ according to $\boldsymbol{q} = \lambda \boldsymbol{q}'$, $\boldsymbol{p} = \lambda \boldsymbol{p}$', $\boldsymbol{s} = \lambda \boldsymbol{s}$' with the consequences

$$1 - s'^2 = \frac{1}{1 + s^2}, \quad d^2s' = \frac{d^2s}{(1 + s^2)^2},$$
$$\delta((\boldsymbol{s}' - \boldsymbol{q}')^2 - 1) = (1 + s^2) \; \delta(q^2 - 2\boldsymbol{sq} - 1),$$
$$\pm 2\boldsymbol{sp} + p^2 > 1, \quad 2\boldsymbol{sq} + q^2 < 1, \quad -2\boldsymbol{sq} + q^2 = 1, \tag{2.3}$$

from which follow $p \geq 1$ and $q \leq 1$ (what makes the energy denominator $q^2 - p^2$ negative) and $\boldsymbol{sq} < 0$, $s \geq \alpha$. Thus

$$X_1 = -\int_0^1 dq \int_1^\infty dp \; \frac{16\;\pi}{p^2 - q^2} \int_{-1}^{+1} \frac{dx}{2} \; \frac{8\;q^2p^2}{(p^2 + q^2)^2 - (2\boldsymbol{qp})^2} \; \times$$

$$\times \int_{s \geq \alpha,\; \cos\varphi_q < 0} \frac{d^2s}{1 + s^2} \; \delta(q^2 - 1 - 2\boldsymbol{sq}) \; \Theta(p^2 - 1 + 2\boldsymbol{sp}) \; \Theta(p^2 - 1 - 2\boldsymbol{sp}). \tag{2.4}$$

Here and in the following of Sec. II the abbreviations

$$\alpha = \frac{1 - q^2}{2q} \geq 0, \quad \beta = \frac{p^2 - 1}{2p} \geq 0, \quad a = \frac{q^2 + p^2}{2qp} \geq 1,$$

$$\frac{t}{s} = \cos\varphi_p = \cos\sphericalangle(\boldsymbol{s}, \boldsymbol{p}), \quad x = \cos\varphi = \cos\sphericalangle(\boldsymbol{q}, \boldsymbol{p}), \quad y = \cos\varphi_q = \cos\sphericalangle(\boldsymbol{q}, \boldsymbol{s}) \tag{2.5}$$

are used:

$$X_1 = -\int_0^1 dq \int_1^\infty dp \; \frac{16\;\pi}{p^2 - q^2} \int_{-1}^{+1} \frac{dx}{a^2 - x^2} \int_{s \geq \alpha,\; \cos\varphi_q < 0} \frac{sds\; d\varphi_q}{1 + s^2} \; \frac{1}{2sq} \; \delta\left(\frac{\alpha}{s} + y\right) \; \Theta(\beta - |t|) \tag{2.6}$$

t is introduced to replace s after having done the φ_q integration. To this purpose the relation between t and s is needed. It follows from $\varphi_q + \varphi_p = \varphi$ or $\varphi_p = \varphi - \varphi_q$, therefore $\cos\varphi_p = \cos(\varphi - \varphi_q) = \cos\varphi\cos\varphi_q + \sin\varphi\sin\varphi_q$, what is in terms of t/s, x, and y (the latter quantity equals $-\alpha/s$ because of the delta function):

$$\frac{t}{s} = x\left(-\frac{\alpha}{s}\right) \pm \sqrt{1 - x^2}\sqrt{1 - \left(\frac{\alpha}{s}\right)^2} \quad \text{or} \quad (t + \alpha x)^2 = (1 - x^2)(s^2 - \alpha^2). \tag{2.7}$$

This has the consequences

$$s^2 - \alpha^2 = \frac{(t+\alpha x)^2}{1-x^2} \ , \tag{2.8}$$

$$\frac{sds}{1+s^2} = \frac{(t+\alpha x)dt}{(t+\alpha x)^2 + (1+\alpha^2)(1-x^2)} \ . \tag{2.9}$$

With the help of Eq. (2.8) the φ_q integration yields a function depending on s (respectively on t). Note that $\cos\varphi_q = -\frac{\alpha}{s}$ has two solutions $\varphi_{q,1}$ and $\varphi_{q,2}$ with $|\sin\varphi_{q,1}| = |\sin\varphi_{q,2}| = \sqrt{1-(\frac{\alpha}{s})^2}$:

$$\int_0^{2\pi} \frac{d\varphi_q}{s} \ [\delta(-\sin\varphi_{q,1}\cdot(\varphi_q - \varphi_{q,1})) + \delta(-\sin\varphi_{q,2}\cdot(\varphi_q-\varphi_{q,2}))]$$

$$= \int_0^{2\pi} \frac{d\varphi_q}{s} \ \frac{\delta(\varphi_q-\varphi_{q,1}) + \delta(\varphi_q - \varphi_{q,2})}{\sqrt{1-(\frac{\alpha}{s})^2}} = \frac{2}{\sqrt{s^2-\alpha^2}} = 2\ \frac{\sqrt{1-x^2}}{|t+\alpha x|} \ . \tag{2.10}$$

Thus

$$\int\limits_{s\geq\alpha,\ \cos\varphi_q} \frac{sds\ d\varphi_q}{1+s^2}\ \frac{1}{2sq}\ \delta\left(\frac{\alpha}{s}+y\right)\ \Theta(\beta-|t|) = \frac{\sqrt{1-x^2}}{q} \int_{t_0}^{t_1} \frac{\mathrm{sign}(t+\alpha x)\ dt}{(t+\alpha x)^2 + (1+\alpha^2)(1-x^2)}$$

$$= \frac{2}{1+q^2} \arctan \frac{t+\alpha x}{\sqrt{(1+\alpha^2)(1-x^2)}}\Bigg|_{t_0}^{t_1} . \tag{2.11}$$

$|t| \leq \beta$ contains the upper limit of the t-integration as $t \leq \beta =: t_1$. What concerns its lower limit t_0, the relation $t \geq -\beta$ competes with $t \geq -\alpha x$, as it follows from Eq. (2.7) for the lower limit α of the s-integration. This means for the ranges of the t- and x-integrations, one has in the (vertical) stripe $q = 0\cdots 1, p = 1\cdots\infty$ of the q-p-plane to distinguish between the two regions, cf. Fig. 3:

(a) region A with $qp \geq 1$ or $\alpha \leq \beta$ or $-\alpha x \geq -\alpha \geq -\beta$, hence $t_0 = -\alpha x$, and

(b) region B with $qp \leq 1$ or $\alpha \geq \beta$ or $-\alpha \leq -\beta$.

In the case (b) one has again to distinguish between

(i) $x = -\beta/\alpha \cdots + \beta/\alpha$, with the consequence $t_0 = -\alpha x$ and

(ii) $x = \beta/\alpha\cdots 1$, with the consequence $t_0 = -\beta$.

Thus it is

$$\int_{-1}^{+1} dx \int_{t_0}^{+\beta} dt = \begin{cases} \int_{-1}^{+1} dx \int_{-\alpha x}^{+\beta} dt & \text{for} \quad qp \geq 1 \text{ or } \alpha \leq \beta, \\ \\ \int_{-\beta/\alpha}^{+\beta/\alpha} dx \int_{-\alpha x}^{+\beta} dt + \int_{+\beta/\alpha}^{+1} dx \int_{-\beta}^{+\beta} dt & \text{for} \quad qp \leq 1 \text{ or } \alpha \geq \beta. \end{cases} \tag{2.12}$$

With the abbreviations (2.5) and

$$f(t,x) = \frac{2}{a^2 - x^2} \ \arctan \frac{t + \alpha x}{\sqrt{(1+\alpha^2)(1-x^2)}} \ , \tag{2.13}$$

Eq. (2.6) can be written as (note that $f(-\alpha x, x) = 0$)

$$X_1 = - \int_0^1 dq \int_1^\infty dp \ \frac{16 \ \pi}{(p^2 - q^2)} \ \frac{1}{(1+q^2)} \ \Bigg\{ \Theta(qp-1) \int_{-1}^{+1} dx \ f(\beta, x) \\ + \Theta(1-qp) \left[\int_{-\beta/\alpha}^{+\beta/\alpha} dx \ f(\beta,x) + \int_{+\beta/\alpha}^{+1} dx \ [f(\beta,x) - f(-\beta,x)] \right] \Bigg\} \ . \tag{2.14}$$

With

$$\int_{+\beta/\alpha}^{+1} dx \ (-1) f(-\beta, x) = \int_{+\beta/\alpha}^{+1} dx \ f(\beta, -x) = \int_{-1}^{-\beta/\alpha} dx \ f(\beta, x) \tag{2.15}$$

the terms for $qp \leq 1$ can be comprised as $\int_{-1}^{+1} dx \ f(\beta, x)$. Therefore

$$X_1 = - \int_0^1 dq \int_1^\infty dp \int_{-1}^{+1} dx \ \frac{16 \ \pi}{p^2 - q^2} \ \frac{1}{1+q^2} \ f(\beta, x) \ . \tag{2.16}$$

The final substitution $p = 1/k$ transforms the region of the last two integrations from the (vertical) stripe $q = 0 \cdots 1, p = 1 \cdots \infty$ to the more simple unit square $q = 0 \cdots 1, k = 0 \cdots 1$. With the abbrevations

$$\alpha = \frac{1-q^2}{2q} \geq 0, \quad \beta = \frac{1-k^2}{2k} \geq 0, \quad a = \frac{1+q^2k^2}{2qk} \geq 1 \tag{2.17}$$

it is:

$$X_1 = - \int_0^1 dq \int_0^1 dk \int_{-1}^{+1} dx \ \frac{16 \ \pi}{1 - q^2k^2} \ \frac{1}{1+q^2} \ f(\beta, x) \approx -30.70598 \cdots \ . \tag{2.18}$$

The coefficients α, β, and a of Eq. (2.17) make the integrand of (2.18) functions of q and k. Mathematica5.2 [17] yields the given figure. It seems to hold

$$X_1 = -\pi^4 \left[\frac{4}{3} \ln 2 - 5 \frac{\zeta(3)}{\pi^2} \right] \approx -30.705985239248893 \cdots \tag{2.19}$$

How to derive this analytically ? Is this possible with the method of ref. [18] ?

III. THE INTEGRAL X_2

The whole procedure of Sec. II is repeated here step by step. In Eq. (A.11), new variables $\boldsymbol{q}' = (\boldsymbol{q}_1 + \boldsymbol{q}_2)/2$, $\boldsymbol{p}' = (\boldsymbol{q}_1 - \boldsymbol{q}_2)/2$, and $\boldsymbol{s}' = \boldsymbol{e} + (\boldsymbol{q}_1 + \boldsymbol{q}_2)/2$ lead to

$$\begin{aligned} \boldsymbol{q}_1 = \boldsymbol{q}' + \boldsymbol{p}', \qquad (\boldsymbol{e} + \boldsymbol{q}_1)^2 &= (\boldsymbol{s}' + \boldsymbol{p}')^2, \quad (\boldsymbol{e} + \boldsymbol{q}_2)^2 = (\boldsymbol{s}' - \boldsymbol{p}')^2, \\ \boldsymbol{q}_2 = \boldsymbol{q}' - \boldsymbol{p}', \quad (\boldsymbol{e} + \boldsymbol{q}_1 + \boldsymbol{q}_2)^2 &= (\boldsymbol{s}' + \boldsymbol{q}')^2, \qquad \boldsymbol{e}^2 = (\boldsymbol{s}' - \boldsymbol{q}')^2. \end{aligned} \tag{3.1}$$

Therefore

$$X_2 = \int d^3q' d^3p' \, \frac{2^3}{q'^2 - p'^2} \, \frac{1}{(\boldsymbol{q}' + \boldsymbol{p}')^2 (\boldsymbol{q}' - \boldsymbol{p}')^2} \times$$

$$\times \int \frac{d^2s'}{\pi} \, \delta((\boldsymbol{s}' - \boldsymbol{q}')^2 - 1) \Theta[(\boldsymbol{s}' + \boldsymbol{q}')^2 - 1] \Theta[1 - (\boldsymbol{s}' + \boldsymbol{p}')^2] \Theta[1 - (\boldsymbol{s}' - \boldsymbol{p}')^2] \; . \tag{3.2}$$

The next step scales $\boldsymbol{q}'$, $\boldsymbol{p}'$, $\boldsymbol{s}'$ with $\lambda = 1/\sqrt{1 - s'^2}$ according to $\boldsymbol{q} = \lambda \boldsymbol{q}'$, $\boldsymbol{p} = \lambda \boldsymbol{p}$', $\boldsymbol{s} = \lambda \boldsymbol{s}$' with the consequences

$$\begin{aligned} 1 - s'^2 = \frac{1}{1 + s^2}, \quad d^2s' = \frac{d^2s}{(1 + s^2)^2}, \\ \delta((\boldsymbol{s}' - \boldsymbol{q}')^2 - 1) = (1 + s^2) \; \delta(q^2 - 2\boldsymbol{s}\boldsymbol{q} - 1), \\ \pm 2\boldsymbol{s}\boldsymbol{p} + p^2 < 1, \quad 2\boldsymbol{s}\boldsymbol{q} + q^2 > 1, \quad -2\boldsymbol{s}\boldsymbol{q} + q^2 = 1, \end{aligned} \tag{3.3}$$

from which follow $q \geq 1$ and $p \leq 1$ (what makes the energy denominator $q^2 - p^2$ positive) and $\boldsymbol{s}\boldsymbol{q} > 0$, $s \geq \bar{\alpha}$. Thus

$$X_2 = \int_1^\infty dq \int_0^1 dp \, \frac{16 \; \pi}{q^2 - p^2} \int_{-1}^{+1} \frac{dx}{2} \, \frac{8 \; q^2 p^2}{(p^2 + q^2)^2 - (2\boldsymbol{q}\boldsymbol{p})^2} \times$$

$$\times \int_{s \geq \bar{\alpha}, \; \cos\varphi_q > 0} \frac{d^2s}{1 + s^2} \; \delta(q^2 - 1 - 2\boldsymbol{s}\boldsymbol{q}) \Theta(1 - p^2 - 2\boldsymbol{s}\boldsymbol{p}) \Theta(1 - p^2 + 2\boldsymbol{s}\boldsymbol{p}). \tag{3.4}$$

Here and in the following of Sec. III the abbreviations

$$\bar{\alpha} = \frac{q^2 - 1}{2q} \geq 0, \quad \bar{\beta} = \frac{1 - p^2}{2p} \geq 0, \quad \bar{a} = \frac{q^2 + p^2}{2qp} \geq 1,$$

$$\frac{t}{s} = \cos\varphi_p = \cos \sphericalangle(\boldsymbol{s}, \boldsymbol{p}), \quad x = \cos\varphi = \cos \sphericalangle(\boldsymbol{q}, \boldsymbol{p}), \quad y = \cos\varphi_q = \cos \sphericalangle(\boldsymbol{q}, \boldsymbol{s}) \tag{3.5}$$

are used:

$$X_2 = \int_1^\infty dq \int_0^1 dp \, \frac{16 \; \pi}{q^2 - p^2} \int_{-1}^{+1} \frac{dx}{\bar{a}^2 - x^2} \int_{s \geq \bar{\alpha}, \; \cos\varphi_q > 0} \frac{sds \; d\varphi_q}{1 + s^2} \, \frac{1}{2sq} \, \delta\left(\frac{\bar{\alpha}}{s} - y\right) \; \Theta(\bar{\beta} - |t|) \; . \tag{3.6}$$

t is introduced to replace s after having done the φ_q integration. To this purpose the relation between t and s is needed. It follows from $\varphi_q + \varphi_p = \varphi$ or $\varphi_p = \varphi - \varphi_q$, therefore $\cos\varphi_p = \cos(\varphi - \varphi_q) = \cos\varphi\cos\varphi_q + \sin\varphi\sin\varphi_q$, what is in terms of t/s, x, and y (which equals $+\bar{\alpha}/s$ because of the delta function):

$$\frac{t}{s} = x\frac{\bar{\alpha}}{s} \pm \sqrt{1-x^2}\sqrt{1-\left(\frac{\bar{\alpha}}{s}\right)^2} \quad \text{or} \quad (t-\bar{\alpha}x)^2 = (1-x^2)(s^2-\bar{\alpha}^2)\ . \tag{3.7}$$

This has the consequences

$$s^2 - \bar{\alpha}^2 = \frac{(t-\bar{\alpha}x)^2}{1-x^2}\ , \tag{3.8}$$

$$\frac{sds}{1+s^2} = \frac{(t-\bar{\alpha}x)dt}{(t-\bar{\alpha}x)^2 + (1+\bar{\alpha}^2)(1-x^2)}\ . \tag{3.9}$$

With the help of Eq. (3.8) the φ_q integration yields a function depending on s (respectively on t). Note that $\cos\varphi_q = +\frac{\bar{\alpha}}{s}$ has two solutions $\varphi_{q,1}$ and $\varphi_{q,2}$ with $|\sin\varphi_{q,1}| = |\sin\varphi_{q,2}| = \sqrt{1-(\frac{\bar{\alpha}}{s})^2}$:

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\varphi_q}{s}\ [\delta(+\sin\varphi_{q,1}\cdot(\varphi_q-\varphi_{q,1})) + \delta(+\sin\varphi_{q,2}\cdot(\varphi_q-\varphi_{q,2}))] \\
&= \int_0^{2\pi} \frac{d\varphi_q}{s}\ \frac{\delta(\varphi_q-\varphi_{q,1}) + \delta(\varphi_q-\varphi_{q,2})}{\sqrt{1-(\frac{\bar{\alpha}}{s})^2}} = \frac{2}{\sqrt{s^2-\bar{\alpha}^2}} = 2\,\frac{\sqrt{1-x^2}}{|t-\bar{\alpha}x|}\ .
\end{aligned} \tag{3.10}$$

Thus

$$\begin{aligned}
\int\limits_{s\geq\bar{\alpha},\ \cos\varphi_q>0} \frac{sds\ d\varphi_q}{1+s^2}\,\frac{1}{2sq}\,\delta\left(\frac{\bar{\alpha}}{s}-y\right)\ \Theta(\beta-|t|) &= \frac{\sqrt{1-x^2}}{q}\int_{t_0}^{t_1} \frac{\text{sign}(t-\bar{\alpha}x)\ dt}{(t-\bar{\alpha}x)^2+(1+\bar{\alpha}^2)(1-x^2)} \\
&= \frac{2}{1+q^2}\arctan\frac{t-\bar{\alpha}x}{\sqrt{(1+\bar{\alpha}^2)(1-x^2)}}\Bigg|_{t_0}^{t_1}
\end{aligned} \tag{3.11}$$

$|t| \leq \bar{\beta}$ contains the upper limit of the t-integration as $t \leq \bar{\beta} =: t_1$. What concerns its lower limit t_0, the relation $t \geq -\bar{\beta}$ competes with $t \geq \bar{\alpha}x$, as it follows from Eq. (3.7) for the lower limit $\bar{\alpha}$ of the s-integration. This means for the ranges of the t- and x-integrations, one has in the (horizontal) stripe $q = 1\cdots\infty, p = 0\cdots 1$ of the q-p-plane to distinguish between the two regions, cf. Fig. 4:

(a) region A with $qp \geq 1$ or $\bar{\alpha} \geq \bar{\beta}$ or $-\bar{\alpha} \leq -\bar{\beta}$, and

(b) region B with $qp \leq 1$ or $\bar{\alpha} \leq \bar{\beta}$ or $\bar{\alpha}x \leq \bar{\alpha} \leq \bar{\beta}$, hence $t_0 = \bar{\alpha}x$.

In the case (a) one has again to distinguish between

(i) $x = -1 \cdots -\bar\beta/\bar\alpha$, with the consequence $t_0 = -\bar\beta$ and

(ii) $x = -\bar\beta/\bar\alpha \cdots + \bar\beta/\bar\alpha$, with the consequence $t_0 = \bar\alpha x$.

Thus it is

$$\int\limits_{-1}^{+1} dx \int\limits_{t_0}^{+\bar\beta} dt = \begin{cases} \int\limits_{-1}^{-\bar\beta/\bar\alpha} dx \int\limits_{-\bar\beta}^{+\bar\beta} dt + \int\limits_{-\bar\beta/\bar\alpha}^{+\bar\beta/\bar\alpha} dx \int\limits_{\bar\alpha x}^{+\bar\beta} dt & \text{for} \quad qp \geq 1 \text{ or } \bar\alpha \geq \bar\beta \,, \\ \int\limits_{-1}^{+1} dx \int\limits_{\bar\alpha x}^{+\bar\beta} dt & \text{for} \quad qp \leq 1 \text{ or } \bar\alpha \leq \bar\beta \,. \end{cases} \tag{3.12}$$

With the abbreviations (3.5) and with

$$\bar f(t, x) = \frac{2}{\bar a^2 - x^2} \ \arctan \frac{t - \bar\alpha x}{\sqrt{(1+\bar\alpha^2)(1-x^2)}} \,, \tag{3.13}$$

Eq. (3.6) can be written as (note that $\bar f(\bar\alpha x, x) = 0$)

$$X_2 = \int\limits_1^\infty dq \int\limits_0^1 dp \ \frac{16\ \pi}{q^2 - p^2} \ \frac{1}{1+q^2} \ \Bigg\{ \Theta(1-qp) \int\limits_{-1}^{+1} dx \ \bar f(\bar\beta, x)$$
$$+ \Theta(qp-1) \left[\int\limits_{-1}^{-\bar\beta/\bar\alpha} dx \ [\bar f(\bar\beta, x) - \bar f(-\bar\beta, x)] + \int\limits_{-\bar\beta/\bar\alpha}^{+\bar\beta/\bar\alpha} dx \ \bar f(\bar\beta, x) \right] \Bigg\} \,. \tag{3.14}$$

With the identity

$$\int\limits_{-1}^{-\bar\beta/\bar\alpha} dx \ (-1) \bar f(-\bar\beta, x) = \int\limits_{-1}^{-\bar\beta/\bar\alpha} dx \ \bar f(\bar\beta, -x) = \int\limits_{+\bar\beta/\bar\alpha}^{+1} dx \ \bar f(\bar\beta, x) \tag{3.15}$$

the terms for $qp \geq 1$ can be comprised as $\int\limits_{-1}^{+1} dx \ \bar f(\bar\beta, x)$. Therefore

$$X_2 = \int\limits_1^\infty dq \int\limits_0^1 dp \int\limits_{-1}^{+1} dx \ \frac{16\ \pi}{q^2 - p^2} \ \frac{1}{1+q^2} \ \bar f(\bar\beta, x) \,. \tag{3.16}$$

This is similar to Eq. (2.16), but there are also differences. The substitution $q = 1/k$ transforms the region of the last two integrations from the (horizontal) stripe $q = 1 \cdots \infty, p = 0 \cdots 1$ to the more simple unit square $k = 0 \cdots 1, p = 0 \cdots 1$. With the abbrevations

$$\bar\alpha = \frac{1-k^2}{2k}, \quad \bar\beta = \frac{1-p^2}{2p}, \quad \bar a = \frac{1+k^2p^2}{2kp} \tag{3.17}$$

it is

$$X_2 = \int_0^1 dk \int_0^1 dp \int_{-1}^{+1} dx \; \frac{16\ \pi}{1-k^2p^2} \; \frac{k^2}{1+k^2} \; \bar{f}(\bar{\beta}, x) \ .$$

Changing finally the notation with $k \to q$ and $p \to k$ makes $\bar{\alpha} = \alpha$, $\bar{\beta} = \beta$, and $\bar{a} = a$, cf. Eq. (2.17). With these identities and with $\bar{f}(t, -x) = f(t, x)$ a further rewriting yields

$$X_2 = \int_0^1 dq \int_0^1 dk \int_{-1}^{+1} dx \; \frac{16\ \pi}{1-q^2k^2} \; \frac{q^2}{1+q^2} \; f(\beta, x) \approx 21.284906\cdots \ . \tag{3.18}$$

This integral differs from Eq. (2.18) 'only' in an additional factor of $-q^2$ in the nominator of the integrand. Mathematica5.2 [17] yields the given figure. It seems to hold

$$X_2 = \pi^4 \left[\frac{2}{3} \ln 2 - 2\ \frac{\zeta(3)}{\pi^2} \right] \approx 21.284905670516334\cdots \ . \tag{3.19}$$

How to derive this analytically ? Is this possible with the method of ref. [18] ?

IV. THE CALCULATION OF X

With Eqs. (2.13), (2.18), (3.18), and with a, α, β being defined in Eq. (2.17) the result for $X = X_1 + X_2$ is

$$X = -\int_0^1 dq \int_0^1 dk \int_{-1}^{+1} dx \; \frac{16\ \pi}{1-q^2k^2} \; \frac{1-q^2}{1+q^2} \frac{2}{a^2-x^2} \; \arctan \frac{\beta + \alpha x}{\sqrt{(1+\alpha^2)(1-x^2)}} \approx -9.42108\cdots \ . \tag{4.1}$$

It seems to hold [18, 19]

$$X = -\pi^4 \left[\frac{2}{3} \ln 2 - 3\ \frac{\zeta(3)}{\pi^2} \right] \approx -9.421079568732553\cdots \ . \tag{4.2}$$

The final result [20]

$$\Sigma_{2\mathrm{x}} = -\frac{(\alpha r_s)^2}{4\pi^4} X = e_{2\mathrm{x}} \quad \text{with} \quad \frac{e_{2\mathrm{x}}}{(\alpha r_s)^2} = \frac{1}{6} \ln 2 - \frac{3}{4} \frac{\zeta(3)}{\pi^2} \approx 0.0241792 \tag{4.3}$$

appears in the weak-correlation limit of the Hugenholtz-van Hove (Luttinger-Ward) theorem [14, 15]. Because of $\mu_{2\mathrm{x}} = e_{2\mathrm{x}}$ it holds the sum rule $\mu_{2\mathrm{x}} = \Sigma_{2\mathrm{x}}$ analogous to $\mu_{\mathrm{x}} = \Sigma_{\mathrm{x}}$. Whether perhaps also the more general expression $\Sigma_{2\mathrm{x}}(k, \omega)$ can be calculated in a similar way, has to be studied.

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[20] This result should be comparable to calculations by E. Shirley [12] if one considers in his Fig. 1 the contribution of the second (exchange) Feynman diagram to the on-shell self-energy $\Sigma(1,1/2)$ in the limit $r_s \to 0$, where the effectively screened Coulomb repulsion $W(12)$ is replaced by the bare Coulomb repulsion $\nu(12)$.

APPENDIX A: DERIVATION OF $\Sigma_{2\mathrm{x}}$

The one-body Green's function of the non-interacting system (ideal Fermi gas)

$$G_0(k,\omega) = \frac{\Theta(k-1)}{\omega - \frac{1}{2}k^2 + \mathrm{i}\delta} + \frac{\Theta(1-k)}{\omega - \frac{1}{2}k^2 - \mathrm{i}\delta} \tag{A.1}$$

(with $\Theta(x)$ = Heaviside step function) and $G(k,\omega)$, the one-body Green's function of the fully interacting system, define the self-energy $\Sigma(k,\omega)$ through

$$G(k,\omega) = G_0(k,\omega) + G_0(k,\omega)\Sigma(k,\omega)G(k,\omega) \ . \tag{A.2}$$

$\Sigma(k,\omega)$ appears in the Hugenholtz-van Hove theorem (in the Luttinger-Ward form $\mu - \mu_0 = \Sigma(1,\mu)$ with μ = chemical potential) [14] and in the Galitskii-Migdal formula [16]

$$v = \frac{1}{2}\int d(k)^3 \int \frac{d\omega}{2\pi\mathrm{i}} \mathrm{e}^{\mathrm{i}\omega\delta} G(k,\omega)\Sigma(k,\omega) \ , \quad \delta \underset{\to}{>} 0 \ . \tag{A.3}$$

v is the potential component of e, the total energy per particle. The contour of the ω-integration is to be closed in the upper complex ω-plane. In lowest order it is $\Sigma_\mathrm{x}(k) = -(1 + \frac{1-k^2}{2k}\ln|\frac{1+k}{1-k}|)$. This makes $v_\mathrm{x} = -\frac{3}{4}\frac{\alpha r_s}{\pi}$, in agreement with $v_\mathrm{x} = e_\mathrm{x}$, what follows from the virial theorem $v = r_s \frac{d}{dr_s} e$.

From the Feynman diagram for the exchange term of the 2nd-order self-energy it follows

$$\begin{aligned}\Sigma_{2\mathrm{x}}(k,\omega) &= \frac{(\alpha r_s)^2}{4\pi^4}\int \frac{d^3q_1 d^3q_2}{q_1^2 q_2^2}\int \frac{d\eta_1 d\eta_2}{(2\pi\mathrm{i})^2}\times \\ &\times\ G_0(|\boldsymbol{k}+\boldsymbol{q}_2|,\omega+\eta_2)G_0(|\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2|,\omega+\eta_1+\eta_2)G_0(|\boldsymbol{k}+\boldsymbol{q}_1|,\omega+\eta_1)\end{aligned} \tag{A.4}$$

Use of (A.1) yields

$$\begin{aligned}\Sigma_{2\mathrm{x}}(k,\omega) = -\frac{(\alpha r_s)^2}{4\pi^4}\int \frac{d^3q_1 d^3q_2}{q_1^2 q_2^2} &\left[\frac{\Theta(|\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2|-1)\Theta(1-|\boldsymbol{k}+\boldsymbol{q}_1|)\Theta(1-|\boldsymbol{k}+\boldsymbol{q}_2|)}{\omega-\frac{1}{2}k^2+\boldsymbol{q}_1\cdot\boldsymbol{q}_2-\mathrm{i}\delta}\right. \\ &\left.+\frac{\Theta(1-|\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2|)\Theta(|\boldsymbol{k}+\boldsymbol{q}_1|-1)\Theta(|\boldsymbol{k}+\boldsymbol{q}_2|-1)}{\omega-\frac{1}{2}k^2+\boldsymbol{q}_1\cdot\boldsymbol{q}_2+\mathrm{i}\delta}\right] .\end{aligned} \tag{A.5}$$

One may check this expression by using it in the Galitskii-Migdal formula (A.3). Its lhs is known from the virial theorem as $v_{2\mathrm{x}} = 2e_{2\mathrm{x}}$ with $e_{2\mathrm{x}}$ = energy in second order of exchange, calculated by Onsager et al. [9]. Its rhs gives with Eqs. (A.1) and (A.5)

$$\begin{aligned}\mathrm{rhs} = -\frac{3(\alpha r_s)^2}{(2\pi)^5}&\left[\int \frac{d^3k d^3q_1 d^3q_2}{q_1^2 q_2^2}\frac{\Theta(1-k)\Theta(1-|\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2|)\Theta(|\boldsymbol{k}+\boldsymbol{q}_1|-1)\Theta(|\boldsymbol{k}+\boldsymbol{q}_2|-1)}{\boldsymbol{q}_1\cdot\boldsymbol{q}_2+\mathrm{i}\delta}\right. \\ &\left.+\int \frac{d^3k d^3q_1 d^3q_2}{q_1^2 q_2^2}\frac{\Theta(k-1)\Theta(|\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2|-1)\Theta(1-|\boldsymbol{k}+\boldsymbol{q}_1|-1)\Theta(1-|\boldsymbol{k}+\boldsymbol{q}_2|)}{\boldsymbol{q}_1\cdot(-\boldsymbol{q}_2)+\mathrm{i}\delta}\right] .\end{aligned} \tag{A.6}$$

It is easy to show with the help of the substitutions $\boldsymbol{q}_1 \to \boldsymbol{q}_1'$, $\boldsymbol{q}_2 \to -\boldsymbol{q}_2'$, $\boldsymbol{k} \to -(\boldsymbol{k}'+\boldsymbol{q}_1')$ that the second term equals the first one. Thus

$$\begin{aligned}\mathrm{Re\ rhs} &= -2\frac{3(\alpha r_s)^2}{(2\pi)^5}\int \frac{d^3k d^3q_1 d^3q_2}{q_1^2 q_2^2}\frac{P}{\boldsymbol{q}_1\cdot\boldsymbol{q}_2}\times \\ &\times\ \Theta(1-k)\Theta(1-|\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2|)\Theta(|\boldsymbol{k}+\boldsymbol{q}_1|-1)\Theta(|\boldsymbol{k}+\boldsymbol{q}_2|-1) = 2e_{2\mathrm{x}} .\end{aligned} \tag{A.7}$$

P means the Cauchy principle value. This is in agreement with the above mentioned relation. That $e_{2\mathrm{x}}$ of Eq. (A.7) really agrees with the integral calculated by Onsager et al. [9] follows from the substitutions $\boldsymbol{k} \to \boldsymbol{k}_1$, $\boldsymbol{q}_1 \to \boldsymbol{q}$, $\boldsymbol{q}_2 \to -(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{q})$. From Eq. (A.5) also follows $n_{2\mathrm{x}}(k)$, the second-order-in-exchange contribution to the momentum distribution. It is again easy to derive the well-known asymptotics $n_{2\mathrm{x}}(k\to\infty) = -\frac{4}{9\pi^2}\frac{(\alpha r_s)^2}{k^8}$.

After this control of $\Sigma_{2\mathrm{x}}(k,\omega)$, the formula for $\Sigma_{2\mathrm{x}} = \mathrm{Re}\ \Sigma_{2\mathrm{x}}(1,1/2)$ follows from Eq. (A.5) as $\Sigma_{2\mathrm{x}} = -\frac{(\alpha r_s)^2}{4\pi^4}(X_1+X_2)$, where $X_{1,2}$ mean the integrals

$$X_1 = \int d^3q_1\ d^3q_2\ \frac{P}{\boldsymbol{q}_1\cdot\boldsymbol{q}_2}\ \frac{1}{q_1^2\ q_2^2}\ \Theta[1-(\boldsymbol{e}+\boldsymbol{q}_1+\boldsymbol{q}_2)^2]\Theta[(\boldsymbol{e}+\boldsymbol{q}_1)^2-1]\Theta[(\boldsymbol{e}+\boldsymbol{q}_2)^2-1]\ , \tag{A.8}$$

$$X_2 = \int d^3q_1\ d^3q_2\ \frac{P}{\boldsymbol{q}_1\cdot\boldsymbol{q}_2}\ \frac{1}{q_1^2\ q_2^2}\Theta[(\boldsymbol{e}+\boldsymbol{q}_1+\boldsymbol{q}_2)^2-1]\Theta[1-(\boldsymbol{e}+\boldsymbol{q}_1)^2]\Theta[1-(\boldsymbol{e}+\boldsymbol{q}_2)^2]\ . \tag{A.9}$$

They contain $\boldsymbol{q}_1\cdot\boldsymbol{q}_2$ as the energy denominator. $1/q_{1,2}^2$ arises from the Coulomb repulsion and the remainder is due to the Pauli principle. $\boldsymbol{e}$ is a unit vector. Note that $\boldsymbol{q}_1\cdot\boldsymbol{q}_2<0$ for X_1, because of $(2\boldsymbol{e}+\boldsymbol{q}_1)\cdot\boldsymbol{q}_1>0$ and $(2\boldsymbol{e}+\boldsymbol{q}_2)\cdot\boldsymbol{q}_2>0$ in combination with $(2\boldsymbol{e}+\boldsymbol{q}_1)\cdot\boldsymbol{q}_1+(2\boldsymbol{e}+\boldsymbol{q}_2)\cdot\boldsymbol{q}_2+2\boldsymbol{q}_1\cdot\boldsymbol{q}_2<0$. This latter inequality enforces $\boldsymbol{q}_1\cdot\boldsymbol{q}_2<0$. Thus $X_1<0$. It follows similarly $X_2>0$. The integrals $X_{1,2}$ do not depend on $\boldsymbol{e}$. Therefore application of $\hat{O}=\int d^3e/(4\pi)\ 2\ \delta(\boldsymbol{e}^2-1)$ does not change them (notice $2\ \delta(\boldsymbol{e}^2-1)=\delta(e-1)$). Following Onsager et al. [9], $\boldsymbol{e}$ is resolved into its components perpendicular to the $\boldsymbol{q}_1-\boldsymbol{q}_2$-plane $\boldsymbol{e}_\perp$, and in the plane $\boldsymbol{e}_\parallel$: $\hat{O}=\int de_\perp/2\ \int d^2e_\parallel/(2\pi)\ 2\ \delta(e_\parallel^2+e_\perp^2-1)$. The integration over $\boldsymbol{e}_\perp$ may be done immediately by means of a change in scale: $\boldsymbol{q}_1=\tilde{\boldsymbol{q}}_1\sqrt{1-e_\perp^2}$, $\boldsymbol{q}_2=\tilde{\boldsymbol{q}}_2\sqrt{1-e_\perp^2}$, $\boldsymbol{e}_\parallel=\tilde{\boldsymbol{e}}_\parallel\sqrt{1-e_\perp^2}$. The results are (denoting $\tilde{\boldsymbol{e}}_\parallel$ as $\boldsymbol{e}$ and deleting also all the other tildes for simplicity)

$$\begin{aligned} X_1 = \quad & \int d^3q_1\ d^3q_2\ \frac{P}{\boldsymbol{q}_1\cdot\boldsymbol{q}_2}\ \frac{1}{q_1^2\ q_2^2}\int\frac{d^2e}{2\pi}\ 2\ \delta(\boldsymbol{e}^2-1)\ \times \\ & \times\Theta[1-(\boldsymbol{e}+\boldsymbol{q}_1+\boldsymbol{q}_2)^2]\Theta[(\boldsymbol{e}+\boldsymbol{q}_1)^2-1]\Theta[(\boldsymbol{e}+\boldsymbol{q}_2)^2-1]\ , \end{aligned} \tag{A.10}$$

$$\begin{aligned} X_2 = & \int d^3q_1\ d^3q_2\ \frac{P}{\boldsymbol{q}_1\cdot\boldsymbol{q}_2}\ \frac{1}{q_1^2\ q_2^2}\ \int\frac{d^2e}{2\pi}\ 2\ \delta(\boldsymbol{e}^2-1)\times \\ & \times\Theta[(\boldsymbol{e}+\boldsymbol{q}_1+\boldsymbol{q}_2)^2-1]\Theta[1-(\boldsymbol{e}+\boldsymbol{q}_1)^2]\Theta[1-(\boldsymbol{e}+\boldsymbol{q}_2)^2]\ . \end{aligned} \tag{A.11}$$

Whereas the two terms of $\Sigma_{2\mathrm{x}}(k,\omega)$, cf. (A.6) each contributes $e_{2\mathrm{x}}$ to $v_{2\mathrm{x}}$ (thus $v_{2\mathrm{x}}=2e_{2\mathrm{x}}$) as shown above, cf. (A.7), X_1 and X_2 contribute different values to $\Sigma_{2\mathrm{x}}(1,1/2)$ as shown in Secs. II and III.

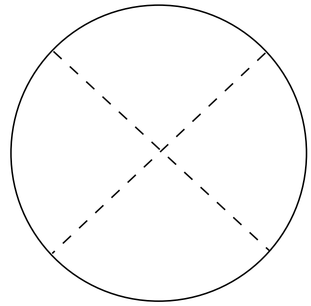

FIG. 1: The Feynman diagram of $e_{2\mathrm{x}}$, analytically calculated by Onsager et al. [9].

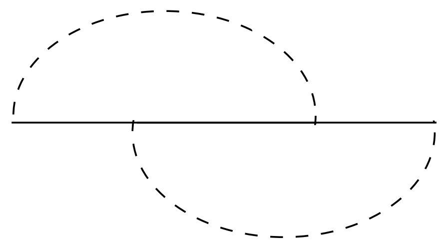

FIG. 2: The Feynman diagram of $\Sigma_{2\mathrm{x}}(k, \omega)$, (semi)analytically calculated in this paper.

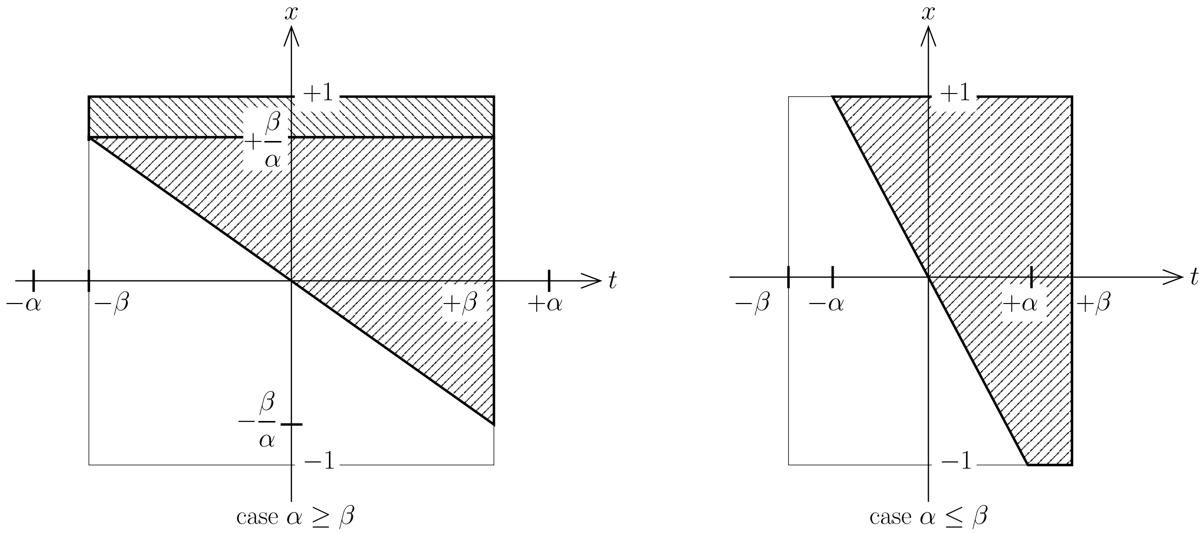

FIG. 3: The dashed area is the region of integration described by Eq. (2.12).

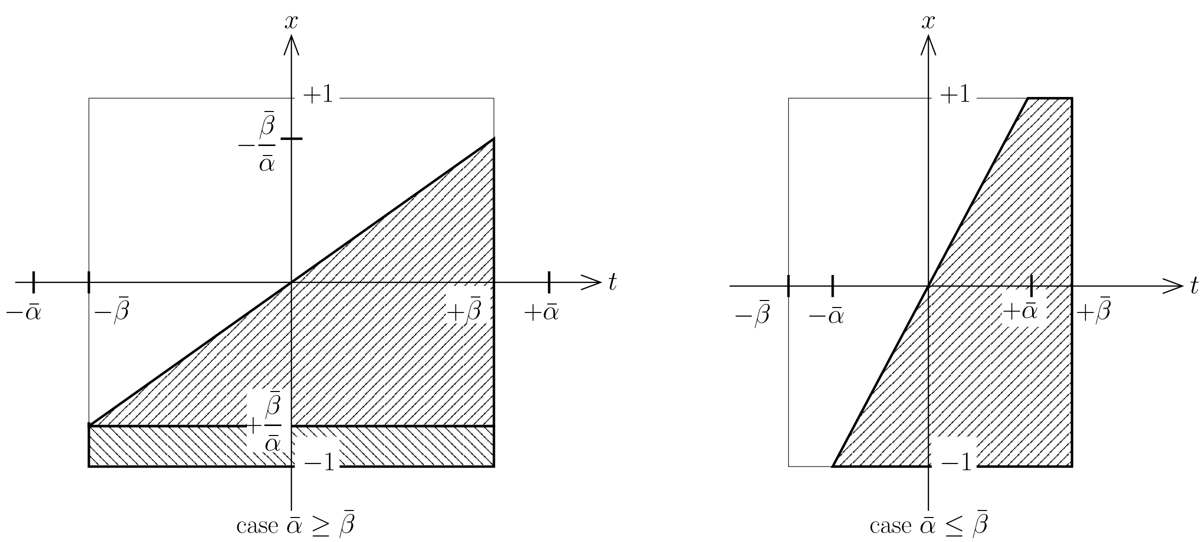

FIG. 4: The dashed area is the region of integration described by Eq. (3.12).