Ring-diagram summations and the self-energy of the uniform electron gas at its weak-correlation limit

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Ring-diagram summations (equivalent to the random-phase approximation) for several properties of the homogeneous electron gas, such as the total energy components t and v, the chemical potential μ, and the quasi-particle weight z_{F}, are reexamined. The ring-diagram summations of the self-energy $\Sigma(k,\omega)$ that yield the correct small-r_s asymptotics of v, μ, and z_{F} are identified with the help of rigorous theorems of Galitskii-Migdal, Hugenholtz-van Hove, and Luttinger-Ward. The lowest-order approximation to the self-energy is given by the product of the non-interacting Green's function G_0 and the static bare Coulomb repulsion v_0, whereas replacing v_0 by the ring-diagram-summed dynamically screened interaction v_{r} yields the proper lowest-order correction to $\Sigma(k,\omega)$. The alternative replacement of G_0 by the ring-diagram-summed G_{r} contributes only to the higher-order terms, providing measures of the correlation strength.

I. INTRODUCTION

Although being an artificial construct, the homogeneous electron gas (HEG) constitutes an important model system for electronic structure theory [1]. The ground state of spin-unpolarized HEG is characterized by only one parameter, namely the radius r_s of the Wigner-Seitz sphere that contains one electron *on average* [2]. It determines the Fermi wave number as $k_{\rm F} = 1/(\alpha r_s)$ [where $\alpha = (4/(9\pi))^{1/3}$], and measures simultaneously the interaction strength and the particle density; high density corresponding to weak interaction and hence weak correlation (for recent papers on this limit see refs. [3]-[6]). One could naively expect that at this weak-correlation limit the bare Coulomb repulsion $v_0(q) = \alpha r_s/q^2$ (where momenta and energies are measured in units of $k_{\rm F}$ and $k_{\rm F}^2$, respectively) can be treated as perturbation. However, already in his early work on the HEG, Heisenberg [7] has shown that ordinary perturbation theory fails in this case. With $e_0 = 3/10$ being the energy per particle of the ideal Fermi gas and $e_{\rm x} = -\frac{3}{4}\frac{\alpha r_s}{\pi}$ being the exchange energy in the lowest (first) order, the total energy $e = e_0 + e_{\rm x} + e_{\rm c}$ defines the correlation energy $e_{\rm c} = e_2 + e_3 + \cdots$, where $e_n \sim (\alpha r_s)^n$ [note that $\tilde{e} = k_{\rm F}^2 e = e/(\alpha r_s)^2$ gives the energy in atomic units]. In the second order, there is a direct term $e_{\rm 2d}$ and an exchange term $e_{\rm 2x}$ so that $e_2 = e_{\rm 2d} + e_{\rm 2x}$. The direct term $e_{\rm 2d}$ diverges logarithmically along the Fermi surface (i.e. for the vanishing transition momenta $q \to 0$, $e_{\rm 2d} \to \ln q$). This failure of perturbation theory has been remedied by Macke [8] with an appropriate partial summation of higher-order terms up to an infinite order that describes screening effects and the collective mode plasmon with a cut-off momentum $q_c = \sqrt{4\alpha r_s/\pi}$. This ring-diagram summation, which is equivalent to the random-phase approximation (RPA), yields $e_{\rm c} = (\alpha r_s)^2(a \ln r_s + {\rm const} + \cdots)$, where $a = (1 - \ln 2)/\pi^2 \approx 0.031091$, for the correlation energy at the weak-correlation limit. This result has been subsequently confirmed by Gell-Mann and Brueckner [9]. The logarithmic behavior of $e_{\rm c}$ at the weak-correlation limit carries over to its kinetic and potential components through the virial theorem [10]

$$t_{\rm c} = -r_s^2 \frac{d}{dr_s}\frac{1}{r_s}\, e_{\rm c} = -(\alpha r_s)^2(a \ln r_s + {\rm const} + \cdots),$$
$$v_{\rm c} = r_s \frac{d}{dr_s}\, e_{\rm c} = (\alpha r_s)^2(2a \ln r_s + {\rm const} + \cdots). \qquad (1.1)$$

Note that $t_0 = e_0$, $t_{\rm x} = 0$, $v_{\rm x} = e_{\rm x}$, and $e_{\rm c} = t_{\rm c} + v_{\rm c}$. It has been shown [5] that these small-r_s non-analyticities result from the ring-diagram summation for the momentum distribution

$n(k)$ [11] and for the static structure factor $S(q)$ [13]. In the lowest order, $n(k)$ diverges along the Fermi surface, $n(k \to 1) \sim \mp 1/(k-1)^2$ for $k \lessgtr 1$, and $S(q)$ diverges for $q \to 0$. This makes $t_{\rm 2d}$ and $v_{\rm 2d}$ diverge correspondingly. The ring-diagram summations remove this unphysical behavior [5, 11, 13]. The chemical potential $\mu = \mu_0 + \mu_{\rm x} + \mu_{\rm c}$, where $\mu_0 = 1/2$ and $\mu_{\rm x} = -\alpha r_s/\pi$, enters our considerations through the Seitz theorem [14] ,

$$\mu_{\rm c} = \left(\frac{5}{3} - \frac{1}{3} r_s \frac{d}{dr_s}\right) e_c = (\alpha r_s)^2 (a \ln r_s + \text{const} + \cdots). \tag{1.2}$$

In the following, we use the term "small-r_s" with the meaning "RPA in the lowest order", i.e. we derive and discuss here only the terms containing $\ln r_s$ or those related to them.

The self-energy $\Sigma(k,\omega)$ is defined by

$$G = G_0 + G_0 \Sigma G, \quad G_0(k,\omega) = \frac{\Theta(1-k)}{\omega - \frac{1}{2}k^2 - {\rm i}\delta} + \frac{\Theta(k-1)}{\omega - \frac{1}{2}k^2 + {\rm i}\delta}, \quad \delta \to 0^+ , \tag{1.3}$$

where G_0 and G are the Green's functions of the ideal Fermi gas and the HEG, respectively. Limiting the summation of the Feynman diagrams for $\Sigma(k,\omega)$t to those terms that afford correct results for $r_s \to 0$ allows one to apply several rigorous theorems, which yield
(i) the condition for μ through the Luttinger theorem Im $\Sigma(1,\mu) = 0$ [15],
(ii) the momentum distribution

$$n(k) = \int \frac{d\omega}{2\pi {\rm i}} {\rm e}^{{\rm i}\omega\delta} G(k,\omega) , \tag{1.4}$$

(iii) the quasi-particle weight (through the Luttinger-Ward formula [16])

$$z_{\rm F} = \frac{1}{1 - {\rm Re}\, \Sigma_{\rm c}'(1,\mu)}, \quad \Sigma_{\rm c}'(k,\omega) = \frac{\partial \Sigma_{\rm c}(k,\omega)}{\partial \omega} , \tag{1.5}$$

(iv) the potential energy (through the Galitskii-Migdal formula [17])

$$v = \frac{1}{2} \int d(k^3) \int \frac{d\omega}{2\pi {\rm i}} {\rm e}^{{\rm i}\omega\delta} G(k,\omega) \Sigma(k,\omega) . \tag{1.6}$$

Note that $\Sigma = \Sigma_{\rm x} + \Sigma_{\rm c}$, $\Sigma_{\rm c} = \Sigma_2 + \Sigma_3 + \cdots$, and $\Sigma_2 = \Sigma_{\rm 2d} + \Sigma_{\rm 2x}$. In the lowest order, one has $\Sigma_{\rm x} = G_0 v_0$ and $v_{\rm x} = G_0 \Sigma_{\rm x}$. With $v_0(q) = \alpha r_s/q^2$ [compare Eq. (A.5)], this produces

$$\Sigma_{\rm x}(k) = -\frac{1}{k}\left(1 + \frac{1-k^2}{2k} \ln\left|\frac{1+k}{1-k}\right|\right) \frac{\alpha r_s}{\pi}, \quad \Sigma_{\rm x}(1) = -\frac{\alpha r_s}{\pi}, \quad v_{\rm x} = -\frac{3}{4}\frac{\alpha r_s}{\pi} . \tag{1.7}$$

Note that $\Sigma_{\rm x}(k)$ does not depend on ω. With $G_{\rm c} = G - G_0$, the correlation part of the potential energy reads

$$v_{\rm c} = (G_0 + G_{\rm c})\Sigma_{\rm c} + G_{\rm c}\Sigma_{\rm x} = G_0 \Sigma_{\rm c} + G_{\rm c}(\Sigma_{\rm x} + \Sigma_{\rm c}) . \tag{1.8}$$

However, our main interest is the Hugenholtz-van Hove (the Luttinger-Ward) theorem [16, 18],

$$\mu_{\rm c} = \Sigma_{\rm c}(1,\mu), \quad \mu = \mu_0 + \mu_{\rm x} + \mu_{\rm c},\ \mu_0 = \frac{1}{2},\ \mu_{\rm x} = -\frac{\alpha r_s}{\pi},\ \mu_{\rm c} = (\alpha r_s)^2 a \ln r_s + \cdots\ . \tag{1.9}$$

The rhs of the above equation depends on r_s through both $\Sigma_{\rm c}(k,\omega)$ and μ. At the limit of $r_s \to 0$, μ can be replaced by $\mu_0 = 1/2$.

The ring-diagram summation is equivalent to setting $v_{\rm r} = v_0 + v_0 Q v_{\rm r}$, where $Q(q,\omega)$ is the polarization propagator [in the lowest order, see Eq. (A.1) in the Appendix]. For the self-energy, this means that $\Sigma^{\rm r} = G_0 v_{\rm r}$. It is easy to show that employing the correlation part $\Sigma^{\rm r}_{\rm c} = G_0(v_{\rm r} - v_0)$ of $\Sigma^{\rm r}$ in conjunction with Eqs. (1.4)and (1.6) results in the RPA approximations for $n^{\rm r}_{\rm c}(k)$ [5, 11] and $v^{\rm r}_{\rm c}$ [5, 13], respectively. In this paper, we show that $\Sigma^{\rm r}_{\rm c}$ is also the proper rhs for Eqs. (1.5) and (1.9) at the limit of $r_s \to 0$, the remainder $\Sigma^{\rm nr}_{\rm c} = \Sigma_{\rm c} - \Sigma^{\rm r}_{\rm c}$ contributing only to the higher-order terms. We also investigate whether $\Sigma^{\rm HF}_{\rm c} = (G - G_0)v_0$, which appears in ref. [21], is an alternative candidate for the rhs of Eq. (1.9). In this case, we find that the "remainder" $\Sigma^{\rm nHF}_{\rm c} = \Sigma_{\rm c} - \Sigma^{\rm HF}_{\rm c} = \Sigma^{\rm r}_{\rm c} + \cdots$ determines the lowest-order terms and $\Sigma^{\rm HF}_{\rm c}$ contributes only to the higher-order ones.

II. THE RING-DIAGRAM SELF-ENERGY $\Sigma^{\rm r}_{\rm c}(k,\omega)$

According to the diagram rules, the ring-diagram-summed self-energy is given by

$$\begin{aligned}\Sigma^{\rm r}_{\rm c}(k,\omega) &= (\alpha r_s)^2\, \frac{2}{\pi^3} \int \frac{d^3q}{q^2} \int \frac{d\eta}{2\pi {\rm i}}\, \frac{Q(q,\eta)}{q^2 + q_c^2 Q(q,\eta)} \times \\ &\times \left[\frac{\Theta(|\boldsymbol{k}+\boldsymbol{q}|-1)}{\omega + \eta - \frac{1}{2}k^2 - \boldsymbol{q}\cdot(\boldsymbol{k} + \frac{1}{2}\boldsymbol{q}) + {\rm i}\delta} + \frac{\Theta(1-|\boldsymbol{k}+\boldsymbol{q}|)}{\omega + \eta - \frac{1}{2}k^2 - \boldsymbol{q}\cdot(\boldsymbol{k} + \frac{1}{2}\boldsymbol{q}) - {\rm i}\delta} \right] .\end{aligned} \tag{2.1}$$

If in the above equation the term $q_c^2 Q(q,\eta)$, which describes the RPA screening of the bare Coulomb repulsion of $\alpha r_s/q^2$, is deleted, $\Sigma^{\rm r}_{\rm c}(k,\omega)$ simplifies to $\Sigma_{\rm 2d}(k,\omega)$. Whereas $\Sigma_{\rm 2d} = {\rm Re}\ \Sigma_{\rm 2d}(1,1/2)$ diverges with an artificial cut-off q_0 according to $(\alpha r_s)^2 \int_{q_0} dq/q$, the ring-diagram sum $\Sigma^{\rm r}_{\rm c} = {\rm Re}\ \Sigma^{\rm r}_{\rm c}(1,1/2)$ is non-divergent, as it effectively replaces q_0 by the "natural" cut-off $q_{\rm c} \sim \sqrt{r_s}$, producing $\Sigma^{\rm r}_{\rm c} \sim (\alpha r_s)^2 \ln r_s$. We follow the procedure of Gell-Mann and Brueckner for the correlation energy [9]. Upon the substitution $\eta = {\rm i}qu$ and contour deformation from the real to the imaginary axis, one arrives at

$$\begin{aligned}
\Sigma_{\mathrm{c}}^{\mathrm{r}} &= -\frac{(\alpha r_s)^2}{\pi^4}\int_0^\infty du \int \frac{d^3q}{q^2}\,\frac{R(q,u)}{q^2+q_c^2R(q,u)}\cdot\frac{2(x+\frac{q}{2})}{u^2+(x+\frac{q}{2})^2} \\
&= -(\alpha r_s)^2\frac{2}{\pi^3}\int_0^\infty du\int_0^\infty dq\,\frac{R(q,u)}{q^2+q_c^2R(q,u)}\cdot\ln\frac{u^2+(\frac{q}{2}+1)^2}{u^2+(\frac{q}{2}-1)^2}\ .
\end{aligned} \tag{2.2}$$

The asymptotic behavior for $r_s \to 0$ is determined by the lower integration limit of $q \to 0$, which allows for the approximate replacements of $R(q,u)$ with $R_0(u)$ [setting $R_0(u) \neq 0$ makes the Coulomb repulsion effectively screened] and $\ln[\cdots]$ with $2q/(1+u^2)$ that yields

$$\Sigma_{\mathrm{c}}^{\mathrm{r}} = (\alpha r_s)^2\left[\left(\frac{2}{\pi^3}\int_0^\infty du\,\frac{R_0(u)}{1+u^2}\right)\ln r_s + \mathrm{const} + \cdots\right]\ . \tag{2.3}$$

[see Eq. (A.4) in the Appendix for the integral]. The resulting $\Sigma_{\mathrm{c}}^{\mathrm{r}} = (\alpha r_s)^2(a\ln r_s + \mathrm{const} + \cdots)$ is in full agreement with the lhs of the Hugenholtz-van Hove theorem [Eq. (1.9)].

The frequency derivative $\Sigma_{\mathrm{c}}^{\mathrm{r}\prime} = \Sigma_{\mathrm{c}}^{\mathrm{r}\prime}(1,1/2)$ can be treated similarly [20],

$$\begin{aligned}
\Sigma_{\mathrm{c}}^{\mathrm{r}\prime} &= \frac{(\alpha r_s)^2}{\pi^4}\int\frac{d^3q}{q^3}\int du\,\frac{R(q,u)}{q^2+q_c^2R(q,u)}\,\frac{\partial}{\partial u}\,\frac{u}{u^2+(x+\frac{q}{2})^2} \\
&= -\frac{(\alpha r_s)^2}{\pi^4}\int\frac{d^3q}{q^3}\int du\,\frac{R(q,u)}{q^2+q_c^2R(q,u)}\,\frac{\partial}{\partial u}\,\frac{1}{2}\left(\arctan\frac{1+\frac{q}{2}}{u}+\arctan\frac{1-\frac{q}{2}}{u}\right)_\delta\ .
\end{aligned} \tag{2.4}$$

Note that a thin layer of vanishing thickness δ has to be deleted along $|\boldsymbol{e}+\boldsymbol{q}|$, allowing integration by parts. The small-q replacements $R(q,u)$ with $R_0(u)$ and $\arctan(1\pm q/2)/u$ with $\arctan 1/u$ yield

$$\Sigma_{\mathrm{c}}^{\mathrm{r}\prime} = \left(\frac{\alpha}{\pi^2}\int_0^\infty du\,\frac{R_0'(u)}{R_0(u)}\arctan\frac{1}{u}\right) r_s + O(r_s^2) \tag{2.5}$$

[see Eq. (A.4) in the Appendix for the integral]. Combining this equation with Eq. (1.5) affords the well known RPA result of $z_{\mathrm{F}} = 1/(1-\Sigma_{\mathrm{c}}^{\mathrm{r}\prime}) = 1 + \Sigma_{\mathrm{c}}^{\mathrm{r}\prime} + \cdots = 1 - 0.18\ r_s + \cdots$ [11].

III. THE HARTEE-FOCK SELF-ENERGY $\Sigma_{\rm c}^{\rm HF}(k)$

Since the bare Coulomb repulsion $v_0(q)$ is a static one, the Hartree-Fock self-energy $\Sigma_{\rm c}^{\rm HF} = (G - G_0)v_0$ is given by the momentum distribution $n(k)$ alone [21]

$$\Sigma_{\rm c}^{\rm HF}(k) = \frac{\alpha r_s}{\pi}\frac{1}{k}\int\limits_0^\infty dk'\; k'\; \ln\left|\frac{k-k'}{k+k'}\right|\; n_{\rm c}(k')\;, \quad n_{\rm c}(k) = n(k) - \Theta(1-k)\;. \tag{3.1}$$

In the above equation, the factor in front of $n(k)$ arises from the Coulomb repulsion. Because $\Sigma_{\rm c}^{\rm HF}(k)$ does not depend on ω, it cannot contribute to the deviations of $n(k)$ from $\Theta(1-k)$ and of $z_{\rm F}$ from 1 according to Eqs. (1.4) and (1.5). Such deviations are caused by the non-HF part $\Sigma_{\rm c}^{\rm nHF} = \Sigma_{\rm c} - \Sigma_{\rm c}^{\rm HF} = \Sigma_{\rm c}^{\rm r} + \cdots$. For $n(k)$ set to $\Theta(1-k)$, the Galitskii-Migdal formula [Eq. (1.6)] yields the lowest-order exchange energy $v_{\rm x} = -\frac{3}{4}\frac{\alpha r_s}{\pi}$, whereas for the actual $n(k)$ it produces the full exchange or Fock energy,

$$v_{\rm F} = -\frac{3}{2}\frac{\alpha r_s}{\pi}\int\limits_0^\infty dk\int\limits_0^\infty dk'\; n(k)n(k')\; kk' \ln\left|\frac{k+k'}{k-k'}\right|\;, \tag{3.2}$$

which constitutes only one component of the exact potential energy v [see Eq. (43) of ref. [12]]. Consider the (dimensionless) pair density $g(r)$ and its cumulant partitioning $g(r) = 1 - \frac{1}{2}f^2(r) - h(r)$ [3], where $f(r)$ is the (dimensionless) one-body density matrix [i.e. the Fourier transform of $n(k)$] and $h(r)$ is the cumulant pair density [i.e. the diagonal part of the cumulant (non-reducible) two-body density matrix]. The potential energy $v = v_{\rm F} + v_{\rm cum}$ follows from the full pair density $g(r)$. The Hartree term $g_0(r) = 1$ is compensated by the positive background, whereas $g_{\rm x}(r) = -\frac{1}{2}f^2(r)$ and $g_{\rm cum}(r) = -h(r)$ give rise to $v_{\rm F}$ of Eq. (3.2) and $v_{\rm cum}$, respectively. Consequently, the knowledge of the non-HF part $\Sigma_{\rm c}^{\rm nHF} = \Sigma_{\rm c}^{\rm r} + \cdots$ is essential for proper evaluation of $n(k)$, $z_{\rm F}$, and v. One may inquire whether it is nevertheless possible to employ the expression (3.1) in Eq. (1.9). Within perturbation theory, the leading term of $n_{\rm c}(k)$ is proportional to r_s^2, requiring that $\Sigma_{\rm c}^{\rm HF}(1) \sim r_s^3$, which contradicts the scaling $\mu_{\rm c} \sim r_s^2$. The following analysis demonstrates that this contradiction remains after the ring-diagram summation, which turns out to yield, respectively, $r_s^2 \ln r_s$ and $r_s^3 \ln r_s$ as the leading terms for the lhs and rhs of Eq. (1.9).

Because of the availability of exact $n_{\rm c}(k)$[11], the rhs of

$$\Sigma_{\rm c}^{\rm HF}(1) = \frac{\alpha r_s}{\pi} I, \ I = \int\limits_0^\infty dk\ n_{\rm c}(k) f(k), \ f(k) = k \ln \left| \frac{1-k}{1+k} \right| \tag{3.3}$$

can be readily computed at the weak-correlation limit of $r_s \to 0$. In the following, the approach previously employed in relating the small-r_s non-analyticities of $t_{\rm c}$ and $v_{\rm c}$ to the peculiarities of $n_{\rm c}(k)$ and the static structure factor $S_{\rm c}(q)$ at the limit of $r_s \to 0$ [5] is used.

The small-r_s behavior of the rhs of Eq. (3.1) is determined by the behavior of $n_c(k)$ near the Fermi surface. As shown by Daniel, Vosko, and Kulik [11], and reiterated in later works [4, 5], two functions are needed to describe this behavior, namely $F(k)$ with the properties

$$F(k \to 0) = 4.11234 + O(k^2)\,, \quad F(k \to \infty) = \frac{8\pi^2}{9} \frac{1}{k^8} + O\left(\frac{1}{k^{10}}\right)\,, \quad F(k \to 1) = \frac{\pi^2}{3} \frac{1 - \ln 2}{k^2 (1-k)^2}\,. \tag{3.4}$$

and $G(x)$ with the asymptotics

$$G(0) = 3.35334, \quad G(x \gg 1) = \frac{\pi}{6} \frac{1 - \ln 2}{x^2} + O\left(\frac{1}{x^4}\right). \tag{3.5}$$

Near $k = 1$, the function $n_c(k)$ is given by

$$n_{\rm c}(k) = \left(\frac{q_{\rm c}^2}{4\pi}\right)^2 \cdot \begin{cases} -F(k)\,, & 0 < k < 1 - \xi \\ -\frac{2\pi}{q_c^2} \frac{1}{k^2} G\left(\frac{1-k}{q_c}\right)\,, & 1 - \xi < k < 1 \\ +\frac{2\pi}{q_c^2} \frac{1}{k^2} G\left(\frac{k-1}{q_c}\right)\,, & 1 < k < 1 + \xi \\ +F(k)\,, & 1 + \xi < k \end{cases} \tag{3.6}$$

where $1 \gg \xi \gg q_c$ [5]. The function $F(k)$ contributes to $I = I_F + I_G$ through the expression

$$\begin{aligned} I_F &= I_F^> + I_F^<, \\ I_F^> &= \left(\frac{q_c^2}{4\pi}\right)^2 \int_{1+\xi}^\infty dk\ F(k) f(k)\,, \quad I_F^< = -\left(\frac{q_c^2}{4\pi}\right)^2 \int_0^{1-\xi} dk\ F(k) f(k)\,. \end{aligned} \tag{3.7}$$

With a fixed positive number A sufficiently small to assure that $F(k)$ can be replaced by its asymptotics (3.4), one obtains

$$\begin{aligned} I_F^> &\approx \left(\frac{q_c^2}{4\pi}\right)^2 \left[\int_{1+A}^\infty dk\ F(k) f(k) \ + \ \frac{\pi^2}{3}(1 - \ln 2) \int_{1+\xi}^{1+A} dk\ \frac{f(k)}{k^2(1-k)^2}\right] \\ &= \qquad O(r_s^2) \qquad + \ q_c^4 \frac{1 - \ln 2}{48} \int_\xi^A dk\ \frac{f(1+k)}{(1+k)^2 k^2}. \end{aligned} \tag{3.8}$$

The result for $I_F^{<}$ is similar, the above integrand being replaced by $-\frac{f(1-k)}{(1-k)^2k^2}$. Therefore

$$I_F \approx O(r_s^2) + q_c^4 \frac{1-\ln 2}{48} \int_\xi^A \frac{dk}{k^2}\, w(k), \quad w(k) = \frac{f(1+k)}{(1+k)^2} - \frac{f(1-k)}{(1-k)^2}. \tag{3.9}$$

The contribution of $G(x)$ to $I = I_F + I_G$ is treated analogously,

$$I_G = I_G^{>} + I_G^{<}\ , \tag{3.10}$$

$$I_G^{>} = \frac{q_c^2}{8\pi} \int_1^{1+\xi} dk\ \frac{1}{k^2} G\left(\frac{|k-1|}{q_c}\right)\ f(k) \quad , \quad I_G^{<} = -\frac{q_c^2}{8\pi} \int_{1-\xi}^{1} dk\ \frac{1}{k^2} G\left(\frac{|k-1|}{q_c}\right) f(k)\ .$$

With a fixed positive number B sufficiently large to assure that $G(x)$ can be replaced by its asymptotics (3.5), it follows that

$$\begin{aligned} I_G^{>} &= \frac{q_c^2}{8\pi} \left[\int_1^{1+q_cB} + \int_{1+q_cB}^{1+\xi}\right] \frac{dk}{k^2} G\left(\frac{|k-1|}{q_c}\right) f(k) \\ &\approx \frac{q_c^3}{8\pi} \int_0^B dx\ G(x)\ \frac{f(1+q_cx)}{(1+q_cx)^2} + q_c^4 \frac{1-\ln 2}{48} \int_{q_cB}^{\xi} \frac{dk}{k^2}\ \frac{f(1+k)}{(1+k)^2}. \end{aligned} \tag{3.11}$$

The result for $I_G^{<}$ is similar, the respective parts of the first and second integrands being replaced by $-\frac{f(1-q_cx)}{(1-q_cx)^2}$ and $-\frac{f(1-k)}{(1-k)^2}$. Therefore,

$$I_G \approx \frac{q_c^3}{8\pi} \int_0^B dx\ G(x)\ w(q_cx) + q_c^4 \frac{1-\ln 2}{48} \int_{q_cB}^{\xi} \frac{dk}{k^2}\ w(k). \tag{3.12}$$

Combining the above estimates, one obtains

$$I \approx O(r_s^2) + \frac{q_c^3}{8\pi} \int_0^B dx\ G(x)\ w(q_cx) + q_c^4 \frac{1-\ln 2}{48} \int_{q_cB}^{A} \frac{dk}{k^2}\ w(k). \tag{3.13}$$

Since for a sufficiently small positive k

$$w(k) = \frac{1}{1+k} \ln\left|\frac{k}{2+k}\right| - \frac{1}{1-k} \ln\left|\frac{k}{1-k}\right| \approx -2k\ln k, \tag{3.14}$$

the integrals of Eq. (3.13) yield the leading terms of

$$\frac{q_c^3}{8\pi} \int_0^B dx\ G(x)\ (-2q_cx)\ln(q_cx) = -q_c^4\ \frac{1-\ln 2}{24}\ [\ln q_c \ln B + C_0 \ln q_c + \frac{1}{2}(\ln B)^2], \tag{3.15}$$

where the constant C_0 does not depend on B, and

$$q_c^4\ \frac{1-\ln 2}{48} \int_{q_cB}^{A} \frac{dk}{k^2}(-2k)\ln k = q_c^4\ \frac{1-\ln 2}{48}\ [(\ln q_c + \ln B)^2 - (\ln A)^2] \tag{3.16}$$

(note that cancellation of the terms dependent on B in the combined integrals). Thus $\Sigma_{\rm c}^{\rm HF}(1) = (\frac{\alpha r_s}{\pi})^3\ \frac{1-\ln 2}{12}\ [(\ln r_s)^2 - 2C_0 \ln r_s] + \cdots$, which clearly demonstrates that for $r_s \to 0$ the non-HF term $\Sigma_{\rm c}^{\rm nHF}(1, 1/2) = \Sigma_{\rm c}^{\rm r}(1, 1/2) + \cdots$ has to be used in the rhs of Eq. (1.9). In summary, the terms that correctly describe the small$-r_s$ behavior are contained in $\Sigma_{\rm c}^{\rm nHF}(k,\omega) = \Sigma_{\rm c}^{\rm r}(k,\omega) + \cdots$ [22]. However, $\Sigma_{\rm c}^{\rm HF}(1)$, together with $v_{\rm F} - v_{\rm x}$, can serve as measures of the correlation strength, see refs. [12] and [23].

IV. CONCLUSIONS

The correct small$-r_s$ behavior of the correlation contribution $\Sigma_{\rm c}(k,\omega)$ to the self-energy is given by the ring-diagram-summed $\Sigma_{\rm c}^{\rm r}(k,\omega)$. The summation eliminates the divergence of $\Sigma_{\rm 2d}(1,1/2) \sim r_s^2 \int\limits_0 dq/q$ and of $n_{\rm 2d}(k)$ at the Fermi surface. Upon application of the Galitskii-Midgal formula, the correct potential energy $v_{\rm c} = 2a(\alpha r_s)^2 \ln r_s + \cdots$ results. The derivative $\partial\Sigma_{\rm c}^{\rm r}(1,\omega)/\partial\omega|_{\omega=1/2}$ used in conjunction with the Luttinger-Ward formula affords the correct $z_{\rm F} = 1-0.18\, r_s+\cdots$ for $r_s \to 0$. Finally, $\Sigma_{\rm c}^{\rm r}(1,1/2) = (\alpha r_s)^2(a\ln r_s+{\rm const}+\cdots)$ is in full agreement with the Hugenholtz-van Hove formula $\mu_{\rm c} = \Sigma_{\rm c}(1,\mu)$ with $\mu \to 1/2$ at the limit of $r_s \to 0$.

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APPENDIX A: THE POLARIZATION PROPAGATOR

The polarization propagator (in the lowest order) is given by

$$Q(q,\eta) \;=\; \int \frac{d^3k}{4\pi}\left[\frac{1}{\boldsymbol{q}(\boldsymbol{k}+\frac{1}{2}\boldsymbol{q})-\eta-\mathrm{i}\delta}+\frac{1}{\boldsymbol{q}(\boldsymbol{k}+\frac{1}{2}\boldsymbol{q})+\eta-\mathrm{i}\delta}\right]\Theta(1-k)\Theta(|\boldsymbol{k}+\boldsymbol{q}|-1). \tag{A.1}$$

For $\eta = \mathrm{i}qu$, a real function of q and u arises [11],

$$R(q,u) = Q(q,\mathrm{i}qu) = \frac{1}{2}\left[1 + \frac{1+u^2-\frac{q^2}{4}}{2q}\ln\frac{(\frac{q}{2}+1)^2+u^2}{(\frac{q}{2}-1)^2+u^2} - u\left(\arctan\frac{1+\frac{q}{2}}{u} + \arctan\frac{1-\frac{q}{2}}{u}\right)\right] , \tag{A.2}$$

which is even in u. The function $R(q,u)$ has the small-q expansion $R(q,u) = R_0(u) + O(q^2)$ with

$$R_0(u) = 1 - u\arctan\frac{1}{u} \ . \tag{A.3}$$

The integrals

$$\int_0^\infty du\ \frac{R_0(u)}{1+u^2} = \frac{\pi}{2}(1-\ln 2) \approx 0.482003 \quad \text{and} \quad \int_0^\infty du\ \frac{R_0'(u)}{R_0(u)}\arctan\frac{1}{u} \approx -3.353337 \tag{A.4}$$

appear in section II of this paper. The integrals

$$\int_0^1 dk'\ k'\ln\left|\frac{k+k'}{k-k'}\right| = 1 + \frac{1-k^2}{2k}\ \ln\left|\frac{1+k}{1-k}\right| \quad \text{and} \quad \int_0^1 dk \int_0^1 dk'\ kk'\ln\left|\frac{k+k'}{k-k'}\right| = \frac{1}{2} \tag{A.5}$$

appear in sections I and III.