INSTABILITY OF TOKAMAKS WITH NON-CIRCULAR CROSS-SECTION

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Abstract. The stability of external axisymmetric ideal MHD modes is considered. For a large aspect ratio equilibrium with linear profiles and with the wall infinitely far, it is shown that an arbitrary deviation from the circular cylinder is unstable. This proves a property which, hitherto, has merely been conjectured by several authors.

An axisymmetric equilibrium is described by the Lüst-Schlüter equation

$$\Delta_* \Psi + F = 0, \quad F(s,\Psi) = I'I + s^2 p', \quad \Delta_* = \frac{\partial^2}{\partial s^2} - \frac{1}{s}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial z^2} = s^2 \nabla \cdot \frac{\nabla}{s^2}$$

for the flux function Ψ. Here, s, ϕ, z are cylindrical coordinates, $p(\Psi)$ is the pressure, and $B_\phi = I(\Psi)/s$ is the toroidal field. The poloidal field $\vec{B}_p$ is related to Ψ by

$$\vec{B}_p = \nabla\phi \times \nabla\Psi.$$

The prime denotes differentiation with respect to Ψ. Let us take $\Psi = 0$ on the plasma boundary S_f, $\Psi > 0$ in the plasma region V_f, and $\Psi < 0$ in the vacuum region V_v. The profile functions $p(\Psi), I(\Psi)$ are chosen such that

$$I'(0) = p'(0) = 0. \tag{1}$$

This choice implies that the current density vanishes everywhere on S_f. If the current density were not zero there, the system could be unstable with respect to peeling modes. If (1) is valid and, in addition, the shear does not vanish, the stability problem for axisymmetric modes can be completely described by the functional

$$\delta W_f \geq \frac{1}{2} \int_{V_f} \frac{d^3\tau}{s^2} \{ |\nabla\xi|^2 + |f|^2 + \gamma p R^2 |g|^2 - \frac{\partial F}{\partial \Psi} |\xi|^2 \},$$

where ξ is the scalar quantity

$$\xi = \vec{\xi} \cdot \nabla\Psi$$

and the surface quantities f, g are defined by

$$f = f(\Psi) = I\Big(\int \frac{1}{s^2} \frac{d^2S}{|\nabla\Psi|}\Big)^{-1} \frac{d}{d\Psi} \int \frac{\xi}{s^2} \frac{d^2S}{|\nabla\Psi|},$$

$$g = g(\Psi) = \left(\int \frac{d^2 S}{|\nabla \Psi|}\right)^{-1} \frac{d}{d\Psi} \int \xi \frac{d^2 S}{|\nabla \Psi|}.$$

If the pressure vanishes on S_f, the energy variation of the system can be written in the form

$$\delta W = \delta W_f + \delta W_v,$$

$$\delta W_v = \frac{1}{2} \int_{V_v} \frac{d^3 \tau}{s^2} |\nabla \hat{\xi}|^2$$

with the boundary conditions

(2) $$\xi = 0 \text{ on the magnetic axis,}$$

(3) $$\hat{\xi} = \xi \text{ on } S_f,$$

and

(4) $$\hat{\xi} = 0 \text{ on } S_w,$$

the perfectly conducting outer wall. The minimum of δW corresponds to the eigenvalue problem [1]

(5) $$\Delta_* \xi + \left[(I'I)' + s^2 p''\right] \xi + (fI)' + \gamma s^2 (pg)' + \sigma \xi = 0 \text{ in } V_f,$$

(6) $$\Delta_* \hat{\xi} = 0 \text{ in } V_v$$

with the boundary conditions (2)-(4) and

(7) $$\nabla \Psi \cdot \nabla (\xi - \hat{\xi}) + fI = 0 \text{ on } S_f.$$

The condition that the lowest eigenvalue be non-negative,

$$\sigma_0 \geq 0,$$

is necessary and sufficient for stability against axisymmetric disturbances.

In this paper the eigenvalue problem (2)-(7) is solved for the straight case if both profiles are linear in Ψ, the position of the wall is infinitely far, and the fluid boundary deviates only slightly from the circle. It is shown that all these configurations are unstable. This contradicts stability investigations dealing with the so-called decay index which yields neutral stability in the large aspect ratio case. So, it has to be concluded that the decay index, which is only a crude estimate of the stability properties, does not give the correct result.
If the profiles $I'I$ and p' are linear in Ψ, conditions (1) imply that

$$I'I = (1 - \beta_p)\lambda\Psi, \quad p' = \beta_p \lambda \Psi$$

with constants $\lambda > 0,\ \beta_p \geq 0$. In polar coordinates r, Θ the equilibrium equation

$$\Delta\Psi + \lambda\Psi = 0, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\Theta^2}$$

has to be solved for the boundary condition

$$\Psi = 0 \ \text{ for } \ r = R(\Theta),$$

where $R(\Theta)$ describes the given contour and λ is considered as an eigenvalue for the equilibrium problem. If the contour deviates only slightly from the circle with radius R_0, a perturbation parameter ϵ can be introduced by

$$R(\Theta) = R_0 + \epsilon R_1(\Theta) + \epsilon^2 R_2(\Theta) + \cdots \tag{8}$$

According to (8) it is assumed that the solutions of the eigenvalue problem can also be expressed as power series in ϵ:

$$\Psi = \Psi_0 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \cdots, \quad \lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots.$$

The solution of the zeroth-order equation

$$\Delta\Psi_0 + \lambda_0\Psi_0 = 0$$

is, in terms of Bessel functions,

$$\Psi_0 = J_0(\rho), \quad \rho = \sqrt{\lambda_0}r, \quad \lambda_0 = \frac{j_{0,1}^2}{R_0^2},$$

where

$$\rho_0 = j_{0,1} = 2.4048255577$$

is the first zero of the Bessel function $J_0(\rho)$. For the discussion of the solubility conditions the notations

$$\overline{\cdots} = \frac{1}{2\pi}\int_0^{2\pi} r \cdots |_{r=R_0} d\Theta$$

$$\langle\cdots\rangle = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{R_0} \cdots r dr d\Theta$$

are introduced. Without restricting the generality it can be assumed that

$$\overline{R_1} = \overline{R_2} = 0, \tag{9}$$

$$\overline{R_1 exp(i\Theta)} = 0, \tag{10}$$

because if these quantities were not zero, they would describe a change of the radius or a shift but not a deformation of the circle. Conditions (10) lead to $\overline{\Psi_1} = 0,\ \lambda_1 = 0$ and in second order to

$$\lambda_2 \langle \Psi_0^2 \rangle = -\Psi_{0,r} (\overline{R_1 \Psi_{1,r}} - \frac{1}{2} R_0^{-1} \overline{R_1^2} \Psi_{0,r})(R_0).$$

The stability problem is also solved as power series in ϵ:

$$\xi = \xi_0 + \epsilon \xi_1 + \epsilon^2 \xi_2 + \cdots, \ \sigma = \sigma_0 + \epsilon \sigma_1 + \epsilon^2 \sigma_2 + \cdots.$$

The interesting eigenvalue $\sigma_0 = 0$ of the unperturbed problem is degenerate, i.e.there are two independent eigenfunctions $\xi_{(1)}, \xi_{(2)}$ belonging to it. These can be chosen as, for instance, the shift in two different directions, or in complex notation as

$$(11) \qquad \begin{cases} \xi_{(1)} = \Psi_{0,r} exp(i\Theta), & \xi_{(2)} = \Psi_{0,r} exp(-i\Theta), \\ \hat{\xi}_{(1)} = \dfrac{R_0 \Psi_{0,r}(R_0)}{r} exp(i\Theta), & \hat{\xi}_{(2)} = \dfrac{R_0 \Psi_{0,r}(R_0)}{r} exp(-i\Theta) \end{cases}$$

with the orthogonality property $\langle \xi^*_{(k)} \xi_{(l)} \rangle = \langle \Psi^2_{0,r} \rangle \delta_{kl}$, where the star denotes the complex conjugate. The unperturbed solution is a superposition

$$\xi_0 = \sum_{l=1}^{2} a_l \xi_{(l)}, \quad \hat{\xi}_0 = \sum_{l=1}^{2} a_l \hat{\xi}_{(l)},$$

whose coefficients a_l are determined in higher order. The solubility condition in first order takes the form

$$\sigma_1 \langle \Psi^2_{0,r} \rangle a_k = -\lambda_0 \sum_{l=1}^{2} \overline{\xi^*_{(k)} \xi_{(l)} R_1} a_l,$$

The system is unstable unless $\overline{R_1 exp(2i\Theta)} = 0$,wich means that the elliptical deformation of the circle must vanish. This corresponds to the result in [2]. If this is satisfied,it follows that $\sigma_1 = 0$ and the second order has to be considered, which yields the two eigenvalues $\sigma_2 = \langle \Psi^2_{0,r} \rangle^{-1} (-d \pm |c|)$ with

$$d = \lambda_0^{\frac{3}{2}} \Psi^2_{0,r}(R_0) \sum_{\nu \neq 0} |\nu| C_\nu|^2 / \rho_0, \quad R_1 = \sum_{\nu=-\infty}^{+\infty} C_\nu exp(i\nu\Theta).$$

This completes the instability proof. More details of the theory will be published elsewhere.

References.

[1] Lortz, D., On the Stability of Axisymmetric MHD Modes, Z. Naturforsch. 43a, (1988)
[2] Haas, F.A. and Papaloizou, J.C., MHD stability of toroidal equilibria to axisymmetric modes, Nuclear Fusion 17, 721-728 (1977)