A complete set of resistive compressible ballooning equations for 2-D flow equilibria

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Abstract: Based on the linearized compressible resistive MHD equations we derive a system of five equations describing the time dependent behaviour of ballooning modes in equilibria with sheared flow. The subsequent ordering scheme is based on a two scale expansion, where the fast varying scale is prescribed by an appropriately chosen eikonal.

1. Introduction: Recently considerable effort has been taken to study the stability behaviour of ballooning modes for MHD flow equilibria. In Refs. 1,2 the ideal MHD equations are investigated, where the equilibrium flow consists of a component parallel to the magnetic field and a rigid toroidal rotation. This ideal limit allows a stability analysis in terms of a single quantity ξ, the Lagrangian displacement vector, which is well known from the energy principle. In their analysis Chun and Hameiri [2] find periodic bursts in the time dependence of the perturbed flow velocity due to a parametric resonance in their equation. Similar conclusions are reached by Cooper [3], who studies the resistive but incompressible case in the framework of the WKB-method [4,5], where periodic bursts in the perturbed flow velocity as a function of time are found numerically. Now, looking at investigations of stability of resistive ballooning modes for static MHD equilibria [6,7], the governing equations are derived in a certain ordering scheme, which is directly applied to the linearized MHD equations. In the following section we will follow this line and derive a set of five ballooning equations for MHD flow equilibria, i.e. containing resistivity as well as compressibility effects.

2. The ballooning equations: We start from the usual linearized MHD equations.
Momentum balance :

$$(1)\quad \rho\frac{\partial\tilde{\mathbf{v}}}{\partial t} = -\rho(\tilde{\mathbf{v}}\cdot\nabla)\mathbf{V} - \rho(\mathbf{V}\cdot\nabla)\tilde{\mathbf{v}} - \tilde{\rho}(\mathbf{V}\cdot\nabla)\mathbf{V} - \nabla\tilde{p} + (\nabla\times\tilde{\mathbf{b}})\times\mathbf{B} + (\nabla\times\mathbf{B})\times\tilde{\mathbf{b}}.$$

Continuity equation :

$$(2)\qquad \frac{\partial\tilde{\rho}}{\partial t} = -\mathbf{V}\cdot\nabla\tilde{\rho} - \tilde{\mathbf{v}}\cdot\nabla\rho - \tilde{\rho}\nabla\cdot\mathbf{V} - \rho\nabla\cdot\tilde{\mathbf{v}}.$$

Energy balance :

$$\frac{\partial \widetilde{p}}{\partial t} = -\mathbf{V}\cdot\nabla\widetilde{p} - \widetilde{\mathbf{v}}\cdot\nabla p - \Gamma\widetilde{p}\nabla\cdot\mathbf{V} - \Gamma p\nabla\cdot\widetilde{\mathbf{v}}. \tag{3}$$

Maxwell's equations:

$$\frac{\partial \widetilde{\mathbf{b}}}{\partial t} = \nabla\times(\widetilde{\mathbf{v}}\times\mathbf{B}) + \nabla\times(\mathbf{V}\times\widetilde{\mathbf{b}}) - \eta\nabla\times(\nabla\times\widetilde{\mathbf{b}}) \tag{4}$$

$$\nabla\cdot\widetilde{\mathbf{b}} = 0 \tag{5}$$

All perturbations are indicated by twidles. To consistently introduce our ordering scheme, we specify two distinct scales denoted by $x_{fast} = x_{\perp}, x_{slow} = x_{\parallel}$ and t_{fast}, t_{slow} for rapid and slow variation of quantities in space and time, respectively. The scales are distinguished by the 'bookkeeping' parameter ε and our requirements on derivatives

$$\nabla_{\perp}\widetilde{a} = O(\varepsilon^{-1}), \quad \nabla_{\parallel}\widetilde{a} = O(1), \quad \nabla_{\perp}A = O(1), \quad \nabla_{\parallel}A = O(1)$$

$$\frac{\partial \widetilde{a}}{\partial t}|_{fast} = O(\varepsilon^{-1}), \quad \frac{\partial \widetilde{a}}{\partial t}|_{slow} = O(1) \tag{6}$$

of perturbations and equilibrium quantities, as well as

$$\eta = O(\varepsilon^2) \tag{7}$$

for the resistivity. The symbols $\nabla_{\perp}$ and $\nabla_{\parallel}$ denote derivatives perpendicular and parallel to the equilibrium magnetic field, respectively. All perturbations are now expanded asymptotically in powers of ε, e.g.

$$\widetilde{a}(x_{\perp}, x_{\parallel}, t_{fast}, t_{slow}) = \widetilde{a}_0(x_{\perp}, x_{\parallel}, t_{fast}, t_{slow}) + \varepsilon\widetilde{a}_1(x_{\perp}, x_{\parallel}, t_{fast}, t_{slow}) + \ldots \tag{8}$$

The scales are separated by means of an eikonal ansatz of the form

$$\widetilde{a}_i = a_i(x_{\perp}, x_{\parallel}, t_{fast}, t_{slow}) exp\frac{i}{\varepsilon}S(x_{\perp}, t_{fast}), \quad i = 0, 1, \ldots \tag{9}$$

Equations (1)-(5) are now consistently solved to order ε^{-1}, if the eikonal satisfies

$$\frac{\partial S}{\partial t}|_{fast} + \mathbf{V}\cdot\nabla_{\perp}S = 0 \tag{10}$$

and the constraints from momentum balance

(11) $$\frac{1}{\varepsilon}\nabla S(p_0 + \mathbf{B}\cdot\mathbf{b}_0) = 0.$$

From the continuity equation ,energy balance and Maxwell's equations we obtain :

(12) $$\frac{1}{\varepsilon}\nabla S\cdot\mathbf{v}_0 = 0, \qquad \frac{1}{\varepsilon}\nabla S\cdot\mathbf{b}_0 = 0 \quad .$$

These constraints are equivalent to the following representations of the perturbations

(13) $$\mathbf{v}_0 = \frac{v_\parallel}{B^2}\mathbf{B} + v_\perp\frac{\mathbf{B}\times\nabla S}{B^2} \quad , \quad \mathbf{b}_0 = -\frac{p_0}{B^2}\mathbf{B} + b_\perp\frac{\mathbf{B}\times\nabla S}{(\nabla S)^2}$$

Physically this ordering scheme implies effectively an incompressible motion on the fast time scale. After some algebra we obtain for the momentum balance equations :

(14) $$\begin{aligned}\rho\frac{\partial v_\parallel}{\partial t} &= -\rho(\mathbf{V}\cdot\nabla_\parallel)v_\parallel + \rho v_\parallel\frac{\mathbf{B}}{B^2}\cdot[(\mathbf{V}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{V}] \\ &+ \rho\frac{v_\perp}{B^2}[(\mathbf{B}\times\nabla S)\cdot(\mathbf{V}\cdot\nabla)\mathbf{B} - \mathbf{B}\cdot[(\mathbf{B}\times\nabla S)\cdot\nabla]\mathbf{V}] \\ &- \rho_0\mathbf{B}\cdot(\mathbf{V}\cdot\nabla)\mathbf{V} - \mathbf{B}\cdot\nabla p_0 + b_\perp\frac{B^2}{(\nabla S)^2}(\nabla\times\mathbf{B})\cdot\nabla S\end{aligned}$$

(15) $$\begin{aligned}\rho(\nabla S)^2\frac{\partial v_\perp}{\partial t} &= 2\rho v_\perp\nabla S\cdot(\nabla S\cdot\nabla)\mathbf{V} - \rho(\nabla S)^2(\mathbf{V}\cdot\nabla_\parallel)v_\perp \\ &+\rho v_\perp\frac{(\nabla S)^2}{B^2}\mathbf{B}\cdot[(\mathbf{V}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{V}] \\ &-\rho v_\perp(\nabla S)^2\nabla\cdot\mathbf{V} - \rho_0(\mathbf{B}\times\nabla S)\cdot(\mathbf{V}\cdot\nabla)\mathbf{V} - 2p_0\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla S) + B^2(\mathbf{B}\cdot\nabla)b_\perp \\ &-\rho v_\parallel\frac{(\mathbf{B}\times\nabla S)}{B^2}\cdot[(\mathbf{V}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{V}].\end{aligned}$$

From Faraday's law we obtain :

(16) $$\frac{\partial b_\perp}{\partial t} = -2\frac{\nabla S\cdot(\nabla S\cdot\nabla)\mathbf{V}}{(\nabla S)^2}b_\perp - \frac{\mathbf{B}}{B^2}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{B}]b_\perp$$

$$+\frac{(\mathbf{B}\times\nabla S)}{B^2}\cdot[(\frac{\mathbf{B}}{B^2}\cdot\nabla)\mathbf{V}-(\mathbf{V}\cdot\nabla)\frac{\mathbf{B}}{B^2}]b_{\|}-(\mathbf{V}\cdot\nabla_{\|})b_{\perp}+\frac{(\nabla S)^2}{B^2}(\mathbf{B}\cdot\nabla)v_{\perp}-\eta n^2(\nabla S)^2 b_{\perp}$$

and from the continuity equation and energy balance :

$$(17)\qquad \frac{\partial \rho_0}{\partial t}=-(\mathbf{V}\cdot\nabla_{\|})\rho_0-\mathbf{v}_0\cdot\nabla\rho-\rho_0\nabla\cdot\mathbf{V}-\rho\nabla_{\|}\cdot\mathbf{v}_0-\rho\nabla_{\perp}\cdot\mathbf{v}_1$$

$$(18)\qquad \frac{\partial p_0}{\partial t}=-(\mathbf{V}\cdot\nabla_{\|})p_0-\mathbf{v}_0\cdot\nabla p-\Gamma p_0\nabla\cdot\mathbf{V}-\Gamma p\nabla_{\|}\cdot\mathbf{v}_0-\Gamma p\nabla_{\perp}\cdot\mathbf{v}_1.$$

Note that the term $\propto \nabla_{\perp}\cdot\mathbf{v}_1$ in the last two equations is of the next to leading order in ε and has to be eliminated by the parallel component of Faraday's law :

$$(19)\ B^2\nabla_{\perp}\cdot\mathbf{v}_1=\frac{\partial p_0}{\partial t}+\mathbf{B}\cdot(\mathbf{b}_0\cdot\nabla)\mathbf{V}-\mathbf{B}\cdot(\mathbf{V}\cdot\nabla_{\|})\mathbf{b}_0+p_0\nabla\cdot\mathbf{V}-\frac{1}{2}(\mathbf{v}_0\cdot\nabla)B^2+\eta n^2(\nabla S)^2 p_0$$

3. Discussion: In the flowless case the system of equations (14)-(19) trivially reproduces the ordinary time dependent ballooning equations of Refs.(6,7). Furthermore we see from eq.(10) that the parametric resonance disappears in the case of parallel flow. In this limit the eikonal remains, as in the static case, constant in time.

One important feature of our ordering scheme is the appearance of slow time derivatives only. Since the system appears to be incompressible on the fast time scale, we do not have to consider effects of the fast magnetoacoustic wave in this ordering.

Numerical solutions of these equations will be presented in a forthcoming publication.

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