MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Calculations of the stellar structure of so-called degenerate stars using a new pressure function

J. Steuerwald and H. Wulff

IPP R/21 Juli 1977

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

IPP R/21

J. Steuerwald
H. Wulff

Calculations of the stellar structure of so-called degenerate stars using a new pressure function

Juli 1977

Abstract:

Masses and radii of degenerate stars were calculated using a pressure function which significantly deviates from the usual one based on the Fermi energy of free electrons. Assuming only a central number density ranging from 10^{30} cm^{-3} to 10^{36} cm^{-3}, the calculations yield masses between those of Jupiter and the sun. The masses were found to be a function of the composition of the elements. The maximum masses and the cosmic abundance of the elements are correlated. The radii come close to those of pulsars at high central densities, while at low densities they are equal to those of white dwarves.

I. Introduction

In a recently published paper /1/ one of the authors gave an energy distribution function for the electrons of a proton--electron plasma which appreciably deviates from the accepted function when degeneracy+) is present. According to the new function, for T = 0° K and $2\pi^2 a_B^3 n > 1$, where $a_B = \hbar^2/me^2$ denotes the Bohr radius and n the number density of the electrons or protons, the pressure of the degenerate electron gas is not given by the usual expression P_{Fermi}, but by

$$P = (2\pi^2 a_B^3 n)^{-1/3} \times P_{Fermi} = \tag{1}$$

$$= (2\pi^2 a_B^3 n)^{-1/3} \times \frac{1}{3}(2\pi^2)^{2/3}\frac{\hbar^2}{m} n^{5/3} = \frac{1}{3}(2\pi^2)^{1/3} e^2 n^{4/3}$$

The requirement $2\pi^2 a_B^3 n > 1$ for the validity of this equation is satisfied at the densities of white dwarves. It is therefore appropriate to use this equation to calculate the masses and radii of such stars. For this purpose it has to be extended to plasmas containing heavy elements in addition to protons and electrons.

/1/ H. Wulff: "On the Electron Energy Distribution in Proton Electron Plasmas with Densities Above the Solid-State Density". Nuovo Cimento, 31B, 92, (1976)

+) It was shown in /1/ that the term "degeneracy" is superfluous. "Degenerate" electrons are bound electrons that differ from those of an atom in that they are no longer bound to an atomic nucleus only. Rather can the stroll around in an ensemble of nuclei: they are conductive.

II. Derivation of the pressure function of a plasma with components of different atomic numbers z

Equation (1), which is valid exclusively for hydrogen plasmas, was obtained in /1/ by the correspondence principle. It was thereby shown that the virial theorem is satisfied with the pressure function according to eq. (1), whereas its existence has hitherto been ignored in deriving the relation $P = P_{Fermi}$.

With Coulomb interaction the virial theorem reads

$$2 \int \overline{E_{kin}}\, dV = - \int \bar{U}\, dV \quad , \tag{2}$$

where $\bar{E}_{kin}$ and $\bar{U}$ respectively denote in terms of Newton mechanics the mean temporal values of the kinetic and potential energies of the electrons in the unit volume. As is known, the quantities in eq. (2) are expressed in terms of quantum mechanics by the expectation values

$$\int \overline{E_{kin}}\, dV = -\frac{\hbar^2}{2m} \int \psi^* \Delta \psi\, d\hat{\tau} \tag{3}$$

and

$$\begin{aligned} \int \bar{U}\, dV &= -\int \psi \psi^* (w \cdot \nabla U)\, d\hat{\tau} \\ &= \int \psi \psi^* U\, d\hat{\tau} \end{aligned} \tag{4}$$

where $d\hat{\tau}$ denotes the volume element in the configuration space of the electrons. With the relation valid at isotropic pressure

$$p = \frac{2}{3} \overline{E_{kin}} \tag{5}$$

and the virial theorem it then also follows that

$$p = -\frac{1}{3}\overline{U} \quad . \tag{6}$$

From eq. (1) it follows direct that eq. (6) is satisfied when the potential energy per electron is

$$e^2/a = e^2(2\pi^2 n)^{1/3} \tag{7}$$

on the average. Whereas eq. (1), which is valid exclusively for proton plasmas, was derived on the correspondence principle, we shall calculate its generalization for arbitrary atomic number z by classical means, i.e. in accordance with the laws of electrostatics. (It was pointed out in /1/ that the energy eigenvalues of the H atom are also obtained by classical mechanics. The quantum mechanical characteristic of the energy values is the discretization of these values. Experimental experience shows that this discretization does not exist at the high densities involved here.) Our point of departure is the Poisson equation of electrostatics for spherical symmetry:

$$\Delta\varphi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = -4\pi\sigma(r) \quad . \tag{8}$$

Let the plasma consist for the time being of just nuclei of atomic number z of number density n_z and with the corresponding electrons, whose charge is assumed to be continuous and uniformly distributed. Let the negative charge density thus be given by

$$\sigma = -e z n_z \quad . \tag{9}$$

In integrating eq. (9), we determine the constants so that the potential and field strength assume the values -eZ/R and eZ/R^2 respectively on the surface of the sphere given by $1/n_Z = \frac{4\pi}{3} R^3$. This yields for the potential of the negative charge

$$\varphi_- = \begin{cases} \frac{4\pi}{3} n_z e Z \left(\frac{1}{2} r^2 - \frac{3}{2} \left(\frac{4\pi}{3} n_z \right)^{-2/3} \right) & r \leq R \\ -eZ/r & r \geq R \end{cases} \quad . \tag{10}$$

To this potential we add the potential of the nuclear (point) charge assumed at the centre of this charge

$$\varphi_+ = eZ/r \quad . \tag{11}$$

To obtain now the potential energy of the electrons per unit volume, we have to calculate the integrals

$$\phi_+ = \int_0^R \varphi_+ \sigma 4\pi r^2 dr \tag{12}$$

and

$$\phi_- = \frac{1}{2} \int_0^R \varphi_- \sigma 4\pi r^2 dr \tag{13}$$

and multiply them by the number of nuclei per unit volume. (Equations (12) and (13) are to be found in, for example, G. Joos: Lehrbuch der Theoretischen Physik, Frankfurt am Main 1959, 12th Edition, eq. (19) on p. 254 and eq. (31) on p. 270, respectively.) The pressure is obtained by means of eq. (6):

$$P_z = -\frac{1}{3} \bar{V} = -\frac{1}{3} n_z \left(\phi_+ + \phi_- \right) \tag{14'}$$

$$= \frac{1}{5} \left(\frac{3^2 \pi}{2} \right)^{1/3} e^2 Z^2 n_z^{4/3} \quad .$$

Obviously, we have not made allowance for any fluctuations in deriving eq. (14'), as the virial theorem requires and as is expressed in the mean value notation.

For z = 1 this equation becomes the equation (1) specially derived for a proton-electron plasma, apart from a factor $F = \frac{5}{3}\left(\frac{4\pi}{9}\right)^{1/3} = 1{,}86$. Better agreement is not to be expected in view of the different principles underlying the two equations and of the assumptions involved. To obtain a fit to eq. (1), we take as a basis for the following calculations instead of eq. (14') the equation

$$p_z = \frac{1}{3}(2\pi^2)^{1/3} e^2 Z^2 n_z^{4/3} \quad , \tag{14}$$

which is also valid for z = 1. The uncertainty involved in determining the pressure will be reconsidered later.

Plasmas with nuclei of different atomic numbers present new aspects. The potential energy on which eq. (14) is based might at first glance lead one to believe that the pressures behave additively like the potentials, e.g. when there are two components, they behave in accordance with the equation

$$\hat{p} = p_{z_1} + p_{z_2} = \frac{1}{3}(2\pi^2)^{1/3} e^2 \left(Z_1^2 n_{z_1}^{4/3} + Z_2^2 n_{z_2}^{4/3}\right) . \tag{15}$$

Here, however, the interaction of the two species, which, taken with a pinch of salt, has its terrestrial analogon in the Stark effect, is not considered. We show this in a purely formal manner for the time being in the case of a plasma which only contains the two (cosmically relatively abundant) element iron and nickel with the atomic numbers

$z_{Fe} = 26$ and $z_{Ni} = 28$. If, in addition, we assume equal densities $n_{Fe} = n_{Ni} = n$ of the two components, we obtain additively according to eq. (15)

$$\tilde{p} = \frac{1}{3}(2\pi^2)^{1/3} e^2 n^{4/3} (Z_{Fe}^2 + Z_{Ni}^2) \approx \frac{1}{3}(2\pi^2)^{1/3} e^2 n^{4/3} 2 Z_{Fe} Z_{Ni}$$
$$= \frac{1}{3}(2\pi^2)^{1/3} e^2 2 Z_{Fe} n_{Fe}^{2/3} Z_{Ni} n_{Ni}^{2/3} . \quad (16)$$

(If $2Z_{Fe} Z_{Ni}$ is substituted for the sum $Z_{Fe}^2 + Z_{Ni}^2$, the error is less than one-third per cent.) On the other hand, it may be assumed that owing to the slight difference in atomic number this plasma consists of a single component with the atomic number $\bar{Z} = \sqrt{Z_{Fe} Z_{Ni}}$ and density $2n$. The pressure is then obtained according to eq. (14):

$$p = \frac{1}{3}(2\pi^2)^{1/3} e^2 2^{4/3} Z_{Fe} n_{Fe}^{2/3} Z_{Ni} n_{Ni}^{2/3} . \quad (17)$$

The additional pressure due to the interaction of the two components is therefore, in accordance with eqs. (16) and (17),

$$p_w = p - \tilde{p} = 2(2^{1/3} - 1)\frac{1}{3}(2\pi^2)^{1/3} Z_{Fe} Z_{Ni} n_{Fe}^{2/3} n_{Ni}^{2/3} . \quad (18)$$

We can thus expect an interaction function which is proportional to $Z_1 Z_2 n_{Z_1}^{2/3} n_{Z_2}^{2/3}$. This is shown in the case of a plasma which consists of a single heavy component of atomic number z and the two cosmically most abundant elements, hydrogen and helium. For this purpose we assume that owing to the field strength of the heavy nucleus the light elements are arranged in the time average around symmetrical surfaces in alternately negative and positive layers, i.e. dipole layers. If we also assume that these layers have uniform separations of

$n_H^{-1/3}$ and $n_{He}^{-1/3}$ respectively, we obtain in this approximation the additional potential energy per unit volume

$$W = n_Z \int_0^R \frac{d\varphi}{dr} \left(e\, n_H^{-1/3} \cdot n_H + 2\, e\, n_{He}^{-1/3} \cdot n_{He} \right) 4\pi r^2 dr \,, \qquad (19)$$

where $R = \left(\frac{4\pi}{3} n_Z \right)^{-1/3}$ is again valid. The expressions $e\, n_H^{-1/3}$ and $2\, e\, n_{He}^{-1/3}$ obviously represent dipole moments whose densities are given by n_H and n_{He} respectively. With $\varphi = \varphi_+ + \varphi_-$ and $P_W = -\frac{1}{3} W$ according to eq. (6), integration of eq. (19) yields the pressure component due to the interaction of the heavy and light elements:

$$P_W = \left(\frac{3}{4} \pi^2 \right)^{1/3} e^2 Z\, n_Z^{2/3} \left(1 \cdot n_H^{2/3} + 2\, n_{He}^{2/3} \right). \qquad (20')$$

As can be seen, this function is of the structure presumed according to eq. (18). The derivation was performed similarly to that of eq. (14'). Just as constant charge density was assumed for this equation, constant dipole density was assumed in deriving eq. (20'). For the sake of uniformity we therefore provide the equation with the factor $F = \frac{5}{3} \left(\frac{4\pi}{9} \right)^{1/3} = 1{,}86$ and calculate in the following discussions with

$$P_W = 5\pi\, 3^{-4/3} e^2 Z\, n_Z^{2/3} \left(1 \cdot n_H^{2/3} + 2\, n_{He}^{2/3} \right). \qquad (20)$$

The transition from eq. (20') to eq. (20) may be ascribed to an increase of the dipole moments $e\, n_H^{-1/3}$ and $2\, e\, n_{He}^{-1/3}$ by a factor F. Although, as already stated, precise determination of the factor in the pressure formulae cannot be expected, it is necessary to subject the model to a critical examination to determine the extent to which the calculated masses and radii

of the stars depend on the choice of model. For this purpose we visualize the order of magnitude of the field strength due to the heavy elements, the metals, relative to that of the protons by taking the field strength, averaged over the volume $1/n_{Fe}$, that is produced by an iron core with its electrons and relating it to that exerted by a proton on the nearest electron. The latter is given by eq. (7):

$$F_H = -(2\pi^2)^{2/3} \varrho\, n_H^{2/3}$$

and the former is calculated with eqs. (10) and (11) from

$$\overline{F_e} = n_z \int_0^R \frac{d}{dr}(\varphi_+ + \varphi_-)\, 4\pi r^2 dr \quad .$$

For the density ratios n_{Fe}/n_H of 10^{-3}, 10^{-2} and 3×10^{-2} this yields the respective field strength ratios

$$\overline{F}_e/F_H = 0.23,\ 1.05 \text{ and } 2.0.$$

These values, or even just the fact that the field exerted by the heavy elements induces the dipoles, suggest that a model other than that with constant dipoles be taken as a basis. In the general case one will have to determine a distribution for which the total energy of the electrons assumes a minimum, and hence the kinetic energy and pressure a maximum. Although the model used certainly needs improvement, we will adhere to the assumption of linear dipoles because the characteristics of the results also appear in this way and refinement of this model can be dispensed with for the present because the pressure function accepted hitherto leads to results which differ in any case from ours.

III. Calculation of the masses and radii of white dwarves as a function of their chemical composition

As usual, our calculations are based on the hydrostatic equation valid for spherical symmetry:

$$-\frac{dP}{dr} = G\,\varrho(r)\,\frac{M(r)}{r^2} \qquad (21)$$

where G is the gravitational constant, ϱ is the mass density, which we calculate for simplicity with

$$\varrho = m_p n_p + \sum_{Z=2} 2\,Z\,m_p\,n_Z \qquad (22)$$

and

$$M(r) = 4\pi \int_0^r \varrho(\hat{r})\,\hat{r}^2\,d\hat{r} \qquad (23)$$

is the mass present in a sphere of radius r. Table 1 shows the radii in km and the masses relative to those of the sun and Jupiter as functions of the central density n(o) for single-component plasmas. They are obtained with the pressure function (14) from eqs. (21), (22) and (23).

As can be seen from table 1 the radii are independent of the type of nuclei except in the case of hydrogen. The formal justification for this correlation is that according to eqs. (14) and (21) both sides of the hydrostatic equation (21) contain Z^2 as a factor, so that the solution of this equation is independent of the atomic number. Therefore, according to eqs. (22) and (23) the mass of the stars increases in proportion to the atomic number. It is also seen from Table 1 that the masses are in-

Table 1

Element	Central density $n(o)$ in cm^{-3}	$M_\odot = 1047\ M_{Q4} = 1{,}98\ 10^{33}$ g $M/M_\odot$	M/M_{Q4}	Radius in km
H	10^{30}	1/223	4,70	4094
	10^{32}			882
	10^{33}			409
	10^{36}			40,9
	10^{37}			19,0
H_e	10^{30}	1/450	2,34	2047
	10^{32}			441
	10^{33}			205
	10^{36}			20,5
	10^{37}			9,5
C	dito	1/150	7,02	dito
Fe	dito	1/34	30,2	dito
z = 60	dito	1/15	68,5	dito

dependent of the central density. This well-known independence is due to the fact that both sides of eq. (21) are changed by the same factors when there is spatial compression or dilatation. The radius of the star is defined as the distance from the centre at which the number density has dropped to the solid-state density $\approx n_B = 1/(2\pi^2 a_B^3) = 3.4 \times 10^{23}\ cm^{-3}$. Down to this density the pressure function is exactly valid only for hydrogen, while it is generally valid, i.e. for arbitrary elements down to the limiting density $n_L = Z^3 n_B$.

It will also be approximately valid at lower densities, as long as the discrete X-ray terms existing under terrestrial conditions are sufficiently perturbed. A more exact pressure function would result in a stronger density drop in the outer regions. As this decrease is in any case already very strong in the stated approximation, as can be seen from Fig. 1, this difference can be neglected. The approximation is ultimately of no significance for calculating the masses since they no longer vary in the third decimal place whenever the density has dropped three powers of ten.

For the following investigations we represent the chemical composition schematically. Besides the cosmically most abundant elements hydrogen and helium, the relatively abundant element iron is taken into account as representative of all heavy elements. With this assumption one then obtains according to eqs. (14) and (20) the pressure function

$$P = \frac{1}{3}(2\pi^2)^{1/3} e^2 \left(1^2 n_H^{4/3} + 2^2 n_{He}^{4/3} + 26^2 n_{Fe}^{4/3}\right) + 5\pi 3^{-4/3} e^2 2\, n_{He}^{2/3}\, n_H^{2/3} + 5\pi 3^{-4/3} e^2 26^{2/3} \left(n_H^{2/3} + n_{He}^{2/3}\right). \qquad (24)$$

This was used to calculate the masses and radii on the assumption that the elements are uniformly mixed. The influence of sedimentation will be dealt with later. The central densities were normalized so that the sum of the number densities of the species of nuclei remains constant at different mixing ratios, i.e. the mass densities are adapted accordingly. In Table 2

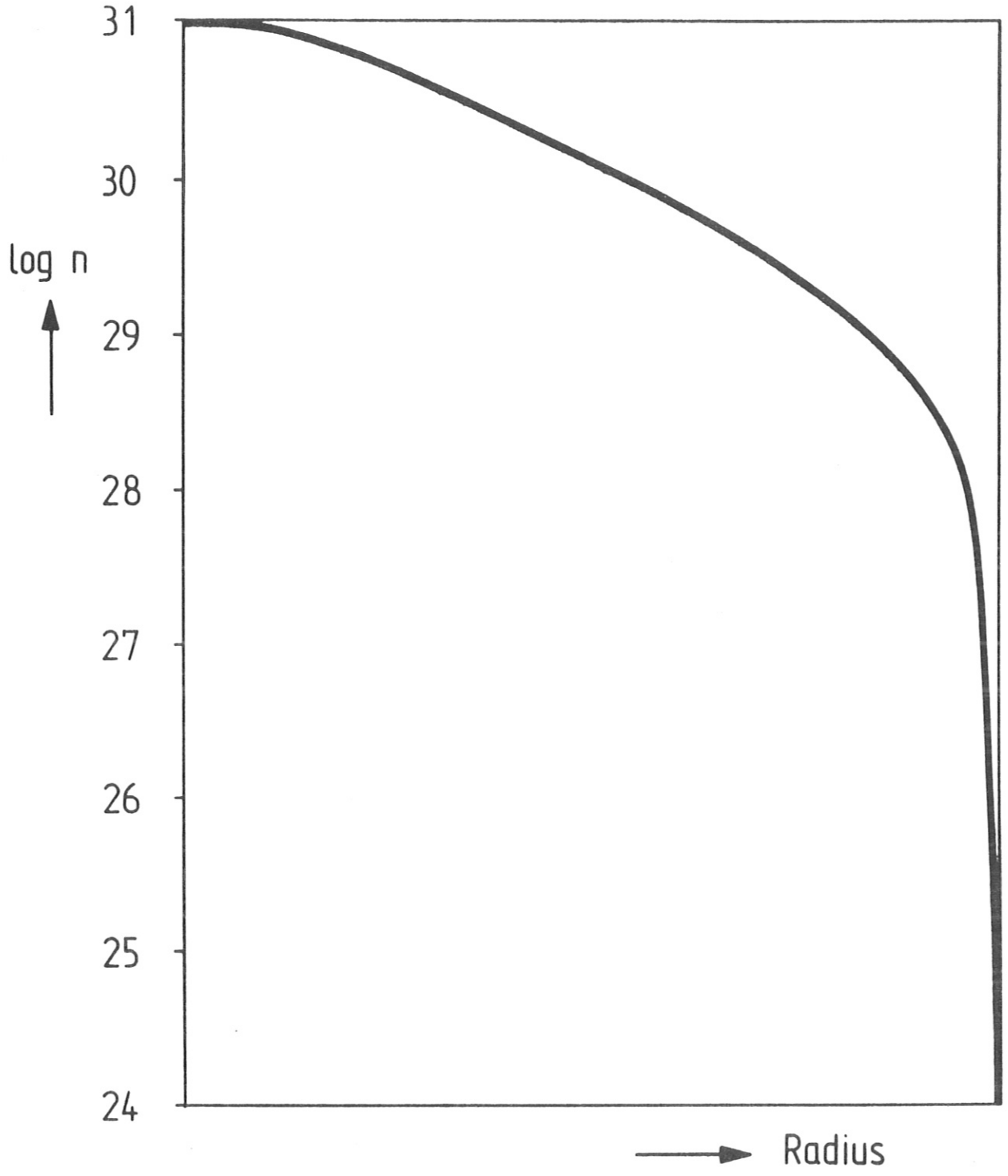

Fig. 1 Number density of the electrons (protons) in a pure hydrogen star as a function of the radial distance. The pressure is assumed to be proportional to $n^{4/3}$ according to eq. (1). The steep pressure drop towards the surface is characteristic. The pressure drop is essentially due to the fact that the force of gravity - represented by the RHS of eq. (22) - becomes very large towards the surface and, owing to the relatively low densities again in accordance with eq. (22), the pressure and hence the density variation have to be very large. This is also valid mutatis mutandis when a temperature pressure due to a temperature $T \neq 0$ is present in addition to the degeneracy pressure.

(Appendix) the masses and radii are given as functions of the mixing ratio for the central density $\sum_{i=H,He,Fe} n_i(0) = 10^{30}$ cm^{-3}. From this table it can be seen that the mass increases more strongly than is in keeping with the replacement of hydrogen nuclei by heavy nuclei. For example, the mass of a star in which 4 % of the protons is replaced by Fe nuclei is a factor of 11.3 as large as that of a pure hydrogen star. On the other hand, this replacement causes the mass density to grow a factor of only 3.2. This growth of the stellar mass is mainly due to the "Stark effect" of the heavy elements, which is represented by the third term of eq. (22). In our representation, which does not make any distinction between the effect of the relatively strong Fe field and the relatively weak He field apart from the z dependence, the maximum of the stellar mass occurs for 4 % and 12 % additions of Fe and He respectively. For all central densities one obtains

$$M_{max} = 0.060_3 \; M_\odot .$$

This mass is a power of ten smaller than the mean mass of white dwarves. On the other hand, these values and those in Table 2 according to S. v.Hoerner and K. Schaifers /2/ are of the same order as those of the invisible components of visible double stars. The radius at M_{max} is 6,475 km. The maximum radius is

---

/2/ Mayers Handbuch über das Weltall, p. 199, Mannheim (1960)

attained for 3 % and 1 % components of He and Fe respectively, being 7,623 km at $M = 0.046\ M_\odot$.

As was discussed on p. 9 , our representation has probably underestimated the influence of the Stark effect of the heavy elements, in this case iron. If only the third term of eq.(24) is increased by, for example, a factor of 3, one obtains instead of the above maximum value for the mass a value 3.3 times as large:

$$M_{max} = 0.2\ M_\odot .$$

This value is also obtained with an He component of 12 % and an Fe component of 3 %.

Finally, in comparing the data with those of directly observable stars, the white dwarves, we have to consider that the matter of which they are composed can no longer be regarded as degenerate according to our theory as opposed to the accepted one since in the case of, for example, a pure hydrogen plasma the characteristic temperature $T_p = P/nk$ is a factor of $(2\pi^2 a_B^3 n)^{-1/3} = (n/3.4 \times 10^{23})^{-1/3}$ lower than the usual Fermi temperature $T_F = P_{Fermi}/nk$. Table 3 lists the characteristic temperatures of a H plasma as a function of the density.

The values in parentheses give only an indication of the temperature since the energy of the electrons should have been calculated relativistically. The self-energy of the electron corresponds in keeping with

$$mc^2 = 3/2\ KT$$

to a temperature of $3.95 \times 10^9\ ^\circ K$.

Table 3

n (o) cm^{-3}	$T_F = P_{Fermi}/n\,k$ ^oK	$T_p = P/nk$ ^oK
10^{30}	2.15×10^9	1.5×10^7
10^{31}	$(1{,}0 \times 10^{10})$	$3{,}24 \times 10^7$
10^{33}	(2.15×10^{11})	1.5×10^8
10^{36}		1.5×10^9
10^{37}		(3.24×10^9)

In /1/ the equation of state for the electrons of a hydrogen plasma generally valid for $T \neq 0$ is given, which for $T \to \infty$ again yields $P_{T\to\infty} = nKT$. Owing to the computing difficulties involved in applying a non-zero temperature function and the necessity for extending it to plasmas of arbitrary composition, we take it into account by multiplying our pressure function by the factor n^{ε}. If we choose $\varepsilon = 7 \times 10^{-3}$, for example, we obtain $n^{\varepsilon} = 10^{30 \times \varepsilon} = 1.62$. The amount in excess of the degeneracy pressure (1.62 - 1) = 0.62 then represents the thermal pressure component. This yields for the centre $T = 0.62\ P/n = 7.8 \times 10^7$ ^oK, assuming an uniform distribution of the thermal energy to all particles - all electrons, 3 % Fe, 12 % He and 85 % H nuclei. This same procedure yields for boundary densities (Σ of the nuclear densities) of 10^{21} and 10^{18} cm^{-3} boundary temperatures of 7.8×10^4 ^oK and 7.8×10^3 ^oK. (as already stated, compared with the boundary density of 10^{24} cm^{-3} these densities produce hardly any change in either the mass or the radius.) This demonstrates that our assumption of the thermal pressure component can be reconciled with the hitherto assumed

thermal energy content of such stars and with their surface temperature. The white dwarf 0^2 Eridiani, for example, has a surface temperature of 13,500 K (see S. v.Hoerner and K. Schaifers /2/, pp. 182/183). White dwarves also have a high surface density compared with "normal" stars; in the spectrum of 0^2 Eridiani the wings of the Balmer lines H_γ and H_δ merge /3/. According to D.R. Inglis and E.Teller /4/, /3/ it can be concluded that the electron density is $n \approx 10^{18}$ cm^{-3}. Other white dwarves have a purely continuous spectrum. It may therefore be presumed that these stars have an even higher surface density.

We have calculated masses and radii by including in the pressure function the temperature factor $n^{\varepsilon} = n^{7/1000}$. Now with 11 % He and 3 % Fe one obtains for a central density of all nuclei of $n(o) = 10^{30}$ cm^{-3} a maximum mass

$$M_{max} = 0.42\ M_{\odot}.$$

The respective radius is 12,602 km. 0^2 Eridiani has a radius of 11,150 km and a mass $M = 0.467\ M_{\odot}$ (see S. v.Hoerner and K. Schaifers /2/, pp 182/183). The masses grow with increasing

---

/3/ See: A. Unsöld: Physik der Sternatmosphären, 2nd Edition, p. 54, Berlin-Heidelberg-Göttingen 1955

/4/ Astroph. Journ. 90, 439 (1939)

central density, which can be ascribed to the fact that the pressure function P is no longer proportional to $n^{4/3}$, but to $n^{4/3+\varepsilon}$. To demonstrate this situation and for later purposes, we list in Table 4 the maximum masses and the respective radii as functions of the central density for temperature factors n^{ε} with $\varepsilon = \pm 7/1000$ and $\varepsilon = 0$.

Table 4

n(o) cm^{-3} 3 % Fe, 11 % He, 86 % H	ε = - 7/1000		ε = 0		ε = + 7/1000	
	$M/M_\odot$	R km	$M/M_\odot$	R km	$M/M_\odot$	R km
10^{30}	0.092	8195	0,196	10 130	0.420	12 558
10^{31}	0.090	3771	"	4 703	0.428	5 897
10^{32}	0.088	1733	"	2 183	0.439	2 759
10^{33}	0.086	800	"	1 013	0.449	1 290
10^{36}	0.08	78	"	101	0.483	132

For comparison we then calculated the mass and radius of a hydrogen star in accordance with the accepted theory, i.e. with the pressure function $P = P_{Fermi}$, for a given mass density in the centre of $\varrho(o) = 1.23 \times 10^5$ g/cm^3, which corresponds to a number density $n(o) = 7.4 \times 10^{28}$ cm^{-3}. This yields $M = 1.38\ M_\odot$ and $R = 31.712$ km. S. Chandrasekhar /5/, on the other hand, obtains $M = 0.88\ M_\odot$ and $R = 27.900$ km. The difference between these values is mainly due

---

/5/ An Introduction to the Study of Stellar Structure, p. 427, Dover Publications, Inc., New York 14, N.Y. (1957)

to the fact that the pressure functions differ by the numerical factors:

$$P_{Chand.} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m}n^{5/3} = 0.0485\,\frac{h^2}{m}n^{5/3} \qquad (25)$$

$$P_{Fermi} = \frac{1}{12}\left(\frac{2}{\pi}\right)^{2/3}\frac{h^2}{m}n^{5/3} = 0.0616\,\frac{h^2}{m}n^{5/3} . \qquad (26)$$

It is again seen how sensitively the calculated masses (more than the radii) depend on the pressure function. (The reason why the numerical factor mostly given in the literature is different as used by Chandrasekhar from our one can easily be explained. This difference is of no significance for the problems involved here.) To a lesser extent one therefore also obtains different masses because Chandrasekhar /5/ takes into account the relativistic degeneracy occurring according to this theory in the central region of the stars. In the relativistic region one accordingly obtains the pressure function (/5/ p.362)

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} h c\, n^{4/3} = 2.4\times10^{-17}\, n^{4/3} .$$

At $n(o) = 7.4 \times 10^{28}$ cm^{-3} the central density has been assumed relatively low in order to neglect the relativistic influence. At high densities we obtain, for example, for $\varrho(o) = 9.67 \times 10^{8}$ gcm^{-3}, which corresponds to a number density $n(o) = 5.8 \times 10^{32}$ cm^{-3}, the mass $M = 5.51\ M_{\odot}$ and the radius $R = 4{,}130$ km. For higher central densities mass and radius values are no longer calculated except for the limiting case $\varrho(o) \to \infty$; this yields almost the the same limiting mass $M = 5.75\ M_{\odot}$ for vanishing radius $R_{\infty} = 0$.

(The rule given by Chandrasekhar /5/ in the footnote to Table 27 on p. 427 for converting the data calculated for a hydrogen star into those for a star of arbitrary composition were verified by our calculations.)

Notes

a) The accepted theory cannot explain the existence of stars with masses smaller than those of white dwarves, not even the masses of these stars, while our calculations also prove the existence of such stars.

b) We have calculated stellar data, assuming sedimentary matter. This always yields smaller radii and masses than when the mixing is assumed to be uniform. During the cooling process, i.e. ε tends to zero, it is thus possible for stars to form with masses smaller than those calculated here, i.e. of planetary dimensions. A look at Table 4 shows that the radii thereby increase if the central density decreases sufficiently during the mass transfer process. This might account for the continental drift.

c) The energy needed to fuse a proton and an electron to form a neutron (inverse ß decay) is 0.78 MeV. This is obtained by the electrons at densities that are between the limiting density valid for our non-relativistic theory $n_e = 10^{36}$ cm^{-3} and the density of neutron stars.

d) As can be seen from Table 4, stars lose mass on cooling, i.e. ε tends to zero. This loss may be a continuous or more or less periodic process. It is noteworthy in this context that one can calculate from our data for d e n s e stars frequencies of shock waves in the star

$$\nu = v_s / R = \sqrt{dP/d\varrho} / R$$

that are in agreement with the repitition frequencies of the radiation emitted by pulsars.

e) The calculated stellar masses assume a maximum at an abundance of elements which is apparently correlated with their cosmic abundance. The dependence of masses and radii on the composition of elements is a characteristic of our theory. This point will now be discussed with reference to the differing principles of our theory and the accepted one.

Theoretical consideration

According to the accepted theory the mass and radius values for a given mass density in the centre depend only on the number of electrons per unit atomic weight; in a pure hydrogen star it is one electron, and in a pure helium star there is one electron to every two units atomic weight. The same applies to a pure iron star[+)]

[+)] See eq. (22), according to which we set $\varrho = 2 Z m_p n_Z$.

or a star of arbitrary composition of elements whose atomic number is $z \geq 2$. Here the interaction of the electrons with the particles in their vicinity does not matter; they may even have zero electric charge. Theory just prescribes that the electrons be Fermi ions. For example, one obtains for both a He star and a Fe star of given central density equal masses and radii, whereas according to our theory the mass of the Fe star is a factor of $Z_{Fe}/Z_{He} = 13$ larger than that of the He star. We now reconsider this important fact in terms of the accepted pressure function according to eq. (25) or (26)

$$p \sim n_e^{5/3} = (\varrho/2m_p)^{5/3}$$

and that according to eq. (14), used here,

$$p \sim e^2 z^2 n_z^{4/3} = e^2 Z^{2/3} (\varrho/2m_p)^{4/3},$$

which, apart from the different power dependence, depends explicitly on the elementary charge and atomic number. This means that the energy of an electron and hence the pressure due to the electron is also a function not only of the density but also of the forces exerted. In terms of the general theory this difference is represented as follows:

To arrive at the relation $P = P_{Fermi}$, one starts from the expression that gives the number of states in phase space:

$$\frac{4\pi p^2 \Delta p V}{h^3} \qquad (27)$$

and uses the Pauli principle by requiering that each quantum cell (spin being ignored) be populated by one electron at most, i.e.

$$\frac{4\pi p^2 \Delta p\, V}{h^3} \geq N(p)\,\Delta p \qquad (28)$$

should be valid. Integration yields

$$\int_0^{p_0} N(p)\,dp = N \leq \frac{4\pi V}{h^3} \int_0^{p_0} p^2 dp = \frac{4\pi}{3}\frac{V}{h^3} p_0^3 \,. \qquad (29)$$

Taking this relation as an equation yields the threshold energy

$$\frac{1}{2m} p_0^2 = \frac{1}{2}\left(\frac{3}{4\pi}\right)^{2/3} \frac{h^2}{m} n^{2/3} \,, \quad n = \frac{N}{V} \,. \qquad (30)$$

The foregoing equations are then used to derive the pressure according to (25). See, for example, S. Chandrasekhar /5/ p. 355 et seq.. Apart from a factor of the order 2 this pressure is already obtained direct from eq. (30); $P = \frac{n}{3m} p_0^2$. Equation (29) is apparently identical with the uncertainty principle in the form

$$\frac{4\pi}{3} p_0^3 \frac{1}{n} \geq h^3 \,. \qquad (31)$$

As is known, the uncertainty principle is obtained direct from the commutation relation

$$(pq - qp)_{ke} = \frac{\hbar}{i} \delta_{kl} \,.$$

It has to be satisfied for each coordinate in configuration space irrespective of wether the electrons are free or bound. The apodictic assignment of such a coordinate to the mean inter-particle distance $\sqrt{\overline{q^2}} \to n^{-1/3}$ - using the Pauli principle leads to the relation (31). This determines the stationary system, particularly its energy, the Fermi energy. In quantum mechanics a stationary system is characterized solely by the Hamilton function $H(q,p)$, which has to be transformed to

principal axes by means of unitary matrices S:

$$S\,H\,S^{-1} = E.$$

This axiomatic requirement cannot be replaced by the given use of the Pauli principle. The exactly solvable single-electron problem of the hydrogen atom leads to, for example, the uncertainty principle in the form

$$|p|^3\, a^3(j) = j^3 \hbar^3 \qquad j = 1, 2, 3 \ldots \,, \tag{34}$$

where $a(j) = a_B j^2 = j^2 \hbar^2 / m e^2$ are agin the Bohr radii and j the principal quantum numbers. It is readily seen that eq. (34) yields the correct expectation values of the kinetic energy

$$\overline{E(j)}_{kin} = \frac{1}{2m} p^2 = \frac{1}{2} \frac{e^2}{a_B j^2}$$

and, because of the virial theorem $2\,\overline{E}_{kin} = -\,\overline{U}$, the correct energy eigenvalues as well:

$$E(j) = \overline{E(j)}_{kin} + \overline{V(j)} = -\frac{1}{2} \frac{e^2}{a_B j^2}$$

A relation to the ion density n is obtained by considering the Bohr orbit with the largest radius. The proximity of the nearest ion limits the orbit radii possible. Using the correspondence principle, one of the authors has shown in /1/ that the maximum available volume 1/n has to be set equal to $2\pi^2 a_B^3 j^6$. One thus obtains from eq. (34) for this limited orbit

$$|p|^3 \frac{1}{2\pi^2 n} = \left(2\pi^2 a_B^3 n\right)^{-1/2} \hbar^3 \,. \tag{35}$$

By means of the virial theorem this equation was also shown to characterize a proton-electron system at lowest energy and densities above solid-state density, i.e. if $2\pi^2 a_B^3 n > 1$ holds. Obviously, eq. (35) is very different from eq. (31) and consequently our pressure function eq.(1) also greatly differs from the usual one $P = P_{Fermi}$ as can be seen from eq. (1).

In terms of wave mechanics the problem can also be viewed as follows:

The ratio of the de Broglie wavelength $\lambda = h/p$ to the mean inter-proton distance $n^{-1/3}$ according to eq. (31) is constant;

$$\lambda n^{1/3} = (4\pi)^{1/3} \quad ,$$

but according to eq. (35) it is

$$\lambda n^{1/3} = (4\pi)^{1/3} (2\pi^2 a_B^3 n)^{1/6} \quad .$$

For high densities, $2\pi^2 a_B^3 n > 1$, the de Broglie wavelength accordingly exeeds the mean inter-proton distance. That is, at these densities there are no longer any single-particle wave functions, i.e. sub-Bohr orbits, as we are well aware from Schrödingers equation. In other words, at such high densities quantum mechanics can only describe systems with more than one electron and this cannot be done by means of single-particle functions such as follow from eq. (31), which is based on the Pauli principle. Doubts as to the proper application of the Pauli principle and even its universal applicability have already been expressed in former times. Heisenberg, for

example, concludes in his paper entitled "Mehrkörperprobleme und Resonanz in der Quantenmechnik" /6/:

"The choice of antisymmetric system in this very case cannot be replaced without force by the requirement that equivalent orbits should not be present (Pauli principle /7/)". Sommerfeld /8/ in his "explanation" of the periodic system refers to the Pauli principle as "rather cabalistic" and concludes the chapter as follows:

"Here we have only applied the Pauli principle to the conditions in a single atom. In keeping with its universal character, however, it is also valid for the totality of electrons in any molecule, even for the much more extensive system of conduction electrons in an arbitrarily expanded metal (and for that of a degenerate star /7/). The question that could already be asked in the case of one atom becomes even more pointed:
How do the electrons manage to communicate to one another what quantum states are to be populated in order to avoid any violation of the Pauli principle? From the corpuscular point of view, it is certainly not possible to answer this question. From the wave-mechanical point of view, the paradoxical nature of the situation is somewhat moderated, but by no means eradicated."

/6/ Zeitschr. f. Phys., 41, 239 (1927)

/7/ Authors' note

/8/ Atombau und Spektrallinien, Vol. I, 6th Edition, p. 167, Braunschweig (1944)

HE %	FE %	M IN G	R IN CM	HE %	FE %	M IN G	R IN CM	HE %	FE %	M IN G	R IN CM
0.0	0.0	0.887E+31	4.094E+08	5.0	0.0	0.199E+32	5.117E+08	10.0	0.0	0.230E+32	5.150E+08
	1.0	7.637E+31	7.315E+08		1.0	0.941E+32	7.600E+08		1.0	0.948E+32	7.401E+08
	2.0	9.338E+31	7.099E+08		2.0	1.111E+32	7.344E+08		2.0	1.126E+32	7.216E+08
	3.0	9.869E+31	6.708E+08		3.0	1.157E+32	6.939E+08		3.0	1.181E+32	6.861E+08
	4.0	9.991E+31	6.336E+08		4.0	1.161E+32	6.554E+08		4.0	1.191E+32	6.509E+08
	5.0	9.950E+31	6.008E+08		5.0	1.148E+32	6.215E+08		5.0	1.181E+32	6.192E+08
	6.0	9.841E+31	5.724E+08		6.0	1.129E+32	5.920E+08		6.0	1.164E+32	5.912E+08
	7.0	9.705E+31	5.477E+08		7.0	1.107E+32	5.662E+08		7.0	1.144E+32	5.665E+08
	8.0	9.560E+31	5.261E+08		8.0	1.086E+32	5.437E+08		8.0	1.123E+32	5.446E+08
	9.0	9.416E+31	5.070E+08		9.0	1.065E+32	5.237E+08		9.0	1.103E+32	5.252E+08
1.0	0.0	0.134E+32	4.648E+08	6.0	0.0	0.208E+32	5.148E+08	11.0	0.0	0.232E+32	5.133E+08
	1.0	0.856E+32	7.549E+08		1.0	0.948E+32	7.570E+08		1.0	0.945E+32	7.351E+08
	2.0	1.023E+32	7.282E+08		2.0	1.119E+32	7.328E+08		2.0	1.124E+32	7.181E+08
	3.0	1.069E+32	6.863E+08		3.0	1.166E+32	6.931E+08		3.0	1.181E+32	6.837E+08
	4.0	1.075E+32	6.471E+08		4.0	1.171E+32	6.552E+08		4.0	1.192E+32	6.493E+08
	5.0	1.065E+32	6.129E+08		5.0	1.158E+32	6.217E+08		5.0	1.184E+32	6.181E+08
	6.0	1.049E+32	5.833E+08		6.0	1.139E+32	5.924E+08		6.0	1.168E+32	5.904E+08
	7.0	1.031E+32	5.577E+08		7.0	1.118E+32	5.668E+08		7.0	1.148E+32	5.660E+08
	8.0	1.013E+32	5.353E+08		8.0	1.096E+32	5.443E+08		8.0	1.128E+32	5.443E+08
	9.0	0.995E+32	5.155E+08		9.0	1.076E+32	5.244E+08		9.0	1.107E+32	5.250E+08
2.0	0.0	0.157E+32	4.857E+08	7.0	0.0	0.215E+32	5.163E+08	12.0	0.0	0.235E+32	5.111E+08
	1.0	0.893E+32	7.606E+08		1.0	0.951E+32	7.535E+08		1.0	0.940E+32	7.300E+08
	2.0	1.059E+32	7.332E+08		2.0	1.124E+32	7.306E+08		2.0	1.121E+32	7.144E+08
	3.0	1.104E+32	6.910E+08		3.0	1.173E+32	6.919E+08		3.0	1.180E+32	6.811E+32
	4.0	1.108E+32	6.515E+08		4.0	1.178E+32	6.546E+08		4.0	1.193E+32	6.475E+08
	5.0	1.096E+32	6.171E+08		5.0	1.166E+32	6.214E+08		5.0	1.186E+32	6.168E+08
	6.0	1.079E+32	5.873E+08		6.0	1.148E+32	5.924E+08		6.0	1.170E+32	5.895E+08
	7.0	1.059E+32	5.614E+08		7.0	1.126E+32	5.671E+08		7.0	1.151E+32	5.653E+08
	8.0	1.039E+32	5.388E+08		8.0	1.105E+32	5.447E+08		8.0	1.131E+32	5.439E+08
	9.0	1.020E+32	5.189E+08		9.0	1.084E+32	5.249E+08		9.0	1.111E+32	5.247E+08
3.0	0.0	0.174E+32	4.984E+08	8.0	0.0	0.221E+32	5.167E+08	13.0	0.0	0.236E+32	5.085E+08
	1.0	0.916E+32	7.623E+08		1.0	0.952E+32	7.494E+08		1.0	0.934E+32	7.247E+08
	2.0	1.083E+32	7.351E+08		2.0	1.126E+32	7.279E+08		2.0	1.117E+32	7.105E+08
	3.0	1.128E+32	6.932E+08		3.0	1.177E+32	6.902E+08		3.0	1.178E+32	6.784E+08
	4.0	1.131E+32	6.539E+08		4.0	1.184E+32	6.536E+08		4.0	1.192E+32	6.455E+08
	5.0	1.118E+32	6.155E+08		5.0	1.173E+32	6.209E+08		5.0	1.187E+32	6.154E+08
	6.0	1.100E+32	5.897E+08		6.0	1.154E+32	5.922E+08		6.0	1.172E+32	5.885E+08
	7.0	1.079E+32	5.638E+08		7.0	1.134E+32	5.671E+08		7.0	1.154E+32	5.646E+08
	8.0	1.059E+32	5.411E+08		8.0	1.112E+32	5.449E+08		8.0	1.134E+32	5.433E+08
	9.0	1.039E+32	5.211E+08		9.0	1.091E+32	5.252E+08		9.0	1.114E+32	5.243E+08
4.0	0.0	0.188E+32	5.065E+08	9.0	0.0	0.226E+32	5.162E+08	14.0	0.0	0.237E+32	5.056E+08
	9.0	0.931E+32	7.618E+08		1.0	0.951E+32	7.449E+08		1.0	0.928E+32	7.192E+08
	2.0	1.099E+32	7.353E+08		2.0	1.127E+32	7.249E+08		2.0	1.113E+32	7.066E+08
	3.0	1.145E+32	6.940E+08		3.0	1.180E+32	6.883E+08		3.0	1.175E+32	6.756E+08
	4.0	1.148E+32	6.550E+08		4.0	1.188E+32	6.524E+08		4.0	1.191E+32	6.434E+08
	5.0	1.135E+32	6.208E+08		5.0	1.178E+32	6.202E+08		5.0	1.187E+32	6.138E+08
	6.0	1.116E+32	5.911E+08		6.0	1.160E+32	5.918E+08		6.0	1.173E+32	5.873E+08
	7.0	1.095E+32	5.653E+08		7.0	1.139E+32	5.669E+08		7.0	1.155E+32	5.637E+08
	8.0	1.074E+32	5.426E+08		8.0	1.118E+32	5.448E+08		8.0	1.136E+32	5.427E+08
	9.0	1.053E+32	5.226E+08		9.0	1.098E+32	5.253E+08		9.0	1.117E+32	5.239E+08

TABLE 2

THE TABLE SHOWS THE MASSES AND RADII OF STARS COMPOSED OF HYDROGEN, HELIUM AND IRON MIXED IN VARIOUS HOMOGENOUS RATIOS. THE HE AND FE COMPONENTS ARE GIVEN AS PERCENTAGES WITH RESPECT TO THE TOTAL NUMBER OF NUCLEI SUM(N(I)), I=H,HE,FE. THE CENTRAL DENSITY OF THE NUCLEI SUM(N(I)) AT R=0 IS GIVEN AS 1.E+30 CM**(-3).