Quadratic-in-spin effects in the orbital dynamics and gravitational-wave energy flux of compact binaries at the 3PN order

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Abstract

We investigate the dynamics of spinning binaries of compact objects at the next-to-leading order in the quadratic-in-spin effects, which corresponds to the third post-Newtonian order (3PN). Using a Dixon-type multipolar formalism for spinning point particles endowed with spin-induced quadrupoles and computing iteratively in harmonic coordinates the relevant pieces of the PN metric within the near zone, we derive the post-Newtonian equations of motion as well as the equations of spin precession. We find full equivalence with available results. We then focus on the far-zone field produced by those systems and obtain the previously unknown 3PN spin contributions to the gravitational-wave energy flux by means of the multipolar post-Minkowskian (MPM) wave generation formalism. Our results are presented in the center-of-mass frame for generic orbits, before being further specialized to the case of spin-aligned, circular orbits. We derive the orbital phase of the binary based on the energy balance equation and briefly discuss the relevance of the new terms.

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I. INTRODUCTION

Coalescing binary systems composed of stellar mass black-holes and/or neutron stars are among the most promising sources for a first direct detection of gravitational waves (GW) by the network of ground-based interferometers formed by GEO-HF [1] and the advanced version of the detectors LIGO [2] and Virgo [3], which should resume their science runs from 2015, approaching gradually their design sensitivity, expected to be better by an order of magnitude than that of the first generation. The cryogenic detector KAGRA [4] will join them in a near future. Further ahead, the space-based observatory eLISA [5, 6] — a serious proposal for the mission recently announced by the European Space Agency — will allow us to scan a different frequency band where we expect to detect, notably, GW emitted by supermassive black-hole binaries before merger.

Extraction of the signal from the noisy data by means of matched filtering techniques and source parameter estimation both require an accurate modeling of the waveform. For binary systems of compact objects, the inspiralling phase of the coalescence can be modeled extremely well by resorting to the perturbative post-Newtonian (PN) scheme (see [7] for a review), in which all quantities of interest are expanded as formal series in powers of $1/c$. For non-spinning (NS) systems, the phase of the waveform is currently known up to the order 3.5PN (i.e. including corrections up to $1/c^7$), whereas the full polarizations have been obtained up to the order 3PN [8] (with the dominant quadrupole and octupole modes in the decomposition of the waveform in spin-weighted spherical harmonics known up to the order 3.5PN [9, 10]).

In recent years, motivated by astrophysical observations suggesting that black holes in our universe can have significant spins, considerable effort has been devoted to investigating higher order corrections to the spin effects in the binary dynamics, mostly restricted to the conservative piece of the body evolution in the near zone. While for the neutron stars observed so far, the largest dimensionless spin magnitude ever measured [11] is only $\chi \sim 0.4$ (and may reasonably be assumed to be much smaller for typical expected observations), the spin of a black-hole might be commonly close to its maximal value [12–15]. Then, its effect on the waveform can be fairly strong and, in particular, for spins misaligned with the orbital angular momentum of the system, the dynamics becomes much more involved as the orbital plane undergoes precession, resulting in large modulations of the waveforms [16, 17]. Even in the simpler case where the spins are aligned with the orbital angular momentum, they significantly affect the inspiral rate of the binary, i.e. the frequency evolution of the signal, starting at the 1.5 PN order (see for instance Ref. [18] for a detailed study of the effect of the spin on the waveform quantified in terms of figures of merit relevant to data analysis). To make all factors $1/c$ appear explicitly in this paper, we rescale the physical spin variable S_{physical} as

$$S = c\, S_{\text{physical}} = G m^2 \chi\,, \tag{1.1}$$

where χ is the dimensionless spin, with value 1 for an extremal Kerr black hole.

The calculation of the spin PN corrections to the conservative part of the dynamics and, to some extent, to the radiation field of the binary beyond the leading order contributions has been tackled using essentially three different approaches: (i) a Hamiltonian approach that strongly relies on the use of the (second) Arnowitt-Deser-Missner (ADM) gauge [19], and in which the dissipative part of the dynamics, demanding a special treatment, is generally discarded (see however Ref. [20]), (ii) an effective field theory (EFT) Lagrangian formalism [21, 22], whose application to binary systems in general relativity has been actively developed since the mid-2000's, and (iii) a post-Newtonian iteration scheme in harmonic coordinates (PNISH), reviewed in Ref. [7], which we follow in the present paper. The existence of those three independent methods permits important checks of calculations that are often tedious, whenever quantities are available at the same order in more than one formalism.

The binary dynamics at the spin-orbit level (i.e. linear-in-spin effects, which will be referred to as SO from now on) are known up to the order 3.5PN in both the PNISH and ADM approaches [23–28], and to the order 2.5PN in the EFT framework [29, 30]. On the other hand, quadratic-in-spin corrections (labeled as SS throughout the paper) have been obtained to the order 2PN in the PNISH formalism [31–33], while in both the ADM and EFT formalisms they are known up to the order 3PN [34–37], and even 4PN for the simpler S_1S_2 interactions [22, 34, 38, 39]. Higher-order-in-spin corrections have also been recently derived [37, 40–42]. As for the spin contributions to the radiation field, they have mostly been computed by using the same usual combination of the MPM and PNISH approaches as in the present paper, although partial results required for the calculation of the 3PN flux [43] and the 2.5PN waveform [44] have been obtained within the EFT approach. The energy flux of gravitational-wave radiation is known up to the order 4PN at the SO level [45–47], whereas at the SS level only the leading order (2PN) terms were known until now [33]. Moreover, the leading order cubic-in-spin terms, which arise at 3.5PN, have been calculated very recently [42].

Our goal here will be to determine, within the PNISH approach, the 3PN (i.e. next-to-leading order) spin-spin corrections entering both the source dynamics (thereby providing an additional confirmation of the ADM and EFT results already available at this order) and most importantly the energy flux, thus completing the knowledge of all the spinning corrections to the phasing formula up to the 3PN order. At the next order 3.5PN, the only remaining unknown terms all come from a SS tail contribution. By contrast, the spin corrections to the full gravitational-wave polarizations are only known to the poorer 2PN accuracy [33, 48] and we postpone to future work the task of obtaining all the corrections up to the order 3PN.

Our source modeling, as well as the one used in the EFT and ADM approaches, consists in representing each compact object as a (spinning) point particle whose internal structure is entirely parametrized by a set of effective multipole moments. The validity of this description, which makes the calculations tractable analytically, relies on (i) the compact character of the bodies, and (ii) the weak influence of their internal dynamics to their "global" motion in general relativity, often referred to as the effacement principle [49]. The foundations of this formalism were laid down in the seminal works of Mathisson [50–52]. Later Papapetrou [53], found a particularly simple form for the evolution equations (which comprise both the equations of motion and of spin precession) for dipolar particles, i.e. at linear order in spins. His derivation was improved and rephrased in the language of distribution theory by Tulczyjew [54], whose method — systematically extensible beyond the dipolar model — has been recently applied at the quadrupolar level [55]. The dynamics of point particles with finite-size effects described by higher multipoles was thoroughly investigated by Dixon [56–59], who constructed an appropriate stress-energy "skeleton" to encode information about the internal structure of the body while, on their side, Bailey & Israel proposed an elegant effective Lagrangian formulation [60]. Recently, Harte [61] showed how the formalism of Dixon could be extended to self-gravitating systems, by constructing appropriate effective momenta and effective multipole moments evolving in some effective metric.

In the present article, we are interested in the quadratic-in-spin contributions arising from the quadrupolar moment of the compact object in the case where it is adiabatically induced by the spin [32, 33, 35, 62], as well as the simpler contributions coming from products of SO corrections. Because, in our source model, we replace extended bodies by point particles within a self-gravitating system, our approach must be regarded as an effective one and supplemented with some UV regularization procedure. A good choice is known to be dimensional regularization, with possible need of renormalization. We find however that, at this order, the so-called pure Hadamard-Schwartz prescription [63] is sufficient, i.e. that dimensional regularization is not necessary.

The paper is organized as follows. In Section II, we explain how the dynamics of a test point particle endowed with a spin-induced quadrupolar structure moving in a curved background spacetime is described in the Dixon-Mathisson-Papapetrou formalism. We also write the equations of evolution for the particle worldline, as well as for the spin, under a convenient explicit form, and we define a spin vector of conserved Euclidian norm in terms of which our PN results shall be written. The validity of the model to describe the body dynamics in self-gravitating binaries is discussed. In Section III, dedicated to the computation of the next-to-leading order SS contributions to the PN equations of motion, we present expressions for the conserved energy in the center-of-mass frame, both for generic orbits and for the restricted case of circular orbits in the absence of precession. Finally, Section IV sketches the derivation of the next-to-leading order SS contributions to the GW flux and includes a discussion of the impact of our newly derived terms on the phase evolution of non-precessing binaries in the frequency band of LIGO and Virgo. Because of the length of the equations, some results are relegated to appendices. Appendix A gives the explicit expressions for the relative acceleration and the precession vector in the center-of-mass frame, and Appendix B shows the relevant SS contributions to the source moments. We also give the explicit transformation between spin vector and spin tensor in Appendix C, as well as the correspondence between our results and the ADM ones in Appendix D.

We use the following conventions henceforth: $\mathcal{O}(n)$ means $\mathcal{O}(1/c^n)$, i.e. represents a contribution of the order $(n/2)$PN at least. Greek indices denote spacetime coordinates, i.e. $\mu = 0, 1, 2, 3$, while Latin indices are used for spatial coordinates, i.e. $i = 1, 2, 3$. Symmetrization and anti-symmetrization are represented by, respectively, parenthesis and brackets around indices. We adopt the signature $(-, +, +, +)$ and keep explicit both Newton's constant G and the speed of light c. Finally the covariant derivative along the worldline is written as $\mathrm{D}/(c\,\mathrm{d}\tau) = u^\mu \nabla_\mu$, where u^μ is the four velocity of the particle, defined such that $u^\mu u_\mu = -1$.

II. DYNAMICS OF QUADRUPOLAR PARTICLES

We shall now introduce the model we have adopted to represent the two spinning compact objects composing the binary as point particles. In Section II A, we display the Dixon-Mathisson-Papapetrou evolution equations for test bodies at quadrupolar order, set the covariant spin supplementary condition, and discuss its consequences. In Section II B, we rewrite the equations of motion in terms of the 4-velocity and introduce a conserved mass. Section II C presents the construction of a spin vector with a conserved Euclidean norm and shows the precession equation it satisfies. Finally, Section II D explains to what extent the Dixon-Mathisson-Papapetrou dynamics can be used for the companions of a self-gravitating binary.

A. The Dixon-Mathisson-Papapetrou framework

When describing the dynamics of a binary system of compact objects with masses m_A, $A = 1, 2$, in the context of the post-Newtonian approximation, it is physically sound to model the two companions as point particles. Indeed, the ratio of the radii $R_A \sim G m_A/c^2$ to the body separation r_{12} is of the order $G m_A/(r_{12} c^2)$, and thus much smaller than 1. The dynamics of test point-like objects including finite size effects has been investigated extensively by Dixon [56–59], who generalized the Mathisson-Papapetrou equations for spinning particles [50, 51, 53, 64] by attaching arbitrary high-order moments to the individual bodies, beyond the monopole and the current dipole also referred to as the particle spin. It can also be derived from an effective Lagrangian-type approach for spinning particles, pioneered by Bailey & Israel [60] (see also an extensive study for special relativity in [65]) and later implemented in EFT [21, 22], where higher-order moments appear as parametrizing couplings in the action to the value of the Riemann tensor and its derivatives on the worldline.

The Dixon-Mathisson-Papapetrou equations of evolution for a spinning particle with quadrupolar structure read:

$$\frac{\mathrm{D}p^\alpha}{c\,\mathrm{d}\tau} = -\frac{1}{2c} R^\alpha{}_{\lambda\mu\nu} u^\lambda S^{\mu\nu} - \frac{c}{3} \nabla_\rho R^\alpha{}_{\lambda\mu\nu} J^{\rho\lambda\mu\nu} \,, \tag{2.1a}$$

$$\frac{\mathrm{D}S^{\alpha\beta}}{c^2\mathrm{d}\tau} = 2 p^{[\alpha} u^{\beta]} + \frac{4c}{3} R^{[\alpha}{}_{\lambda\mu\nu} J^{\beta]\lambda\mu\nu} \,, \tag{2.1b}$$

where p^α is the 4-momentum of the particle and $u^\lambda = \mathrm{d}x^\lambda/(c\,\mathrm{d}\tau)$ the 4-velocity along the world-line. The anti-symmetric spin tensor $S^{\mu\nu}$ represents the effective 4-angular momentum of the object, while the (effective) mass and current type quadrupoles are encoded into the Dixon quadrupolar tensor $J^{\rho\lambda\mu\nu}$, which is only constrained at this stage to have the same symmetry properties as $R^{\rho\lambda\mu\nu}$.

The stress-energy tensor $T^{\alpha\beta}$ of the model can be constructed after the Tulczyjew procedure, by making the only assumption that its support is point like with at most two derivatives acting on the Dirac distributions, in the three following steps [55]: (i) write the most general symmetric tensor that involves up to two (covariant) derivatives of the particle scalar density

$$n = \int_{-\infty}^{+\infty} c\,\mathrm{d}\tau' \frac{\delta^4(x - y(\tau'))}{\sqrt{-g}} \,, \tag{2.2}$$

where $\delta^4(x - y(\tau))$ is a 4-dimensional Dirac delta, with $y(\tau)$ the particle worldline and x the field point; (ii) derive the hierarchy of equations verified by the coefficients of n in $T^{\mu\nu}$ due to the conservation equation $\nabla_\nu T^{\mu\nu} = 0$; (iii) constrain those coefficients by solving all algebraic equations, which leaves two sets of ordinary differential equations. Identifying these two equations to Eqs. (2.1) yields the expression of $T^{\mu\nu}$ in terms of p^μ, $S^{\mu\nu}$ and $J^{\rho\lambda\mu\nu}$:

$$\begin{aligned} T^{\mu\nu} = n &\left[p^{(\mu} u^{\nu)} c + \frac{1}{3} R^{(\mu}{}_{\lambda\rho\sigma} J^{\nu)\lambda\rho\sigma} c^2 \right] \\ &- \nabla_\rho \left[n\, S^{\rho(\mu} u^{\nu)} \right] - \frac{2}{3} \nabla_\rho \nabla_\sigma \left[n\, c^2 J^{\rho(\mu\nu)\sigma} \right] \,. \end{aligned} \tag{2.3}$$

It can be recovered with a smaller amount of calculation, further assuming that the system dynamics is governed by the effective Lagrangian of Bailey & Israel [60], by differentiating the resulting action with respect to the metric [42].

As the spin tensor $S^{\mu\nu}$ is anti-symmetric, it actually contains six degrees of freedom. Moreover, for an isolated body, the space-time components J^{0i} of the total angular momentum $J^{\mu\nu} = S^{\mu\nu}/c$ in an appropriate asymptotically Minkowkian gauge represent the mass-type dipole of the object, and can thus always be taken to be zero. Similarly, for a test particle moving in a gravitational background, three degrees of freedom among those contained in the effective spin tensor are expected to be non-dynamical. They may be eliminated by fixing the "center-of-body" reference point with the help of three independent space-time equations, globally referred to as the spin supplementary condition (SSC). The three remaining degrees of freedom correspond to the spatial components of the spin vector S^{μ}. Various choices of SSC are possible (see for instance [66]). Here we shall adopt, in keeping with previous works, the covariant (or Tulczyjew [54]) condition

$$S^{\mu\nu} p_{\nu} = 0\,. \tag{2.4}$$

Assuming that the rotating bodies are always at equilibrium, we can reasonably expect their moments to depend on their masses, spins, as well as possible dimensionless parameters that characterize the internal structures. Notably, the spins may induce mass quadrupoles as they do for Kerr black holes. This effect produces spin square contributions that must be crucially taken into account at quadratic order in the spin variables. Tidal fields inside the bodies may also generate $\ell \geq 2$ multipoles, but their leading order contribution to the acceleration would be $\sim (R_{1,2}/r_{12})^5 = \mathcal{O}(10)$ for a compact binary, so that they can safely be neglected in the present work.

As not all degrees of freedom in the Dixon quadrupole are physical, its value as a function of time cannot be uniquely determined by the internal dynamics of the body. In the adiabatic approximation, there exists a relation, valid along the particle worldline, between $J^{\rho\lambda\mu\nu}$, the 4-velocity u^{μ} and the spin tensor $S^{\mu\nu}$. It can derived from an effective Lagrangian L_{SS} built to be the most general Lagrangian — modulo perturbative redefinitions of the gravitational field, terms in the form of a total time derivative, terms that vanish under some given SSC, and $\mathcal{O}(S^3)$ remainders — with the properties of: (i) being quadratic in $S^{\mu\nu}$, (ii) depending on u^{μ}, the metric $g_{\mu\nu}$, (derivatives of) the Riemann tensor, as well as some parameters characterizing the object [21, 33, 35]. After redefining p^{μ}, $S^{\mu\nu}$, we find that the stress-energy tensor associated with L_{SS} coincides with that of Eq. (2.3) provided $J^{\rho\lambda\mu\nu}$ is given by

$$J^{\rho\lambda\mu\nu} = \frac{3\kappa}{\tilde{m}\,c^4} S^{\sigma[\rho} u^{\lambda]} S_{\sigma}^{\ [\mu} u^{\nu]}\,, \tag{2.5}$$

at any instant. The above expression properly describes the presence of a non-vanishing spin-induced quadrupole, with the source dependent constant κ representing the quadrupolar polarisability. The mass parameter $\tilde{m}$ is defined by $p^2 \equiv p_{\alpha}p^{\alpha} = -\tilde{m}^2 c^2$. Notice that $\tilde{m}$ is not a priori conserved. In fact, as shown below, its time derivative is quadratic in spin and cannot be consistently ignored at our accuracy level.

The (contravariant) 4-momentum and 4-velocity of the particle are proportional when terms beyond linear order in the spins are neglected: $p^{\alpha} = \tilde{m}\,c\,u^{\alpha} + \mathcal{O}(S^2)$. Our first step will consist in expressing p^{α} as a function of u^{α} to quadratic order in the spin. We impose that the derivative along the worldine of the SSC (2.4) is zero, insert the equations of motion (2.1a) and (2.1b) into the resulting identity, and use the fact that $J^{\rho\lambda\mu\nu} \sim \mathcal{O}(S^2)$ whereas $S^{\alpha\beta}u_{\beta} \sim \mathcal{O}(S^3)$. This yields, at quadratic order in spin,

$$p^{\alpha} = \tilde{m}\,c\,u^{\alpha} - \frac{S^{\alpha\beta}S^{\mu\nu}}{2\tilde{m}\,c^3} u^{\lambda} R_{\beta\lambda\mu\nu} + \frac{4c}{3} u_{\beta} R^{[\alpha}{}_{\lambda\mu\nu} J^{\beta]\lambda\mu\nu} + \mathcal{O}(S^3)\,. \tag{2.6}$$

We are now in position to write the spin evolution equation in a more explicit way. In the Lagrangian formalism, the effective linear and angular momenta are defined in a way that guarantees the conservation of the spin magnitude [42, 62]. This conservation law is a remarkable feature of the spinning-particle dynamics. In our context, it will follow from Eq. (2.1) for some class of supplementary conditions. In fact, it can indeed be derived explicitly from those equations, for the form (2.5) of the quadrupole moment and the covariant SSC (2.4). By substituting the 4-momentum (2.6) into equation (2.1b) we get

$$\begin{aligned}\frac{\mathrm{D}S^{\alpha\beta}}{c^2\mathrm{d}\tau} = \frac{4c}{3}\left[R^{[\alpha}{}_{\lambda\mu\nu}J^{\beta]\lambda\mu\nu} + u_\gamma u^{[\beta}R^{\alpha]}{}_{\lambda\mu\nu}J^{\gamma\lambda\mu\nu} - u_\gamma u^{[\beta}J^{\alpha]\lambda\mu\nu}R^{\gamma}{}_{\lambda\mu\nu}\right] \\ - u^\lambda R_{\gamma\lambda\mu\nu}u^{[\beta}\frac{S^{\alpha]\gamma}S^{\mu\nu}}{\tilde{m}\,c^3} + \mathcal{O}(S^3)\,. \end{aligned} \tag{2.7}$$

If we contract this expression with $S_{\alpha\beta}$, we obtain $S_{\alpha\beta}\mathrm{D}S^{\alpha\beta}/(c\mathrm{d}\tau) \sim \mathcal{O}(S^4)$ and, therefore, defining the spin magnitude as $s^2 = S_{\alpha\beta}S^{\alpha\beta}/2$,

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} \sim \mathcal{O}(S^3)\,. \tag{2.8}$$

This demonstrates that the spin magnitude is actually conserved at order $\mathcal{O}(S^2)$.

B. Conserved mass and evolution equations

Our next task is to investigate the issue of mass conservation at quadratic order in spins. For this purpose, let us compute the time derivative of the mass parameter $\tilde{m}$. Using the equation of motion (2.1a) and the Bianchi identities, we can write

$$-\tilde{m}\,c^2\frac{\mathrm{d}\tilde{m}}{c\,\mathrm{d}\tau} = p_\alpha\frac{\mathrm{D}p^\alpha}{c\,\mathrm{d}\tau} = -\frac{\tilde{m}\,c^2}{6}u^\beta\nabla_\beta R_{\rho\lambda\mu\nu}J^{\rho\lambda\mu\nu} + \mathcal{O}(S^3)\,. \tag{2.9}$$

As the time dependence of $J^{\rho\lambda\mu\nu}$ is through the 4-velocity and the spin tensor, i.e. $J^{\rho\lambda\mu\nu}(\tau) = J^{\rho\lambda\mu\nu}(u^\beta(\tau), S^{\alpha\beta}(\tau))$, the fact that $\mathrm{D}u^\beta/(c\,\mathrm{d}\tau) \sim \mathcal{O}(S)$ and $\mathrm{D}S^{\alpha\beta}/(c\,\mathrm{d}\tau) \sim \mathcal{O}(S^2)$ implies the approximate conservation of the Dixon quadrupole: $\mathrm{D}J^{\rho\lambda\mu\nu}/(c\,\mathrm{d}\tau) \sim \mathcal{O}(S^3)$. Now, substituting $u^\beta\nabla_\beta$ with $\mathrm{D}/(c\,\mathrm{d}\tau)$, we can write down the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\tilde{m} - \frac{1}{6}R_{\rho\lambda\mu\nu}J^{\rho\lambda\mu\nu}\right] = \mathcal{O}(S^3)\,, \tag{2.10}$$

which finally allows us to define a conserved quantity m as

$$m \equiv \tilde{m} - \frac{1}{6}R_{\rho\lambda\mu\nu}J^{\rho\lambda\mu\nu}\,. \tag{2.11}$$

Hereafter, the constant parameter m will be regarded as the effective mass of the particle. This mass is the one that appears in all our post-Newtonian results. By construction, it is conserved, like the spin magnitude. Substituting the expression (2.9) into Eq. (2.6) gives us the link between the 4-momentum and the 4-velocity:

$$p^\alpha = m\,c\,u^\alpha + \frac{c}{6}u^\alpha R_{\rho\lambda\mu\nu}J^{\rho\lambda\mu\nu} - \frac{S^{\alpha\beta}S^{\mu\nu}}{2m\,c^3}u^\lambda R_{\beta\lambda\mu\nu} + \frac{4c}{3}u_\beta R^{[\alpha}{}_{\lambda\mu\nu}J^{\beta]\lambda\mu\nu} + \mathcal{O}(S^3)\,, \tag{2.12}$$

where m was just shown to be a constant parameter at order $\mathcal{O}(S^2)$. We are then in position to rewrite the evolution equations for spinning particles to quadratic order in the spins, using the 4-velocity instead of the 4-momentum. Those are:

$$\frac{\mathrm{D}u^\alpha}{\mathrm{d}\tau} = \frac{u^\rho}{2}\frac{\mathrm{D}R_{\beta\rho\mu\nu}}{\mathrm{d}\tau}\frac{S^{\alpha\beta}S^{\mu\nu}}{m^2c^4} - \frac{1}{2}R^\alpha{}_{\lambda\mu\nu}u^\lambda\frac{S^{\mu\nu}}{m\,c} - \frac{c}{3}\nabla_\rho R^\alpha{}_{\lambda\mu\nu}\frac{J^{\rho\lambda\mu\nu}}{m} - \frac{u^\alpha}{6}\frac{\mathrm{D}R_{\lambda\rho\mu\nu}}{\mathrm{d}\tau}\frac{J^{\lambda\rho\mu\nu}}{m} - \frac{4u_\beta}{3}\frac{\mathrm{D}R^{[\alpha}{}_{\rho\mu\nu}}{\mathrm{d}\tau}\frac{J^{\beta]\rho\mu\nu}}{m} + \mathcal{O}(S^3)\,, \tag{2.13a}$$

$$\frac{\mathrm{D}S^{\alpha\beta}}{\mathrm{d}\tau} = \frac{4c^3}{3}\left[u^{[\beta}R^{\alpha]}{}_{\lambda\mu\nu}u_\gamma J^{\gamma\lambda\mu\nu} - u^{[\beta}J^{\alpha]\lambda\mu\nu}u_\gamma R^\gamma{}_{\lambda\mu\nu} + R^{[\alpha}{}_{\lambda\mu\nu}J^{\beta]\lambda\mu\nu}\right] - R_{\gamma\lambda\mu\nu}u^\lambda u^{[\beta}\frac{S^{\alpha]\gamma}S^{\mu\nu}}{m\,c} + \mathcal{O}(S^3)\,. \tag{2.13b}$$

C. Definition of a spin vector and equation of precession

From the anti-symmetric spin tensor $S^{\alpha\beta}$, we define the spin 4-covector $\tilde{S}_\alpha$ as

$$\tilde{S}_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\frac{p^\beta}{m\,c}S^{\mu\nu}\,, \tag{2.14}$$

where $\epsilon_{\alpha\beta\mu\nu} = \sqrt{-g}\,\eta_{\alpha\beta\mu\nu}$ denotes the covariant Levi-Civita tensor, with $\eta_{\alpha\beta\mu\nu}$ being the completely anti-symmetric symbol that verifies $\eta_{0123} = 1$, and where $g = \det g_{\mu\nu}$ is the determinant of the metric tensor in generic coordinates. The tilde on this covariant spin vector will allow us to distinguish it from the Euclidean conserved-norm spin vector we shall introduce below. Notice that $\tilde{S}_\alpha$ automatically satisfies $\tilde{S}_\alpha p^\alpha = 0$ and thus carries 3 degrees of freedom as required. If we contract the above equation with $\epsilon^{\alpha\gamma\rho\sigma}p_\gamma$ and use the SSC $p_\nu S^{\mu\nu} = 0$, we can invert Eq. (2.14) and obtain the spin tensor in terms of $\tilde{S}_\alpha$:

$$S^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}\frac{p_\mu}{m\,c}\tilde{S}_\nu + \mathcal{O}(S^3)\,. \tag{2.15}$$

Remembering that $p^\beta = m\,c\,u^\beta + \mathcal{O}(S^2)$ at the linear-in-spin level, it is straightforward to check that $\tilde{S}_\alpha\tilde{S}^\alpha = s^2$, by virtue of the relation $\epsilon_{\alpha\beta\mu\nu}\epsilon^{\lambda\rho\sigma\tau} = -4!\delta^{[\lambda}_{\alpha}\delta^{\rho}_{\beta}\delta^{\sigma}_{\mu}\delta^{\tau]}_{\nu}$.

To derive the evolution equation for the spin 4-covector, we differentiate Eq. (2.14) with respect to the proper time, which yields

$$\frac{\mathrm{D}\tilde{S}_\alpha}{\mathrm{d}\tau} = \epsilon_{\alpha\beta\mu\nu}\left[\frac{1}{4}R^\beta{}_{\lambda\sigma\rho}u^\lambda\frac{S^{\sigma\rho}S^{\mu\nu}}{m\,c} - \frac{2c^3}{3}R^\mu{}_{\lambda\sigma\rho}J^{\nu\lambda\sigma\rho}u^\beta\right] + \mathcal{O}(S^3)\,. \tag{2.16}$$

In what follows, we shall explicitly resort to our particular form (2.5) for $J^{\mu\nu\alpha\beta}$, relevant in the case of a spin-induced quadrupole. It will be convenient to investigate each term on the right-hand side of Eq. (2.16) individually. With our definition of $\tilde{S}_\mu$, the first term there reads

$$\frac{1}{4}\epsilon_{\alpha\beta\mu\nu}R^\beta{}_{\lambda\sigma\rho}u^\lambda\frac{S^{\sigma\rho}S^{\mu\nu}}{m\,c} = -\frac{1}{2}u_\kappa\epsilon^{\sigma\rho\kappa\gamma}\frac{\tilde{S}_\gamma}{m\,c}R^\beta{}_{\lambda\sigma\rho}\tilde{S}_\beta u^\lambda u_\alpha + \mathcal{O}(S^4)\,. \tag{2.17}$$

At this stage, it is useful to introduce the gravitomagnetic part of the Bel decomposition of the Riemann tensor

$$H_{\mu\nu} = 2\,{}^*R_{\mu\kappa\nu\lambda}u^\kappa u^\lambda\,. \tag{2.18}$$

where ${}^{*}R$ is the self-dual Riemann tensor defined by

$$ {}^{*}R_{\alpha\beta\mu\nu} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\kappa\gamma}\, R_{\kappa\gamma\mu\nu} \,. \tag{2.19} $$

Physically, the tensor $H_{\mu\nu}$ represents the tidal current-type quadrupole in the relativistic theory of tides. We can now put Eq. (2.16) in the form

$$ \frac{1}{4}\epsilon_{\alpha\beta\mu\nu}R^{\beta}{}_{\lambda\sigma\rho}u^{\lambda}\frac{S^{\sigma\rho}S^{\mu\nu}}{m\,c} = \frac{1}{2}H^{\gamma\beta}u_{\alpha}\frac{\tilde{S}_{\gamma}\tilde{S}_{\beta}}{m\,c} + \mathcal{O}(S^3)\,. \tag{2.20} $$

Let us focus next on the second expression on the right-hand side of Eq. (2.16). After substituting the value for $J^{\nu\lambda\sigma\rho}$ therein, we rewrite the resulting expression in terms of the gravitoelectric part of the Bel decomposition of the Riemann tensor

$$ G_{\mu\nu} = -R_{\mu\lambda\nu\rho}u^{\lambda}u^{\rho}\,, \tag{2.21} $$

which is nothing but the tidal mass-type quadrupole generalizing that of Newtonian gravity (up to a factor $1/c^2$). Next, we directly replace the spin tensors with their corresponding spin covectors in Eq. (2.16), hence:

$$ \frac{\mathrm{D}\tilde{S}_{\alpha}}{\mathrm{d}\tau} = \frac{1}{2}H^{\gamma\beta}u_{\alpha}\frac{\tilde{S}_{\gamma}\tilde{S}_{\beta}}{m\,c} - \kappa\,\epsilon_{\alpha\beta\mu\nu}u^{\beta}G^{\mu\sigma}\frac{\tilde{S}_{\sigma}\tilde{S}^{\nu}}{m\,c} + \mathcal{O}(S^3)\,. \tag{2.22} $$

Finally, after setting

$$ \tilde{\Omega}_{\alpha\beta} = \frac{\tilde{S}_{\lambda}}{m\,c}\left[u_{[\alpha}H_{\beta]}{}^{\lambda} - \kappa\,\epsilon_{\alpha\beta\mu\nu}u^{\mu}G^{\nu\lambda}\right]\,, \tag{2.23} $$

the spin precession equation for the covariant spin vector takes the form

$$ \frac{\mathrm{D}\tilde{S}_{\alpha}}{\mathrm{d}\tau} = \tilde{\Omega}_{\alpha\beta}\tilde{S}^{\beta} + \mathcal{O}(S^3)\,. \tag{2.24} $$

The anti-symmetric tensor $\tilde{\Omega}_{\alpha\beta}$ may be interpreted as a spin-precession frequency tensor.

It remains to construct a spin 3-vector S^i with conserved Euclidean norm. A "canonical" construction is already explained in Section 2.1 of Ref. [25], to which the reader may refer for further details. The precession vector governing the evolution of S^i differs from that of Ref. [25], derived in the SO approximation, by additional terms that are quadratic in spins.

The passage to spin 3-vectors is achieved by introducing a direct orthonormal tetrad $e_{\underline{\alpha}}{}^{\mu}$. The underlined index represents the vector label, which we may be viewed as the tetrad index, spacetime indices being represented by Greek letters and spatial indices by Latin letters as usual. Posing $e_{\underline{0}}{}^{\mu} = u^{\mu}$, we see that

$$ \tilde{S}_{\underline{0}} = \tilde{S}_{\mu}e_{\underline{0}}{}^{\mu} = \mathcal{O}(S^3)\,, \tag{2.25} $$

which means that $\tilde{S}_{\underline{0}}$ may be neglected. The squared Euclidean norm of $\tilde{S}_{\underline{a}}$ is then given by

$$ \delta_{\underline{ab}}\tilde{S}^{\underline{a}}\tilde{S}^{\underline{b}} = \gamma_{\mu\nu}\tilde{S}^{\mu}\tilde{S}^{\nu} = \tilde{S}_{\mu}\tilde{S}^{\mu} = s^2\,, \tag{2.26} $$

with $\gamma_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$. In words, the spin vector $\tilde{S}_{\underline{a}}$ has a conserved Euclidean norm. To define the spin variable uniquely in some coordinate grid, we still need to specify the choice to be made for the spatial part of the tetrad. Considering that $\delta^{\underline{ab}} e_{\underline{a}i} e_{\underline{b}j} = \gamma_{ij}$, a natural choice is to take for $e_{\underline{a}i}$ the unique symmetric positive-definite square root (in the matrix sense) of γ_{ij}. The complete expression for the tetrad is

$$e_{\underline{a}}{}^{\mu} = \left(\gamma^{\mu i} - \gamma^{\mu 0} \frac{v^i}{c}\right) e_{\underline{a}i} \,. \tag{2.27}$$

with $v^\mu \equiv= c\, u^\mu / u^0$ denoting the coordinate velocity. After projection on the basis vectors (2.27), the precession equation for the spin vector becomes

$$\frac{\mathrm{d}\tilde{S}_{\underline{\alpha}}}{\mathrm{d}\tau} = \left(\tilde{\omega}_{\underline{\alpha\beta}} + \tilde{\Omega}_{\underline{\alpha\beta}}\right) \tilde{S}^{\underline{\beta}} \,, \tag{2.28}$$

where we have introduced the rotation coefficients for the tetrad

$$\tilde{\omega}_{\underline{\alpha\beta}} = -e_{\underline{\alpha}}{}^{\mu} \frac{\mathrm{D}e_{\underline{\beta}\mu}}{\mathrm{d}\tau} \,, \tag{2.29}$$

and where $\tilde{\Omega}_{\underline{\alpha\beta}} = \tilde{\Omega}_{\mu\nu} e_{\underline{\alpha}}{}^{\mu} e_{\underline{\beta}}{}^{\nu}$. Now, as $\mathrm{d}/\mathrm{d}\tau = u^0 \mathrm{d}/\mathrm{d}t$, it is convenient to define an anti-symmetric precession frequency tensor associated with the coordinate time as

$$\Omega_{\underline{\alpha\beta}} = \frac{1}{u^0} \left(\tilde{\omega}_{\underline{\alpha\beta}} + \tilde{\Omega}_{\underline{\alpha\beta}}\right) \,. \tag{2.30}$$

Since $\tilde{S}^{\underline{0}}$ is negligible, the precession equation reduces to

$$\frac{\mathrm{d}\tilde{S}_{\underline{i}}}{\mathrm{d}t} = \Omega_{\underline{ij}} \tilde{S}^{\underline{j}} + \mathcal{O}(S^3) \,. \tag{2.31}$$

Moreover, from the equality $e_{\underline{0}}{}^{\mu} = u^\mu$, it follows that the first term on the right-hand side of Eq. (2.23) vanishes when projected on spatial tetrad indices, so that

$$\tilde{\Omega}_{\underline{ij}} = -\kappa\, \varepsilon_{\underline{ijk}} G^{\underline{kl}} \frac{\tilde{S}_{\underline{l}}}{m\, c} \,, \tag{2.32}$$

where $\varepsilon_{\underline{ijk}}$ or ε_{ijk} (indifferently) denote the Euclidean Levi-Civita symbol, with normalization $\varepsilon^{123} = \varepsilon_{123} = 1$, which is linked to the four dimensional Levi-Civita tensor by the relation $\epsilon_{\underline{0ijk}} = \varepsilon_{\underline{ijk}}$.

In the rest of the paper, we shall use a conserved Euclidean spin vector $\boldsymbol{S}$ with spatial components S^i in harmonic coordinates such that

$$S^i \equiv \tilde{S}^{\underline{i}} \,. \tag{2.33}$$

Because of the anti-symmetric character of $\Omega_{\underline{ij}}$, we can finally rewrite the precession equation in terms of a precession vector $\Omega^i = -\varepsilon^{\underline{ijk}} \Omega_{\underline{jk}}/2$ as

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = \varepsilon_{ijk} \Omega^j S^k + \mathcal{O}(S^3) \,. \tag{2.34}$$

It is the above precession vector Ω^i that will be computed, along with the equation of motion, in Section III. Our results will be displayed either in terms of the vector S^i or of the spatial components of the spin tensor S^{ij}.

D. Application to self-gravitating binary systems

Although the evolution equations (2.1) originally obtained by Dixon are only suitable to describe the dynamics of test particles, their rederivation based on the method of Tulczyjew or the Lagrangian approach of Bailey & Israel, regarded as effective field schemes, holds for self-gravitating N point-like body systems. Nonetheless, the validity of the point particle model breaks down at UV scales where the post-Newtonian expansion cannot be applied, i.e. for $r_A \sim R_A$, with r being the distance between the particle representing the body A and the field point $\boldsymbol{x}$. In particular, some infinities arise when computing the gravitational field iteratively due to divergences at the particle positions $\boldsymbol{y}_A$. The situation is even worse as we make $\boldsymbol{x}$ tend towards $\boldsymbol{y}_A$.

As usual, those infinities are cured thanks to dimensional regularization, which preserves the invariance under diffeomorphism of general relativity, combined with some renormalization procedure. For an appropriate choice of the space dimension d, the field remains weak near $r_A = 0$ and can be computed perturbatively in the post-Newtonian approximation. We are confident that this leads to the correct PN dynamics because: (i) the result for the acceleration is unambiguous up to the order 3.5PN for binaries of spinning compact objects, (ii) it is equivalent to that obtained from other methods (see the review paper [7] for references), and notably from the approach à la Einstein-Infeld-Hoffmann used by Itoh [67] in the case of spinless bodies where no regularization is needed. Those cautions being taken, a self-gravitating system of N spinning bodies endowed with a quadrupolar structure may be modeled by means of the following effective stress-energy tensor, which generalizes that of Eq. (2.3):

$$T^{\mu\nu} = \sum_{A=1,2} \bigg[n_A \left(p_A^{(\mu} u_A^{\nu)} c + \frac{1}{3} R^{(\mu}{}_{\lambda\rho\sigma} J_A^{\nu)\lambda\rho\sigma} c^2 \right) - \nabla_\rho \left(n_A S_A^{\rho(\mu} u_A^{\nu)} \right) - \frac{2}{3} \nabla_\rho \nabla_\sigma \left(n_A c^2 J_A^{\rho(\mu\nu)\sigma} \right) \bigg] \,, \tag{2.35}$$

where the subscripts A indicate the particle label.

The presence of poles $\propto \varepsilon^{-k}$ in the metric at a given post-Newtonian order, with $\varepsilon \equiv d-3$ and k being a positive integer, may generate contributions in the source for the next order that could not be recovered by resorting to a purely three dimensional regularization. However, in the absence of such subtleties, the so-called pure Hadamard-Schwartz regularization [68] is sufficient to get the correct result. This prescription essentially relies on a specific use of the Hadamard partie finie regularization, which we shall briefly discuss now (the reader will find more details in Ref. [69]).

Let us consider a function $F(\boldsymbol{x})$ with the same regularity properties as those arising in our problem, i.e smooth everywhere except at some singular points $\boldsymbol{y}_A$ $(A = 1, 2, ..., N)$ in the neighborhood of which its admits an expansion of the form

$$F(\boldsymbol{x}) = \sum_{p_0 \leq p \leq P} r_A^p \underset{A}{f_p}(\boldsymbol{n}_A) + o(r_A^P) \tag{2.36}$$

for any integer P, with $\boldsymbol{n}_A = (\boldsymbol{x} - \boldsymbol{y}_A)/r_A$. Such a function is said to be of class $\mathcal{F}$. Its Hadamard partie finie $(F)_A$ is then defined as the angular average of the finite part ${}_A f_0(\boldsymbol{n}_A)$:

$$(F)_A = \int \frac{\mathrm{d}\Omega_A}{4\pi} \underset{A}{f_0}(\boldsymbol{n}_A) \,, \tag{2.37}$$

where $d\Omega_A$ denotes the elementary solid angle with direction $\boldsymbol{n}_A$ centered on $\boldsymbol{y}_A$. The operation of taking the Hadamard partie finie is not distributive with respect to multiplication in the sense that, for another generic function $G(\boldsymbol{x})$ of class $\mathcal{F}$, $(F)_A(G)_A \neq (FG)_A$ in general. Moreover, it does not respect the Lorentz invariance. Because of the first of those two unpleasant features, the so-defined regularization is fundamentally ambiguous as such. Howbeit, it can still be used in practical computations provided it is supplemented by some additional prescription. In the PNISH approach, the post-Newtonian metric is constructed iteratively with the help of PN potentials. Those are elementary bricks satisfying a wave-type equation (more details are provided in Section III). A convenient prescription is to define the value of a product FG of two potentials (or potential derivatives) evaluated at point $\boldsymbol{y}_A$ as $(F)_A(G)_A$. Similarly, the regularized product of a potential F and an arbitrary smooth function $\alpha(\boldsymbol{x})$ will be given by $\alpha(\boldsymbol{y}_A)F_A$.

Divergent integrals are cured by applying another kind of Hadamard partie finie regularization. The regularized value of an integral with class-$\mathcal{F}$ integrand is calculated in three main steps: (i) balls of radius η centered on the singular points are extracted from the integration domain; (ii) terms that diverge near $\eta = 0$ are removed; (iii) one goes to the limit $\eta \to 0$. The singularities that generate poles in dimensional regularization produce logarithmic divergences in the Hadamard one. Those are associated with cutoff parameters s_A entering terms such as $\ln(\eta/s_A)$. For consistency between the two kinds of Hadamard regularizations, all derivatives must be evaluated in the sense of distributions [69]. The action of the three dimensional Dirac delta $\delta_A \equiv \delta^3(\boldsymbol{x} - \boldsymbol{y}_A)$ on test functions must also be generalized to $\mathcal{F}$-class functions by posing $F\,\delta_A = (F)_A\,\delta_A$.

In this context, the pure Hadamard-Schwartz regularization is an ensemble of prescriptions designed to yield results that are "as close as possible" to those obtained through dimensional regularization. Those prescriptions demand: (i) to evaluate monomials of the form $\alpha(\boldsymbol{x})F_1...F_n$, where $\alpha(\boldsymbol{x})$ is a smooth function and the F_k's are (derivatives of) $\mathcal{F}$-class potentials, as $\alpha(\boldsymbol{y}_A)(F_1)_A...(F_n)_A$; (ii) to evaluate divergent integrals by means of the Hadamard partie regularization for integrals; (iii) to extend the definition of δ_A as explained above; (iv) to compute all derivatives in the sense of Schwartzian distributions.

The absence of logarithmic cut-offs in the SS piece of the metric up to the order 3PN suggests that dimensional regularization may safely be swapped for the pure Hadamard-Schwartz one at this accuracy level. The insensitivity of the calculations to the choice of regularization procedure has been checked explicitly by evaluating source terms of the type $FG\,\delta_A$ in the stress-energy tensor as $(F)_A(G)_A\,\delta_A$, thus violating the pure Hadamard-Schwartz prescription. The results have always turned out to be unaffected by such modifications.

III. NEXT-TO-LEADING ORDER CONTRIBUTIONS TO THE POST-NEWTONIAN EVOLUTION

We now turn to the computation of the dynamics of a binary system in the post-Newtonian approximation, at next-to-leading order for the quadratic-in-spin effects, i.e. at order $1/c^6$ (or 3PN) in the equations of motion and at order $1/c^5$ in the equations of precession. We will recover the results for the dynamics obtained in the ADM [62, 70–73] and EFT [21, 22, 35, 38, 39, 74] approaches, and extend them towards the completion of the calculation of the gravitational waves energy flux.

We start with some general definitions in Section III A. Next, we introduce a set of potentials parametrizing the PN metric in Section III B, and express the quantities of interest in terms of these potentials. In Section III C, we present their computation, and finally in Section III D the results obtained for the dynamics as well as various tests of their correctness. The lengthier calculations are all performed by means of the algebraic computing software Mathematica® supplemented by the tensor calculus package xAct [75].

A. General definitions

The two objects are represented as quadrupolar point particles as explained above. An important ingredient of the formalism is the treatment of the infinite self-field of the point particles, essentially represented by means of Dirac deltas, through the pure Hadamard-Schwartz regularization procedure discussed in Section II D. The distributional contributions yielded by derivatives are handled by using the Gel'fand-Shilov formula [76]. We found that at this order in spin, we have to keep track of distributional contributions in the metric itself to obtain the correct result for the wave generation formalism, as will be detailed in Section IV B.

The general structure of the equations of motion and precession is as follows:

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{A}^{\text{N}}_{\text{NS}} + \frac{1}{c^2}\boldsymbol{A}^{\text{1PN}}_{\text{NS}} + \frac{1}{c^4}\boldsymbol{A}^{\text{2PN}}_{\text{NS}} + \frac{1}{c^5}\boldsymbol{A}^{\text{2.5PN}}_{\text{NS}} + \frac{1}{c^6}\boldsymbol{A}^{\text{3PN}}_{\text{NS}} \\ &\quad + \frac{1}{c^3}\boldsymbol{A}^{\text{N}}_{\text{SO}} + \frac{1}{c^5}\boldsymbol{A}^{\text{1PN}}_{\text{SO}} + \frac{1}{c^4}\boldsymbol{A}^{\text{N}}_{\text{SS}} + \frac{1}{c^6}\boldsymbol{A}^{\text{1PN}}_{\text{SS}} + \mathcal{O}(7)\,, \qquad (3.1a)\\ \boldsymbol{\Omega} &= \frac{1}{c^2}\boldsymbol{\Omega}^{\text{N}}_{\text{NS}} + \frac{1}{c^4}\boldsymbol{\Omega}^{\text{1PN}}_{\text{NS}} + \frac{1}{c^3}\boldsymbol{\Omega}^{\text{N}}_{\text{SO}} + \frac{1}{c^5}\boldsymbol{\Omega}^{\text{1PN}}_{\text{SO}} + \mathcal{O}(6)\,, \qquad (3.1b)\end{aligned}$$

where the spin order in Eq. (3.1b) indicates the contribution in $\boldsymbol{\Omega}$ itself, rather than in $\dot{\boldsymbol{S}} = \boldsymbol{\Omega}\times\boldsymbol{S}$ (notably the SO terms feature the constants $\kappa_{1,2}$ and actually correspond to SS terms in $\dot{\boldsymbol{S}}$). The 2.5PN NS terms in the acceleration are the first manifestation of radiation reaction.

We use the same notations as in previous works. Three-dimensional indices are represented with Latin letters $a, b, ...$ or $i, j, ...$, and are risen or lowered with the Euclidean metric δ_{ij}; we do not distinguish between upper and lower indices. We sometimes use boldface for Euclidean vectors. The positions and velocities of the two bodies are denoted by y_1^i, y_2^i and v_1^i, v_2^i. Apart from the separation distance $r_{12} = |\boldsymbol{y}_{12}| = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$ which we have already defined, we shall need the separation direction $n_{12}^i = (y_1^i - y_2^i)/r_{12}$. The symbol $1 \leftrightarrow 2$ indicates the same expression as the one before it, with the label of the two particles exchanged. The results are expressed in terms of the spatial components S_1^{ij}, S_2^{ij} of the spin tensor $S^{\mu\nu}$, as well as the spin vectors S_1^i, S_2^i of conserved Euclidean norm as defined above, in Section II C. The mixed components S^{0i} of the spin tensors can always be eliminated with the help of the spin supplementary condition (2.4). We allow repeated indices in scalars quantities enclosed by parenthesis, in the absence of a risk of confusion.

In harmonic (or DeDonder) gauge, the gravitational field equations can be rewritten as

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}\left[h\right] \equiv \frac{16\pi G}{c^4}\tau^{\mu\nu}\,, \qquad (3.2)$$

where the stress-energy pseudo-tensor $\tau^{\mu\nu}$ includes both matter and field contributions through $T^{\mu\nu}$ and $\Lambda^{\mu\nu}$, the latter source term being at least quadratic in $h^{\mu\nu}$. The field equations (3.2), when iterated order by order, yield a solution expressed formally in terms of a hierarchy of potentials of increasing complexity and post-Newtonian order (see Refs. [77, 78] for the precise definition of this iteration in the near-zone).

Since on the one hand we are working at order $1/c^6$ in the equations of motion, and on the other hand the spin contributions always come at relative $1/c$ order at least, only the so-called 2PN metric and potentials (i.e. necessary for the 2PN non-spinning case) are required. In fact, we will see below that, among the potentials arising at the order 2PN, only $\hat{X}$ turns out to be needed. For completeness, we quote here the result of the iteration for the 2PN metric, which reads

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_i V_i + \frac{V^3}{6}\right) + \mathcal{O}(8)\,, \tag{3.3a}$$

$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i + \mathcal{O}(7)\,, \tag{3.3b}$$

$$g_{ij} = \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2\right] + \frac{4}{c^4}\hat{W}_{ij} + \mathcal{O}(6)\,. \tag{3.3c}$$

The potentials therein are defined as[1]

$$V = \Box^{-1}_{\mathcal{R}}[-4\pi G\,\sigma]\,, \tag{3.4a}$$

$$V_i = \Box^{-1}_{\mathcal{R}}[-4\pi G\,\sigma_i]\,, \tag{3.4b}$$

$$\hat{X} = \Box^{-1}_{\mathcal{R}}\Big[-4\pi G\,V\sigma_{ii} + \hat{W}_{ij}\partial_{ij}V + 2V_i\partial_t\partial_i V + V\partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i\Big]\,, \tag{3.4c}$$

$$\hat{R}_i = \Box^{-1}_{\mathcal{R}}\left[-4\pi G\,(V\sigma_i - V_i\sigma) - 2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\right]\,, \tag{3.4d}$$

$$\hat{W}_{ij} = \Box^{-1}_{\mathcal{R}}\left[-4\pi G\,(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V\partial_j V\right]\,, \tag{3.4e}$$

where the σ, σ_i, σ_{ij} quantities are convenient matter source densities defined as

$$\sigma = \frac{1}{c^2}(T^{00} + T^{ii})\,,\quad \sigma_i = \frac{1}{c}T^{0i}\,,\quad \sigma_{ij} = T^{ij}\,, \tag{3.5}$$

while $\Box^{-1}_{\mathcal{R}}$ stands for the PN-expanded retarded d'Alembertian operator acting on a function $f(\boldsymbol{x},t)$ as [77, 78]

$$(\Box^{-1}_{\mathcal{R}} f)(\boldsymbol{x},t) = -\frac{1}{4\pi}\sum_{n\geq 0}\frac{(-1)^n}{n!}\mathrm{FP}_{B=0}\int \mathrm{d}^3x'(|\boldsymbol{x}'|/r_0)^B|\boldsymbol{x}-\boldsymbol{x}'|^{n-1}f^{(n)}(\boldsymbol{x}',t)\,. \tag{3.6}$$

Here $\mathrm{FP}_{B=0}$ denotes the so-called Finite Part regularization, and r_0 is an associated arbitrary length scale. This regularization is used and described in Section IV A for the wave generation formalism, but in Eq. (3.6) it cures the divergences of the near-zone post-Newtonian metric at infinity rather than the divergences of the multipolar far-zone expansion at the origin. At the order we are considering here, it does not matter for the equations of motion. In particular the final results are all independent of the scale r_0[2].

[1] Possible contributions to the metric of non-linear tail terms, which are not made of (products of) elementary potentials defined by means of the operator $\Box^{-1}_{\mathcal{R}}$, do not arise below the order 4PN [77, 78].

[2] At the 3PN non-spinning order, the scale r_0 does appear in the final results for the dynamics, but it disappears when considering gauge-invariant expressions such as $E(\omega)$, the conserved energy as a function of the orbital frequency.

B. Matter source and equations of motion in terms of potentials

In this section, we introduce convenient additional definitions for the matter source in the PN context. In the covariant expression (2.35), the worldline integration contained in the particle densities n_A [see Eq. (2.2)] can be performed explicitly in a definite coordinate grid $(t, \boldsymbol{x})$. This results in

$$T^{\alpha\beta} = \sum_{A=1,2} \left[U_A^{\alpha\beta}\delta_A + \nabla_\mu\Big(U_A^{\alpha\beta\mu}\delta_A\Big) + \nabla_\mu\nabla_\nu\Big(U_A^{\alpha\beta\mu\nu}\delta_A\Big)\right], \tag{3.7}$$

where we have defined

$$U_A^{\alpha\beta} = \frac{1}{u_A^0\sqrt{-g}}\Big(c\, p_A^{(\alpha}u_A^{\beta)} + \frac{1}{3}R^{(\alpha}_{A\,\lambda\mu\nu}J_A^{\beta)\lambda\mu\nu}\Big)\,, \tag{3.8a}$$

$$U_A^{\alpha\beta\mu} = \frac{1}{u_A^0\sqrt{-g}}u_A^{(\alpha}S_A^{\beta)\mu}\,, \tag{3.8b}$$

$$U_A^{\alpha\beta\mu\nu} = \frac{1}{u_A^0\sqrt{-g}}\Big(-\frac{2}{3}c^2 J_A^{\mu(\alpha\beta)\nu}\Big)\,. \tag{3.8c}$$

Here $u^0 = 1/\sqrt{-g^A_{\mu\nu}v^\mu v^\nu/c^2}$, $v^\mu = (c, v^i)$ (so that $u^\mu = u^0 v^\mu/c$), and the label index A on metric-dependent quantities means that they are to be regularized according to Hadamard regularization at the location of the particle A. In terms of partial derivatives, we have

$$T^{\alpha\beta} = \sum_{A=1,2} \left[\mathcal{T}_A^{\alpha\beta}\delta_A + \frac{1}{\sqrt{-g}}\partial_\mu\Big(\mathcal{T}_A^{\alpha\beta\mu}\delta_A\Big) + \frac{1}{\sqrt{-g}}\partial_{\mu\nu}\Big(\mathcal{T}_A^{\alpha\beta\mu\nu}\delta_A\Big)\right], \tag{3.9}$$

with

$$\begin{aligned}\mathcal{T}_A^{\alpha\beta} &= U_A^{\alpha\beta} + 2\Gamma^{(\alpha}_{A\,\mu\nu}U_A^{\beta)\mu\nu} + (\partial_\lambda\Gamma^{(\alpha}_{\mu\nu})_A\, U_A^{\beta)\lambda\mu\nu} \\ &\quad + \Gamma^{(\alpha}_{A\,\rho\lambda}(U_A^{\beta)\lambda\mu\nu}\Gamma^\rho_{A\,\mu\nu} - \Gamma^{\beta)}_{A\,\mu\nu}U_A^{\rho\lambda\mu\nu})\,,\end{aligned} \tag{3.10a}$$

$$\mathcal{T}_A^{\alpha\beta\mu} = \sqrt{-g}(U_A^{\alpha\beta\mu} + \Gamma^\mu_{A\,\nu\lambda}U_A^{\alpha\beta\nu\lambda} - 2\Gamma^{(\alpha}_{A\,\nu\lambda}U_A^{\beta)\mu\nu\lambda})\,, \tag{3.10b}$$

$$\mathcal{T}_A^{\alpha\beta\mu\nu} = \sqrt{-g}U_A^{\alpha\beta\mu\nu}\,. \tag{3.10c}$$

By using Eqs. (3.8), and the definitions of σ, σ_i, σ_{ij} given in Eqs. (3.5), we arrive at the following expressions in terms of metric potentials

$$\begin{aligned}\sigma = m_1\delta_1\bigg[&1 + \frac{1}{c^2}\Big(\frac{3}{2}v_1^2 - V\Big) + \frac{1}{c^4}\Big(\frac{7}{8}v_1^4 - 4\big(v_1^aV_a\big) - 2\hat{W} + \frac{1}{2}v_1^2V + \frac{1}{2}V^2\Big) \\ &+ \frac{1}{m_1c^5}\Big(2(S_1^{ab}v_1^a\partial_bV) - 4(S_1^{ab}\partial_bV_a)\Big) + \frac{1}{c^6}\Big(-8(\hat{R}_av_1^a) + \frac{11}{16}v_1^6 \\ &\quad - 10v_1^2(v_1^aV_a) - 4(V_aV_a) + 2(v_1^av_1^b\hat{W}_{ab}) - 3v_1^2\hat{W} - 8\hat{Z} + \frac{33}{8}v_1^4V \\ &\quad - 4(v_1^aV_a)V + 2\hat{W}V + \frac{11}{4}v_1^2V^2 - \frac{1}{6}V^3 - 4\hat{X} - \frac{\kappa_1}{2m_1^2}(S_1^{ba}S_1^{bi}\partial_{ia}V)\Big)\bigg] \\ &+ \frac{1}{\sqrt{-g}}\partial_k\bigg\{\delta_1\bigg[-\frac{2S_1^{ka}v_1^a}{c^3} + \frac{S_1^{ka}}{c^5}\Big(-4Vv_1^a + 4V_a\Big)\end{aligned}$$

$$
\begin{aligned}
&\qquad + \frac{\kappa_1}{m_1 c^6}\Big(S_1^{ab}S_1^{kb}\partial_a V - (S_1^{ab}S_1^{ab})\partial_k V\Big)\Big]\Big\} \\
&+ \frac{\kappa_1}{2m_1c^6\sqrt{-g}}\partial_t^2\Big(\delta_1 S_1^{ab}S_1^{ab}\Big) + \frac{\kappa_1}{m_1c^6\sqrt{-g}}\partial_t\partial_k\Big[\delta_1\Big(-S_1^{ab}S_1^{kb}v_1^a + \big(S_1^{ab}S_1^{ab}\big)v_1^k\Big)\Big] \\
&+ \frac{\kappa_1}{m_1\sqrt{-g}}\partial_{kl}\Big\{\delta_1\Big[\frac{S_1^{ka}S_1^{la}}{2c^4} + \frac{1}{c^6}\Big(S_1^{ka}S_1^{la}(\frac{3}{4}v_1^2 + \frac{3}{2}V) - \frac{1}{2}S_1^{ka}S_1^{lb}v_1^a v_1^b \\
&\qquad + \frac{1}{2}(S_1^{ab}S_1^{ab})v_1^k v_1^l + S_1^{ab}(-S_1^{lb}v_1^a v_1^k - S_1^{kb}v_1^a v_1^l)\Big)\Big]\Big\} \\
&+ 1 \leftrightarrow 2 + \mathcal{O}\Big(\frac{1}{c^7}\Big)\,,
\end{aligned} \tag{3.11a}
$$

$$
\begin{aligned}
\sigma_i = m_1\delta_1\Big[&v_1^i + \frac{1}{c^2}\Big(\frac{1}{2}v_1^2v_1^i - Vv_1^i\Big) - \frac{S_1^{ia}\partial_a V}{2m_1c^3} \\
&+ \frac{1}{c^4}\Big(\frac{3}{8}v_1^4 v_1^i - 4(v_1^aV_a)v_1^i - 2\hat{W}v_1^i + \frac{3}{2}v_1^2Vv_1^i + \frac{1}{2}V^2v_1^i\Big)\Big] \\
&+ \frac{1}{2c^3\sqrt{-g}}\partial_t\Big(\delta_1 S_1^{ia}v_1^a\Big) + \frac{1}{\sqrt{-g}}\partial_k\Big[\delta_1\Big(\frac{S_1^{ik}}{2c} - \frac{S_1^{ka}v_1^a v_1^i}{2c^3}\Big)\Big] \\
&- \frac{\kappa_1}{2m_1c^4\sqrt{-g}}\partial_t\partial_k\Big(\delta_1 S_1^{ia}S_1^{ka}\Big) \\
&+ \frac{\kappa_1}{m_1c^4\sqrt{-g}}\partial_{kl}\Big[\delta_1\Big(\frac{1}{2}S_1^{ka}S_1^{la}v_1^i - \frac{1}{4}S_1^{ia}S_1^{la}v_1^k - \frac{1}{4}S_1^{ia}S_1^{ka}v_1^l\Big)\Big] \\
&+ 1 \leftrightarrow 2 + \mathcal{O}\Big(\frac{1}{c^5}\Big)\,,
\end{aligned} \tag{3.11b}
$$

$$
\begin{aligned}
\sigma_{ij} = m_1\delta_1\Big[&v_1^i v_1^j + \frac{1}{c^2}\Big(\frac{1}{2}v_1^2 v_1^i v_1^j - Vv_1^i v_1^j\Big)\Big] \\
&+ \frac{1}{c\sqrt{-g}}\partial_k\Big[\delta_1\Big(\frac{1}{2}S_1^{jk}v_1^i + \frac{1}{2}S_1^{ik}v_1^j\Big)\Big] + 1 \leftrightarrow 2 + \mathcal{O}\Big(\frac{1}{c^3}\Big)\,.
\end{aligned} \tag{3.11c}
$$

Here we have dropped the indices on the metric potentials, as we recall that, according to the pure Hadamard-Schwartz regularization, we can indifferently consider any quantity in factor of a Dirac delta as regularized, according to the rule $F\delta_1 = (F)_1\delta_1$. Notice however that the $1/\sqrt{-g}$ prefactors must *not* be evaluated at point $\boldsymbol{y}_1$, i.e. they are still functions of the field point $\boldsymbol{x}$. These matter sources display explicit factors with spins and other without spin, but we should always keep in mind that there are secondary spin contributions coming from the potentials themselves.

Let us now turn to the expression of the equations of evolution in terms of the metric potentials (3.4). If we pose $P_\mu = p_\mu/m$, using $\mathrm{d}/\mathrm{d}\tau = u^0\mathrm{d}/\mathrm{d}t$, the covariant equations of motion (2.1a) may be put in the form

$$
\frac{\mathrm{d}P_\mu}{\mathrm{d}t} = F_\mu \equiv \Gamma^\rho{}_{\mu\nu}v^\nu\frac{p_\rho}{m} - \frac{1}{2mc}R_{\mu\nu\rho\sigma}v^\nu S^{\rho\sigma} - \frac{c^2}{6mu^0}J^{\lambda\nu\rho\sigma}\nabla_\mu R_{\lambda\nu\rho\sigma} + \mathcal{O}(S^3)\,, \tag{3.12}
$$

where $P_\mu = p_\mu/m$ can be read from Eq. (2.12). For a lower spatial index $\mu = i$, we decompose P_μ as (coming back to notations for the body 1)

$$
P_{1i} = P_{1i}^{\mathrm{NS}} + P_{1i}^{\mathrm{SS}}\,,
$$

$$F_{1i} = F_{1i}^{\rm NS} + F_{1i}^{\rm SO} + F_{1i}^{\rm SS}\,. \tag{3.13}$$

Here, the order in spin refers to the order in the spin tensor as it reads in the formulas (2.12) and (2.1a), but we should recall that there are also spin contributions coming from the potentials themselves, as well as from the replacement of accelerations using the equations of motion. We see from Eq. (2.12) that P_i has no SO part in this sense, and its NS part comes from $P_i = u^0 g_{i\nu} v^\nu / c$. The NS part of F_i comes from the usual connexion term in the geodesic equation, the first term in Eqs. (3.12). As the NS and SO parts can already be found e.g. in Eqs. (2.12) of Ref. [79] and in Eqs. (3.7) of Ref. [24], we only display here the SS pieces:

$$\begin{aligned}
P_{1i}^{(\rm S^2)} = \frac{1}{m_1^2 c^6}\bigg\{ &- S_1^{ab} S_1^{ib} \partial_t \partial_a V + 2 S_1^{bj} S_1^{ia} \partial_{ja} V_b + S_1^{ib}\big(S_1^{bj} v_1^a \partial_{ja} V - 2 S_1^{aj} v_1^a \partial_{jb} V\big) \\
&+ \kappa_1 \bigg[\frac{3}{2}(S_1^{ai} S_1^{ab} \partial_{ib} V) v_1^i - S_{1ab} S_1^{ib} \partial_t \partial_a V \\
&\qquad + (S_1^{ab} S_1^{ab}) \partial_t \partial_i V + (S_1^{ab} S_1^{ab}) v_1^a \partial_{ia} V + S_1^{bj} S_1^{ib} v_1^a \partial_{ja} V \\
&\qquad + 2 S_1^{ab} S_1^{bj} v_1^a \partial_{ji} V + S_1^{aj} S_1^{ab} (-2 \partial_{jb} V_i + 2 \partial_{ji} V_b)\bigg]\bigg\}\,,
\end{aligned} \tag{3.14a}$$

$$\begin{aligned}
F_{1i}^{(\rm S^2)} = {} & \frac{\kappa_1}{2 m_1^2 c^4} S_1^{aj} S_1^{ab} \partial_{ji} \partial_b V + \frac{\kappa_1}{m_1^2 c^6}\bigg[\big(S_1^{ab} S_1^{ab}\big) v_1^a \partial_t \partial_{ia} V \\
&+ \frac{1}{2}\big(S_1^{ab} S_1^{ab}\big) \partial_t^2 \partial_i V - \frac{3}{2}\big(S_1^{ai} S_1^{ab} \partial_{ib} V\big) \partial_i V + \big(S_1^{ab} S_1^{ab}\big) \partial_a V \partial_{ia} V \\
&+ \frac{1}{2}\big(S_1^{ab} S_1^{ab}\big) v_1^a v_1^b \partial_{iba} V - S_1^{bj} S_1^{ib} \partial_a V \partial_{ja} V + S_1^{ab} S_1^{bj} \Big(v_1^a \partial_t \partial_{ji} V \\
&\qquad + 4 \partial_a V \partial_{ji} V\Big) + S_1^{aj} S_1^{ab} \Big(2 \partial_t \partial_{ji} V_b + \frac{3}{4} v_1^2 \partial_{jib} V + \frac{1}{2} V \partial_{jib} V\Big) \\
&+ S_1^{aj} \Big(2 S_1^{jk} v_1^a v_1^b \partial_{kib} V - \frac{1}{2} S_1^{bk} v_1^a v_1^b \partial_{kji} V\Big) \\
&+ S_1^{bk} S_1^{bj} \big(2 v_1^a \partial_{kia} V_j - 2 v_1^a \partial_{kji} V_a\big)\bigg]\,.
\end{aligned} \tag{3.14b}$$

In these formulas, the potentials and their derivatives are to be understood as regularized at the location of body 1.

For the equation of precession of the conserved-norm spin, we decompose similarly the precession vector into a NS and a SO part, before replacement of the potentials. We obtain

$$\Omega_1^i = \big(\Omega_1^i\big)_{\rm NS} + \big(\Omega_1^i\big)_{\rm SO}\,, \tag{3.15}$$

where $(\Omega_1^i)_{\rm NS}$ is given by Eq. (2.19) in [25] and where

$$\begin{aligned}
\big(\Omega_1^i\big)_{\rm SO} = {} & \frac{\kappa_1}{m_1 c^3} S_1^a \partial_{ia} V + \frac{1}{m_1 c^5}\bigg\{ v_1^i\,\big(S_1^a v_1^b \partial_{ba} V\big) - \frac{1}{2}\big(S_1^a v_1^a\big) \partial_t \partial_i V + S_1^a v_1^b \partial_{ab} V_i \\
&- S_1^a (v_1^a v_1^a) \partial_{ia} V + \frac{1}{2}\,(S_1^a v_1^a)\, v_1^a \partial_{ia} V - (S_1^a v_1^a)\, \partial_{ia} V_a + S_1^a v_1^b \partial_{ia} V_b \\
&+ \kappa_1 \bigg[- \big(S_1^a \partial_t \partial_a V\big) v_1^i - \frac{3}{2}\big(S_1^a v_1^b \partial_{ba} V\big) v_1^i - \big(S_1^a v_1^a\big) \partial_t \partial_i V
\end{aligned}$$

$$+ S_1^i\Big(2(v_1^a\partial_t\partial_a V) + (\partial_a V\partial_a V) + (v_1^a v_1^b\partial_{ba}V) + \partial_t^2 V\Big)$$
$$- 3\big(S_1^a\partial_a V\big)\partial_i V - \frac{3}{2}\big(S_1^a v_1^a\big)v_1^a\partial_{ia}V + S_1^a\Big(2\partial_t\partial_a V_i + 2\partial_t\partial_i V_a$$
$$+ 2v_1^b\partial_{ba}V_i + \frac{3}{2}v_1^2\partial_{ia}V - 3V\partial_{ia}V - 4v_1^b\partial_{ia}V_b + 2v_1^b\partial_{ib}V_a\Big)\Big]\Big\}\,. \tag{3.16}$$

The contributions featuring κ_1 come directly from the second term in Eq. (2.30), while the other contributions come from the first term there. The time derivatives of the velocities that enter the definition of the tetrad are replaced by the expression of the acceleration in terms of potentials, which include SO terms (as given for instance in Section 3 of [24]).

C. Spin contributions in the metric potentials

We now investigate the spin contributions to the metric potentials introduced in Section III A. As we are effectively working at the next-to-leading order, calculating these contributions from the results already presented above will be rather straightforward and we will only need to resort to well-known techniques.

By inspection of the matter sources (3.11), one can see that the SO and SS contributions to the metric potentials start at the following PN orders:[3]

$$\begin{aligned}
V^{\rm SO} &= \mathcal{O}(3)\,, & V^{\rm SS} &= \mathcal{O}(4)\,,\\
V_i^{\rm SO} &= \mathcal{O}(1)\,, & V_i^{\rm SS} &= \mathcal{O}(4)\,,\\
\hat{X}^{\rm SO} &= \mathcal{O}(1)\,, & \hat{X}^{\rm SS} &= \mathcal{O}(2)\,,\\
\hat{R}_i^{\rm SO} &= \mathcal{O}(1)\,, & \hat{R}_i^{\rm SS} &= \mathcal{O}(4)\,,\\
\hat{W}_{ij}^{\rm SO} &= \mathcal{O}(3)\,, & \hat{W}_{ij}^{\rm SS} &= \mathcal{O}(4)\,.
\end{aligned} \tag{3.17}$$

From Eqs. (3.14) and (3.16), we see that it is sufficient to compute the new SS contributions of the potentials V at the order 3PN (the leading order contribution at the order 2PN being already known, see e.g. Ref. [33]), V_i at the order 2PN, and $\hat{X}$ at the order 1PN.

We turn now to the caculation of $V^{\rm SS}$ at the 3PN order which actually corresponds to the relative 1PN order. Truncating Eq. (3.6) appropriately, we have (dropping the $\mathrm{FP}_{B=0}$ regularization, which plays no role for compact sources)

$$V^{\rm SS} = \bigg[\int\frac{\mathrm{d}^3x'}{|x-x'|}\sigma(t,x') - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}^3x'\sigma(t,x') + \frac{1}{2c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int\mathrm{d}^3x'|x-x'|\sigma(t,x')\bigg]_{\rm SS} + \mathcal{O}(8)\,, \tag{3.18}$$

Because we are working at the next-to-leading order, various indirect contributions appear. Aside from the SS terms generated directly by the SS terms of σ given in Eq. (3.11a), there are contributions from the SO 0.5PN part of V_i in the SO 2.5PN part of σ, from the SS 2PN part of V in the NS 2PN part of σ, and from the acceleration replacement featuring the SS 2PN part of a^i in the second time derivative of the integral of the NS Newtonian part of σ (in the third term above).

[3] It is implicitly understood that the orders n showed in the terms $\mathcal{O}(n)$ below take their highest possible values.

In the following, to present the result in a more compact form, we adopt short cut notations: for any vectors a, b and spin tensors $S_{A,B}$, we define the scalars $(ab) = a^i b^i$, $(S_A S_B) \equiv S_A^{ij} S_B^{ij}$, $(a S_A b) \equiv a^i S_A^{ij} b^j$, and $(a S_A S_B b) \equiv a^i S_A^{ik} S_B^{jk} b^j$ (beware of the convention for the order of the indices on the spin tensors).

We replace in Eq. (3.18) the full expression (3.11a) for σ, perform integration by parts when derivatives of Dirac deltas occur, and compute the resulting integrals using Hadamard regularization, i.e. $\int \mathrm{d}^3 x F \delta_1 = (F)_1$. The metric potentials can be considered as regularized when appearing in factor of a Dirac delta in the integrals, according to the pure Hadamard-Schwartz rule [63] $F\delta_1 = (F_1)\delta_1$.

An important point is that the derivatives have to be treated in a distributional sense. For the first time in our formalism, we have to take into account an essential distributional term in the potential V itself. The leading order result is indeed

$$V^{\mathrm{SS}} = \frac{G\kappa_1}{2c^4 m_1} S_1^{ik} S_1^{jk} \partial_{ij}\left(\frac{1}{r_1}\right) + 1 \leftrightarrow 2 + \mathcal{O}(6)\,, \tag{3.19}$$

which, along with a non-distributional contribution, yields a distributional term given by

$$V^{\mathrm{SS}}_{\mathrm{distr}} = \frac{G\kappa_1}{2c^4 m_1 r_1^3} S_1^{ij} S_1^{ij} \frac{4\pi}{3}\delta_1 + 1 \leftrightarrow 2 + \mathcal{O}(6)\,. \tag{3.20}$$

This distributional term will play no role in the derivation of the equations of motion themselves, but it will produce a net contribution when computing the mass source quadrupole moment, as explained below in Section IV B. Because, in this computation of the quadrupole moment, the V potential is only needed at the 2PN order, we will not need to consider possible distributional terms at the higher 3PN order.

Gathering the different non-distributional contributions we obtain

$$\begin{aligned}
V^{\mathrm{SS}}_{\text{non distr}} = {} & \frac{G\kappa_1}{2c^4 m_1 r_1^3}\left(3(n_1 S_1 S_1 n_1) - (S_1 S_1)\right) \\
& + \frac{G\kappa_1}{c^6 m_1}\Bigg[(S_1 S_1)\left(\frac{3(n_1 v_1)^2}{4r_1^3} - \frac{(v_1 v_1)}{r_1^3} - \frac{3Gm_2(n_{12}n_1)}{4r_{12}^4} + \frac{3Gm_2(n_{12}n_2)}{4r_{12}^4} + \frac{3Gm_2}{2r_1 r_{12}^3}\right. \\
& \qquad \left. - \frac{3Gm_2(n_{12}n_1)}{4r_1^2 r_{12}^2} - \frac{Gm_2}{2r_1^3 r_{12}} + \frac{Gm_2}{2r_2 r_{12}^3}\right) - \frac{3(n_1 S_1 v_1)^2}{2r_1^3} \\
& \qquad + (n_{12} S_1 S_1 n_{12})\left(\frac{15Gm_2(n_{12}n_1)}{4r_{12}^4} - \frac{15Gm_2(n_{12}n_2)}{4r_{12}^4} - \frac{9Gm_2}{2r_1 r_{12}^3} - \frac{3Gm_2}{2r_2 r_{12}^3}\right) \\
& \qquad + (n_1 S_1 S_1 n_1)\left(-\frac{15(n_1 v_1)^2}{4r_1^3} + \frac{3(v_1 v_1)}{r_1^3} + \frac{3Gm_2(n_{12}n_1)}{4r_1^{2r_{12}^2}} + \frac{3Gm_2}{2r_1^3 r_{12}}\right) \\
& \qquad + (n_{12} S_1 S_1 n_1)\left(-\frac{3Gm_2}{2r_{12}^4} + \frac{3Gm_2}{2r_1^2 r_{12}^2}\right) + \frac{(v_1 S_1 S_1 v_1)}{r_1^3} + \frac{3Gm_2(n_{12} S_1 S_1 n_2)}{2r_{12}^4}\Bigg] \\
& + \frac{G}{c^6}\Bigg[(n_1 S_1 S_2 n_{12})\left(-\frac{3G}{2r_{12}^4} - \frac{2G}{r_1^2 r_{12}^2}\right) + (n_{12} S_1 S_2 n_{12})\left(\frac{15G(n_{12}n_1)}{2r_{12}^4} - \frac{6G}{r_1 r_{12}^3}\right) \\
& \qquad + (S_1 S_2)\left(\frac{-3G(n_{12}n_1)}{2r_{12}^4} + \frac{G}{r_1 r_{12}^3}\right) - \frac{3G(n_{12} S_1 S_2 n_1)}{2r_{12}^4}\Bigg] + 1 \leftrightarrow 2 + \mathcal{O}(8)\,.
\end{aligned} \tag{3.21}$$

For the calculation of $V_i^{\rm SS}$ at its leading order 2PN, we proceed similarly as for V, but keeping only the first term in the expansion (3.18). The calculation is simpler, with no indirect SS contributions. We get

$$V_i^{\rm SS} = \frac{G\kappa_1}{2m_1c^4r_1^3}\left(3(n_1S_1S_1n_1) - (S_1S_1)\right)v_1^i - \frac{G\kappa_1}{2m_1c^4}\frac{4\pi}{3}(S_1S_1)\delta_1 + 1 \leftrightarrow 2 + \mathcal{O}(6)\,, \tag{3.22}$$

where we have included for completeness a distributional term completely analogous to the one discussed above for the potential $V^{\rm SS}$, but that will not contribute in the rest of our calculations.

The computation of $\hat{X}^{\rm SS}$ is different, as it involves non-compact support terms. From Eqs. (3.4), we see that the only 1PN SS contribution in $\hat{X}$ is

$$\hat{X}^{\rm SS} = \Box_{\mathcal{R}}^{-1}\left[-2\partial_i V_j^{\rm SO}\partial_j V_i^{\rm SO}\right] + \mathcal{O}(4)\,, \tag{3.23}$$

with the leading order SO part of V_i given by

$$V_i^{\rm SO} = \frac{G}{2c}S_1^{ij}\partial_j\frac{1}{r_1} + 1 \leftrightarrow 2 + \mathcal{O}(3)\,. \tag{3.24}$$

We are working at leading order here, so that we keep only the first term in the expanded inverse d'Alembertian operator, which is just an inverse Laplacian. For the cross term, with derivatives of both $1/r_1$ and $1/r_2$, we use the function $g = \ln\left[r_1 + r_2 + r_{12}\right]$ which satisfies $\Delta g = 1/(r_1r_2)$ (including the distributional part of the derivatives). With the notations $\partial_i^1 = \partial/\partial y_1^i$, $\partial_i^2 = \partial/\partial y_2^i$, we can write[4]

$$\Delta^{-1}\left[\partial_{ij}\left(\frac{1}{r_1}\right)\partial_{kl}\left(\frac{1}{r_2}\right)\right] = \partial_{ij}^1\partial_{kl}^2\left[\Delta^{-1}\left(\frac{1}{r_1r_2}\right)\right] = \partial_{ij}^1\partial_{kl}^2 g\,. \tag{3.25}$$

For the "self" terms, we can "factorize" the derivatives as explained in [81]. Since we may ignore contributions of the form $\Delta^{-1}\left(\hat{n}_1^L/r_1^p\delta_1\right)$ for $\ell + p$ even, we disregard possible distributional terms generated by space or time differentiation. After factorizing the derivatives, we transform them into derivatives with respect to $y_{1,2}^i$ and apply Δ^{-1} straightforwardly on the argument. The relevant formula is

$$\begin{aligned}\Delta^{-1}\left[\partial_{ij}\left(\frac{1}{r_1}\right)\partial_{kl}\left(\frac{1}{r_1}\right)\right] = \frac{1}{128}\Big[&3\partial_{ijkl}\left(\ln r_1\right) - 5\left(\delta_{kl}\partial_{ij} + \delta_{ij}\partial_{kl}\right)\left(\frac{1}{r_1^2}\right)\\ &+3\left(\delta_{ik}\partial_{jl} + \delta_{il}\partial_{jk} + \delta_{jk}\partial_{il} + \delta_{jk}\partial_{il}\right)\left(\frac{1}{r_1^2}\right)\\ &+\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\Delta\left(\frac{1}{r_1^2}\right)\Big]\,.\end{aligned} \tag{3.26}$$

Gathering these contributions, we find the following simple expression for the leading-order SS part of the potential $\hat{X}$:

$$\hat{X}^{\rm SS} = \frac{G^2}{2c^2r_1^4}\left[\frac{1}{4}(S_1S_1) - (n_1S_1S_1n_1)\right] - \frac{G^2}{2c^2}S_1^{ij}S_2^{kl}\partial_{jk}^1\partial_{il}^2 g + 1 \leftrightarrow 2 + \mathcal{O}(4)\,, \tag{3.27}$$

where we keep the derivatives in the second term unexpanded.

[4] Including properly the regularization $\mathrm{FP}_{B=0}$ would yield an additional constant contribution [80], which would vanish after applying the derivatives.

D. Results for the evolution equations

Using the results of the previous section and the NS and SO parts of the metric potentials that are already known, we are in position to complete the calculation of the equations of motion and precession (3.14) and (3.16). The results for the accelerations $\boldsymbol{a}_{1,2}$ and the precession vectors $\boldsymbol{\Omega}_{1,2}$ must pass several tests checking their validity.

The first one is to make sure of the existence of a set of conserved quantities, in the absence of reaction reaction at this order, associated with the Poincaré invariance of the problem: a conserved energy E, an angular momentum $\boldsymbol{J}$, a linear momentum $\boldsymbol{P}$, and a center-of-mass integral $\boldsymbol{G}$. We were actually able to construct all those quantities explicitly by guess work. The higher-order terms in the precession equations intervene only in the conservation of the angular momentum, whereas the higher-order terms in the equations of motion intervene in all other conservation relations. We shall exhibit below the expression of the conserved energy, which will be later used to control the phase evolution of the binary in the case of circular orbits through the balance equation as explained in Section IV C.

Another test consists in checking the Lorentz invariance of the dynamics, which must be manifest since the harmonic gauge choice is Lorentz-preserving. We use the same method as in Refs. [23, 24], to which we refer the reader for more details, and find that our results pass this second test.

As the 3PN SS dynamics has been already investigated in both the EFT [21, 22, 35, 38, 39, 74] and the ADM [62, 70–73] approaches, we must be able to recover their results in our scheme. The equivalence between the ADM and EFT description has been shown to hold in Refs. [36, 37, 73], so that we will only compare our results to the ADM ones, in keeping with our previous works. We present this comparison, and the resulting transformation from harmonic to ADM variables, in Appendix D. The agreement with the ADM results also validates the test-mass limit of ours.

Because the expressions produced are rather lengthy, we will give directly their reduced version in the center-of-mass (CM) frame. As in our previous works, this frame is defined as the one where the center-of-mass integral G_i (which is such that $\mathrm{d}\boldsymbol{G}/\mathrm{d}t = \boldsymbol{P}$ and hence $\mathrm{d}^2\boldsymbol{G}/\mathrm{d}t^2 = 0$) vanishes. We define $\boldsymbol{x} = r\boldsymbol{n} = \boldsymbol{y}_1 - \boldsymbol{y}_2$ the separation vector of the binary, $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ the relative velocity, $m = m_1 + m_2$ the total mass, $\nu = m_1 m_2/m^2$ the symmetric mass ration and $\delta = (m_1 - m_2)/m$ the mass difference. We also use, for convenience, the same spin variables as in the previous works [23, 45], namely

$$\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2\,, \quad \boldsymbol{\Sigma} = m\left(\frac{\boldsymbol{S}_2}{m_2} - \frac{\boldsymbol{S}_1}{m_1}\right)\,. \tag{3.28}$$

The vectors $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are the conserved-norm vectors constructed in Section II C. Additionally, we will use the notation $\kappa_+ = \kappa_1 + \kappa_2$ and $\kappa_- = \kappa_1 - \kappa_2$.

The positions of the two bodies the new frame read

$$\boldsymbol{y}_1 = \frac{m_2}{m}\boldsymbol{x} + \frac{1}{c^2}\boldsymbol{z}\,, \quad \boldsymbol{y}_2 = -\frac{m_1}{m}\boldsymbol{x} + \frac{1}{c^2}\boldsymbol{z}\,, \tag{3.29}$$

with $\boldsymbol{z}$ being a vector related to the center-of-mass integral $\boldsymbol{G}$. In general, when working at the nPN order, only the $(n-1)$PN expression of $\boldsymbol{G}$ (or $\boldsymbol{z}$) is required. This can be checked explicitly from the Newtonian expressions in the general frame of the quantities of interest, as explained for instance in Ref. [25]. Thus, we would only need the SS 2PN expression of $\boldsymbol{G}$ in principle, but it turns out that there is no such contribution in $\boldsymbol{G}$. We can therefore translate our results to the CM frame using simply the same rules as in previous works: namely, we need the NS 1PN and the SO 1.5PN terms in $\boldsymbol{z}$, as given in the Section 3 of Ref. [25].

For the SS contributions to the conserved energy, we find

$$E_{\rm SS} = \frac{G\nu}{c^4 r^3}\left[\frac{1}{4}e_4^0 + \frac{1}{c^2}\left(\frac{1}{8}e_6^0 + \frac{1}{4}\frac{Gm}{r}e_6^1\right)\right] , \tag{3.30}$$

with

$$\begin{aligned}
e_4^0 &= S^2\left(-2\kappa_+ - 4\right) + (S\Sigma)\left(-2\delta\kappa_+ - 4\delta + 2\kappa_-\right) + \Sigma^2\left(\left(\delta\kappa_- - \kappa_+\right) + \nu\left(2\kappa_+ + 4\right)\right) \\
&\quad + (nS)^2\left(6\kappa_+ + 12\right) + (n\Sigma)^2\left(\left(3\kappa_+ - 3\delta\kappa_-\right) + \nu\left(-6\kappa_+ - 12\right)\right) \\
&\quad + (nS)(n\Sigma)\left(6\delta\kappa_+ + 12\delta - 6\kappa_-\right) , \\
e_6^0 &= S^2\left[(nv)^2\left(\left(6\delta\kappa_- - 6\kappa_+ + 24\right) + \nu\left(6\kappa_+ + 12\right)\right)\right. \\
&\qquad\qquad \left. + v^2\left(\left(-2\delta\kappa_- + 8\kappa_+ - 28\right) + \nu\left(2\kappa_+ + 4\right)\right)\right] \\
&\quad + (S\Sigma)\left[(nv)^2\left(\left(-12\delta\kappa_+ + 48\delta + 12\kappa_-\right) + \nu\left(6\delta\kappa_+ + 12\delta - 30\kappa_-\right)\right)\right. \\
&\qquad\qquad \left. + v^2\left(\left(10\delta\kappa_+ - 52\delta - 10\kappa_-\right) + \nu\left(2\delta\kappa_+ + 4\delta + 6\kappa_-\right)\right)\right] \\
&\quad + \Sigma^2\left[(nv)^2\Big(\left(6\delta\kappa_- - 6\kappa_+ + 24\right) + \nu\left(-9\delta\kappa_- + 21\kappa_+ - 72\right) + \nu^2\left(-6\kappa_+ - 12\right)\Big)\right. \\
&\qquad\qquad \left. + v^2\Big(\left(-5\delta\kappa_- + 5\kappa_+ - 24\right) + \nu\left(\delta\kappa_- - 11\kappa_+ + 76\right) + \nu^2\left(-2\kappa_+ - 4\right)\Big)\right] \\
&\quad + (nS)^2\left[(nv)^2\nu\left(-30\kappa_+ - 60\right) + v^2\left(\left(60 - 18\kappa_+\right) + \nu\left(-6\kappa_+ - 12\right)\right)\right] \\
&\quad + (nS)(vS)(nv)\left(\left(-18\delta\kappa_- + 18\kappa_+ - 84\right) + \nu\left(12\kappa_+ + 24\right)\right) \\
&\quad + (vS)^2\left(6\delta\kappa_- - 6\kappa_+ + 28\right) \\
&\quad + (n\Sigma)^2\left[(nv)^2\Big(\nu\left(15\delta\kappa_- - 15\kappa_+\right) + \nu^2\left(30\kappa_+ + 60\right)\Big)\right. \\
&\qquad\qquad \left. + v^2\Big(\left(9\delta\kappa_- - 9\kappa_+ + 48\right) + \nu\left(3\delta\kappa_- + 15\kappa_+ - 156\right) + \nu^2\left(6\kappa_+ + 12\right)\Big)\right] \\
&\quad + (n\Sigma)(v\Sigma)(nv)\Big(\left(-18\delta\kappa_- + 18\kappa_+ - 72\right) + \nu\left(12\delta\kappa_- - 48\kappa_+ + 228\right) + \nu^2\left(-12\kappa_+ - 24\right)\Big) \\
&\quad + (v\Sigma)^2\left(\left(6\delta\kappa_- - 6\kappa_+ + 24\right) + \nu\left(-6\delta\kappa_- + 18\kappa_+ - 76\right)\right) \\
&\quad + (nS)(n\Sigma)\left[(nv)^2\nu\left(-30\delta\kappa_+ - 60\delta + 30\kappa_-\right)\right. \\
&\qquad\qquad \left. + v^2\left(\left(-18\delta\kappa_+ + 108\delta + 18\kappa_-\right) + \nu\left(-6\delta\kappa_+ - 12\delta + 6\kappa_-\right)\right)\right] \\
&\quad + (n\Sigma)(vS)(nv)\left(\left(18\delta\kappa_+ - 72\delta - 18\kappa_-\right) + \nu\left(6\delta\kappa_+ + 12\delta + 30\kappa_-\right)\right) \\
&\quad + (nS)(v\Sigma)(nv)\left(\left(18\delta\kappa_+ - 84\delta - 18\kappa_-\right) + \nu\left(6\delta\kappa_+ + 12\delta + 30\kappa_-\right)\right) \\
&\quad + (vS)(v\Sigma)\left(\left(-12\delta\kappa_+ + 52\delta + 12\kappa_-\right) + \nu\left(-24\kappa_-\right)\right) , \\
e_6^1 &= S^2\left(-3\delta\kappa_- + 5\kappa_+ + 8\right) + (S\Sigma)\left(\left(8\delta\kappa_+ + 8\delta - 8\kappa_-\right) + \nu\left(12\kappa_-\right)\right) \\
&\quad + \Sigma^2\left(\left(4\kappa_+ - 4\delta\kappa_-\right) + \nu\left(3\delta\kappa_- - 11\kappa_+ - 10\right)\right) + (nS)^2\left(9\delta\kappa_- - 15\kappa_+ - 36\right) \\
&\quad + (n\Sigma)^2\left(\left(12\delta\kappa_- - 12\kappa_+\right) + \nu\left(-9\delta\kappa_- + 33\kappa_+ + 30\right)\right) \\
&\quad + (nS)(n\Sigma)\left(\left(-24\delta\kappa_+ - 32\delta + 24\kappa_-\right) + \nu\left(-36\kappa_-\right)\right) .
\end{aligned} \tag{3.31}$$

The corresponding expressions for the relative acceleration $\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2$ and the precession vectors $\boldsymbol{\Omega}_{1,2}$ are provided in Appendix A.

Finally, we further specialize our results to the case of circular, non-precessing orbits. As discussed in Ref. [33], we have in fact three classes of orbits for the conservative dynamics. The CM expression are valid for general orbits, for which we make no assumption on the presence of precession and/or eccentricity. Quasi-circular precessing orbits correspond to the case where we allow a generic orientation of the spins, but assume that the separation is constant at the SO level; as soon as SS and higher-order-in-spin terms are included the radius and orbital frequency become also variable on an orbital timescale. In Ref. [33] the definition of such orbits was investigated by perturbing orbital averaged quantities. The third and simplest class of orbits is that of the circular orbits with spins aligned with the orbital angular momentum and where precession is absent. As working at the next-to-leading order makes their determination more complicated, we leave the investigation of quasi-circular orbits for future work and focus on the circular, spin-aligned, non-precessing case.

To present results for circular orbits, we use the same definitions as in previous works. We introduce a moving basis $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell})$, with $\boldsymbol{n}$ denoting the unit vector along the separation vector, $\boldsymbol{x} = r\boldsymbol{n}$, $\boldsymbol{\ell} = \boldsymbol{n} \times \boldsymbol{v}/|\boldsymbol{n} \times \boldsymbol{v}|$ the normal to the orbital plane, and $\boldsymbol{\lambda}$ completing the triad. When neglecting both radiation reaction and spin precession and assuming the spins aligned with $\boldsymbol{\ell}$ the expressions for the relative velocity and acceleration become $\boldsymbol{v} = r\omega\boldsymbol{\lambda}$ and $\boldsymbol{a} = -r\omega^2\boldsymbol{n}$, with ω the orbital frequency defined by $\dot{\boldsymbol{n}} = \omega\boldsymbol{\lambda}$. For the projected value of the (aligned or anti-aligned) spins along $\boldsymbol{\ell}$, we use the notation $S_\ell = \boldsymbol{S} \cdot \boldsymbol{\ell}$. We also introduce the usual PN parameters $\gamma = Gm/rc^2$ and $x = (Gm\omega/c^3)^{2/3}$, both of order 1PN. In the following, we only display the SS terms, and refer the reader to Sections 9.3 and 11.3 of Ref. [7] for NS and SO contributions, and to Ref. [42] for the newly computed cubic-in-spin contributions.

First, we relate r to ω by means of the equations of motion. We obtain the following SS terms for the PN generalization of Kepler's law:

$$
\begin{aligned}
\gamma_{\text{SS}} = \frac{x}{G^2m^4}\Bigg\{ & x^2\left[S_\ell^2\left(-\frac{\kappa_+}{2}-1\right) + S_\ell\Sigma_\ell\left(-\frac{\delta\kappa_+}{2} - \delta + \frac{\kappa_-}{2}\right)\right. \\
& \left. +\Sigma_\ell^2\left(\left(\frac{\delta\kappa_-}{4} - \frac{\kappa_+}{4}\right) + \nu\left(\frac{\kappa_+}{2}+1\right)\right)\right] \\
& + x^3\left[S_\ell^2\left(\left(-\frac{11\delta\kappa_-}{12} - \frac{11\kappa_+}{12} + \frac{14}{9}\right) + \nu\left(-\frac{\kappa_+}{6} - \frac{1}{3}\right)\right)\right. \\
& + S_\ell\Sigma_\ell\left(\left(\frac{5\delta}{3}\right) + \nu\left(-\frac{\delta\kappa_+}{6} - \frac{\delta}{3} + \frac{23\kappa_-}{6}\right)\right) \\
& \left. +\Sigma_\ell^2\left(1 + \nu\left(\delta\kappa_- - \kappa_+ - 2\right) + \nu^2\left(\frac{\kappa_+}{6} + \frac{1}{3}\right)\right)\right] + \mathcal{O}(8)\Bigg\} \,. \qquad (3.32)
\end{aligned}
$$

The result for the energy for circular, spin-aligned orbits is then

$$
\begin{aligned}
E_{\text{SS}} = -\frac{1}{2}m\nu c^2 x \frac{1}{G^2m^4}\Bigg\{ & x^2\left[S_\ell^2\left(-\kappa_+ - 2\right) + S_\ell\Sigma_\ell\left(-\delta\kappa_+ - 2\delta + \kappa_-\right)\right. \\
& \left. +\Sigma_\ell^2\left(\left(\frac{\delta\kappa_-}{2} - \frac{\kappa_+}{2}\right) + \nu\left(\kappa_+ + 2\right)\right)\right] \\
& + x^3\left[S_\ell^2\left(\left(-\frac{5\delta\kappa_-}{3} - \frac{25\kappa_+}{6} + \frac{50}{9}\right) + \nu\left(\frac{5\kappa_+}{6} + \frac{5}{3}\right)\right)\right. \\
& + S_\ell\Sigma_\ell\left(\left(-\frac{5\delta\kappa_+}{2} + \frac{25\delta}{3} + \frac{5\kappa_-}{2}\right) + \nu\left(\frac{5\delta\kappa_+}{6} + \frac{5\delta}{3} + \frac{35\kappa_-}{6}\right)\right) \\
& +\Sigma_\ell^2\left(\left(\frac{5\delta\kappa_-}{4} - \frac{5\kappa_+}{4} + 5\right) + \nu\left(\frac{5\delta\kappa_-}{4} + \frac{5\kappa_+}{4} - 10\right)\right.
\end{aligned}
$$

$$+\nu^2\left(-\frac{5\kappa_+}{6}-\frac{5}{3}\right)\Bigg)\Bigg]\Bigg\} + \mathcal{O}(8)\,. \tag{3.33}$$

This expression can be shown to be in agreement, in the test-mass limit, with the energy of a test particle in circular equatorial orbits around a Kerr black hole [82]. It is crucial to control the phase evolution through the balance equation (see Section IV C).

IV. NEXT-TO-LEADING ORDER CONTRIBUTIONS TO THE POST-NEWTONIAN GRAVITATIONAL WAVES ENERGY FLUX

We now move to the computation of the 3PN spin-spin contribution to the energy flux radiated by the system. We start by briefly reviewing in Section IV A the basic elements of the wave generation formalism that we need here, before providing in Section IV B some intermediate results useful in the calculation of the source multipole moments that are required to this order. The explicit results for the moments in the CM frame are relegated to Appendix B because of their length. Our explicit result for the GW flux is presented in Section IV C for general orbits in the center of mass in the system and then reduced to the case of circular orbits in the configuration where the spins are aligned with the orbital angular momentum.

A. Formalism

We perform our calculation in the framework of the multipolar post-Newtonian approach to gravitational radiation. This formalism has been developed over many years, see e.g. [77, 83–87]. Since we will only use a simplified version of the full formalism, as we are working at next-to-leading order, we will refer the reader to [7] for a review, and give only a brief overview.

The asymptotic waveform is defined from the transverse-tracefree (TT) projection of the metric perturbation, in a suitable radiative coordinate system $X^\mu = (cT, \mathbf{X})$, as its leading-order term in the $1/R$ expansion when the distance $R = |\mathbf{X}|$ to the source tends to infinity (keeping the retarded time $T_R \equiv T - R/c$ fixed). It can be parametrized using two sets of symmetric and trace-free (STF) *radiative* multipole moments, U_L of mass type and V_L of current type as

$$h^{\rm TT}_{ij} = \frac{4G}{c^2R}\,\mathcal{P}^{\rm TT}_{ijkl}(\mathbf{N})\sum_{\ell=2}^{+\infty}\frac{N_{L-2}}{c^\ell \ell!}\left[U_{klL-2}(T_R) - \frac{2\ell}{c(\ell+1)}\,N_m\,\varepsilon_{mn(k}\,V_{l)nL-2}(T_R)\right] + \mathcal{O}\left(\frac{1}{R^2}\right)\,, \tag{4.1}$$

where we denote by $L = i_1 ... i_\ell$ a multi-index composed of ℓ multipolar spatial indices i_1, ..., i_ℓ ranging from 1 to 3. Similarly $L-1 = i_1...i_{\ell-1}$ and $kL-2 = ki_1...i_{\ell-2}$; $N_L = N_{i_1}...N_{i_\ell}$ is the product of ℓ spatial vectors N_i. The transverse-traceless (TT) projection operator is denoted $\mathcal{P}^{\rm TT}_{ijkl} = \mathcal{P}_{ik}\mathcal{P}_{jl} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{kl}$ where $\mathcal{P}_{ij} = \delta_{ij} - N_iN_j$ is the projector orthogonal to the unit direction $\mathbf{N} = \mathbf{X}/R$ of the radiative coordinate system. Like in the rest of this paper, the quantity ε_{ijk} is the Levi-Civita anti-symmetric symbol such that $\varepsilon_{123} = 1$. The symmetric-trace-free (STF) projection is indicated using brackets or a hat. Thus $U_L = \hat{U}_L = U_{\langle L\rangle}$ and $V_L = \hat{V}_L = V_{\langle L\rangle}$ for STF moments. We denote time derivatives with a superscript (n).

In terms of these radiative moments, the energy flux into gravitational waves then reads

$$\mathcal{F} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\,\ell!(2\ell+1)!!} U_L^{(1)} U_L^{(1)} + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} V_L^{(1)} V_L^{(1)} \right]. \quad (4.2)$$

The U_L and V_L can be expressed as (non-linear) functions of two sets of intermediate source rooted so-called *canonical* moments M_L and S_L which are themselves related by a gauge transformation to a set of two so-called *source* multipole moments I_L, J_L (plus 4 gauge STF moments) which parametrize the most general solution to the Einstein equations outside the source. The differences between M_L and I_L (and similarly between J_L and S_L) arise at the 2.5PN order (see for instance [8]) and, since we are interested in SS effects which always add at least a factor $1/c^2$, we can safely ignore their differences. Using the same argument, we only need to consider the terms in the relation between the radiative moments and the canonical ones up to the order 2PN. Furthermore, we can also neglect the tail terms, which will only generate SS contributions at the order 3.5PN, so we finally have the simple relation

$$(U_{ij})_{\rm SS} = (I_{ij}^{(2)})_{\rm SS} + \mathcal{O}(7)\ , \quad (4.3)$$

$$(V_{ij})_{\rm SS} = (J_{ij}^{(2)})_{\rm SS} + \mathcal{O}(7)\ , \quad (4.4)$$

$$(U_{ijk})_{\rm SS} = (I_{ijk}^{(3)})_{\rm SS} + \mathcal{O}(7)\ . \quad (4.5)$$

Noticing additionally that the leading order spin-spin contribution to any of the I_L or J_L (and their time derivatives) is of the order 2PN (as will be clear from the expressions in the next section), we can express the spin-spin flux in terms of the relevant source moments as

$$\mathcal{F}_{\rm SS} = \frac{G}{c^5} \left\{ \frac{1}{5} I_{ij}^{(3)} I_{ij}^{(3)} + \frac{1}{c^2} \left[\frac{1}{189} I_{ijk}^{(4)} I_{ijk}^{(4)} + \frac{16}{45} J_{ij}^{(3)} J_{ij}^{(3)} \right] + \frac{1}{c^4} \left[\frac{1}{84} J_{ijk}^{(4)} J_{ijk}^{(4)} \right] \right\}_{\rm SS} + \mathcal{O}(7)\ , \quad (4.6)$$

which requires computing the SS parts of I_{ij} to the order 3PN and of J_{ij} and I_{ijk} to the order 2PN. We also need the NS parts of I_{ij} up to the order 1PN and of J_{ij} and I_{ijk} at the Newtonian order, as well as the SO contributions in I_{ij} and J_{ij} up to the order 1.5PN and the leading 0.5PN SO contribution to J_{ijk}, all of which are known from previous works. Remember that the spin-orbit contributions to mass (resp. current) type moments start at 1.5PN (resp. 0.5PN) order, and that time derivatives of non-spinning (resp. spin-orbit) expressions generate spin-spin contributions with an additional order 2PN (resp. 1.5PN) at least.

The matching procedure at the core of the formalism finally allows us to express the source moments as closed-form integrals over space [87]. Instead of reproducing here the general expressions which can be found in Eq. (123) of Ref. [7], we directly display below the terms that contribute to the spin-spin corrections at the required orders. They read

$$(I_{ij})_{\rm SS} = \mathop{\rm FP}_{B=0} \int d^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \left\{ \hat{x}_{ij} \left[\overline{\Sigma} + \frac{r^2}{14c^2} \overline{\Sigma}^{(2)} \right] - \frac{20}{21c^2} \hat{x}_{qij} \overline{\Sigma}_q^{(1)} \right\}_{\rm SS} + \mathcal{O}\left(\frac{1}{c^7}\right)\ , \quad (4.7a)$$

$$(J_{ij})_{\rm SS} = \mathop{\rm FP}_{B=0} \int d^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \varepsilon_{ab<j_\ell} \hat{x}_{i>a} \overline{\Sigma}_b^{\rm SS} + \mathcal{O}\left(\frac{1}{c^5}\right)\ , \quad (4.7b)$$

$$(I_{ijk})_{\rm SS} = \mathop{\rm FP}_{B=0} \int d^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \hat{x}_{ijk} \overline{\Sigma}^{\rm SS} + \mathcal{O}\left(\frac{1}{c^5}\right)\ , \quad (4.7c)$$

where $\mathrm{FP}_{B=0}$ denotes a finite part operation defined by analytic continuation in the complex plane for the parameter B, which deals here with the infrared divergences at infinity. An arbitrary scale r_0 is introduced, which will play no role in the present calculation and has to disappear from gauge-invariant results. The basic "building blocks" Σ, Σ_i and Σ_{ij} entering the integrands are defined as

$$\Sigma \equiv \frac{\tau^{00} + \tau^{ii}}{c^2}\,, \qquad \Sigma_i \equiv \frac{\tau^{0i}}{c}\,, \qquad \Sigma_{ij} \equiv \tau^{ij}\,, \tag{4.8}$$

where $\tau^{\mu\nu}$ has been defined in (3.2), and the overline indicates a post-Newtonian (near-zone) expansion. In identifying the relevant terms in (4.7a), we slightly anticipated on the results of the next subsection (see Eq. (4.9a)) and used the fact that the SS contributions to Σ, Σ_i and Σ_{ij} all start at the 2PN order at least.

B. Computation of the source moments

To obtain the relevant SS contributions to the source moments, we first express the sources $\overline{\Sigma}$, $\overline{\Sigma}_i$ and $\overline{\Sigma}_{ij}$ in terms of the potentials parametrizing the metric and the matter sources σ, σ_i and σ_{ij} defined in (3.5) (the complete relations can be found, generalized to d dimensions, in [88]). Taking into account the order of the spin corrections in these quantities, the only terms that yield spin-spin contributions to the orders we are interested in are

$$\overline{\Sigma}^{\mathrm{SS}} = \left\{\left[1 + \frac{4V}{c^2}\right]\sigma - \frac{1}{\pi G c^2}\,\partial_i V \partial_i V + \frac{2}{\pi G c^4}\partial_i V_j \partial_j V_i\right\}_{\mathrm{SS}} + \mathcal{O}(7)\;, \tag{4.9a}$$

$$\overline{\Sigma}_i^{\mathrm{SS}} = \left\{\left[1 + \frac{4V}{c^2}\right]\sigma_i - \frac{1}{\pi G c^2}\partial_k V \partial_k V_i\right\}_{\mathrm{SS}} + \mathcal{O}(5)\;, \tag{4.9b}$$

$$\overline{\Sigma}_{ij}^{\mathrm{SS}} = \mathcal{O}(3)\;. \tag{4.9c}$$

The integrals in Eq. (4.7a) can now be performed using the standard techniques described in [68, 69], handling the UV divergences of the integral through the Hadamard regularization and the IR divergences through the finite part operation $\mathrm{FP}_{B=0}$.

We highlight here that the distributional parts of the sources have to be treated with care. In particular, for the first time, we encountered the situation where such contributions in the metric itself (more precisely in the potential V), and not just those coming from derivatives applied to the metric, have to be crucially taken into account.

More specifically, the spin-spin leading order contribution in the potential V was computed in Eqs. (3.19) and (3.20) and contains a term proportional to δ_1 which has to be accounted for when integrating the $\partial_i V \partial_i V$ term of (4.9a) in (4.7a). In order to illustrate this further, let us focus on the second and third terms in $\overline{\Sigma}_{\mathrm{SS}}$

$$\overline{\Sigma}_V^{\mathrm{SS}} = \frac{4V}{c^2}\sigma - \frac{1}{\pi G c^2}\,\partial_i V \partial_i V\,, \tag{4.10}$$

which we can rewrite using the identity $2\partial_i A \partial_i B = \Delta(AB) - A\Delta B - B\Delta A$, and the fact that $\Delta V = -4\pi G\sigma$ at leading order, as

$$\overline{\Sigma}_V^{\mathrm{SS}} = -\frac{1}{2}\frac{1}{\pi G c^2}\Delta[V^2]\,. \tag{4.11}$$

By reinjecting this second form into (4.7a), integrating by parts, using $\Delta\hat{x}_{ij} = 0$ and treating the surface terms as explained in the Section IV D of [68], we readily see that $\overline{\Sigma}_V^{\rm SS}$ actually gives a vanishing contribution to I_{ij}. If on the other hand one uses (4.10) without including the distributional part of V, one obtains an incorrect non-zero result.

Our explicit results for the SS contributions to the source moments reduced to the center of mass are presented in Appendix B.

C. Gravitational waves energy flux

Using equation (4.6), our results for the source moments and the equations of motion and precession obtained in Section III D to compute time derivatives, we can finally compute explicitly the gravitational wave flux. We will give the result already reduced in the center-of-mass frame, and we use the same notations as already introduced in Section III D. We obtain

$$\mathcal{F}_{\rm SS} = \frac{G^3 m^2 \nu^2}{5 c^9 r^6} \left[\frac{1}{3} f_4^0 + \frac{1}{21 c^2} \left(f_6^0 + \frac{G m}{r} f_6^1 + \frac{G^2 m^2}{r^2} f_6^2 \right) \right] , \tag{4.12}$$

with

$$\begin{aligned}
f_4^0 = {}& S^2 \left[(nv)^2 \left(-312\kappa_+ - 624\right) + v^2 \left(288\kappa_+ + 576\right) \right] \\
& + (S\Sigma) \left[(nv)^2 \left(-312\delta\kappa_+ - 624\delta + 312\kappa_-\right) + v^2 \left(288\delta\kappa_+ + 576\delta - 288\kappa_-\right) \right] \\
& + \Sigma^2 \left[(nv)^2 \left((156\delta\kappa_- - 156\kappa_+ + 18) + \nu \left(312\kappa_+ + 624\right) \right) \right. \\
& \qquad \left. + v^2 \left((-144\delta\kappa_- + 144\kappa_+ + 6) + \nu \left(-288\kappa_+ - 576\right) \right) \right] \\
& + (nS)^2 \left[(nv)^2 \left(1632\kappa_+ + 3264\right) + v^2 \left(-1008\kappa_+ - 2016\right) \right] \\
& + (nS)(vS)(nv) \left(-696\kappa_+ - 1392\right) + (vS)^2 \left(144\kappa_+ + 288\right) \\
& + (n\Sigma)^2 \left[(nv)^2 \left((-816\delta\kappa_- + 816\kappa_+ + 18) + \nu \left(-1632\kappa_+ - 3264\right) \right) \right. \\
& \qquad \left. + v^2 \left((504\delta\kappa_- - 504\kappa_+) + \nu \left(1008\kappa_+ + 2016\right) \right) \right] \\
& + (n\Sigma)(v\Sigma)(nv) \left((348\delta\kappa_- - 348\kappa_+ - 12) + \nu \left(696\kappa_+ + 1392\right) \right) \\
& + (v\Sigma)^2 \left((-72\delta\kappa_- + 72\kappa_+ + 2) + \nu \left(-144\kappa_+ - 288\right) \right) \\
& + (nS)(n\Sigma) \left[(nv)^2 \left(1632\delta\kappa_+ + 3264\delta - 1632\kappa_-\right) + v^2 \left(-1008\delta\kappa_+ - 2016\delta + 1008\kappa_-\right) \right] \\
& + (n\Sigma)(vS)(nv) \left(-348\delta\kappa_+ - 696\delta + 348\kappa_-\right) + (nS)(v\Sigma)(nv) \left(-348\delta\kappa_+ - 696\delta + 348\kappa_-\right) \\
& + (vS)(v\Sigma) \left(144\delta\kappa_+ + 288\delta - 144\kappa_-\right) , \\
f_6^0 = {}& S^2 \left[(nv)^4 \left((2274\delta\kappa_- + 12918\kappa_+ + 35436) + \nu \left(-14112\kappa_+ - 28224\right) \right) \right. \\
& \qquad + (nv)^2 v^2 \left((-2592\delta\kappa_- - 17544\kappa_+ - 51984) + \nu \left(17928\kappa_+ + 35856\right) \right) \\
& \qquad \left. + v^4 \left((366\delta\kappa_- + 5034\kappa_+ + 18276) + \nu \left(-4584\kappa_+ - 9168\right) \right) \right] \\
& + (S\Sigma) \left[(nv)^4 \left((10644\delta\kappa_+ + 50652\delta - 10644\kappa_-) \right. \right. \\
& \qquad \qquad \left. + \nu \left(-14112\delta\kappa_+ - 28224\delta + 5016\kappa_-\right) \right) \\
& \qquad + (nv)^2 v^2 \left((-14952\delta\kappa_+ - 69672\delta + 14952\kappa_-) \right. \\
& \qquad \qquad \left. + \nu \left(17928\delta\kappa_+ + 35856\delta - 7560\kappa_-\right) \right) \\
& \qquad \left. + v^4 \left((4668\delta\kappa_+ + 20812\delta - 4668\kappa_-) + \nu \left(-4584\delta\kappa_+ - 9168\delta + 3120\kappa_-\right) \right) \right] \\
& + \Sigma^2 \left[(nv)^4 \left((-5322\delta\kappa_- + 5322\kappa_+ + 9714) + \nu \left(4782\delta\kappa_- - 15426\kappa_+ - 64788\right) \right. \right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad +\nu^2\left(14112\kappa_+ + 28224\right)\big)\\
&\quad +(nv)^2v^2\left((7476\delta\kappa_- - 7476\kappa_+ - 14286)\right.\\
&\qquad\qquad \left.+\nu\left(-6372\delta\kappa_- + 21324\kappa_+ + 86316\right) + \nu^2\left(-17928\kappa_+ - 35856\right)\right)\\
&\quad +v^4\left((-2334\delta\kappa_- + 2334\kappa_+ + 3796) + \nu\left(1926\delta\kappa_- - 6594\kappa_+ - 23336\right)\right.\\
&\qquad\qquad \left.+\nu^2\left(4584\kappa_+ + 9168\right)\right)\big]\\
&+ (nS)^2\big[(nv)^4\left((12930\delta\kappa_- - 90570\kappa_+ - 81570) + \nu\left(71520\kappa_+ + 143040\right)\right)\\
&\quad +(nv)^2v^2\left((-8124\delta\kappa_- + 81636\kappa_+ + 65220) + \nu\left(-62976\kappa_+ - 125952\right)\right)\\
&\quad +v^4\left((570\delta\kappa_- - 14778\kappa_+ - 6546) + \nu\left(13632\kappa_+ + 27264\right)\right)\big]\\
&+ (nS)(vS)\big[(nv)^3\left((-19752\delta\kappa_- + 51816\kappa_+ + 16464) + \nu\left(-29184\kappa_+ - 58368\right)\right)\\
&\quad +(nv)v^2\left((9522\delta\kappa_- - 19890\kappa_+ - 180) + \nu\left(4092\kappa_+ + 8184\right)\right)\big]\\
&+ (vS)^2\big[(nv)^2\left((6378\delta\kappa_- - 9114\kappa_+ + 7794) + \nu\left(5100\kappa_+ + 10200\right)\right)\\
&\quad +v^2\left((-1668\delta\kappa_- - 324\kappa_+ - 6478) + \nu\left(120\kappa_+ + 240\right)\right)\big]\\
&+ (n\Sigma)^2\big[(nv)^4\left((51750\delta\kappa_- - 51750\kappa_+ + 18420)\right.\\
&\qquad\qquad \left.+\nu\left(-48690\delta\kappa_- + 152190\kappa_+ + 60960\right) + \nu^2\left(-71520\kappa_+ - 143040\right)\right)\\
&\quad +(nv)^2v^2\left((-44880\delta\kappa_- + 44880\kappa_+ - 8112)\right.\\
&\qquad\qquad \left.+\nu\left(39612\delta\kappa_- - 129372\kappa_+ - 79608\right) + \nu^2\left(62976\kappa_+ + 125952\right)\right)\\
&\quad +v^4\left((7674\delta\kappa_- - 7674\kappa_+ + 3090) + \nu\left(-7386\delta\kappa_- + 22734\kappa_+ + 10884\right)\right.\\
&\qquad\qquad \left.+\nu^2\left(-13632\kappa_+ - 27264\right)\right)\big]\\
&+ (n\Sigma)(v\Sigma)\big[(nv)^3\left((-35784\delta\kappa_- + 35784\kappa_+ - 33534)\right.\\
&\qquad\qquad \left.+\nu\left(34344\delta\kappa_- - 105912\kappa_+ + 48858\right) + \nu^2\left(29184\kappa_+ + 58368\right)\right)\\
&\quad +(nv)v^2\left((14706\delta\kappa_- - 14706\kappa_+ + 7782) + \nu\left(-11568\delta\kappa_- + 40980\kappa_+ - 7794\right)\right.\\
&\qquad\qquad \left.+\nu^2\left(-4092\kappa_+ - 8184\right)\right)\big]\\
&+ (v\Sigma)^2\big[(nv)^2\left((7746\delta\kappa_- - 7746\kappa_+ + 14124) + \nu\left(-8928\delta\kappa_- + 24420\kappa_+ - 36432\right)\right.\\
&\qquad\qquad \left.+\nu^2\left(-5100\kappa_+ - 10200\right)\right)\\
&\quad +v^2\left((-672\delta\kappa_- + 672\kappa_+ - 2242) + \nu\left(1608\delta\kappa_- - 2952\kappa_+ + 9788\right)\right.\\
&\qquad\qquad \left.+\nu^2\left(-120\kappa_+ - 240\right)\right)\big]\\
&+ (nS)(n\Sigma)\big[(nv)^4\left((-103500\delta\kappa_+ - 70920\delta + 103500\kappa_-)\right.\\
&\qquad\qquad \left.+\nu\left(71520\delta\kappa_+ + 143040\delta - 123240\kappa_-\right)\right)\\
&\quad +(nv)^2v^2\left((89760\delta\kappa_+ + 718808\delta - 89760\kappa_-)\right.\\
&\qquad\qquad \left.+\nu\left(-62976\delta\kappa_+ - 125952\delta + 95472\kappa_-\right)\right)\\
&\quad +v^4\left((-15348\delta\kappa_+ - 8664\delta + 15348\kappa_-) + \nu\left(13632\delta\kappa_+ + 27264\delta - 15912\kappa_-\right)\right)\big]\\
&+ (n\Sigma)(vS)\big[(nv)^3\left((35784\delta\kappa_+ - 15402\delta - 35784\kappa_-)\right.\\
&\qquad\qquad \left.+\nu\left(-14592\delta\kappa_+ - 29184\delta + 54096\kappa_-\right)\right)\\
&\quad +(nv)v^2\left((-14706\delta\kappa_+ + 8190\delta + 14706\kappa_-) + \nu\left(2046\delta\kappa_+ + 4092\delta - 21090\kappa_-\right)\right)\big]\\
&+ (nS)(v\Sigma)\big[(nv)^3\left((35784\delta\kappa_+ - 240\delta - 35784\kappa_-)\right.\\
&\qquad\qquad \left.+\nu\left(-14592\delta\kappa_+ - 29184\delta + 54096\kappa_-\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad +(nv)v^2\left((-14706\delta\kappa_+ - 5124\delta + 14706\kappa_-) + \nu\,(2046\delta\kappa_+ + 4092\delta - 21090\kappa_-)\right)\Big] \\
&+ (vS)(v\Sigma)\,\Big[(nv)^2\big((-15492\delta\kappa_+ + 23052\delta + 15492\kappa_-) \\
&\qquad\qquad +\nu\,(5100\delta\kappa_+ + 10200\delta - 30612\kappa_-)\big) \\
&\qquad +v^2\left((1344\delta\kappa_+ - 8188\delta - 1344\kappa_-) + \nu\,(120\delta\kappa_+ + 240\delta + 6552\kappa_-)\right)\Big] \\
f_6^1 = {}& S^2\,\Big[(nv)^2\left((-2772\delta\kappa_- + 23844\kappa_+ + 48872) + \nu\,(-1320\kappa_+ - 2640)\right) \\
&\qquad +v^2\left((2572\delta\kappa_- - 21028\kappa_+ - 41832) + \nu\,(720\kappa_+ + 1440)\right)\Big] \\
&+ (S\Sigma)\,\Big[(nv)^2\left((26616\delta\kappa_+ + 55176\delta - 26616\kappa_-) + \nu\,(-1320\delta\kappa_+ - 2640\delta + 12408\kappa_-)\right) \\
&\qquad +v^2\left((-23600\delta\kappa_+ - 48872\delta + 23600\kappa_-) + \nu\,(720\delta\kappa_+ + 1440\delta - 11008\kappa_-)\right)\Big] \\
&+ \Sigma^2\,\Big[(nv)^2\big((-13308\delta\kappa_- + 13308\kappa_+ + 1208) + \nu\,(3432\delta\kappa_- - 30048\kappa_+ - 61736) \\
&\qquad\qquad +\nu^2\,(1320\kappa_+ + 2640)\big) \\
&\qquad +v^2\big((11800\delta\kappa_- - 11800\kappa_+ - 4408) + \nu\,(-2932\delta\kappa_- + 26532\kappa_+ + 56288) \\
&\qquad\qquad +\nu^2\,(-720\kappa_+ - 1440)\big)\Big] \\
&+ (nS)^2\,\Big[(nv)^2\left((28788\delta\kappa_- - 135588\kappa_+ - 300528) + \nu\,(2028\kappa_+ + 4056)\right) \\
&\qquad +v^2\left((-12380\delta\kappa_- + 77736\kappa_+ + 182752) + \nu\,(-1992\kappa_+ - 3984)\right)\Big] \\
&+ (nS)(vS)(nv)\left((-20472\delta\kappa_- + 64056\kappa_+ + 123000) + \nu\,(1932\kappa_+ + 3864)\right) \\
&+ (vS)^2\left((4664\delta\kappa_- - 14652\kappa_+ - 27240) + \nu\,(-168\kappa_+ - 336)\right) \\
&+ (n\Sigma)^2\,\Big[(nv)^2\big((82188\delta\kappa_- - 82188\kappa_+ - 5604) + \nu\,(-29802\delta\kappa_- + 194178\kappa_+ + 264672) \\
&\qquad\qquad +\nu^2\,(-2028\kappa_+ - 4056)\big) \\
&\qquad +v^2\big((-45058\delta\kappa_- + 45058\kappa_+ + 9700) + \nu\,(13376\delta\kappa_- - 103492\kappa_+ - 185532) \\
&\qquad\qquad +\nu^2\,(1992\kappa_+ + 3984)\big)\Big] \\
&+ (n\Sigma)(v\Sigma)(nv)\big((-42264\delta\kappa_- + 42264\kappa_+ - 9808) + \nu\,(19506\delta\kappa_- - 104034\kappa_+ - 69804) \\
&\qquad\qquad +\nu^2\,(-1932\kappa_+ - 3864)\big) \\
&+ (v\Sigma)^2\big((9658\delta\kappa_- - 9658\kappa_+ + 4784) + \nu\,(-4580\delta\kappa_- + 23896\kappa_+ + 9808) \\
&\qquad\qquad +\nu^2\,(168\kappa_+ + 336)\big) \\
&+ (nS)(n\Sigma)\,\Big[(nv)^2\big((-164376\delta\kappa_+ - 282444\delta + 164376\kappa_-) \\
&\qquad\qquad +\nu\,(2028\delta\kappa_+ + 4056\delta - 117180\kappa_-)\big) \\
&\qquad +v^2\left((90116\delta\kappa_+ + 183716\delta - 90116\kappa_-) + \nu\,(-1992\delta\kappa_+ - 3984\delta + 51512\kappa_-)\right)\Big] \\
&+ (n\Sigma)(vS)(nv)\left((42264\delta\kappa_+ + 46432\delta - 42264\kappa_-) + \nu\,(966\delta\kappa_+ + 1932\delta + 39978\kappa_-)\right) \\
&+ (nS)(v\Sigma)(nv)\left((42264\delta\kappa_+ + 50744\delta - 42264\kappa_-) + \nu\,(966\delta\kappa_+ + 1932\delta + 39978\kappa_-)\right) \\
&+ (vS)(v\Sigma)\left((-19316\delta\kappa_+ - 18480\delta + 19316\kappa_-) + \nu\,(-168\delta\kappa_+ - 336\delta - 18488\kappa_-)\right)\,, \\
f_6^2 = {}& S^2\left((16\delta\kappa_- + 144\kappa_+ + 368) + \nu\,(-576\kappa_+ - 1152)\right) \\
&+ (S\Sigma)\left((128\delta\kappa_+ + 224\delta - 128\kappa_-) + \nu\,(-576\delta\kappa_+ - 1152\delta + 512\kappa_-)\right) \\
&+ \Sigma^2\big((-64\delta\kappa_- + 64\kappa_+ + 24) + \nu\,(272\delta\kappa_- - 400\kappa_+ - 384) + \nu^2\,(576\kappa_+ + 1152)\big) \\
&+ (nS)^2\left((-48\delta\kappa_- - 432\kappa_+ - 936) + \nu\,(1728\kappa_+ + 3456)\right) \\
&+ (n\Sigma)^2\big((192\delta\kappa_- - 192\kappa_+ - 16) + \nu\,(-816\delta\kappa_- + 1200\kappa_+ + 928) \\
&\qquad\qquad\qquad\qquad +\nu^2\,(-1728\kappa_+ - 3456)\big)
\end{aligned}
$$

$$+ (nS)(n\Sigma)\left((-384\delta\kappa_+ - 784\delta + 384\kappa_-) + \nu\left(1728\delta\kappa_+ + 3456\delta - 1536\kappa_-\right)\right) . \tag{4.13}$$

After reduction to the case of spin-aligned, circular orbits, using the notations already introduced in Section III D for the energy, we obtain

$$\begin{aligned}\mathcal{F}_{\text{SS}} = \frac{32\nu^2}{5}\frac{c^5x^5}{G}\frac{1}{G^2m^4}\Bigg\{ & x^2\Bigg[S_\ell^2\left(2\kappa_+ + 4\right) + S_\ell\Sigma_\ell\left(2\delta\kappa_+ + 4\delta - 2\kappa_-\right) \\ & +\Sigma_\ell^2\left(\left(-\delta\kappa_- + \kappa_+ + \frac{1}{16}\right) + \nu\left(-2\kappa_+ - 4\right)\right)\Bigg] \\ & +x^3\Bigg[S_\ell^2\left(\left(\frac{41\delta\kappa_-}{16} - \frac{271\kappa_+}{112} - \frac{5239}{504}\right) + \nu\left(-\frac{43\kappa_+}{4} - \frac{43}{2}\right)\right) \\ & +S_\ell\Sigma_\ell\Bigg(\left(-\frac{279\delta\kappa_+}{56} - \frac{817\delta}{56} + \frac{279\kappa_-}{56}\right) \\ & \qquad +\nu\left(-\frac{43\delta\kappa_+}{4} - \frac{43\delta}{2} + \frac{\kappa_-}{2}\right)\Bigg) \\ & +\Sigma_\ell^2\Bigg(\left(\frac{279\delta\kappa_-}{112} - \frac{279\kappa_+}{112} - \frac{25}{8}\right) + \nu\left(\frac{45\delta\kappa_-}{16} + \frac{243\kappa_+}{112} + \frac{344}{21}\right) \\ & \qquad +\nu^2\left(\frac{43\kappa_+}{4} + \frac{43}{2}\right)\Bigg)\Bigg] + \mathcal{O}(7)\Bigg\} .\end{aligned} \tag{4.14}$$

Using this result as well as the expression of the orbital energy (3.33), we can write the balance equation $\mathcal{F} = -\mathrm{d}E/\mathrm{d}t$ for circular orbits to obtain the phase evolution of the binary. Different ways of mixing analytical and numerical integration give rise to different approximants (see for instance [89] for a comparison of these different approximants). For simplicity, we will give here only the phasing formula for the TaylorT2 approximant: we re-expand $\mathrm{d}\phi = 2\omega\mathrm{d}t = 2\omega(-\mathcal{F}/(\mathrm{d}E/\mathrm{d}t))$ and integrate term by term to obtain the phase of the wave ϕ (here ϕ is the phase of the leading 22 mode, hence the factor of 2) as a function of ω or equivalently of x. We get for the SS contributions

$$\begin{aligned}(\phi)_{\text{SS}} = -\frac{x^{-5/2}}{32\nu}\frac{1}{G^2m^4}\Bigg\{ & x^2\Bigg[S_\ell^2\left(-25\kappa_+ - 50\right) + S_\ell\Sigma_\ell\left(-25\delta\kappa_+ - 50\delta + 25\kappa_-\right) \\ & +\Sigma_\ell^2\left(\left(\frac{25\delta\kappa_-}{2} - \frac{25\kappa_+}{2} - \frac{5}{16}\right) + \nu\left(25\kappa_+ + 50\right)\right)\Bigg] \\ & +x^3\Bigg[S_\ell^2\left(\left(\frac{2215\delta\kappa_-}{48} + \frac{15635\kappa_+}{84} - \frac{31075}{126}\right) + \nu\left(30\kappa_+ + 60\right)\right) \\ & +S_\ell\Sigma_\ell\Bigg(\left(\frac{47035\delta\kappa_+}{336} - \frac{9775\delta}{42} - \frac{47035\kappa_-}{336}\right) \\ & \qquad +\nu\left(30\delta\kappa_+ + 60\delta - \frac{2575\kappa_-}{12}\right)\Bigg) \\ & +\Sigma_\ell^2\Bigg(\left(-\frac{47035\delta\kappa_-}{672} + \frac{47035\kappa_+}{672} - \frac{410825}{2688}\right) \\ & \qquad +\nu\left(-\frac{2935\delta\kappa_-}{48} - \frac{4415\kappa_+}{56} + \frac{23535}{112}\right) \\ & \qquad +\nu^2\left(-30\kappa_+ - 60\right)\Bigg)\Bigg] + \mathcal{O}(7)\Bigg\} .\end{aligned} \tag{4.15}$$

The known NS and SO contributions are summarized in Sections 9.3 and 11.3 of [7], and

LIGO/Virgo	$10M_\odot + 1.4M_\odot$	$10M_\odot + 10M_\odot$
Newtonian	3558.9	598.8
1PN	212.4	59.1
1.5PN	$-180.9 + 114.0\chi_1 + 11.7\chi_2$	$-51.2 + 16.0\chi_1 + 16.0\chi_2$
2PN	$9.8 - 10.5\chi_1^2 - 2.9\chi_1\chi_2$	$4.0 - 1.1\chi_1^2 - 2.2\chi_1\chi_2 - 1.1\chi_2^2$
2.5PN	$-20.0 + 33.8\chi_1 + 2.9\chi_2$	$-7.1 + 5.7\chi_1 + 5.7\chi_2$
3PN	$2.3 - 13.2\chi_1 - 1.3\chi_2$ $-1.2\chi_1^2 - 0.2\chi_1\chi_2$	$2.2 - 2.6\chi_1 - 2.6\chi_2$ $-0.1\chi_1^2 - 0.2\chi_1\chi_2 - 0.1\chi_2^2$
3.5PN	$-1.8 + 11.1\chi_1 + 0.8\chi_2 +$ (SS) $-0.7\chi_1^3 - 0.3\chi_1^2\chi_2$	$-0.8 + 1.7\chi_1 + 1.7\chi_2 +$ (SS) $-0.05\chi_1^3 - 0.2\chi_1^2\chi_2 - 0.2\chi_1\chi_2^2 - 0.05\chi_2^3$
4PN	(NS) $- 8.0\chi_1 - 0.7\chi_2 +$ (SS)	(NS) $- 1.5\chi_1 - 1.5\chi_2 +$ (SS)

TABLE I. Number of cycles associated to the different PN terms in the phasing formula, between the starting frequency for advanced detectors (10Hz) and a cut-off chosen to be the Scwarzschild ISCO $x = 1/6$. We show the result for typical black hole/neutron star and black hole/black hole systems. Spin-aligned, circular orbits are assumed, and we use the dimensionless spins χ_A such that $S_{A\ell} = Gm_A^2\chi_A$. We ignore contributions that are at least quadratic in the spin of the neutron star. We gather all contributions known to date, the ones still unknown are indicated in parenthesis.

additional cubic-in-spin 3.5PN contributions can be found in [42]. We give in table I the number of cycles of the signal resulting from each term in the phasing formula, for the frequency band of advanced LIGO/Virgo detectors. Notice however that these results are illustrative, as they are specific to the TaylorT2 approximant and as these number of cycles give only a rough idea of the relevance of these terms in actual data analysis applications.

We can check that our result (4.14) is in agreement, in limit of a test particle orbiting a Kerr black hole, with the result of [90] obtained in the framework of black hole perturbation theory. We leave for future work the comparison of our results with the so far incomplete results (given only at the level of the multipole moments) of [43, 44]. Natural extensions of the work presented here include the investigation of quasi-circular precessing orbits, the computation of the spherical harmonic decomposition of the waveform (or, equivalently, the full polarizations $h_{+,\times}$), and the implementation of these results for the factorized waveforms of the Effective-One-Body formalism with spins (see e.g. [91–94]).

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Appendix A: Explicit results for the equations of evolution

We gather in this appendix explicit results for the equations of motion and precession obtained in Section III D that are too long to be shown in the main text. We present them already reduced to the center-of-mass frame.

For the precession vectors $\Omega^i_{1,2}$, we have found simpler to keep the variables $S_{1,2}$ and $\kappa_{1,2}$ instead of S, Σ and $\kappa_\pm$. We get

$$\left(\Omega_1^i\right)_{\text{SO}} = \frac{G}{c^3 r^3}\left[w^i_{3,0} + \frac{1}{c^2}\left(w^i_{5,0} + \frac{Gm}{r} w^i_{5,1}\right)\right] + \mathcal{O}(7)\,, \tag{A1}$$

with

$$\begin{aligned}
w^i_{3,0} &= 3(nS_2)n^i - S_2^i + 3\kappa_1 (nS_1) n^i \left(\frac{1-\delta}{2\nu} - 1\right)\,,\\
w^i_{5,0} &= S_2^i \left[(nv)^2\left(\frac{3\delta}{4} - \frac{3\nu}{2} + \frac{15}{4}\right) + v^2\left(-\frac{3\delta}{4} - \frac{\nu}{2} - \frac{9}{4}\right)\right]\\
&\quad + n^i\left[(nS_1)(nv)^2\kappa_1\left(-\frac{15\nu}{2}\right) + (nS_2)(nv)^2\left(-\frac{15\delta}{4} + \frac{15\nu}{2} - \frac{15}{4}\right)\right.\\
&\qquad +(nS_1)v^2\left(\left(-\frac{3\delta}{2} - 3\frac{1-\delta}{2\nu} + \frac{9}{2}\right) + \kappa_1\left(-\frac{3\delta}{2} - \frac{3\nu}{2} + \frac{9}{2}\frac{1-\delta}{2\nu} - 3\right)\right)\\
&\qquad +(nS_2)v^2\left(\frac{9\delta}{4} + \frac{3\nu}{2} + \frac{15}{4}\right) + (nv)(vS_2)\left(\frac{3\delta}{4} - \frac{3\nu}{2} - \frac{9}{4}\right)\\
&\qquad \left. +(nv)(vS_1)\left(\left(\frac{3\delta}{4} + \frac{3}{2}\frac{1-\delta}{2\nu} - \frac{9}{4}\right) + \kappa_1\left(\frac{3\nu}{2} - \frac{9}{2}\frac{1-\delta}{2\nu} + \frac{9}{2}\right)\right)\right]\\
&\quad + v^i\left[(nS_1)(nv)\left(\left(\frac{3\delta}{2} + 3\frac{1-\delta}{2\nu} - \frac{9}{2}\right) + \kappa_1\left(\frac{3\nu}{2} - \frac{9}{2}\frac{1-\delta}{2\nu} + \frac{9}{2}\right)\right)\right.\\
&\qquad\quad +(nS_2)(nv)\left(-\frac{3\delta}{4} - \frac{3\nu}{2} - \frac{9}{4}\right) + (vS_2)\left(\frac{\delta}{2} + 2\right)\\
&\qquad\quad \left. +(vS_1)\left(\left(-\frac{3\delta}{4} - \frac{3}{2}\frac{1-\delta}{2\nu} + \frac{9}{4}\right) + \kappa_1\left(3\frac{1-\delta}{2\nu} - 3\right)\right)\right]\,,\\
w^i_{5,1} &= n^i\left[(nS_1)\left(\left(\frac{5\delta}{4} + \frac{\nu}{2} - \frac{5}{4}\right) + \kappa_1\left(-\frac{9\delta}{4} - 12\frac{1-\delta}{2\nu} + \frac{57}{4}\right)\right)\right.\\
&\qquad \left. +(nS_2)\left(-\frac{3\delta}{4} + \frac{\nu}{2} - \frac{39}{4}\right)\right] + S_2^i\left[\frac{13}{4} + \frac{\delta}{4} - \frac{\nu}{2}\right]\,.
\end{aligned} \tag{A2}$$

For the relative acceleration $a^i = a^i_1 - a^i_2$, we obtain (coming back to the S, Σ and $\kappa_\pm$ variables)

$$\left(a^i\right)_{\text{SS}} = \frac{G}{c^4 r^4 m}\left[\frac{1}{4}\alpha^i_{4,0} + \frac{1}{c^2}\left(\frac{1}{8}\alpha^i_{6,0} + \frac{1}{4}\frac{Gm}{r}\alpha^i_{6,1}\right)\right] + \mathcal{O}(8)\,, \tag{A3}$$

with

$$\begin{aligned}
\alpha^i_{4,0} &= S^i\Big[(nS)\left(-12\kappa_+ - 24\right) + (n\Sigma)\left(-6\delta\kappa_+ - 12\delta + 6\kappa_-\right)\Big]\\
&\quad + \Sigma^i\Big[(nS)\left(-6\delta\kappa_+ - 12\delta + 6\kappa_-\right) + (n\Sigma)\left(\left(6\delta\kappa_- - 6\kappa_+\right) + \nu\left(12\kappa_+ + 24\right)\right)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ n^i\Big[S^2\left(-6\kappa_+ - 12\right) + \Sigma^2\left(\left(3\delta\kappa_- - 3\kappa_+\right) + \nu\left(6\kappa_+ + 12\right)\right) + (nS)^2\left(30\kappa_+ + 60\right)\\
&+ (S\Sigma)\left(-6\delta\kappa_+ - 12\delta + 6\kappa_-\right) + (n\Sigma)^2\left(\left(15\kappa_+ - 15\delta\kappa_-\right) + \nu\left(-30\kappa_+ - 60\right)\right)\\
&+ (nS)(n\Sigma)\left(30\delta\kappa_+ + 60\delta - 30\kappa_-\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\alpha^i_{6,0} = &\; S^i\Big[(nS)\big((nv)^2\left(\left(60\kappa_+ - 60\delta\kappa_-\right) + \nu\left(60\kappa_+ + 120\right)\right)\\
&\qquad + v^2\left(\left(12\delta\kappa_- - 36\kappa_+ - 48\right) + \nu\left(-72\kappa_+ - 144\right)\right)\big)\\
&+ (n\Sigma)\big((nv)^2\left(\left(60\delta\kappa_+ - 120\delta - 60\kappa_-\right) + \nu\left(30\delta\kappa_+ + 60\delta + 90\kappa_-\right)\right)\\
&\qquad + v^2\left(\left(24\kappa_- - 24\delta\kappa_+\right) + \nu\left(-36\delta\kappa_+ - 72\delta + 12\kappa_-\right)\right)\big)\\
&+ (vS)(nv)\left(\left(30\delta\kappa_- - 30\kappa_+ + 84\right) + \nu\left(-12\kappa_+ - 24\right)\right)\\
&+ (v\Sigma)(nv)\left(\left(-30\delta\kappa_+ + 132\delta + 30\kappa_-\right) + \nu\left(-6\delta\kappa_+ - 12\delta - 54\kappa_-\right)\right)\Big]\\
&+ \Sigma^i\Big[(nS)(nv)^2\left(\left(60\delta\kappa_+ - 60\kappa_-\right) + \nu\left(30\delta\kappa_+ + 60\delta + 90\kappa_-\right)\right)\\
&+ (n\Sigma)(nv)^2\big(\left(-60\delta\kappa_- + 60\kappa_+ - 120\right) + \nu\left(30\delta\kappa_- - 150\kappa_+ + 240\right) + \nu^2\left(-60\kappa_+ - 120\right)\big)\\
&+ (nv)(vS)\left(\left(-30\delta\kappa_+ + 48\delta + 30\kappa_-\right) + \nu\left(-6\delta\kappa_+ - 12\delta - 54\kappa_-\right)\right)\\
&+ (nv)(v\Sigma)\big(\left(30\delta\kappa_- - 30\kappa_+ + 96\right) + \nu\left(-24\delta\kappa_- + 84\kappa_+ - 276\right) + \nu^2\left(12\kappa_+ + 24\right)\big)\\
&+ (nS)v^2\left(\left(-24\delta\kappa_+ - 24\delta + 24\kappa_-\right) + \nu\left(-36\delta\kappa_+ - 72\delta + 12\kappa_-\right)\right)\\
&+ (n\Sigma)v^2\big(\left(24\delta\kappa_- - 24\kappa_+ + 24\right) + \nu\left(24\delta\kappa_- + 24\kappa_+\right) + \nu^2\left(72\kappa_+ + 144\right)\big)\Big]\\
&+ n^i\Big[(nS)^2(nv)^2\nu\left(-210\kappa_+ - 420\right) + (nv)^2S^2\left(\left(30\delta\kappa_- - 30\kappa_+ + 120\right) + \nu\left(30\kappa_+ + 60\right)\right)\\
&+ (nS)(n\Sigma)(nv)^2\nu\left(-210\delta\kappa_+ - 420\delta + 210\kappa_-\right) + (vS)^2\left(6\delta\kappa_- - 6\kappa_+ + 84\right)\\
&+ (n\Sigma)^2(nv)^2\big(\nu\left(105\delta\kappa_- - 105\kappa_+\right) + \nu^2\left(210\kappa_+ + 420\right)\big)\\
&+ (nv)^2(S\Sigma)\left(\left(-60\delta\kappa_+ + 240\delta + 60\kappa_-\right) + \nu\left(30\delta\kappa_+ + 60\delta - 150\kappa_-\right)\right)\\
&+ (nv)^2\Sigma^2\big(\left(30\delta\kappa_- - 30\kappa_+ + 120\right) + \nu\left(-45\delta\kappa_- + 105\kappa_+ - 360\right) + \nu^2\left(-30\kappa_+ - 60\right)\big)\\
&+ (nS)(nv)(vS)\left(\left(-30\delta\kappa_- + 30\kappa_+ - 420\right) + \nu\left(60\kappa_+ + 120\right)\right)\\
&+ (n\Sigma)(nv)(vS)\left(\left(30\delta\kappa_+ - 240\delta - 30\kappa_-\right) + \nu\left(30\delta\kappa_+ + 60\delta + 30\kappa_-\right)\right)\\
&+ (nS)(nv)(v\Sigma)\left(\left(30\delta\kappa_+ - 420\delta - 30\kappa_-\right) + \nu\left(30\delta\kappa_+ + 60\delta + 30\kappa_-\right)\right)\\
&+ (n\Sigma)(nv)(v\Sigma)\big(\left(-30\delta\kappa_- + 30\kappa_+ - 240\right) + \nu\left(900 - 60\kappa_+\right) + \nu^2\left(-60\kappa_+ - 120\right)\big)\\
&+ (vS)(v\Sigma)\left(\left(-12\delta\kappa_+ + 132\delta + 12\kappa_-\right) + \nu\left(-24\kappa_-\right)\right)\\
&+ (v\Sigma)^2\left(\left(6\delta\kappa_- - 6\kappa_+ + 48\right) + \nu\left(-6\delta\kappa_- + 18\kappa_+ - 180\right)\right)\\
&+ (nS)^2v^2\left(\left(60\kappa_+ + 120\right) + \nu\left(180\kappa_+ + 360\right)\right)\\
&+ S^2v^2\left(\left(-6\delta\kappa_- - 6\kappa_+ - 48\right) + \nu\left(-36\kappa_+ - 72\right)\right)\\
&+ (nS)(n\Sigma)v^2\left(\left(60\delta\kappa_+ + 120\delta - 60\kappa_-\right) + \nu\left(180\delta\kappa_+ + 360\delta - 180\kappa_-\right)\right)\\
&+ (n\Sigma)^2v^2\big(\left(30\kappa_+ - 30\delta\kappa_-\right) + \nu\left(-90\delta\kappa_- + 30\kappa_+ - 120\right) + \nu^2\left(-180\kappa_+ - 360\right)\big)\\
&+ (S\Sigma)v^2\left(\left(-72\delta\right) + \nu\left(-36\delta\kappa_+ - 72\delta + 60\kappa_-\right)\right)\\
&+ \Sigma^2v^2\big(-24 + \nu\left(24\delta\kappa_- - 24\kappa_+ + 96\right) + \nu^2\left(36\kappa_+ + 72\right)\big)\Big]\\
&+ v^i\Big[(nS)^2(nv)\left(\left(-240\kappa_+\right) + \nu\left(120\kappa_+ + 240\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ (nv)S^2\left((-12\delta\kappa_- + 60\kappa_+ - 48) + \nu\,(-24\kappa_+ - 48)\right) \\
&+ (nS)(n\Sigma)(nv)\left((-240\delta\kappa_+ + 240\delta + 240\kappa_-) + \nu\,(120\delta\kappa_+ + 240\delta - 120\kappa_-)\right) \\
&+ (n\Sigma)^2(nv)\Big((120\delta\kappa_- - 120\kappa_+ + 240) + \nu\,(-60\delta\kappa_- + 300\kappa_+ - 480) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \nu^2\,(-120\kappa_+ - 240)\Big) \\
&+ (nv)(S\Sigma)\left((72\delta\kappa_+ - 144\delta - 72\kappa_-) + \nu\,(-24\delta\kappa_+ - 48\delta + 72\kappa_-)\right) \\
&+ (nv)\Sigma^2\Big((-36\delta\kappa_- + 36\kappa_+ - 96) + \nu\,(24\delta\kappa_- - 96\kappa_+ + 240) + \nu^2\,(24\kappa_+ + 48)\Big) \\
&+ (nS)(vS)\left((6\delta\kappa_- + 90\kappa_+ + 84) + \nu\,(-36\kappa_+ - 72)\right) \\
&+ (n\Sigma)(vS)\left((42\delta\kappa_+ - 42\kappa_-) + \nu\,(-18\delta\kappa_+ - 36\delta + 6\kappa_-)\right) \\
&+ (nS)(v\Sigma)\left((42\delta\kappa_+ + 36\delta - 42\kappa_-) + \nu\,(-18\delta\kappa_+ - 36\delta + 6\kappa_-)\right) \\
&+ (n\Sigma)(v\Sigma)\Big((-42\delta\kappa_- + 42\kappa_+ - 48) + \nu\,(12\delta\kappa_- - 96\kappa_+ + 12) + \nu^2\,(36\kappa_+ + 72)\Big)\Big]\,, \\
\alpha^i_{6,1} &= S^i\Big[(nS)\left((-24\delta\kappa_- + 72\kappa_+ + 164) + \nu\,(36\kappa_+ + 72)\right) \\
&+ (n\Sigma)\left((48\delta\kappa_+ + 72\delta - 48\kappa_-) + \nu\,(18\delta\kappa_+ + 36\delta + 30\kappa_-)\right)\Big] \\
&+ \Sigma^i\Big[(nS)\left((48\delta\kappa_+ + 84\delta - 48\kappa_-) + \nu\,(18\delta\kappa_+ + 36\delta + 30\kappa_-)\right) \\
&+ (n\Sigma)\Big((48\kappa_+ - 48\delta\kappa_-) + \nu\,(6\delta\kappa_- - 102\kappa_+ - 148) + \nu^2\,(-36\kappa_+ - 72)\Big)\Big] \\
&+ n^i\Big[(nS)^2\left((48\delta\kappa_- - 192\kappa_+ - 420) + \nu\,(-96\kappa_+ - 192)\right) \\
&+ S^2\left((-8\delta\kappa_- + 40\kappa_+ + 72) + \nu\,(20\kappa_+ + 40)\right) \\
&+ (nS)(n\Sigma)\left((-240\delta\kappa_+ - 396\delta + 240\kappa_-) + \nu\,(-96\delta\kappa_+ - 192\delta - 96\kappa_-)\right) \\
&+ (n\Sigma)^2\Big((120\delta\kappa_- - 120\kappa_+) + \nu\,(240\kappa_+ + 372) + \nu^2\,(96\kappa_+ + 192)\Big) \\
&+ (S\Sigma)\left((48\delta\kappa_+ + 72\delta - 48\kappa_-) + \nu\,(20\delta\kappa_+ + 40\delta + 12\kappa_-)\right) \\
&+ \Sigma^2\Big((24\kappa_+ - 24\delta\kappa_-) + \nu\,(-2\delta\kappa_- - 46\kappa_+ - 72) + \nu^2\,(-20\kappa_+ - 40)\Big)\Big]\,. \qquad \text{(A4)}
\end{aligned}
$$

Appendix B: Explicit results for the source multipole moments

We list in this appendix explicit results for the newly computed SS contributions to the source moments, the computation of which is described in Section IV B. We recall that the brackets indicate the STF projection.

For the mass quadrupole moment, we obtain

$$
\left(I^{ij}\right)_{\rm SS} = \frac{1}{mc^4}\left[\frac{i^{ij}_{4,0}}{2} + \frac{\nu}{84c^2}\left(i^{ij}_{6,0} + \frac{Gm}{r}i^{ij}_{6,1}\right)\right]\,, \qquad \text{(B1a)}
$$

with

$$
\begin{aligned}
i^{ij}_{4,0} &= -\,S^{<i}S^{j>}\,(\delta\kappa_- + \kappa_+) + 4\,S^{<i}\Sigma^{j>}\nu\kappa_- + \,\Sigma^{<i}\Sigma^{j>}\nu(\delta\kappa_- - \kappa_+)\,, \\
i^{ij}_{6,0} &= 29\,S^{<i}S^{j>}v^2\,(\delta\kappa_- - \kappa_+) + 58\,S^{<i}\Sigma^{j>}v^2\,((\kappa_- - \delta\kappa_+) - 2\nu\kappa_-) \\
&\quad + \,\Sigma^{<i}\Sigma^{j>}v^2\,(29(\delta\kappa_- - \kappa_+) + \nu(-29\delta\kappa_- + 87\kappa_+ + 140)) + 66\,(Sv)S^{<i}v^{j>}\,(\kappa_+ - \delta\kappa_-) \\
&\quad + 66\,(\Sigma v)S^{<i}v^{j>}\,((\delta\kappa_+ - \kappa_-) + 2\nu\kappa_-) + 66\,(Sv)\Sigma^{<i}v^{j>}\,((\delta\kappa_+ - \kappa_-) + 2\nu\kappa_-)
\end{aligned}
$$

$$
\begin{aligned}
&+ 6\,(\Sigma v)\Sigma^{<i}v^{j>}\,(11(\kappa_+ - \delta\kappa_-) + \nu(11\delta\kappa_- - 33\kappa_+ - 28)) + 22\,S^2\,v^{<i}v^{j>}\,(\delta\kappa_- - \kappa_+)\\
&+ 44\,(S\Sigma)v^{<i}v^{j>}\,((\kappa_- - \delta\kappa_+) - 2\nu\kappa_-)\\
&+ 2\,\Sigma^2\,v^{<i}v^{j>}\,(11(\delta\kappa_- - \kappa_+) + \nu(-11\delta\kappa_- + 33\kappa_+ + 14))\ ,\\
i^{ij}_{6,1} &= 6\,(nS)^2 n^{<i}n^{j>}\,((7\delta\kappa_- + 18\kappa_+ + 36) - 40\nu(\kappa_+ + 2))\\
&+ 2\,S^2\,n^{<i}n^{j>}\,((-11\delta\kappa_- + 8\kappa_+ - 48) + 30\nu(\kappa_+ + 2))\\
&+ 6\,(nS)(n\Sigma)n^{<i}n^{j>}\,((11\delta\kappa_+ + 36\delta - 11\kappa_-) + 4\nu(-10\delta\kappa_+ - 20\delta + 3\kappa_-))\\
&+ 3\,(n\Sigma)^2 n^{<i}n^{j>}\,\big(11(\kappa_+ - \delta\kappa_-) + 2\nu(13\delta\kappa_- - 24\kappa_+ - 36) + 80\nu^2(\kappa_+ + 2)\big)\\
&+ 2\,(S\Sigma)n^{<i}n^{j>}\,((19\delta\kappa_+ - 48\delta - 19\kappa_-) + 2\nu(15\delta\kappa_+ + 30\delta + 7\kappa_-))\\
&+ \ \Sigma^2\,n^{<i}n^{j>}\,\big(19(\kappa_+ - \delta\kappa_-) - 2\nu(4\delta\kappa_- + 15\kappa_+ + 8) - 60\nu^2(\kappa_+ + 2)\big)\\
&+ 12\,(nS)n^{<i}S^{j>}\,((2\delta\kappa_- - 13\kappa_+ - 22) + 5\nu(\kappa_+ + 2))\\
&+ 2\,(n\Sigma)n^{<i}S^{j>}\,(5(-9\delta\kappa_+ - 2\delta + 9\kappa_-) + 3\nu(5\delta\kappa_+ + 10\delta - 13\kappa_-))\\
&+ 2\,S^{<i}S^{j>}\,(17\delta\kappa_- + 109\kappa_+)\\
&+ 2\,(nS)n^{<i}\Sigma^{j>}\,((-45\delta\kappa_+ - 122\delta + 45\kappa_-) + 3\nu(5\delta\kappa_+ + 10\delta - 13\kappa_-))\\
&+ 6\,(n\Sigma)n^{<i}\Sigma^{j>}\,\big(15(\delta\kappa_- - \kappa_+) + \nu(-9\delta\kappa_- + 39\kappa_+ + 44) - 10\nu^2(\kappa_+ + 2)\big)\\
&+ 8\,S^{<i}\Sigma^{j>}\,(23(\delta\kappa_+ - \kappa_-) - 17\nu\kappa_-)\\
&+ 2\,\Sigma^{<i}\Sigma^{j>}\,(46(\kappa_+ - \delta\kappa_-) + \nu(-17\delta\kappa_- - 75\kappa_+ + 56))\ . \qquad \text{(B1b)}
\end{aligned}
$$

The current quadrupole moment reads at leading order (see also Ref. [42] for leading order expressions at any multipolar order)

$$
\begin{aligned}
\big(J^{ij}\big)_{\text{SS}} = \frac{\nu}{2c^4 m}\big(&-2\kappa_- S^{<i}\epsilon^{j>ab}S_a v_b + (-3 - \delta\kappa_- + \kappa_+)S^{<i}\epsilon^{j>ab}\Sigma_a v_b\\
&+ (-\delta\kappa_- + \kappa_+)\Sigma^{<i}\epsilon^{j>ab}S_a v_b + (-\kappa_- + \delta\kappa_+ + 2\kappa_-\nu)\Sigma^{<i}\epsilon^{j>ab}\Sigma_a v_b\big)\ . \qquad \text{(B2)}
\end{aligned}
$$

Finally, for the mass octupole moment, we find [42]

$$
\big(I^{ijk}\big)_{\text{SS}} = \frac{3\nu r}{2c^4 m} n^{<i}\left(-2\kappa_- S^j S^{k>} + 2(\kappa_+ - \delta\kappa_-)S^j\Sigma^{k>} + (\delta\kappa_+ - \kappa_- + 2\kappa_-\nu)\Sigma^j\Sigma^{k>}\right)\ . \qquad \text{(B3)}
$$

Appendix C: Correspondence between the spin vector and spin tensor variables

This appendix provides the link between the spin tensor and the conserved-norm spin vector variables which we use to present our PN results. We recall that the spin tensor variable S^{ij} is the spatial part, in harmonic coordinates, of the spin tensor introduced in Section II A, and that the spin vector variable has been defined in Section II C as $S^i = \tilde{S}^{\underline{i}}$, with $\tilde{S}_\mu$ given by Eq. (2.14) and $\underline{i}$ being a spatial index referring to the tetrad $e_{\underline{\alpha}}{}^{\mu}$ constructed in the same section.

We display below the SS contributions to the expression of the spin vector in terms of the spin tensor, in the general frame. These contributions complete those computed at the SO order in Ref. [25], Eqs. (B.1) (notice that the spin tensor components there were denoted as $\tilde{S}^{ij}$ instead of S^{ij}). We have

$$
\big(S_1^i\big)_{\text{SS}} = \frac{G}{c^5 r_{12}^2}\bigg[2n_{12}^a v_1^b S_2^{aj} S_1^{jk}\varepsilon^{ibk} + \frac{1}{2}v_1^i\left(n_{12}^a S_2^{aj} S_1^{bi}\varepsilon^{bij}\right) - \frac{1}{2}v_2^i\left(n_{12}^a S_2^{aj} S_1^{bi}\varepsilon^{bij}\right)
$$

$$-\frac{1}{2}n_{12}^a S_2^{ia}\left(v_1^a S_1^{bi}\varepsilon^{abi}\right) - S_1^{ab}\varepsilon^{iab}\left(n_{12}^a v_1^b S_2^{ab}\right) + \frac{1}{2}n_{12}^a S_2^{ia}\left(v_2^a S_1^{bi}\varepsilon^{abi}\right)$$
$$+S_1^{ab}\varepsilon^{iab}\left(n_{12}^a v_2^b S_2^{ab}\right)\Bigg] + \mathcal{O}(7)\,, \quad \text{(C1)}$$

where we have authorized the repetition of indices appearing in scalar quantities enclosed with parenthesis. At this order, there appear $S_1 S_2$ terms only and thus no S_1^2, S_2^2 terms.

Appendix D: Equivalence with ADM results for the dynamics

In this appendix, we compare our results for the dynamics with those previously obtained in the ADM [62, 70–73] and EFT [21, 22, 35, 38, 39, 74] approaches. As the equivalence of ADM and EFT results has been already demonstrated in Refs. [36, 37, 73], we actually restrict ourselves to the comparison of our findings with the ADM ones, in line with our previous works.

The two results have been obtained in different gauges and the spin variables differ in their definition. It is thus important to take properly into account the transformation of the particle positions and spins from one formalism to the other. In the following, we will denote the ADM variables with an overbar and resort to the convenient notation $\overline{\boldsymbol{\pi}}_A = \overline{\boldsymbol{p}}_A/m_A$. Let us now introduce the contact transformation $\boldsymbol{Y}_A(\overline{\boldsymbol{x}},\overline{\boldsymbol{p}},\overline{\boldsymbol{S}})$ and the rotation vector $\boldsymbol{\theta}_A(\overline{\boldsymbol{x}},\overline{\boldsymbol{p}},\overline{\boldsymbol{S}})$ such that the harmonic variables are related to the ADM ones by

$$\boldsymbol{y}_A = \boldsymbol{Y}_A(\overline{\boldsymbol{x}},\overline{\boldsymbol{p}},\overline{\boldsymbol{S}}) + \mathcal{O}(7)\,, \quad \text{(D1a)}$$
$$\boldsymbol{S}_A = \overline{\boldsymbol{S}}_A + \boldsymbol{\theta}_A(\overline{\boldsymbol{x}},\overline{\boldsymbol{p}},\overline{\boldsymbol{S}}) \times \overline{\boldsymbol{S}}_A + \mathcal{O}(6)\,. \quad \text{(D1b)}$$

The ADM spin variables and ours have the same Euclidean norm $\boldsymbol{S}_A\cdot\boldsymbol{S}_A = \overline{\boldsymbol{S}}_A\cdot\overline{\boldsymbol{S}}_A = s_A^2$, which is precisely the conserved norm introduced in Section II A. Since the first corrections enter as $\boldsymbol{\theta}_A = \mathcal{O}(4)$, we see that the transformation for the spins necessarily takes this form.

Now, if we denote by $\boldsymbol{A}_A(\overline{\boldsymbol{x}},\overline{\boldsymbol{p}},\overline{\boldsymbol{S}})$ and $\boldsymbol{\Omega}_A(\overline{\boldsymbol{x}},\overline{\boldsymbol{p}},\overline{\boldsymbol{S}})$ the function that converts to ADM variables the harmonic-coordinate acceleration and precession vector, and by $\overline{\boldsymbol{\Omega}}_A$ the precession vector of the ADM spins, such that $\mathrm{d}\overline{\boldsymbol{S}}_A/\mathrm{d}t = \overline{\boldsymbol{\Omega}}_A \times \overline{\boldsymbol{S}}_A$, the two relations to impose for the dynamics to be equivalent are

$$\boldsymbol{A}_A = \left\{\left\{\boldsymbol{Y}_A, H_{\mathrm{ADM}}\right\}, H_{\mathrm{ADM}}\right\} + \mathcal{O}(7)\,, \quad \text{(D2a)}$$
$$\left\{\boldsymbol{\theta}_A, H_{\mathrm{ADM}}\right\} + \boldsymbol{\theta}_A \times \overline{\boldsymbol{\Omega}}_A = \boldsymbol{\Omega}_A\left(\overline{\boldsymbol{x}},\overline{\boldsymbol{p}},\overline{\boldsymbol{S}}\right) - \overline{\boldsymbol{\Omega}}_A + \mathcal{O}(6)\,, \quad \text{(D2b)}$$

where H_{ADM} is the ADM Hamiltonian (which can be found for instance in Section 6.2 of Ref. [62]) and $\{,\}$ is the usual Poisson brackets extended to spin variables. Here the term $\boldsymbol{\theta}_A \times \overline{\boldsymbol{\Omega}}_A$ is actually negligible, for $\boldsymbol{\theta}_A = \mathcal{O}(2)$ and $\overline{\boldsymbol{\Omega}}_A = \mathcal{O}(4)$.

We find that there are no contributions at leading order in the transformations (D1), i.e. $(\boldsymbol{Y}_A)_{\mathrm{SS}} = \mathcal{O}(6)$ and $(\boldsymbol{\theta}_A)_{\mathrm{SO}} = \mathcal{O}(5)$. Using the method of undetermined coefficients then leads to a unique solution for the higher-order terms in the transformations. For the rotation vector we obtain

$$(\boldsymbol{\theta}_1)_{\mathrm{SO}} = \frac{G}{c^5\overline{r}_{12}^2}\left[\frac{m_2}{m_1}\boldsymbol{\theta}_1^{5,1} + \boldsymbol{\theta}_1^{5,2}\right] + \mathcal{O}(7)\,, \quad \text{(D3)}$$

with (adopting the same notations as in the rest of the paper for scalar products)

$$\boldsymbol{\theta}_1^{5,1} = -\frac{3\kappa_1}{2}\overline{\boldsymbol{n}}_{12}\left[(\overline{\pi}_2\overline{S}_1) + (\overline{n}_{12}\overline{\pi}_2)(\overline{n}_{12}\overline{S}_1)\right] - \frac{3\kappa_1}{2}\overline{\boldsymbol{\pi}}_2(\overline{n}_{12}\overline{S}_1) - \frac{1}{2}\overline{\boldsymbol{\pi}}_1(\overline{n}_{12}\overline{S}_1)\,,$$

$$\boldsymbol{\theta}_1^{5,2} = -\frac{1}{2}\overline{\boldsymbol{S}}_2(\overline{n}_{12}\overline{\pi}_2) + \frac{1}{2}\overline{\boldsymbol{n}}_{12}\left[(\overline{\pi}_2\overline{S}_2) - 3(\overline{n}_{12}\overline{\pi}_2)(\overline{n}_{12}\overline{S}_2)\right] + \frac{1}{2}\overline{\boldsymbol{\pi}}_2(\overline{n}_{12}\overline{S}_2)\,. \tag{D4}$$

We recall that SO terms in θ actually correspond to SS effects in the dynamics. For the contact transformation, we arrive at the simple expression

$$(\boldsymbol{Y}_1)_{\text{SS}} = \frac{Gm_2}{2m_1^2c^6\overline{r}_{12}^2}\left[\overline{\boldsymbol{S}}_1(\overline{n}_{12}\overline{S}_1) - \overline{\boldsymbol{n}}_{12}(\overline{S}_1\overline{S}_1)\right] + \mathcal{O}(8)\,. \tag{D5}$$

The relevant NS and SO contributions to these transformations are given for instance in Ref. [24] and Refs. [24, 25].

The existence of a solution relating our variables to the ADM ones validates our results, the problem of finding such transformations being largely over-constrained.

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