# ON THE LOCAL POWER ABSORPTION OF HF WAVES IN HOT INHOMOGENEOUS PLASMAS

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1. - The power dissipated per unit volume by a plane wave propagating in a uniform plasma is [1]:

$$P_{abs} = \frac{\omega}{8\pi} \vec{E}_0^* \cdot \underline{\underline{\epsilon}}^A \cdot \vec{E}_0 \tag{1}$$

where $\underline{\underline{\epsilon}}^A$ is the antihermitean part of the dielectric tensor. $P_{abs}$ differs from $\langle \vec{J} \cdot \vec{E} \rangle$ by the divergence of a vector having the dimensions of a power flux. This kinetic correction to the Poynting flux,

$$\vec{T} = \frac{\omega}{16\pi} \vec{E}_0^* \cdot \frac{\partial \underline{\underline{\epsilon}}^H}{\partial \vec{k}} \cdot \vec{E}_0 \tag{2}$$

is due to space dispersion. In a hot plasma the mean collisional free path of charged particles typically exceeds the wavelength by a large factor: as a consequence the h.f. current $\vec{J}$ is a nonlocal functional of $\vec{E}$, a fact reflected in the dependence of the dielectric tensor $\underline{\underline{\epsilon}}$ on the wavevector $\vec{k}$.

Due to space dispersion, Maxwell equations in non-homogeneous plasmas are generally integro-differential. Only in plane-stratified or cylindrical geometry, assuming the perpendicular wavelength to be much greater than the average Larmor radius, it is possible to obtain purely differential approximations to the wave equations. Finite Larmor Radius wave equations have recently received much attention, as they provide the theoretical description of mode conversion and absorption in ICR heating of fusion oriented [2]-[5]. To obtain quantitative information about power deposition profiles, it is also necessary to know the appropriate generalisation of (1) to the non-homogeneous case.

Even the knowledge of an adequate approximation for the constitutive relation, however, is not sufficient to establish the form of the local power dissipated per unit volume. Indeed, the wave equations do not offer any clue to distinguish within $\langle \vec{J} \cdot \vec{E} \rangle$ the irreversible part $P_{abs}$ from the reversible kinetic flux div$\vec{T}$. The necessity of reverting to first principles to evaluate $P_{abs}$ has been recognised by McVey et al. [6] (cfr. also [7]), who suggest to start from:

$$\int_{-\infty}^{t} P_{abs}^{\alpha} dt' = Z_\alpha e \int d\vec{v} \Big\langle \int_{-\infty}^{t} dt' \left\{ \mathrm{Re}\left[\vec{E}(\vec{r}\,', t')\right] \cdot \vec{v}\,' \right\} \left\{ \mathrm{Re}\left[f_\alpha(\vec{v}\,', \vec{r}\,', t')\right] \right\} \Big\rangle_t \tag{3}$$

Here $f_\alpha$ is the solution of the linearised Vlasov equation for species $\alpha$; $\vec{r}\,'$, $\vec{v}\,'$ are the characteristics of this equation, i.e. represent the unperturbed trajectory of a particle having velocity $\vec{v}$ at point $\vec{r}$ at time $t$. The r.h. side is easily recognised to be the total energy gained by all particles of species $\alpha$ up to time $t$, averaged over the fast time scale $\omega^{-1}$. It is assumed that the field has been adiabatically switched on, $|\vec{E}|^2 \to 0$ for $t \to -\infty$. In the case of a time harmonic wave, this means $\gamma = \mathrm{Im}\omega > 0$, $\gamma \to 0+$, which leads to the correct prescription to handle the singularities due to resonant particles on the r.h. side of (3). It should also be clear that to apply (3) one must assume sufficient collisions to ensure that particle trapping by the wave negligible; this is at the same time the condition for the validity of the linearised Vlasov equation.

2. - The result obtained in [6] by applying Eq. (3) to a plane-stratified plasma in the small Larmor approximation can be generalised to all situations in which $\vec{r}'$, $\vec{v}'$ can be evaluated in the drift approximation. Under this quite unrestrictive condition

$$\begin{aligned}\int_{-\infty}^{t} P^{\alpha}_{abs}dt' = \frac{Z_\alpha^2 e^2}{2m_\alpha}\int d\vec{v}\, e^{2\gamma t}\mathrm{Re}\Bigg\{ \int_{-\infty}^{t} dt' e^{-i\omega^*(t-t')}\left(\vec{E}^*(\vec{r}')\cdot\vec{v}'\right)\cdot \\ \int_{-\infty}^{t'} dt'' e^{i\omega(t-t'')}\left[\left(\vec{E}(\vec{r}'')+\frac{\vec{v}''}{c}\times\vec{B}(\vec{r}'')\right)\cdot\frac{\partial F_\alpha}{\partial\vec{v}''}\right]\Bigg\}\end{aligned} \tag{4}$$

Using the group properties of the particle trajectories and the fact that the unperturbed distributions $F_\alpha$ depend only on constants of the motion, and assuming for simplicity local thermal equilibrium, this can be simplified to

$$\begin{aligned}P^{\alpha}_{abs} = \frac{\omega}{4\pi}\frac{\omega^2_{p\alpha}}{v^2_{th\alpha}}\int d\vec{v}\, F_{M\alpha}(v^2,\vec{r}_g)\lim_{\gamma\to 0}\frac{2\gamma}{\omega}\mathrm{Re}\Bigg\{\int_{-\infty}^{t} dt' e^{-i\omega^*(t-t')}\left(\vec{E}^*(\vec{r}')\cdot\vec{v}'\right) \\ \int_{-\infty}^{t'} dt'' e^{i\omega(t-t'')}\left[\vec{E}(\vec{r}'')\cdot\left(\vec{v}''-\frac{v^2_{th\alpha}}{2\Omega_{c\alpha}}\vec{b}\times\vec{\nabla}\log F_{M\alpha}\right)\right]\Bigg\}\end{aligned} \tag{5}$$

where $\vec{b} = \vec{B}_o/B_o$, $F_{M\alpha}$ is the Maxwell distribution, and $\vec{r}_g = \vec{r} + \left(\vec{v}\times\vec{b}\right)/\Omega_{c\alpha}$ is the particle guiding center. The second term in the $t''$ integral is easily recognised as the one responsible for low frequency drift instabilities ($\omega << \Omega_c$); in the ICR frequency range it can be consistently neglected. An integration by parts then allows to recast $P^{\alpha}_{abs}$ in the form:

$$P^{\alpha}_{abs} = \frac{\omega}{8\pi}\frac{\omega^2_{p\alpha}}{\omega^2}\int d\vec{v}\, F_{M\alpha}(v^2,\vec{r}_g)\lim_{\gamma\to 0}\frac{2\gamma}{\omega}\left|-i\omega\int_0^{\infty} d\tau e^{i\omega\tau}\left(\vec{E}(\vec{r}')\cdot\frac{\vec{v}'}{v_{th\alpha}}\right)\right|^2 \tag{6}$$

Here the integrand is positive defined: in this approximation *energy flows always from the waves to the particles at each point of space.* This result seems to contraddict recent analysis of the generalised Plasma Dispersion Function in tokamak geometry [8], [9].

The difficulty is easily removed by realising that the locally negative contibutions to $\langle \vec{J} \cdot \vec{E} \rangle$ found by these Authors belong to div$\vec{T}$ rather than to $P^{\alpha}_{abs}$.

Although physically plausible, the above expression for $P^{\alpha}_{abs}$ is only acceptable if it is in agreement with the formal conservation theorem of Maxwell-Vlasov equations. To prove that this is the case it is necessary and sufficient to show that the difference between $P^{\alpha}_{abs}$ and $\langle \vec{J}_\alpha \cdot \vec{E} \rangle$ is the divergence of a vector. Indeed it can be shown that [10]

$$\langle \vec{J}_\alpha \cdot \vec{E} \rangle = P^{\alpha}_{abs} + \mathrm{div}\vec{T}^{\alpha} \tag{7}$$

with

$$\vec{T}^{\alpha} = \frac{1}{8\pi} \frac{\omega^2_{p\alpha}}{\omega^2} \int d\vec{v}\, \vec{v} F_{M\alpha}(v^2, \vec{r}_g) \left| -i\omega \int_0^\infty d\tau e^{i\omega\tau} \left\{ \vec{E}(\vec{r}') \cdot \frac{\vec{v}'}{v_{th\alpha}} \right\} \right|^2 \tag{8}$$

3. - It is instructive to apply Eq. (6) in the homogeneous limit. In this case the most general time-harmonic field can be written

$$\vec{E}(\vec{r}, t) = \int dk_{\parallel} \int d\psi\, \vec{E}(k_{\parallel}, \psi) e^{-i\left[k_{\parallel} z + k_{\perp}(x \cos\psi + y \sin\psi) - \omega t\right]} \tag{9}$$

where $k_\perp = k_\perp(\omega, k_\parallel)$ has to satisfy the dispersion relation, but does not depend on $\psi = \mathrm{atan}(k_y/k_x)$; parallel and perpendicular refer to the direction of the static magnetic field. Eq. (6) then becomes:

$$\begin{aligned} P^{\alpha}_{abs} = \frac{\omega}{8\pi} \frac{\omega^2_{p\alpha}}{\omega^2} \int dk_{\parallel} \int dk'_{\parallel} \int d\psi \int d\psi'\ \mathrm{Re}\Bigg\{ e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} \\ \int d\vec{v} F_{M\alpha}(v^2) \mathrm{exp} i \left[ \frac{k_\perp v_\perp}{\Omega_{c\alpha}} \sin(\phi+\psi) - \frac{k'_\perp v_\perp}{\Omega_{c\alpha}} \sin(\phi+\psi') \right] \\ \lim_{\gamma\to 0} \frac{2\gamma}{\omega} \left[ \sum_n \frac{\omega}{\omega - i\gamma - n\Omega_{c\alpha} - k_\parallel v_\parallel} \left( \vec{V}^{\alpha}_n(\vec{k}) \cdot \vec{E}(k_\parallel, \psi) \right) \right] \cdot \\ \left[ \sum_{n'} \frac{\omega}{\omega + i\gamma - n'\Omega_{c\alpha} - k'_\parallel v_\parallel} \left( \vec{V}^{\alpha}_{n'}(\vec{k}') \cdot \vec{E}^*(k'_\parallel, \psi') \right) \right] \Bigg\} \end{aligned} \tag{10}$$

with

$$\vec{V}^{\alpha}_n \cdot \vec{E} = v_\perp \left[ E_x \frac{n\Omega_{c\alpha}}{k_\perp v_\perp} J_n\left(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}\right) + iE_y J'_n\left(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}\right) \right] + v_\parallel E_z J_n\left(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}\right) \tag{11}$$

Now it is not difficult to show that

$$\begin{aligned} \lim_{\gamma\to 0} \frac{2\gamma}{\omega} \left\{ \frac{\omega}{\omega - i\gamma - n\Omega_{c\alpha} - k_\parallel v_\parallel} \right\} \left\{ \frac{\omega}{\omega + i\gamma - n'\Omega_{c\alpha} - k'_\parallel v_\parallel} \right\} = \\ 2\pi\delta\left( \frac{\omega - n\Omega_{c\alpha} - k_\parallel v_\parallel}{\omega} \right) \delta\left(k_\parallel - k'_\parallel\right) \delta_{n,n'} \end{aligned} \tag{12}$$

so that finally

$$P^{\alpha}_{abs} = \frac{\omega}{8\pi}\frac{\omega^2_{p\alpha}}{\omega^2}\int dk_\parallel \int d\psi \int d\vec{v}\, F_{M\alpha}(v^2) \sum_n 2\pi\delta\left(\frac{\omega - n\Omega_{c\alpha} - k_\parallel v_\parallel}{\omega}\right) |\vec{V}^{\alpha}_n \cdot \vec{E}|^2 \quad (13)$$

According to this equation, $P^{\alpha}_{abs}$ has two remarkable properties (not independent from each other): *a) it is constant along the static magnetic field*; and *b) it is the sum of independent contributions from each partial wave*. Both properties follow from the fact that $P^{\alpha}_{abs}$ bears no memory of phase relations between partial waves. They can easily be shown to hold also for the FLR wave equations in a plane-stratified plasma.

By contrast, $\langle\vec{J}\cdot\vec{E}\rangle$ is generally a function of $z$, showing the interference between partial waves normally to be expected for quadratic forms in the field amplitude, and is not positive definite. The averaging along $\vec{B}_o$ leading to (13) is clearly due to the free streaming of charged particles, and this equation can only be valid if the collisional mean free path is large compared with the parallel wavelength of all partial waves, as assumed above. On the other hand, the wave must have a stationary amplitude for times longer than the collision time: for shorter transients the distinction between $P^{\alpha}_{abs}$ and div$\vec{T}$ loses sharpness, as indicated by the existence of echo effects in weakly collisional plasmas.

Finally the case of a plane wave in a homogeneous plasma is recovered by narrowing the spectral width of the wavepacket (9) so that

$$|\vec{E}(k_\parallel, \psi)|^2 \rightarrow |\vec{E}_0|^2 \delta\left(k_\parallel - k_{\parallel 0}\right) \delta\left(\psi - \psi_0\right) \quad (14)$$

It is straightforword to check that in this limit (13) reduces to (1), as it should.

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