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Spontaneously Broken Yang-Mills-Einstein Supergravities as Double Copies

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Abstract

Color/kinematics duality and the double-copy construction have proved to be systematic tools for gaining new insight into gravitational theories. Extending our earlier work, in this paper we introduce new double-copy constructions for large classes of spontaneously-broken Yang-Mills-Einstein theories with adjoint Higgs fields. One gauge-theory copy entering the construction is a spontaneously-broken (super-)Yang-Mills theory, while the other copy is a bosonic Yang-Mills-scalar theory with trilinear scalar interactions that display an explicitly-broken global symmetry. We show that the kinematic numerators of these gauge theories can be made to obey color/kinematics duality by exhibiting particular additional Lie-algebraic relations. We discuss in detail explicit examples with $\mathcal{N} = 2$ supersymmetry, focusing on Yang-Mills-Einstein supergravity theories belonging to the generic Jordan family in four and five dimensions, and identify the map between the supergravity and double-copy fields and parameters. We also briefly discuss the application of our results to $\mathcal{N} = 4$ supergravity theories. The constructions are illustrated by explicit examples of tree-level and one-loop scattering amplitudes.

Contents

1 Introduction

Einstein's theory of gravity and spontaneously-broken gauge theory are two of the pillars of our current understanding of the known fundamental interactions of Nature. While supersymmetric field theories that combine gravitational interactions and spontaneous symmetry breaking have been studied extensively at the Lagrangian level, the perturbative S matrices of these theories have largely been unexplored.

Modern work on scattering amplitudes in matter-coupled gravitational theories has been largely focused on pure supergravities and on cases in which additional matter consists of abelian vectors (i.e. Maxwell-Einstein supergravities) or fermion/scalar fields. A key tool has been the double-copy construction [1, 2], which has led to a dramatic simplification of perturbative calculations. For example, explicit expressions of one-, two-, three- and four-loop amplitudes have been obtained for $\mathcal{N} = 4$, $\mathcal{N} = 5$ and $\mathcal{N} = 8$ supergravities in refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. For the case of $\mathcal{N} \leq 4$, one-loop four-point superamplitudes have been obtained for the generic Jordan family of $\mathcal{N} = 2$ Maxwell-Einstein supergravity (MESG) theories [15, 16, 17], for pure supergravities with $\mathcal{N} \leq 4$ [15, 18, 17], and for orbifolds thereof [15, 19].

The double-copy construction assumes the existence of presentations of gauge-theory scattering amplitudes that exhibit color/kinematics duality. The duality states that, in an amplitude's Feynman-like diagrammatic expansion, one can find numerator factors that obey Lie-algebraic kinematic relations mirroring the relations satisfied by the corresponding gauge-group color factors. Once found, the numerators may play the role of these color factors in any gauge theory amplitude, and upon substitution one obtains valid gravitational amplitudes. There is by now extensive evidence for the duality and for the double-copy construction in wide classes of Yang-Mills (YM) theories and in the associated (super)gravity theories. Examples where color/kinematics duality has been demonstrated include: pure super-Yang-Mills (SYM) theories [1, 2, 20, 21, 22], SYM theories with adjoint matter [15, 18, 17], self-dual Yang-Mills theory [23, 24], QCD and super-QCD [25, 26], YM coupled to ϕ^3 theory [27], and YM theory extended by a higher-dimensional operator [28]. It has also been observed that the duality is not limited to YM gauge theories, but it also applies to certain Chern-Simons-matter theories [29, 30, 31], as well as to the non-linear sigma model/chiral Lagrangian [32] and to the closed (heterotic) sector of string theory [33].

Amplitudes in Maxwell-Einstein supergravities are obtained by a double-copy construction of the form (pure SYM)$\otimes$(YM coupled to scalars). Subgroups of the global symmetries of Maxwell-Einstein supergravities can be gauged.[1] In the resulting theories some of the vector fields become gauge fields of the chosen gauge group and transform in its adjoint representation. Therefore, the only subgroups of the global symmetry that can be gauged are

[1]While gauging part of the R-symmetry group is very interesting, here we will focus on gaugings that only affect the other global symmetries.

those whose adjoint representation is smaller than the number of vector fields that transform non-trivially under the global symmetry group. In five dimensions, gauging only a subgroup of the global symmetry group in $\mathcal{N} = 2$ Maxwell-Einstein supergravity theories does not introduce a potential for the scalar fields and hence the resulting theory is guaranteed to have a Minkowski vacuum state.

The double-copy construction of a wide class of Yang-Mills-Einstein supergravity (YMESG) theories was given in [27], where it was shown that one of the two gauge-theory factors is a pure SYM theory, and the other is a bosonic YM theory coupled to scalars that transforms in the adjoint representation of both the gauge group and a global symmetry group. The latter theory has trilinear ϕ^3 couplings, and hence we refer to it as YM + ϕ^3 theory. Through the double-copy construction, the global symmetry of the non-supersymmetric gauge-theory factor becomes a local symmetry, and the trilinear scalar couplings generate the minimal couplings of the corresponding gauge fields. The gravitational supersymmetry is directly inherited from the SYM theory, thus accommodating $\mathcal{N} = 1, 2, 4$ YMESG theories and $\mathcal{N} = 0$ Yang-Mills-Einstein (YME) theories. Earlier work [34] introduced the same type of construction for single-trace tree-level YME amplitudes. Recent work on YME amplitudes takes several different approaches, see refs. [35, 36, 37, 38, 39, 40, 41].

It is essential to explore the validity of the double-copy construction away from the origin of the moduli space. In particular, a natural and physically-motivated extension is to consider cases in which the supergravity gauge symmetry is spontaneously broken through the Brout-Englert-Higgs mechanism. We will present such an extension in the present paper. As a key result, we find that one of the two gauge-theory factors is the spontaneously-broken pure SYM theory (or, alternatively stated, the Coulomb branch of pure SYM theory), while the other is a particular massive deformation that explicitly breaks the global symmetries of the YM + ϕ^3 theory.

Identifying the relation between asymptotic states of the supergravity theory and the corresponding states of the gauge-theory factors is an important aspect of the double-copy construction. For gauge theories with only adjoint fields, the double copy gives a supergravity state for every tensor product of gauge-theory states (not counting the degeneracy of the representation). In cases in which the gauge-theory matter transforms in non-adjoint representations of the gauge group, the double-copy construction allows for better tuning of the matter content of the gravitational theory, since only certain tensor products of the gauge-theory matter are allowed.

In ref. [19] color/kinematics duality was extended to non-adjoint representations in the context of orbifolds of $\mathcal{N} = 4$ SYM, and the associated double copies were found to be matter-coupled supergravity theories. The construction required that: (1) the gauge groups of the two gauge theories should be identified, and (2) supergravity states correspond to gauge-invariant bilinears that can be formed out of the gauge-theory states. This construction correlates gauge- and global-group representations appearing in the resulting gauge theories.

In ref. [17] color/kinematics duality was extended to theories with fields in the fundamental representation and used to construct pure $\mathcal{N} \leq 4$ supergravity theories as well as matter-coupled theories. In this construction, the necessary condition for the double copy to be valid is that the kinematic matter-dependent numerators obey the same relations as the corresponding color factors with fundamental representations. Upon replacing the color factors with kinematic numerator factors one similarly obtains a double copy that correlates the representations of the states of the two gauge-theory sides.

For the double-copy constructions of supergravity theories with spontaneously-broken gauge symmetry, the identification of the asymptotic states will follow closely the non-adjoint or fundamental cases. However, the details of the kinematic algebra obeyed by the numerators will differ substantially compared to previous situations. The kinematic Jacobi identities and commutation relations will be extended by additional identities which are inherited from the Jacobi relations of the theory with unbroken gauge symmetry. We stress that our construction works well with – but does not require – supersymmetry, and similarly works in all dimensions in which the theories are defined, as it is expected for color/kinematics duality.

The paper is organized as follows. In section 2 we review color/kinematics duality, and identify matter-coupled gauge theories with fields in several different representations of the gauge group and specific cubic and quartic couplings which obey the duality. We extend color/kinematics duality and the double-copy construction to massive field theories, as well as to field theories with spontaneously-broken gauge symmetry, paying close attention to the construction of asymptotic states. In particular subsection 2.6 discusses extensions of the double-copy construction and contains our main results of this generalization.

In section 3 we review, from the Lagrangian perspective, the Higgs mechanism in four- and five-dimensional $\mathcal{N} = 2$ Yang-Mills-Einstein supergravities. Such theories are uniquely specified by their cubic interactions and provide simple examples of our construction. In particular, we identify the four-dimensional symplectic frame in which the amplitudes from the spontaneously-broken Yang-Mills-Einstein supergravity Lagrangian reproduce the ones from the double-copy construction.

In section 4, we compute tree-level scattering amplitudes in the gauge theories discussed in section 2 and in the supergravity theories discussed in section 3. We find the constraints imposed by color/kinematics duality on the cubic and quartic couplings of the gauge theories, identify the precise map between supergravity states and gauge-invariant billinears of gauge-theory states, and give the relation between the gauge-theory and supergravity parameters.

In section 5, we discuss loop-level calculations in theories formulated in the earlier sections. Section 6, discusses briefly spontaneously-broken $\mathcal{N} = 4$ Yang-Mills-Einstein supergravity theories. We review the bosonic part of their Lagrangians in five dimensions and discuss how their amplitudes can be obtained through the double-copy construction with a straightforward extension of the results obtained for $\mathcal{N} = 2$ theories.

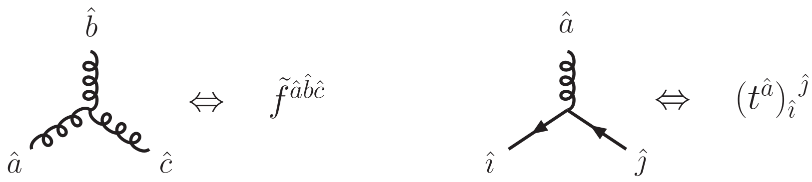

Figure 1: The two cubic types of interactions for fields in adjoint representation and a generic complex representation. We organize the amplitudes around cubic graphs with these two types of vertices, and the corresponding color factors are contractions of the structure constants and the generators.

2 Color/kinematics duality and double copy

In this section, we review the color/kinematics duality applied to gauge theories that have fields in complex representations of the gauge group. Giving concrete examples, we write down Lagrangians of several gauge theories where the duality should be present. We then spontaneously (and explicitly) break the symmetries of these theories, and in the process generalize color/kinematics duality to such situations. Finally, we give the double-copy prescription for spontaneously- and explicitly-broken theories.

2.1 Review: color/kinematics duality for complex representations

The scattering amplitudes in a gauge theory with fields in both the adjoint representation and some generic complex representation[2] U of a Lie group can be organized in terms of cubic graphs.[3] At L loops and in D dimensions, such amplitude has the following form[4]

$$\mathcal{A}_n^{(L)} = i^{L-1} g^{n-2+2L} \sum_{i\in\text{cubic}} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{c_i n_i}{D_i}\,, \tag{2.1}$$

where c_i are color factors, n_i are kinematic numerators and D_i are denominators encoding the propagator structure of the cubic graphs. The denominators may contain masses, corresponding to massive fields in the representation U. The S_i are standard symmetry factors that also appear in Feynman loop diagrams.

The cubic form (2.1) directly follows the organization of the color factors c_i, which are constructed from two cubic building blocks. These are the structure constants $\tilde{f}^{\hat{a}\hat{b}\hat{c}}$ for

[2]By generic complex representation, we mean a representation that only has quadratic and cubic invariants $\overline{U}U$, and $\overline{U}\,(\text{Adj})\,U$, respectively. A canonical example of such an U is the fundamental representation.

[3]Quartic and higher-degree interactions are absorbed into the numerators of the cubic graphs. This corresponds to having introduced suitably-chosen auxiliary fields to make the Lagrangian cubic.

[4]We use a different numerator normalization compared to ref. [2]. Relative to that work, we absorb one factor of i into the numerator, giving a uniform overall i^{L-1} to the gauge and gravity amplitudes.

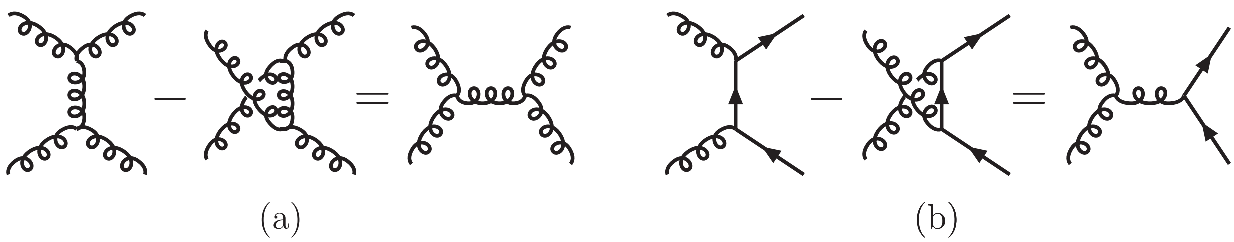

Figure 2: Pictorial form of the basic color and kinematic Lie-algebraic relations: (a) the Jacobi relations for fields in the adjoint representation, and (b) the commutation relation for fields in a generic complex representation.

vertices linking three adjoint fields and the generators $(t^{\hat{a}})_{\hat{\imath}}{}^{\hat{\jmath}}$ for the U-$\overline{U}$-adjoint vertices, as shown in figure 1. When isolating color from kinematics, the crossing symmetry of a vertex only holds up to signs dependent on the signature of the permutation. These signs are apparent in the total antisymmetry of $\tilde{f}^{\hat{a}\hat{b}\hat{c}}$ and may be made uniform by defining the generators in the representation U to have a similar antisymmetry:

$$(t^{\hat{a}})^{\hat{\jmath}}{}_{\hat{\imath}} \equiv -(t^{\hat{a}})_{\hat{\imath}}{}^{\hat{\jmath}} \quad \Leftrightarrow \quad \tilde{f}^{\hat{c}\hat{a}\hat{b}} = -\tilde{f}^{\hat{b}\hat{a}\hat{c}} \, . \tag{2.2}$$

The effect of such a relabeling is that any color factor picks a minus sign, $c_i \to -c_i$, under the permutation of any two graph edges meeting at a vertex.

The color factors obey simple linear relations arising from the Jacobi identities and commutation relations of the gauge group,

$$\left.\begin{array}{r} \tilde{f}^{\hat{d}\hat{a}\hat{c}}\tilde{f}^{\hat{c}\hat{b}\hat{e}} - \tilde{f}^{\hat{d}\hat{b}\hat{c}}\tilde{f}^{\hat{c}\hat{a}\hat{e}} = \tilde{f}^{\hat{a}\hat{b}\hat{c}}\tilde{f}^{\hat{d}\hat{c}\hat{e}} \\ (t^{\hat{a}})_{\hat{\imath}}{}^{\hat{k}}\,(t^{\hat{b}})_{\hat{k}}{}^{\hat{\jmath}} - (t^{\hat{b}})_{\hat{\imath}}{}^{\hat{k}}\,(t^{\hat{a}})_{\hat{k}}{}^{\hat{\jmath}} = \tilde{f}^{\hat{a}\hat{b}\hat{c}}\,(t^{\hat{c}})_{\hat{\imath}}{}^{\hat{\jmath}} \end{array}\right\} \quad \Rightarrow \quad c_i - c_j = c_k \ ; \tag{2.3}$$

these relations are shown diagrammatically in figure 2. The identity $c_i - c_j = c_k$ is understood to hold for triplets of diagrams (i, j, k) that differ only by the subgraphs in figure 2 and otherwise have common graph structure. The linear relations among the color factors c_i imply that the corresponding kinematic parts of the graphs, n_i/D_i, are in general not unique. This should be expected, given that individual (Feynman) diagrams are gauge-dependent quantities.

It was observed by Bern, Carrasco and one of the current authors (BCJ) [1, 2] that, within the gauge freedom of individual graphs, there exist particularly nice amplitude presentations that make the kinematic numerator factors n_i obey the same general algebraic identities as the color factors c_i. In the present context, this implies that there is a numerator relation for every color Jacobi or commutation relation (2.3) and a numerator sign flip for every color factor sign flip (2.2):

$$\begin{aligned} n_i - n_j = n_k \qquad &\Leftrightarrow \qquad c_i - c_j = c_k \, , \\ n_i \to -n_i \qquad &\Leftrightarrow \qquad c_i \to -c_i \, . \end{aligned} \tag{2.4}$$

In a more general context, there could exist color identities beyond the Jacobi or commutation relation, which would justify the introduction of corresponding kinematic numerator identities. Indeed, we will encounter this in section 2.4 after introducing additional (bi-fundamental) complex representations of the gauge group.

Amplitudes built out of numerators that satisfy the same general identities as the color factors are said to exhibit color/kinematics duality manifestly. Theories whose amplitudes can be presented in a form that exhibits this property are said to obey the color/kinematics duality.

It is interesting to note that eq. (2.4) defines a kinematic algebra in terms of the numerators, which suggests the existence of an underlying Lie algebra. While not much is known about this kinematic Lie algebra, it should be infinite-dimensional due to the momentum-dependence of the numerators. In the restricted case of self-dual YM theory the kinematic algebra has been shown to be isomorphic to that of the area-preserving diffeomorphisms [23] (see also ref. [42]).

A central aspect of the color/kinematics duality is that, once numerators have been found to obey the duality, they can replace the color factors in eq. (2.1). This gives a double-copy construction for amplitudes of the form

$$\mathcal{M}_n^{(L)} = i^{L-1}\left(\frac{\kappa}{2}\right)^{n-2+2L} \sum_{i\in\text{cubic}} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{D_i}\,, \tag{2.5}$$

which describe scattering in a gravitational theory.[5] The tilde notation is necessary since the two copies of numerators may not be identical. The two sets of numerators entering the double-copy construction may belong to different gauge theories, and at most one set is required to manifestly obey the duality [1, 2].

While the double copy discussed here strictly applies to the construction of a gravitational amplitude using the scattering amplitudes of two gauge theory as building blocks, it is often convenient the shorten the description using the notation gravity = gauge $\otimes$ $\widetilde{\text{gauge}}$. This emphasizes the tensor structure of the asymptotic states of the double copy, and at the same time gives essential information about the theories that enter the construction. The notation is also motivated by the observations that the double copy appears to have extensions beyond perturbation theory [23, 43, 44, 45].

As examples of double copies, we note that pure Yang-Mills theory "squares" to gravity coupled to a dilaton and a two-index anti-symmetric tensor: GR + ϕ + $B^{\mu\nu}$ = YM $\otimes$ YM [46, 47]. Pure Einstein gravity may be obtained by removing these extra particles via a ghost-like double-copy prescription for massless quarks [17]. An asymmetrical double copy, YM $\otimes$ (YM + ϕ^3), is needed for the amplitudes that couple Yang-Mills theory to gravity [27]. For the double copies of YM theories with matter in a complex representation U, such as described in eq. (2.5), one obtains amplitudes that involve gravitons, dilatons,

[5]If vector contributions are absent in either n_i or $\tilde{n}_i$, then eq. (2.5) describes a non-gravitational sector.

two-index antisymmetric tensors and matter fields [17]. In supersymmetric extensions of these theories, superamplitudes are labeled by the corresponding supermultiplets; the tensor product of two supermultiplets is typically reducible to a sum of smaller multiplets of the resulting supersymmetry algebra.

While the color/kinematics duality has a conjectural status at loop level, amplitudes up to four loops for diverse theories (with and without additional matter) have been explicitly constructed in forms consistent with the duality and the double copy [2, 3, 4, 6, 15, 24, 10, 48, 18, 49, 19, 17, 11, 50, 51, 52].

At tree level, the double-copy construction restricted to fields in the adjoint representation is known [1, 53] to be equivalent to the field-theory limit of the Kawai-Lewellen-Tye (KLT) relations [46, 47] between open- and closed-string amplitudes. Color/kinematics duality has been used to derive a number of impressive results for string-theory amplitudes [20, 21, 54, 55, 56, 57, 33]; more generally, the duality combined with string-theory methods provides powerful new tools for field theory [22, 16, 58, 59, 60, 13, 14, 61, 62]. Recently, the double-copy construction has been extended to express certain Kerr-Schild-type solutions of general relativity in terms of classical solutions of the Yang-Mills equations of motion [44, 45]. The duality implies the BCJ amplitude relations [1] that limit the number of independent tree amplitudes to $(n-3)!$ in the purely adjoint case, and otherwise to $(n-3)!(2k-2)/k!$ when $k > 1$ fundamental-antifundamental pairs are present [25]. The BCJ amplitude relations have a close connection to the scattering equations and to the associated string-like formulae for gauge and gravity tree amplitudes [63, 64, 65, 66, 67, 68, 69, 70, 71, 72].

2.2 Scalar ϕ^3 theories

As a warm-up exercise, consider a simple scalar model that exhibits the properties described in the previous section where all fields transform either in the adjoint of a group G_c or in a generic complex representation U of this group (and corresponding conjugate $\overline{U}$).

Suppressing all G_c indices, assume we have a family of real massless scalars transforming in the adjoint representation, labeled as ϕ^a. And, similarly, a family of identical-mass complex scalars transforming in the U ($\overline{U}$) representation, labeled as φ_i ($\overline{\varphi}^i$). For a scalar theory with at most cubic interactions the Lagrangian is then[6]

$$g^2\,\mathcal{L}_{\text{scalar}} = \operatorname{Tr}\Big(\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a+\frac{i}{3!}\lambda\,F^{abc}[\phi^a,\phi^b]\phi^c\Big)+\partial_\mu\overline{\varphi}^i\partial^\mu\varphi_i-m^2\overline{\varphi}^i\varphi_i+\lambda\,T^{a}_{\;i}{}^{j}\,(\overline{\varphi}^i\phi^a\varphi_j)\,. \tag{2.6}$$

Note that the indices $a, b, c, \ldots$ and $i, j, \ldots$ are not G_c indices, but rather labels that distinguish fields in the same representation (see appendix A for a summary of notation). The coefficients F^{abc} and $T^{a}_{\;j}{}^{i}$ are arbitrary couplings between these fields, and λ is a dimension-

[6]Scalar and gauge-theory Lagrangians are written in mostly-minus spacetime signature, whereas gravity Lagrangians use mostly-plus signature.

one constant (in four dimensions) such that all terms in $\mathcal{L}_{\text{scalar}}$ have uniform dimension. For later convenience we have also introduced a dimensionless coupling g.

Denoting by $(t^{\hat{a}})_{\hat{\imath}}{}^{\hat{\jmath}}$ the generators[7] of G_c in the representation U and expressing the adjoint fields as $\phi^a = t^{\hat{a}}\,\phi^{a\hat{a}}$, a more explicit form of the Lagrangian can be obtained,

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2}\partial_\mu\phi^{a\hat{a}}\partial^\mu\phi^{a\hat{a}} + \frac{1}{3!}g\lambda F^{abc}f^{\hat{a}\hat{b}\hat{c}}\phi^{a\hat{a}}\phi^{b\hat{b}}\phi^{c\hat{c}} + \partial_\mu\overline{\varphi}^i\partial^\mu\varphi_i - \overline{\varphi}^i m^2\varphi_i + g\lambda T^a_{\ i}{}^{j}\phi^{a\hat{a}}\,\overline{\varphi}^i t^{\hat{a}}\varphi_j\ . \tag{2.7}$$

Here $f^{\hat{a}\hat{b}\hat{c}} = -i\text{Tr}([t^{\hat{a}}, t^{\hat{b}}]t^{\hat{c}})$ are the structure constants of the group G_c, the coupling constant g has been moved to the cubic interactions via the redefinition $\phi \to g\phi$, $\varphi \to g\varphi$, and the indices of the complex representation U remain suppressed.

The symmetry G_c can be gauged, as we will do in the next section. Even before gauging, scattering amplitudes from $\mathcal{L}_{\text{scalar}}$ have the same form as eq. (2.1), with the coefficients c_i given in terms of the generators and structure constants of G_c. Anticipating the gauging of G_c we can constrain the Lagrangian (2.7) such that amplitudes expressed in this form have numerators n_i that obey the duality (2.4), in one-to-one correspondence with those obeyed by the group-theoretic factors c_i. This simple theory has no derivative couplings, and therefore the numerator factors n_i have no momentum dependence, they are only built out of the couplings F^{abc} and $T^a_{\ i}{}^{j}$. An inspection of the Lagrangian shows that the duality holds if the couplings are in one-to-one correspondence with the structure constants and generators of G_c,

$$F^{abc} \ \Leftrightarrow\ f^{\hat{a}\hat{b}\hat{c}} \qquad \text{and} \qquad T^a_{\ i}{}^{j} \ \Leftrightarrow\ (t^{\hat{a}})_{\hat{\imath}}{}^{\hat{\jmath}}\,, \tag{2.8}$$

in the sense that the pair (F^{abc}, T^a) obeys the same general algebraic relations as $(f^{\hat{a}\hat{b}\hat{c}}, t^{\hat{a}})$.

This implies that

1. $(T^a)_i{}^j \equiv T^a_{\ i}{}^{j}$ are the generators of a generic complex representation U' of a "kinematic" Lie algebra[8] of some group G_k. They can be taken to be normalized as $\text{Tr}(T^aT^b) = \delta^{ab}$.

2. F^{abc} are the structure constants of that algebra; given by $F^{abc} = -i\text{Tr}([T^a, T^b]T^c)$.

3. The ranges of indices $a, b, c, \ldots$ and $i, j, k, \ldots$ are the dimensions of the adjoint representation of G_k and its representation U', respectively.

The resulting theory describes a $G_c \otimes G_k$ invariant scalar field theory, with massless scalars $\phi^{a\hat{a}}$ in the "bi-adjoint" representation and massive complex scalar fields $\varphi^{i\hat{\imath}}$ in the representation $U \otimes U'$. This is one of the simplest realizations of a theory that exhibits a duality of the type described in section 2.1 which is manifest in the Lagrangian.

[7]We normalize the generators as $\text{Tr}(t^{\hat{a}}t^{\hat{b}}) = \delta^{\hat{a}\hat{b}}$.

[8]This name is convenient because, once the G_c symmetry is gauged, the Lie algebra of G_k becomes a subalgebra of the full kinematic algebra obeyed by the numerator factors.

Note that it is straightforward to modify the mass spectrum of the theory while preserving the duality. If the G_c representation U and/or the G_k representation U' are reducible, the mass m in eq. (2.7) can carry labels identifying the irreducible components of U and U'. Hence, the $U \otimes U'$ representations can be decomposed into irreps of $G_c \otimes G_k$, each with a different mass term in the Lagrangian.

As a concrete example of this generalization, take the kinematic algebra to be $G_k = SU(N_k)$, and let the representation U' be N_f copies of the fundamental representation; these copies are labeled by the *flavor* indices $m, n = 1, \ldots, N_f$. Next take $G_c = SU(N_c)$, and let the representation U be its fundamental representation. With these choices, the scalar theory takes the form

$$\begin{aligned}\mathcal{L}'_{\text{scalar}} \;&=\; \frac{1}{2}\partial_\mu \phi^{a\hat{a}} \partial^\mu \phi^{a\hat{a}} + \frac{1}{3!} g\lambda F^{abc} f^{\hat{a}\hat{b}\hat{c}} \phi^{a\hat{a}} \phi^{b\hat{b}} \phi^{c\hat{c}} \\ &\quad + \partial_\mu \overline{\varphi}^{im} \partial^\mu \varphi_{im} - (m^2)_m{}^n \overline{\varphi}^{im} \varphi_{in} + g\lambda T^a{}_i{}^j \phi^{a\hat{a}} \overline{\varphi}^{im} t^{\hat{a}} \varphi_{jm} \;,\end{aligned} \tag{2.9}$$

where $t^{\hat{a}}$ and T^a are generators in the fundamental representation of respective group. The fundamental SU(N_c) indices $\hat{\imath}, \hat{\jmath}$ are not shown explicitly. The mass matrix is assumed to be diagonalized, $m_m{}^n = \delta_m{}^n m_n$ (no sum), corresponding to the mass eigenstates: $\varphi_{i\hat{\imath}n}$ and $\overline{\varphi}^{i\hat{\imath}n}$. In the limit that $m_n \to 0$ (or $m_n \to m$) this theory has $SU(N_c) \times SU(N_k) \times SU(N_f)$ symmetry, where $SU(N_f)$ is the flavor group. For generic m_n the flavor group is explicitly broken to $SU(N_f) \to U(1)^{N_f}$. The case $N_f = 0$ is that of the pure bi-adjoint ϕ^3 theory,

$$\mathcal{L}_{\phi^3} = \frac{1}{2}\partial_\mu \phi^{a\hat{a}} \partial^\mu \phi^{a\hat{a}} + \frac{1}{3!} g\lambda F^{abc} f^{\hat{a}\hat{b}\hat{c}} \phi^{a\hat{a}} \phi^{b\hat{b}} \phi^{c\hat{c}} \;, \tag{2.10}$$

which was identified in refs. [34, 27] to be useful for obtaining amplitudes in gravity theories coupled to non-abelian gauge fields with $SU(N_k)$ symmetry.[9] See also refs. [73, 74] for other applications of this theory in the context of color/kinematics duality.

2.3 Yang-Mills-scalar theories: gauging G_c

Let us now gauge the symmetry group G_c and include the self-interactions of the corresponding non-abelian gauge fields. In eq. (2.6) we may replace all derivatives by covariant derivatives in the representation U, $\partial_\mu \to D_\mu$, and add the standard pure-Yang-Mills Lagrangian with gauge group G_c.

Gauging the G_c symmetry is not sufficient for the resulting theory to obey color/kinematics duality; indeed, it is known from the $N_f = 0$ case [27, 75] as well as from the case of fundamental and orbifold field theories [19, 17] that quartic scalar terms like ϕ^4, $\phi^2\overline{\varphi}\varphi$ and $(\overline{\varphi}\varphi)^2$ are required. For the particular theories discussed in this subsection, color/kinematics duality will uniquely dictate the ϕ^4 and $\phi^2\overline{\varphi}\varphi$ terms, whereas all terms of $(\overline{\varphi}\varphi)^2$ type will be

[9] Compared to the notation used in ref. [27], we have renamed the two couplings: $\mathrm{g} \to g$, $g' \to \lambda$.

unconstrained. However, if $\overline{\varphi}$ and φ are in special complex representations for which the color factors obey extra identities, then the $(\overline{\varphi}\varphi)^2$ terms may be constrained by color/kinematics duality. We will see that these special representations include the ones arising from the spontaneous symmetry breaking of a larger gauge group.

In ref. [27] we showed that the specific ϕ^4 term that is consistent with color/kinematics duality is

$$\mathcal{L}_{\phi^4} = -\frac{g^2}{4} f^{\hat{a}\hat{b}\hat{e}} f^{\hat{e}\hat{c}\hat{d}} \phi^{a\hat{a}} \phi^{b\hat{b}} \phi^{a\hat{c}} \phi^{b\hat{d}} \ . \tag{2.11}$$

In section 4 we will compute four-point amplitudes in the YM-scalar theories and see that they obey color/kinematics duality only if the Lagrangian also contains the term

$$\mathcal{L}_{\phi^2\overline{\varphi}\varphi} = -g^2 \phi^{a\hat{a}} \phi^{a\hat{b}} \overline{\varphi}^i t^{\hat{a}} t^{\hat{b}} \varphi_i \ . \tag{2.12}$$

There are several terms involving four fields in complex representations that can in principle be freely added; we find that the combination

$$\mathcal{L}_{(\overline{\varphi}\varphi)^2} = -g^2 \overline{\varphi}^i t^{\hat{a}} \varphi_j \, \overline{\varphi}^j t^{\hat{a}} \varphi_i + \frac{g^2}{2} \overline{\varphi}^i t^{\hat{a}} \varphi_i \, \overline{\varphi}^j t^{\hat{a}} \varphi_j \tag{2.13}$$

is particularly natural as it is in a certain sense (discussed in section 2.5.1) the complex generalization of the adjoint contact term (2.11).

Thus, the Lagrangian with local symmetry G_c and global symmetry G_k, giving Yang-Mills theory coupled to scalar fields, takes the following form:

$$\mathcal{L}_{\text{YM+scalar}} = -\frac{1}{4} F^{\hat{a}}_{\mu\nu} F^{\mu\nu\hat{a}} + \mathcal{L}_{\text{scalar}}\Big|_{\partial\to D} + \mathcal{L}_{\phi^4} + \mathcal{L}_{\phi^2\overline{\varphi}\varphi} + \mathcal{L}_{(\overline{\varphi}\varphi)^2} \ . \tag{2.14}$$

For the particular choices of groups and representations that led to the theory (2.9), the Lagrangian is

$$\begin{aligned}
\mathcal{L}'_{\text{YM+scalar}} \quad = \quad & -\frac{1}{4} F^{\hat{a}}_{\mu\nu} F^{\mu\nu\hat{a}} + \frac{1}{2} (D_\mu \phi^a)^{\hat{a}} (D^\mu \phi^a)^{\hat{a}} + \frac{1}{3!} g\lambda F^{abc} f^{\hat{a}\hat{b}\hat{c}} \phi^{a\hat{a}} \phi^{b\hat{b}} \phi^{c\hat{c}} \\
& + \overline{D_\mu \varphi^{im}} D^\mu \varphi_{im} - (m^2)_m^{\ n} \, \overline{\varphi}^{im} \varphi_{in} + g\lambda T^{a\ j}_{\ i} \phi^{a\hat{a}} \overline{\varphi}^{im} t^{\hat{a}} \varphi_{jm} \\
& + \frac{g^2}{4} f^{\hat{a}\hat{b}\hat{e}} f^{\hat{e}\hat{c}\hat{d}} \phi^{a\hat{a}} \phi^{b\hat{b}} \phi^{a\hat{c}} \phi^{b\hat{d}} - g^2 \phi^{a\hat{a}} \phi^{a\hat{b}} \overline{\varphi}^{im} t^{\hat{a}} t^{\hat{b}} \varphi_{im} \\
& - g^2 \overline{\varphi}^{im} t^{\hat{a}} \varphi_{jn} \, \overline{\varphi}^{jn} t^{\hat{a}} \varphi_{im} + \frac{g^2}{2} \overline{\varphi}^{im} t^{\hat{a}} \varphi_{im} \, \overline{\varphi}^{jn} t^{\hat{a}} \varphi_{jn} \, .
\end{aligned} \tag{2.15}$$

This theory has a local symmetry $SU(N_c)$, a global symmetry $SU(N_k)$, and a broken flavor symmetry $SU(N_f) \to U(1)^{N_f}$ generically (for special choices of mass matrix, it is broken to some subgroup $SU(N_f)$). We will derive the Lagrangian (2.15) in section 2.5.1 as a particular truncation of a gauge theory with broken global symmetry. We expect that it obeys color/kinematics duality, at least at tree level, as it should inherit this property from the broken theory considered in section 2.5. The corresponding BCJ relations for tree-level amplitudes in the theories (2.14) and (2.15) should be the same as those of QCD [25]. Note that theories (2.14) and (2.15) do not admit obvious supersymmetric extensions unless $\lambda = 0$.

In the next two sections we consider spontaneous symmetry breaking for dimensionally-reduced YM theories (obtained by setting $\lambda = 0$ and $N_f = 0$) including supersymmetric extensions, and, similarly, explicit symmetry breaking in a YM + ϕ^3 theory (obtained by setting $N_f = 0$).

2.4 Adjoint Higgs mechanism: breaking G_c

Here we briefly review Yang-Mills theories for which the gauge symmetry is spontaneously broken by an adjoint Higgs field, and introduce the color/kinematics duality in this setting. This a necessary ingredient in the double-copy construction of Yang-Mills-Einstein supergravity theories with spontaneously-broken gauge symmetry. While supersymmetry is not required by the construction, its presence facilitates the identification of gravitational theories generated by the double-copy prescription.

Consider a YM-scalar theory that is the dimensional reduction of pure YM theory,

$$\mathcal{L}_{\mathrm{YM_{DR}}} = -\frac{1}{4}\mathcal{F}^{\hat{A}}_{\mu\nu}\mathcal{F}^{\mu\nu\hat{A}} + \frac{1}{2}(\mathcal{D}_\mu\phi^a)^{\hat{A}}(\mathcal{D}^\mu\phi^a)^{\hat{A}} - \frac{g^2}{4}f^{\hat{A}\hat{B}\hat{E}}f^{\hat{C}\hat{D}\hat{E}}\phi^{a\hat{A}}\phi^{b\hat{B}}\phi^{a\hat{C}}\phi^{b\hat{D}}\,, \tag{2.16}$$

where $\hat{A},\hat{B},\ldots$ are adjoint gauge indices, and $a,b,\ldots$ are global symmetry indices. The indices $a,b = 0,1,\ldots N'_\phi - 1$ run over the different real scalar fields in the theory. For example, considering the particular cases $N'_\phi = 2$ or $N'_\phi = 6$ in $D = 4$ dimensions, we obtain the bosonic part of the $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM Lagrangians, respectively. In the $\mathcal{N} = 4$ case, the scalars transform in the anti-symmetric tensor representation of the R-symmetry group $SU(4)$, and in $\mathcal{N} = 2$ theories the scalars carry a charge only under the $U(1)$ part of the full R-symmetry group $SU(2) \times U(1)$.

It is well-known that the Coulomb-branch vacua of this theory are described by constant scalar fields solving

$$[\phi^a, \phi^b] = 0\,, \quad \phi^a \equiv \phi^{\hat{A}a}t^{\hat{A}}\,, \tag{2.17}$$

where $t^{\hat{A}}$ are the generators of the gauge group. We choose a vacuum with scale V such that the vacuum expectation value (VEV) of the field ϕ^0 is proportional to a single gauge group generator t^0,

$$\langle\phi^a\rangle = V t^0\,\delta^{a0}\,. \tag{2.18}$$

With this choice, we can interpret the theory with $N'_\phi = 2$ as the dimensional reduction of a spontaneously-broken half-maximal SYM theory in five dimensions where ϕ^0 is the scalar of the vector multiplet. The fact that our construction uplifts to $D = 5$ dimensions will be useful when identifying the corresponding supergravity Lagrangian obtained by the double-copy construction. Similarly, for $N'_\phi = 6$, the theory can be uplifted to the spontaneously-broken maximally-supersymmetric YM theory in $D \leq 9$. For convenience of presentation, in the following we will ignore terms containing fermions in the supersymmetric Lagrangians.

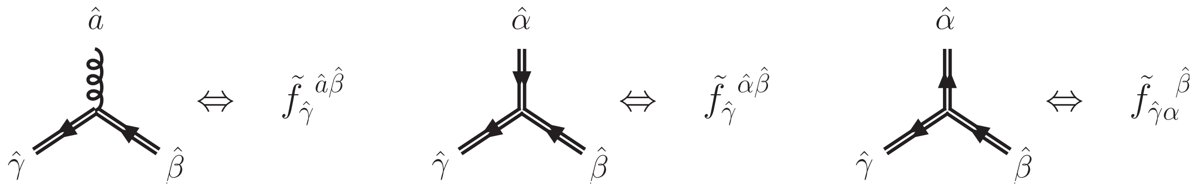

Figure 3: Additional types of cubic interactions that are obtained after the gauge symmetry is spontaneously-broken in a purely adjoint theory. The resulting amplitudes are organized around cubic graphs where these vertices are included. The corresponding color factors are contractions of the various types of structure constants.

The Higgsed Lagrangian corresponding to (2.16) is obtained by splitting the scalar and vector fields as

$$A^{\hat A}_\mu = \left(A^{\hat a}_\mu, W_{\mu\hat\alpha}, \overline{W}^{\hat\alpha}_\mu\right) , \qquad \phi^{\hat A a} = \left(\phi^{a\hat a}, \varphi^a{}_{\hat\alpha}, \overline{\varphi}^{a\hat\alpha}\right) , \tag{2.19}$$

so that the index $\hat a$ runs over the adjoint representation of the unbroken part of the gauge group and the index $\hat\alpha$ runs over the matter-like non-adjoint complex representations. Under this split, the only non-zero entries of the structure constants $f^{\hat A\hat B\hat C}$ are[10]

$$f^{\hat a\hat b\hat c} = -i\mathrm{Tr}([t^{\hat a}, t^{\hat b}]t^{\hat c}) , \quad f^{\hat a}{}_{\hat\beta}{}^{\hat\alpha} = -i\mathrm{Tr}([t^{\hat a}, (t^{\hat\beta})^\dagger]t^{\hat\alpha}) , \quad f^{\hat\alpha}{}_{\hat\beta}{}^{\hat\gamma} = (f_{\hat\gamma}{}^{\hat\beta}{}_{\hat\alpha})^\dagger = -i\mathrm{Tr}([t^{\hat\alpha}, (t^{\hat\beta})^\dagger]t^{\hat\gamma}) , \tag{2.20}$$

which are the structure constants of the unbroken part of the gauge group, the generators and Clebsch-Gordan coefficients for the matter-like representations. Since the scalar VEV has been taken along the gauge group generator t^0, the mass matrix has the expression

$$m_{\hat\alpha}{}^{\hat\beta} = igV f^0{}_{\hat\alpha}{}^{\hat\beta} . \tag{2.21}$$

Expanding the original covariant derivative $(D_\mu\phi^a)^{\hat A} = \partial_\mu\phi^{Aa} + gf^{\hat A\hat B\hat C}A^{\hat B}_\mu\phi^{\hat C a}$ and the covariant field strengths around the scalar VEV, and decomposing these objects in representations of the unbroken part of the gauge group leads to[11]

$$(\mathcal{D}_\mu\phi^a)^{\hat A} = \begin{pmatrix} (D_\mu\phi^a)^{\hat a} + g\overline{W}_\mu f^{\hat a}\varphi^a - g\overline{\varphi}^a f^{\hat a} W_\mu \\ (D_\mu\varphi^a)_{\hat\alpha} - i\delta^{a0}(mW_\mu)_{\hat\alpha} + g\phi^{a\hat a}(f^{\hat a}W_\mu)_{\hat\alpha} + g\overline{W}_\mu f_{\hat\alpha}\varphi^a - g\overline{\varphi}^a f_{\hat\alpha}W_\mu + g\varphi^a f_{\hat\alpha}W_\mu \\ (\overline{D_\mu\varphi^a})^{\hat\alpha} + i\delta^{a0}(\overline{W}_\mu m)^{\hat\alpha} - g\phi^{a\hat a}(\overline{W}_\mu f^{\hat a})^{\hat\alpha} + g\overline{W}_\mu f^{\hat\alpha}\varphi^a - g\overline{\varphi}^a f^{\hat\alpha}W_\mu - g\overline{W}_\mu f^{\hat\alpha}\overline{\varphi}^a \end{pmatrix} , \tag{2.22}$$

[10]Note that we may freely cyclicly permute the indices, e.g. $f^{\hat\alpha}{}_{\hat\beta}{}^{\hat\gamma} = f_{\hat\beta}{}^{\hat\gamma\hat\alpha} = f^{\hat\gamma\hat\alpha}{}_{\hat\beta}$.

[11] We use the shorthand notation

$$\overline{V}^{\hat\beta} f^{\hat a}{}_{\hat\beta}{}^{\hat\gamma} U_{\hat\gamma} \to \overline{V} f^{\hat a} U , \quad \overline{V}^{\hat\beta} f^{\hat\alpha}{}_{\hat\beta}{}^{\hat\gamma} U_{\hat\gamma} \to \overline{V} f^{\hat\alpha} U , \quad V_{\hat\beta} f^{\hat\beta}{}_{\hat\alpha}{}^{\hat\gamma} U_{\hat\gamma} \to V f_{\hat\alpha} U , \quad \overline{V}^{\hat\beta} f_{\hat\beta}{}^{\hat\alpha}{}_{\hat\gamma} \overline{U}^{\hat\gamma} \to \overline{V} f^{\hat\alpha} \overline{U} .$$

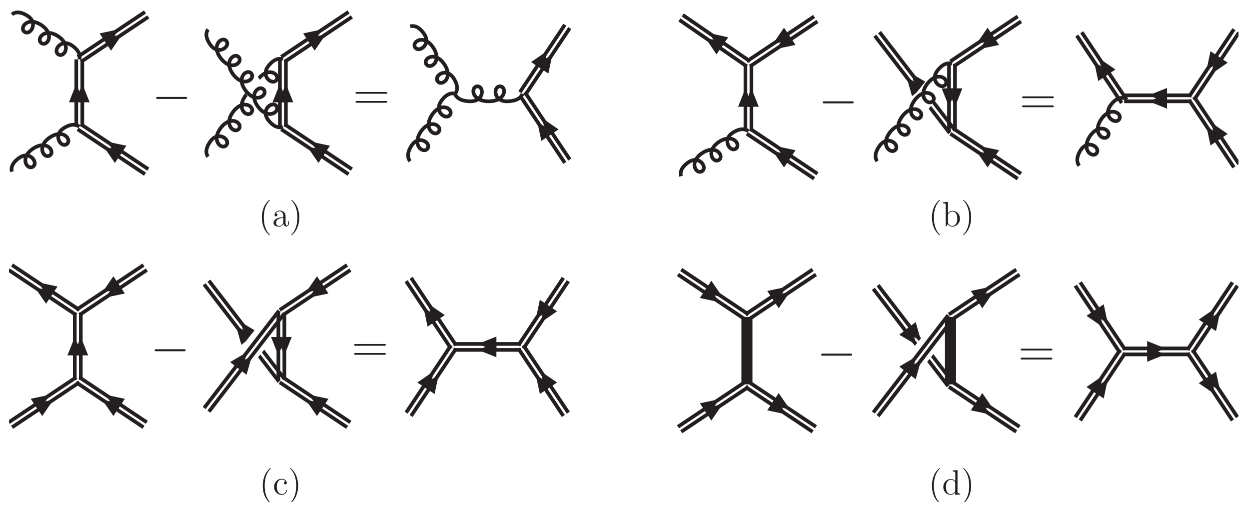

Figure 4: Pictorial representation of additional color Lie-algebra relations that are obtained after the gauge symmetry spontaneously-broken in a purely adjoint theory. These are also pictorial representations of the kinematic algebra that should be imposed on diagram numerators in the context of color/kinematics duality. The relations are generalizations of the Jacobi identity. Curly lines represent unbroken adjoint states (massless fields) and double lines represent broken non-hermitian states (massive fields). Solid fat lines in (d) represent sums over all three types of states (the massless and two conjugates of the massive ones), giving seven terms in the (d) identity.

$$\mathcal{F}^{\hat{A}}_{\mu\nu} = \begin{pmatrix} F^{\hat{a}}_{\mu\nu} + 2g\overline{W}_{[\mu} f^{\hat{a}} W_{\nu]} \\ 2(D_{[\mu}W_{\nu]})_{\hat{\alpha}} + 2g\overline{W}_{[\mu} f_{\hat{\alpha}} W_{\nu]} - gW_{\mu} f_{\hat{\alpha}} W_{\nu} \\ 2(\overline{D_{[\mu}W_{\nu]}})^{\hat{\alpha}} + 2g\overline{W}_{[\mu} f^{\hat{\alpha}} W_{\nu]} - g\overline{W}_{\mu} f^{\hat{\alpha}} \overline{W}_{\nu} \end{pmatrix} . \tag{2.23}$$

In general, the matrix $m_{\hat{\alpha}}{}^{\hat{\beta}}$ is block diagonal, with each block corresponding to different irreducible representations. As usual, the mass of the scalar fields in the matter-like representation corresponding to the generator t^0 ($\varphi^{0\hat{\alpha}}$) depends on the choice of gauge. In the unitary gauge its mass is infinite and this field decouples. In this gauge the Lagrangian is

$$\begin{aligned} \mathcal{L}_{\cancel{\mathrm{YM}}_{\mathrm{DR}}} &= -\frac{1}{4}\mathcal{F}^{\hat{A}}_{\mu\nu}\,\mathcal{F}^{\mu\nu\hat{A}} + \frac{1}{2}\left(\mathcal{D}_\mu\phi^a\right)^{\hat{A}}\left(\mathcal{D}^\mu\phi^a\right)^{\hat{A}} - \frac{g^2}{4} f^{\hat{a}\hat{b}\hat{e}} f^{\hat{c}\hat{d}\hat{e}} \phi^{a\hat{a}}\phi^{b\hat{b}}\phi^{a\hat{c}}\phi^{b\hat{d}} - \overline{\varphi}^{a\hat{\alpha}}(m^2)_{\hat{\alpha}}{}^{\hat{\beta}}\varphi^{a}{}_{\hat{\beta}} \\ &\quad - 2igf^{\hat{a}}{}_{\hat{\alpha}}{}^{\hat{\gamma}} m_{\hat{\gamma}}{}^{\hat{\beta}} \phi^{0\hat{a}}\overline{\varphi}^{a\hat{\alpha}}\varphi^{a}_{\hat{\beta}} + V_4\big(\phi,\varphi\big) \ , \end{aligned} \tag{2.24}$$

where we have written explicitly the cubic term in the scalar potential of eq. (2.16).

Focusing on the $N'_\phi = 2$ case (corresponding to the bosonic part of $\mathcal{N} = 2$ SYM) in the unitary gauge, we have only one family of massive real scalars $\varphi_{\hat{\alpha}} \equiv \varphi^1{}_{\hat{\alpha}}$; the other family, $\varphi^0{}_{\hat{\alpha}}$, becomes the longitudinal component of the W bosons.

The structure constants, generators and Clebsch-Gordan coefficients obey relations inherited from the Jacobi relations of the original gauge group. A first set of relations is

$$\begin{aligned}
f^{\hat{d}\hat{a}\hat{c}} f^{\hat{c}\hat{b}\hat{e}} - f^{\hat{d}\hat{b}\hat{c}} f^{\hat{c}\hat{a}\hat{e}} &= f^{\hat{a}\hat{b}\hat{c}} f^{\hat{d}\hat{c}\hat{e}} , \\
f^{\hat{a}}{}_{\hat{\gamma}}{}^{\hat{\beta}} f^{\hat{b}}{}_{\hat{\alpha}}{}^{\hat{\gamma}} - f^{\hat{b}}{}_{\hat{\gamma}}{}^{\hat{\beta}} f^{\hat{a}}{}_{\hat{\alpha}}{}^{\hat{\gamma}} &= f^{\hat{a}\hat{b}\hat{c}} f^{\hat{c}}{}_{\hat{\alpha}}{}^{\hat{\beta}} , \\
f^{\hat{a}}{}_{\hat{\epsilon}}{}^{\hat{\gamma}} f^{\hat{\epsilon}}{}_{\hat{\delta}}{}^{\hat{\beta}} - f^{\hat{a}}{}_{\hat{\epsilon}}{}^{\hat{\beta}} f^{\hat{\epsilon}}{}_{\hat{\delta}}{}^{\hat{\gamma}} &= f^{\hat{a}}{}_{\hat{\delta}}{}^{\hat{\epsilon}} f^{\hat{\gamma}}{}_{\hat{\epsilon}}{}^{\hat{\beta}} .
\end{aligned} \tag{2.25}$$

These relations are necessary to ensure gauge invariance in any gauge theory (with or without massive vectors). Since they are components of the structure constants of a larger group, and since they control the gauge invariance of massive vector interactions, the Clebsch-Gordan coefficients $f^{\hat{\gamma}}{}_{\hat{\epsilon}}{}^{\hat{\beta}}$ need to obey two further identities:

$$\begin{aligned}
f^{\hat{\alpha}}{}_{\hat{\epsilon}}{}^{\hat{\gamma}} f^{\hat{\epsilon}}{}_{\hat{\delta}}{}^{\hat{\beta}} - f^{\hat{\alpha}}{}_{\hat{\epsilon}}{}^{\hat{\beta}} f^{\hat{\epsilon}}{}_{\hat{\delta}}{}^{\hat{\gamma}} &= f^{\hat{\alpha}}{}_{\hat{\delta}}{}^{\hat{\epsilon}} f^{\hat{\gamma}}{}_{\hat{\epsilon}}{}^{\hat{\beta}} , \\
\left(f^{\hat{\beta}}{}_{\hat{\gamma}}{}^{\hat{\epsilon}} f_{\hat{\epsilon}}{}^{\hat{\alpha}}{}_{\hat{\delta}} + f^{\hat{\alpha}}{}_{\hat{\delta}}{}^{\hat{\epsilon}} f_{\hat{\epsilon}}{}^{\hat{\beta}}{}_{\hat{\gamma}} + f^{\hat{a}}{}_{\hat{\gamma}}{}^{\hat{\beta}} f^{\hat{a}}{}_{\hat{\delta}}{}^{\hat{\alpha}} \right) - (\hat{\alpha} \leftrightarrow \hat{\beta}) &= f^{\hat{\alpha}}{}_{\hat{\epsilon}}{}^{\hat{\beta}} f_{\hat{\delta}}{}^{\hat{\epsilon}}{}_{\hat{\gamma}} .
\end{aligned} \tag{2.26}$$

It is important to note that, for a given assignment of external masses, at most three terms of the above seven-term identity can be non-zero. Hence, the seven-term identity can be thought of as a compact notation for a set of distinct three-term identities. These three-term identities will be the ones imposed on the numerator factors in a duality-satisfying amplitude presentation.

We should also note that, depending on the field content, the relations in eq. (2.26) could be relaxed, in the sense of replacing $f^{\hat{\gamma}}{}_{\hat{\epsilon}}{}^{\hat{\beta}}$ by another (more general) solution to eq. (2.25). This is the case when the fields transforming in the matter representations are scalars or/and fermions. However, if massive vectors transform in matter representations of the unbroken gauge group, then these extra relations are required by the consistency of the theory (as the massive vectors can arise only through a Higgs mechanism).

Color/kinematics duality for YM theories with gauge symmetry spontaneously broken by an adjoint Higgs field is implemented by requiring that the kinematic numerators of scattering amplitudes in these theories obeys identities that mirror the color identities in eq. (2.25) and (2.26). Except for the Jacobi identity, these kinematic identities are pictorially shown in figure 4. Note that since these identities always break up into three-term identities, they can in practice be mapped to the usual three-term numerator identities considered in the framework of color/kinematics duality. Indeed, as it is well known (e.g. see ref. [76]), amplitudes in SYM theory on the Coulomb branch can be reinterpreted as amplitudes in a $(D+1)$ dimensional unbroken SYM theory (see appendix B.1 for a Lagrangian derivation of this). For a SYM theory on the Coulomb branch the kinematic identities in figure 4 are simply obtained through a decomposition of the usual $(D+1)$ dimensional kinematic Jacobi identity into states with zero (massless states) and positive/negative (massive states) momentum in the $(D+1)$ direction.[12]

[12]Note that the type of kinematic algebra introduced here is more general; it need not be inherited from $(D+1)$ dimensions. For example, it applies to the explicitly broken YM + ϕ^3 theory considered in section 2.5.

An important consideration for color/kinematics duality to give well-behaved double copies, is that that we construct amplitude presentations valid for arbitrary gauge groups and arbitrary breaking patterns. This is to prevent the color factors from obeying accidental algebraic relations beyond those of eqs. (2.25) and (2.26), as it might happen for particular choices of gauge groups and gauge-symmetry breakings.

It is useful to note that in the basis in which the mass matrix is diagonal, $m_{\hat{\alpha}}^{\hat{\beta}} = \delta_{\hat{\alpha}}^{\hat{\beta}} m_{\hat{\alpha}}$ (no sum), there is a direct correspondence between the masses of the complex fields and the non-vanishing structure constants involving the broken generators,

$$
\begin{aligned}
f^{\hat{\alpha}}{}_{\hat{\beta}}{}^{\hat{\gamma}} \neq 0 \qquad &\Leftrightarrow \qquad m_{\hat{\alpha}} + m_{\hat{\gamma}} = m_{\hat{\beta}}\,, \\
f^{\hat{a}}{}_{\hat{\beta}}{}^{\hat{\gamma}} \neq 0 \qquad &\Leftrightarrow \qquad m_{\hat{\gamma}} = m_{\hat{\beta}}\,.
\end{aligned}
\tag{2.27}
$$

Such relations arise from the proportionality relation between the mass and the charge with respect to the preferred $U(1)$ generator in (2.21). As a trivial consequence of (2.21), mass and charge obey the same three-term identities

$$
\begin{aligned}
q_{\hat{\alpha}} + q_{\hat{\gamma}} = q_{\hat{\beta}} \qquad &\Leftrightarrow \qquad m_{\hat{\alpha}} + m_{\hat{\gamma}} = m_{\hat{\beta}}\,, \\
q_{\hat{\gamma}} = q_{\hat{\beta}} \qquad &\Leftrightarrow \qquad m_{\hat{\gamma}} = m_{\hat{\beta}}\,,
\end{aligned}
\tag{2.28}
$$

which can be seen as charge/mass conservation for the trilinear interactions. The double-copy construction that we will spell out in section 2.6 requires the masses in the (super)gravity theory to be equal to the ones in the two gauge-theory factors and relies on the charge/mass conservation at each vertex. Interestingly, the close relationship between masses and charges is similar to the double copy framework discovered in refs. [44, 45], where charges of gauge-theory classical solutions were interchanged with masses of classical gravitational solutions.

In sections 4 and 5, we present tree- and loop-level amplitudes in spontaneously-broken SYM that exhibit color/kinematics duality.

2.4.1 $SU(N)$ Examples

For the purpose of illustration, in this subsection we include two simple examples of spontaneous symmetry breaking. The simplest breaking pattern is

$$
SU(N_1+N_2) \to SU(N_1) \times SU(N_2) \times U(1)\ . \tag{2.29}
$$

This pattern can be obtained by giving a VEV

$$
\langle \phi^0 \rangle = V \begin{pmatrix} \frac{1}{N_1} I_{N_1} & 0 \\ 0 & -\frac{1}{N_2} I_{N_2} \end{pmatrix}\,, \tag{2.30}
$$

where we have absorbed a normalization constant in the VEV. As discussed, we denote the corresponding generator as t^0 (with a proper normalization factor) and the generators of the unbroken subgroup that commutes with t^0 as $t^{\hat{a}}$, with $a = 1, 2, \ldots, N_1^2 + N_2^2 - 2$. The remaining "non-hermitian" generators can be divided into two conjugate sets,

$$(t^{(kl)})_{\hat{\imath}}{}^{\hat{\jmath}} = \delta^k_{\hat{\imath}} \delta^{\hat{\jmath}}_l \qquad \text{and} \qquad (t_{(kl)})_{\hat{\imath}}{}^{\hat{\jmath}} = \delta^l_{\hat{\imath}} \delta^{\hat{\jmath}}_k \,, \tag{2.31}$$

where we introduced the composite index $\alpha = (kl)$ with $k = 1, \ldots, N_1$ and $l = N_1 + 1, \ldots, N_1 + N_2$. With this choice, the mass matrix is diagonal and has a single eigenvalue:

$$m = gV\Big(\frac{1}{N_1} + \frac{1}{N_2}\Big) \,. \tag{2.32}$$

The theory can be represented by a quiver diagram with two nodes and two lines with opposite orientations connecting them. In the supersymmetric case each node corresponds to a massless adjoint vector multiplet and each link corresponds to a massive bifundamental vector multiplet.

The simplest example with several masses involves the breaking pattern

$$SU(N_1 + N_2 + N_3) \to SU(N_1) \times SU(N_2) \times SU(N_3) \times U(1)^2 \,. \tag{2.33}$$

It can be realized by choosing a scalar VEV with three diagonal blocks

$$\langle \phi^0 \rangle = V \left(\begin{array}{ccc} \frac{v_1}{N_1} I_{N_1} & 0 & 0 \\ 0 & \frac{v_2}{N_2} I_{N_2} & 0 \\ 0 & 0 & -\frac{v_1+v_2}{N_3} I_{N_3} \end{array} \right) \,. \tag{2.34}$$

In this case, the broken generators can be divided into six sets. The three upper-diagonal sets of generators are

$$(t^{(kl)})_{\hat{\imath}}{}^{\hat{\jmath}} = \delta^k_{\hat{\imath}} \delta^{\hat{\jmath}}_l \,, \qquad (t^{(kr)})_{\hat{\imath}}{}^{\hat{\jmath}} = \delta^k_{\hat{\imath}} \delta^{\hat{\jmath}}_r \,, \qquad (t^{(lr)})_{\hat{\imath}}{}^{\hat{\jmath}} = \delta^l_{\hat{\imath}} \delta^{\hat{\jmath}}_r \,, \tag{2.35}$$

with $k = 1, \ldots, N_1$; $l = 1 + N_1, \ldots, N_1 + N_2$; $r = 1 + N_1 + N_2, \ldots, N_1 + N_2 + N_3$, and the corresponding eigenvalues are

$$m_1 = gV\Big(\frac{v_1}{N_1} - \frac{v_2}{N_2}\Big) \,, \qquad m_2 = gV\Big(\frac{v_1}{N_1} + \frac{v_1 + v_2}{N_3}\Big) \,, \qquad m_3 = gV\Big(\frac{v_2}{N_2} + \frac{v_1 + v_2}{N_3}\Big) \,, \tag{2.36}$$

with $m_2 = m_1 + m_3$. In this case, the quiver diagram has three nodes and six links pairwise connecting the nodes.

2.5 Explicit breaking of the global group G_k

Returning to the YM-scalar theories, in ref. [27] amplitudes in the generic Jordan family YMESG theories were constructed by double-copying the pure $\mathcal{N} = 2$ SYM theory with a bosonic YM + ϕ^3 theory (setting $N_f = 0$), where the latter is described by the Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{YM}+\phi^3} \;=\; & -\frac{1}{4} F^{\hat{a}}_{\mu\nu} F^{\mu\nu\hat{a}} + \frac{1}{2} (D_\mu \phi^A)^{\hat{a}} (D^\mu \phi^A)^{\hat{a}} + \frac{1}{3!} \lambda g F^{ABC} f^{\hat{a}\hat{b}\hat{c}} \phi^{A\hat{a}} \phi^{B\hat{b}} \phi^{C\hat{c}} \\ & - \frac{g^2}{4} f^{\hat{a}\hat{b}\hat{e}} f^{\hat{e}\hat{c}\hat{d}} \phi^{A\hat{a}} \phi^{B\hat{b}} \phi^{A\hat{c}} \phi^{B\hat{d}} \,. \end{aligned} \tag{2.37}$$

The global G_k symmetry acting on the A, B, C indices becomes, through the double copy, a local (gauge) symmetry in the resulting supergravity theory. Since our goal is to describe the latter theory with broken gauge symmetry, it is natural to discuss the breaking of the G_k symmetry before the double copy is taken.

To reduce the global symmetry, $G_k \to G_k^{\text{red.}}$, while preserving the G_k symmetry at high energies,[13] we shall follow a pattern similar to the adjoint Higgs mechanism discussed in the previous section and break the symmetry by adding to the Lagrangian terms with dimension smaller than four (in $D = 4$) − i.e. quadratic and cubic terms. To this end, we single out one generator, T^0, define $G_k^{\text{red.}}$ to be spanned by the generators of G_k that commute with T^0, and decompose the adjoint representation of G_k in representations of $G_k^{\text{red.}}$,

$$\phi^{A\hat{a}} = (\phi^{a\hat{a}}, \overline{\varphi}^{\alpha\hat{a}}, \varphi^{\hat{a}}{}_{\alpha}) \,. \tag{2.38}$$

The first field transforms in the adjoint representation of $G_k^{\text{red.}}$ and the latter two transform in conjugate complex representations of $G_k^{\text{red.}}$. Note that these latter fields carry an adjoint index of the G_c gauge group and an index of a complex representation of $G_k^{\text{red.}}$ and are thus different from the fields $\varphi^{i\hat{\imath}}$ which appeared in section 2.3.

With this decomposition, the symmetry-breaking terms we introduce are

$$\delta_1\mathcal{L} = -(m^2)_\alpha{}^\beta\, \overline{\varphi}^{\alpha\hat{a}} \varphi_\beta{}^{\hat{a}} \;, \qquad \delta_2\mathcal{L} \propto F^0{}_\beta{}^\alpha f^{\hat{b}\hat{a}\hat{c}} \phi^{0\hat{a}} \overline{\varphi}^{\alpha\hat{b}} \varphi_\beta{}^{\hat{c}} \;. \tag{2.39}$$

We take the mass matrix to be

$$m_\alpha{}^\beta = \frac{i}{2} \rho\lambda F^0{}_\beta{}^\alpha \;, \tag{2.40}$$

where ρ is a free real parameter, $F^0{}_\alpha{}^\beta = -i\text{Tr}([T^0, (T^\alpha)^\dagger]\, T^\beta)$, and T^α are (non-hermitian) generators of G_k that do not commute with the T^0.[14] For the normalization of the cubic term, it is convenient to introduce the diagonal matrix

$$\Delta^{ab} = \delta^{ab} + (\sqrt{1+\rho^2} - 1)\delta^{a0}\delta^{0b} \;. \tag{2.41}$$

[13]This is necessary as, on the one hand, the unbroken and the spontaneously-broken phases of a supergravity theory (or any theory) are, from the perspective of the integrand, the same at high energy and on the other the high-energy limit of the supergravity integrand is given by the high-energy limit of the integrands of the two gauge theories.

[14]This pattern is akin to that in which a symmetry is spontaneously broken; the main difference here is that none of the fields in eq. (2.37), including the one corresponding to T^0, have a vacuum expectation value.

With this notation, the Lagrangian with broken G_k symmetry that we will use is

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YM}+\not\phi^3} \;=\;& -\frac{1}{4}F^{\hat a}_{\mu\nu}F^{\mu\nu\hat a} + \frac{1}{2}(D_\mu\phi^a)^{\hat a}(D^\mu\phi^a)^{\hat a} + (\overline{D_\mu\varphi^\alpha})^{\hat a}(D^\mu\varphi_\alpha)^{\hat a} - (m^2)_\alpha{}^\beta\,\overline{\varphi}^{\alpha\hat a}\varphi_\beta{}^{\hat a} \\
& + \frac{1}{3!}g\lambda F^{abc}f^{\hat a\hat b\hat c}\phi^{a\hat a}\phi^{b\hat b}\phi^{c\hat c} + g\lambda\Delta^{ab}F^a_{\ \alpha}{}^\beta f^{\hat b\hat a\hat c}\phi^{b\hat a}\overline{\varphi}^{\alpha\hat b}\varphi_\beta{}^{\hat c} \\
& + \frac{1}{2}g\lambda F^\alpha_{\ \beta}{}^\gamma f^{\hat a\hat b\hat c}\varphi_\alpha{}^{\hat a}\overline{\varphi}^{\beta\hat b}\varphi_\gamma{}^{\hat c} + \frac{1}{2}g\lambda F_\alpha{}^\beta{}_\gamma f^{\hat a\hat b\hat c}\overline{\varphi}^{\alpha\hat a}\varphi_\beta{}^{\hat b}\overline{\varphi}^{\gamma\hat c} \\
& - \frac{g^2}{4}f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\big(\phi^{a\hat a}\phi^{a\hat c} + 2\overline{\varphi}^{\alpha\hat a}\varphi_\alpha{}^{\hat c}\big)\big(\phi^{b\hat b}\phi^{b\hat d} + 2\overline{\varphi}^{\beta\hat b}\varphi_\beta{}^{\hat d}\big) \\
& + \frac{g^2}{2}f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\overline{\varphi}^{\alpha\hat a}\varphi_\alpha{}^{\hat b}\overline{\varphi}^{\beta\hat c}\varphi_\beta{}^{\hat d}\,,
\end{aligned}
\tag{2.42}
$$

where the structure constants $F^a_{\ \alpha}{}^\beta$ and $F^\alpha_{\ \beta}{}^\gamma$ are defined in the usual way,

$$
F^a_{\ \alpha}{}^\beta = -i\mathrm{Tr}([T^a,(T^\alpha)^\dagger]\,T^\beta) \qquad F^\alpha_{\ \beta}{}^\gamma = (F_\gamma{}^\beta{}_\alpha)^\dagger = -i\mathrm{Tr}([T^\alpha,(T^\beta)^\dagger]\,T^\gamma)\;, \tag{2.43}
$$

with T^a being hermitian generators of G_k^{red}.

We will motivate the symmetry-breaking terms through calculations in section 4 and appendix B.2. In section 4 we calculate amplitudes and show that color-kinematics duality requires that these terms be present. Moreover, in appendix B.2 this Lagrangian is derived as the dimensional compactification/reduction and truncation of the unbroken $(D+1)$ theory. This does not imply that the amplitudes of the explicitly broken theory are equivalent to $(D+1)$-dimensional amplitudes; indeed they are not, as there are no massive vectors in the Lagrangian (2.42). See appendix B.2 for more details.

Returning to the color/kinematics duality, we expect that it should be possible find amplitude presentations such that for each three-term Jacobi identity of G_c there exists a three-term numerator identity. The latter requires G_k relations which are decompositions of the Jacobi identity (decomposed following eq. (2.38)). Thus we have the following correspondences:

$$
c_i - c_j = c_k \qquad\Leftrightarrow\qquad n_i - n_j = n_k\,,
$$

$$
\tilde f^{\hat d\hat a\hat c}\tilde f^{\hat c\hat b\hat e} - \tilde f^{\hat d\hat b\hat c}\tilde f^{\hat c\hat a\hat e} = \tilde f^{\hat a\hat b\hat c}\tilde f^{\hat d\hat c\hat e} \quad\Leftrightarrow\quad \left\{\begin{array}{l}
F^{dac}F^{cbe} - F^{dbc}F^{cae} = F^{abc}F^{dce} \\
F^a_{\ \gamma}{}^\beta F^b_{\ \alpha}{}^\gamma - F^b_{\ \gamma}{}^\beta F^a_{\ \alpha}{}^\gamma = F^{abc}F^c_{\ \alpha}{}^\beta \\
F^a_{\ \epsilon}{}^\gamma F^\epsilon_{\ \delta}{}^\beta - F^a_{\ \epsilon}{}^\beta F^\epsilon_{\ \delta}{}^\gamma = F^a_{\ \delta}{}^\epsilon F^\gamma_{\ \epsilon}{}^\beta \\
F^\alpha_{\ \epsilon}{}^\gamma F^\epsilon_{\ \delta}{}^\beta - F^\alpha_{\ \epsilon}{}^\beta F^\epsilon_{\ \delta}{}^\gamma = F^\alpha_{\ \delta}{}^\epsilon F^\gamma_{\ \epsilon}{}^\beta \\
4F^\epsilon_{\ [\delta}{}^{[\alpha}F_{\gamma]}{}^{\beta]}{}_\epsilon + 2F^a_{\ [\delta}{}^\alpha F^a_{\ \gamma]}{}^\beta = F^\alpha_{\ \epsilon}{}^\beta F_\delta{}^\epsilon{}_\gamma
\end{array}\right\}. \tag{2.44}
$$

The last identity for the F's has seven terms, but given fixed assignments of the free indices at most three terms contribute (the integer factors in the seven-term relation compensates for the antisymmetrization over indices).[15]

[15] The fact that the seven-term identity is not affected by the symmetry-breaking terms when insisting on color-kinematics duality is a rather non-trivial fact, as we shall see in section 4.

The Lagrangian in eq. (2.42) can be generalized to fields $\varphi_\alpha{}^{\hat a}$ that transform in a complex representation of the gauge group while preserving the numerator relations and, simultaneously, replacing the color Jacobi relations by the five identities in eqs. (2.25) and (2.26). This can be done in several different ways, all leading to the same formal expression for the Lagrangian but each emphasizing different properties. For example, labeling by greek hatted indices a complex potentially reducible representation of G_c, one may simply replace $\varphi_\alpha{}^{\hat a} \to \varphi_\alpha{}^{\hat\alpha}$ and $\bar\varphi^{\alpha\hat a} \to \bar\varphi^\alpha{}_{\hat\alpha}$ while making the corresponding replacements of indices on the $f^{\hat a\hat b\hat c}$ structure constants and requiring that the resulting coefficients obey the relations in eqs. (2.25) and (2.26). This construction, which is quite general, does not require any correlation between the representation of the gauge and global symmetry groups. One may also decompose the adjoint color indices into adjoint and complex representations of a subgroup (the latter being denoted by hatted greek letters), assign to complex gauge and global indices, $\hat\alpha$ and α, the $U(1)$ charge corresponding to the diagonal of the preferred generators $f^0{}_{\hat\beta}{}^{\hat\alpha}$ and $F^0{}_\beta{}^\alpha$ and project onto the fields with vanishing charge. This restricts the gauge group to the chosen subgroup and introduces a correlation between the irreducible representations under this and global group, i.e. the fields carry only certain combinations of the irreducible components of the representations denoted by the $\hat\alpha$ and α indices. By construction, the remaining components of the $f^{\hat a\hat b\hat c}$ structure constants obey the relations in eqs. (2.25) and (2.26). Due to the closer similarity between the gauge and global symmetry representations carried by fields one may interpret the resulting Lagrangian[16] as a more refined example of color/kinematics duality:

$$
\begin{aligned}
\mathcal{L}'_{\mathrm{YM}+\cancel{\phi^3}} \;=\;& -\frac{1}{4}F^{\hat a}_{\mu\nu}F^{\mu\nu\hat a} + \frac{1}{2}(D_\mu\phi^a)^{\hat a}(D^\mu\phi^a)^{\hat a} + (\overline{D_\mu\varphi^\alpha})_{\hat\alpha}(D^\mu\varphi_\alpha)^{\hat\alpha} - (m^2)_\alpha{}^\beta\,\overline{\varphi}^\alpha_{\hat\alpha}\varphi_\beta{}^{\hat\alpha} \\
& + \frac{1}{3!}g\lambda F^{abc}f^{\hat a\hat b\hat c}\phi^{a\hat a}\phi^{b\hat b}\phi^{c\hat c} + g\lambda\Delta^{ab}F^a_\alpha{}^\beta f^{\hat a}_{\hat\gamma}{}^{\hat\beta}\phi^{b\hat a}\overline{\varphi}^\alpha{}_{\hat\beta}\varphi_\beta{}^{\hat\gamma} \\
& + \frac{1}{2}g\lambda F^\alpha_\beta{}^\gamma f_{\hat\alpha}{}^{\hat\beta}{}_{\hat\gamma}\varphi_\alpha{}^{\hat\alpha}\overline{\varphi}^\beta_{\hat\beta}\varphi_\gamma{}^{\hat\gamma} + \frac{1}{2}g\lambda F_\alpha{}^\beta{}_\gamma f^{\hat\alpha}{}_{\hat\beta}{}^{\hat\gamma}\overline{\varphi}^\alpha{}_{\hat\alpha}\varphi_\beta{}^{\hat\beta}\overline{\varphi}^\gamma{}_{\hat\gamma} \\
& - \frac{g^2}{4}f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\phi^{a\hat a}\phi^{a\hat c}\phi^{b\hat b}\phi^{b\hat d} - g^2 f^{\hat a}{}_{\hat\alpha}{}^{\hat\gamma}f^{\hat b}{}_{\hat\gamma}{}^{\hat\beta}\phi^{a\hat a}\phi^{a\hat b}\overline{\varphi}^\alpha{}_{\hat\beta}\varphi_\alpha{}^{\hat\alpha} \\
& - g^2 f^{\hat\alpha}{}_{\hat\epsilon}{}^{\hat\beta}f_{\hat\gamma}{}^{\hat\epsilon}{}_{\hat\delta}\overline{\varphi}^\alpha{}_{\hat\alpha}\varphi_\alpha{}^{\hat\gamma}\overline{\varphi}^\beta{}_{\hat\beta}\varphi_\beta{}^{\hat\delta} + \frac{g^2}{2}f^{\hat e}{}_{\hat\beta}{}^{\hat\alpha}f^{\hat e}{}_{\hat\delta}{}^{\hat\gamma}\overline{\varphi}^\alpha{}_{\hat\alpha}\varphi_\alpha{}^{\hat\beta}\overline{\varphi}^\beta{}_{\hat\gamma}\varphi_\beta{}^{\hat\delta}\,,
\end{aligned}
\tag{2.45}
$$

Indeed, explicit calculations summarized in section 4 confirm that the tree-level scattering amplitudes following from this Lagrangian obey color/kinematics duality. Note that the kinematic numerators of the amplitudes coming from the theory (2.45) can be chosen to be

[16] Alternatively, we could have constructed this Lagrangian directly as a YM-scalar Lagrangian which (1) has the same gauge group as the unbroken gauge group of a spontaneously-broken theory of the form (2.24), (2) contains additional scalar fields with the same masses as the fields in eq (2.24) and conjugate gauge-group representations, (3) has cubic couplings analogous to the ones of (2.37), and (4) has the cubic and quartic couplings selected by requiring relations between kinematic numerators that mirror relations between color factors.

the same as those of the theory (2.42), since the change of G_c representations only affects the color factors of the amplitudes.[17]

It is important to note that the introduction of the Lagrangian (2.45) together with a specific correlation between the irreducible components of the complex gauge and global indices can be motivated from the double-copy construction between a spontaneously broken SYM and the current theory that we will define in section 2.6. To guarantee the required properties of the fields φ, that they are double-copied with $\mathcal{N}=2$ SYM fields with the same mass, it is necessary to impose the condition

$$2V f^{0}{}_{\hat{\beta}}{}^{\hat{\alpha}} \varphi_{\alpha}{}^{\hat{\beta}} = \lambda\rho F^{0}{}_{\alpha}{}^{\beta} \varphi_{\beta}{}^{\hat{\alpha}} \, . \tag{2.46}$$

The general solution to this equation is that the fields $\varphi_{\alpha}{}^{\hat{\alpha}}$ have a block structure in the space of irreducible components of the α and $\hat{\alpha}$ indices and all fields inside each block have equal mass. Identifying the gauge groups of the YM+$\cancel{\phi^3}$ and $\mathcal{N}=2$ SYM theories, eq. (2.46) together with eqs. (2.21) and (2.40), implies that fields in the same representation of the gauge group have equal masses.

2.5.1 $SU(N)$ Example

An interesting example involves the generation of several different (flavored) massive scalars. The global symmetry group is broken as

$$SU(N_k+N_f) \to SU(N_k)\times U(1)^{N_f} \, . \tag{2.47}$$

Different flavors carry different charges under the $U(1)$ factors. Such a breaking can be obtained by choosing T^0 to be

$$T^0 = \begin{pmatrix} v_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & v_{N_f} & 0 \\ 0 & \cdots & 0 & v_0 I_{N_k} \end{pmatrix} \; ; \tag{2.48}$$

tracelessness requires that $v_0 = -\sum_{n=1}^{N_f} v_n/N_k$ and the v_i are normalized as $\mathrm{Tr}(T^0T^0)=1$.

The symmetry generators of the original $\mathrm{SU}(N_k+N_f)$ symmetry group can be divided into six sets (the generators in the second through fifth sets are broken):

$$\begin{array}{lll} SU(N_k)\ \text{adjoint} & T^a\,, & (a=1,\dots,N_k^2-1) \\ SU(N_k)\ \text{fund. \& flavored} & (T^{(kn)})_i{}^j = \delta_i^k\delta_n^j\,, & \\ SU(N_k)\ \overline{\text{fund.}}\ \text{\& flavored} & (T_{(kn)})_i{}^j = \delta_i^n\delta_k^j\,, & \\ U(1)^{N_f}\ \text{bi-flavored} & (T^{(nm)})_i{}^j = \delta_i^n\delta_m^j\,, & (n<m) \\ U(1)^{N_f}\ \text{bi-flavored} & (T_{(nm)})_i{}^j = \delta_i^m\delta_n^j\,, & (n<m) \\ U(1)^{N_f}\ \text{un-flavored} & (T^{(nn)})_i{}^j = \delta_i^n\delta_n^j - \frac{1}{N_k}I_{N_k}\,, & (\text{no sum}) \end{array} \tag{2.49}$$

[17]This is particularly clear in the construction of (2.45) through projection.

where $k = N_f + 1, \ldots, N_f + N_k$ are fundamental indices, and $n, m = 1, \ldots, N_f$ are flavor indices. The eigenvalues of the corresponding mass matrix are

$$m_{(kn)} = \rho\lambda(v_0 - v_n)\,, \qquad m_{(nm)} = \rho\lambda(v_n - v_m)\,, \tag{2.50}$$

where we use the convention that conjugate representations have masses of opposite signs just like the charges (all physical quantities depend only on the squared masses).

Lagrangians discussed in earlier sections may be obtained from the Lagrangian (2.42) and the generators (2.49) by restricting to a subset of its fields. While this truncation is not always technically consistent (in the sense that the equations of motion of the fields that are truncated away contain sources depending only on the remaining fields) we may nevertheless define such a restricted theory. Its tree-level S matrix however cannot be obtained from that of the parent by simply restricting the external states to those of the daughter theory; rather, it is necessary to also eliminate all the (Feynman) graphs with truncated fields appearing on the internal lines.

By truncating away the $U(1)$-flavored modes corresponding to the generators $T^{(nm)}$, $T_{(nm)}$, $T^{(nn)}$ and T^0 we can recover a theory that is very similar to the one in eq. (2.15). Under this truncation the only surviving structure constants are

$$F^{abc} \qquad \text{and} \qquad F^{a}{}_{(im)}{}^{(jn)} = (T^a)_i{}^j\,\delta^n_m - (T^a)_m{}^n\,\delta^j_i \to (T^a)_i{}^j\,\delta^n_m\,, \tag{2.51}$$

where in the second structure constant the term, $-(T^a)_m{}^n\,\delta^j_i$, has been dropped due to the truncation. The only difference between this theory and the one described by the Lagrangian in eq. (2.15) is that while there the complex scalars in eq. (2.15) transform in the fundamental representation of the gauge group, here the complex scalars are in the adjoint. The two Lagrangians may nevertheless be mapped into each other by identifying the gauge group generators in the adjoint representation and replacing them with the ones in the fundamental representation,[18] e.g. $\overline{\varphi}^{\alpha\hat{a}}\tilde{f}^{\hat{a}\hat{c}\hat{b}}\varphi_\beta^{\hat{b}} \to \overline{\varphi}^\alpha t^{\hat{c}}\varphi_\beta$. This is straightforward except for the quartic terms for which the color/kinematics-satisfying result is obtained as:

$$\begin{aligned}
&\frac{g^2}{4} f^{\hat{a}\hat{b}\hat{e}} f^{\hat{e}\hat{c}\hat{d}}\Big[\big(\phi^{a\hat{a}}\phi^{a\hat{c}} + 2\overline{\varphi}^{\alpha\hat{a}}\varphi_\alpha{}^{\hat{c}}\big)\big(\phi^{b\hat{b}}\phi^{b\hat{d}} + 2\overline{\varphi}^{\beta\hat{b}}\varphi_\beta{}^{\hat{d}}\big) - 2\overline{\varphi}^{\alpha\hat{a}}\varphi_\alpha{}^{\hat{b}}\overline{\varphi}^{\beta\hat{c}}\varphi_\beta{}^{\hat{d}}\Big] \\
&\quad = \frac{g^2}{4} f^{\hat{a}\hat{b}\hat{e}} f^{\hat{e}\hat{c}\hat{d}}\Big[\phi^{a\hat{a}}\phi^{a\hat{c}}\phi^{b\hat{b}}\phi^{b\hat{d}} - 4\overline{\varphi}^{\alpha\hat{a}}\phi^{a\hat{b}}\phi^{a\hat{c}}\varphi_\alpha{}^{\hat{d}} - 4\overline{\varphi}^{\alpha\hat{a}}\varphi_\beta{}^{\hat{b}}\overline{\varphi}^{\beta\hat{c}}\varphi_\alpha{}^{\hat{d}} + 2\overline{\varphi}^{\alpha\hat{a}}\varphi_\alpha{}^{\hat{b}}\overline{\varphi}^{\beta\hat{c}}\varphi_\beta{}^{\hat{d}}\Big] \\
&\quad \to \frac{g^2}{4} f^{\hat{a}\hat{b}\hat{e}} f^{\hat{e}\hat{c}\hat{d}}\phi^{a\hat{a}}\phi^{a\hat{c}}\phi^{b\hat{b}}\phi^{b\hat{d}} - g^2\overline{\varphi}^\alpha\phi^a\phi^a\varphi_\alpha - g^2\overline{\varphi}^\alpha t^{\hat{e}}\varphi_\beta\overline{\varphi}^\beta t^{\hat{e}}\varphi_\alpha + \frac{g^2}{2}\overline{\varphi}^\alpha t^{\hat{e}}\varphi_\alpha\overline{\varphi}^\beta t^{\hat{e}}\varphi_\beta\,.
\end{aligned} \tag{2.52}$$

On the second line, a Jacobi relation was used to reorganize the $(\overline{\varphi}\varphi)^2$ terms. Using $\alpha = (im)$, $\beta = (jn)$, the manipulations above lead exactly to the Lagrangian in eq. (2.15) with massive fundamental scalars and symmetry $SU(N_c) \times SU(N_k) \times U(1)^{N_f}$.

[18]Such a replacement can also be done at the level of (Feynman) graphs.

2.6 The double copy for spontaneously-broken theories

Here we spell out a double-copy construction which combines the ingredients introduced in the previous sections to produce amplitudes in Yang-Mills-Einstein supergravities, some of which have spontaneously-broken gauge symmetry. The case of unbroken Yang-Mills-Einstein supergravities is a review of ref. [27].

Let us assume that expressions for gauge-theory scattering amplitudes are available such that the kinematic numerators $\tilde{n}_i$ satisfy the same general Lie-algebraic relations as the corresponding color factors c_i. By general Lie-algebraic relations we mean relations that are not specific to a given gauge group or symmetry-breaking pattern, but are more generally valid, such as the Jacobi identity and commutation relation in figure 2, and the kinematic relations for theories with broken symmetry in figure 4. The double-copy construction states that, regardless of the spacetime dimension, a valid (super)gravity amplitude is obtained by replacing color factors with numerators in a gauge-theory amplitude, and by replacing the gauge coupling with its gravitational counterpart:

$$c_i \to \tilde{n}_i \qquad \text{and} \qquad g \to \frac{\kappa}{2}\,. \tag{2.53}$$

This statement can be taken as a conjecture to which we will give non-trivial supporting evidence in the case of spontaneously-broken Yang-Mills-Einstein supergravities.

We note that if two different gauge theories are considered, and the numerators of the first theory are replacing the color factors of the second theory,[19] then it is convenient to take the two gauge groups, and thus the color factors, to be identical. Since the double copy does not depend on the details of the color factors, there is no loss of generality.

A familiar property of the double copy, which also holds for spontaneously-broken theories, is that it is sufficient for one set of numerators $\tilde{n}_i$ to be manifestly duality-satisfying, while the other needs not to obey the duality manifestly. This is because, once color factors are replaced by kinematic factors with the same algebraic properties, the second kinematic numerators can in principle [27] be brought to a duality-satisfying form through generalized gauge transformations [1, 2].

Using eq. (2.53) gravity scattering amplitude will take the same general form already given in eq. (2.5); however, the details will differ depending on the gauge theory and whether it is unbroken or (spontaneously) broken. To understand the precise outcome of this prescription, it is essential to identify the proper tensor products of the asymptotic states that appear in the various theories introduced.

Before we discuss the explicit theories introduced in previous sections let us look at the asymptotic states from a uniform formal perspective. In particular, we have introduced gauge

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[19] As indicated by their common graph label, if $\tilde{n}_i$ and c_i belong to different theories they still need to dress the same cubic-diagram specified by the poles $1/D_i$ in their respective amplitudes, thus ensuring that the mass spectra of the two theories are aligned.

theories where fields transform in massless adjoint representations and massive complex (conjugate) representations. Let us assemble the fields into sets, or multiplets, corresponding to these three types:

$$\Big(\mathcal{V}, V(m^2), \overline{V}(m^2)\Big)\,, \tag{2.54}$$

where $\mathcal{V}$ is the set of massless fields and m^2 labels the massive ones. By this notation it is understood that there are distinct massive multiplets $V(m^2)$ and $\overline{V}(m^2)$ for each allowed mass m in the spectrum. The gauge-group indices of the fields have been suppressed, and since they are asymptotic fields for all practical purposes we may think of them as having been stripped of their color dependence.

The asymptotic fields produced by the double copy (2.53) are then given by the gauge-invariant subset of tensor products of gauge-theory fields of the left (L) and right (R) theories. We obtain the supergravity states

$$\Big(\mathcal{V}_L \otimes \mathcal{V}_R, V_L(m^2) \otimes V_R(m^2), \overline{V}_L(m^2) \otimes \overline{V}_R(m^2)\Big)\Big|_{\text{gauge invariant}}\,, \tag{2.55}$$

where for each allowed m^2 there is a distinct pair of tensor products that contribute to the supergravity spectrum. It is important to note that the tensor-product structure of eq. (2.55) is not an independent prescription but rather follows from eq. (2.53). This is because the gauge-theory asymptotic states already have a double-copy structure, between the kinematic and color wave functions (e.g. $A^{\mu\hat{a}} \sim \varepsilon^\mu c^{\hat{a}}$ and $W^\mu_{\hat{\alpha}} \sim \varepsilon^\mu c_{\hat{\alpha}}$). After the replacement (2.53) the gravitational theory inherits such a structure.

Connecting to previous work, one can associate to each field a charge that is uniform within the multiplets $\mathcal{V}, V(m^2)$, $\overline{V}(m^2)$ but otherwise distinct, as was explicitly done in theories constructed through orbifold projections in ref. [15, 19]. For example, in our case this charge may be given in terms of the t^0 generator. From this point of view it is convenient to take the fields of the left and right theory to have opposite charges. The consistency of the construction through orbifold projection then requires the set of supergravity states to be given by the set of zero-charge bilinears constructed from the states of the two gauge theories, as in eq. (2.55).

Let us now be concrete and describe the asymptotic states that enter the double copy (2.55) for each of the theories of interest.

2.6.1 GR + YM = YM ⊗ (YM + ϕ^3)

The case of Yang-Mills-Einstein supergravities was first treated in [27], here we give a summary of that construction. The massive multiplets $V(m^2)$ are absent in the unbroken case. Following the discussion above, the massless multiplets of the pure-adjoint unbroken left

Gravity coupled to YM	Left gauge theory	Right gauge theory
$\mathcal{N}=4$ YMESG theory	$\mathcal{N}=4$ SYM	YM + ϕ^3
$\mathcal{N}=2$ YMESG theory (gen.Jordan)	$\mathcal{N}=2$ SYM	YM + ϕ^3
$\mathcal{N}=1$ YMESG theory	$\mathcal{N}=1$ SYM	YM + ϕ^3
$\mathcal{N}=0$ YME + dilaton + $B^{\mu\nu}$	YM	YM + ϕ^3
$\mathcal{N}=0$ $\mathrm{YM}_{\mathrm{DR}}$-E + dilaton + $B^{\mu\nu}$	$\mathrm{YM}_{\mathrm{DR}}$	YM + ϕ^3

Table 1: The double-copy constructions that appeared in ref. [27]. These give amplitudes in YME gravity theories for various amounts of supersymmetry, corresponding to different choices of the left gauge theory. The right theory labeled by YM + ϕ^3 corresponds to the YM + scalar theory with $N_f \to 0$. The $\mathcal{N}=1$ YMESG theory is a particular truncation of a generic Jordan family $\mathcal{N}=2$ YMESG theory in which the scalar and one fermion is dropped from every nonabelian vector multiplet together with the vector field and one of the gravitini in the graviton multiplet. The last row corresponds to dimensional reductions of a higher-dimensional left gauge theory; this row has the same bosonic content as the previous cases, given that the original theory lived in $D = 10, 6, 4, 4$ dimensions, respectively.

gauge theories,

$$\begin{array}{ll} \mathcal{N}=4 \text{ SYM}: & \mathcal{V}_L = A^\mu \oplus \lambda^{1,2,3,4} \oplus \phi^{0,1,2,3,4,5}\,, \\ \mathcal{N}=2 \text{ SYM}: & \mathcal{V}_L = A^\mu \oplus \lambda^{1,2} \oplus \phi^{0,1}\,, \\ \mathcal{N}=1 \text{ SYM}: & \mathcal{V}_L = A^\mu \oplus \lambda\,, \\ \text{pure YM}: & \mathcal{V}_L = A^\mu\,, \\ \mathrm{YM}_{\mathrm{DR}}: & \mathcal{V}_L = A^\mu \oplus \phi^{a'}\,, \end{array} \tag{2.56}$$

are to be double copied (2.55) with the right theory

$$\mathrm{YM}+\phi^3: \qquad \mathcal{V}_R = A^\mu \oplus \phi^a\,. \qquad . \tag{2.57}$$

We recall that $\mathrm{YM}_{\mathrm{DR}}$ stands for the dimensional reduction of some higher-dimensional pure YM theory. As explained in ref. [27] the double copy of these left and right theories gives rise to amplitudes in (super)gravity coupled to pure Yang-Mills theory. The supersymmetric $\mathcal{N}=4,2$ theories can be uplifted to $D=10,6$ dimensions, respectively, without spoiling the construction. Similarly the bosonic theories can be considered in any dimension.

The tensor product between $\mathcal{V}_L$ corresponding to $\mathrm{YM}_{\mathrm{DR}}$ (for some higher dimension) and $\mathcal{V}_R$ corresponding to $\mathrm{YM}+\phi^3$ is part of all of these gravitational theories; it is given by

$$\mathcal{V}_L \otimes \mathcal{V}_R \to \left(h^{\mu\nu}, \phi, B^{\mu\nu}, A^{\mu a}, A^{\mu a'}, \phi^{a a'}\right), \tag{2.58}$$

where a is an G_k index and a' is either a R-symmetry index or an additional global index. The construction is summarized in table 1.

Gravity coupled to $\not{\text{YM}}$	Left gauge theory	Right gauge theory
$\mathcal{N}=4$ $\not{\text{YM}}$ESG	$\mathcal{N}=4$ S$\not{\text{YM}}$	YM + $\not{\phi}^3$
$\mathcal{N}=2$ $\not{\text{YM}}$ESG (gen.Jordan)	$\mathcal{N}=2$ S$\not{\text{YM}}$	YM + $\not{\phi}^3$
$\mathcal{N}=0$ $\not{\text{YM}}_{\text{DR}}$-E + dilaton + $B^{\mu\nu}$	$\not{\text{YM}}_{\text{DR}}$	YM + $\not{\phi}^3$

Table 2: New double-copy constructions corresponding to spontaneously-broken YME gravity theories for different amounts of supersymmetry. The dimensionally-reduced YM_{DR} theory must have at least one scalar to provide the VEV responsible for spontaneous symmetry breaking. See the caption of table 1 for further details.

2.6.2 GR + $\not{\text{YM}} = \not{\text{YM}} \otimes (\text{YM} + \not{\phi}^3)$

The multiplets of the pure-adjoint spontaneously-broken left gauge theories are as follows:

$$\begin{array}{lll}
\mathcal{N}=4\ \text{S}\not{\text{YM}}: & \mathcal{V}_L = A^\mu \oplus \lambda^{1,2,3,4} \oplus \phi^{0,1,2,3,4,5}\,, & V_L(m^2) = W^\mu \oplus \Lambda^{1,2,3,4} \oplus \varphi^{1,2,3,4,5}\,, \\
\mathcal{N}=2\ \text{S}\not{\text{YM}}: & \mathcal{V}_L = A^\mu \oplus \lambda^{1,2} \oplus \phi^{0,1}\,, & V_L(m^2) = W^\mu \oplus \Lambda^{1,2} \oplus \varphi^{1}\,, \\
\not{\text{YM}}_{\text{DR}}: & \mathcal{V}_L = A^\mu \oplus \phi^{0,a}\,, & V_L(m^2) = W^\mu \oplus \varphi^{a'}\,,
\end{array} \tag{2.59}$$

where for brevity have suppressed the mass dependence of the component fields in $V_L(m^2)$. Similarly, for brevity, we do not display the set of conjugate fields $\overline{V}_L(m^2)$ since it gives no additional information.

The above fields are to be double copied with the right theory asymptotic fields

$$\text{YM} + \not{\phi}^3: \qquad \mathcal{V}_R = A^\mu \oplus \phi^a\,, \quad V_R(m^2) = \varphi_\alpha\,. \tag{2.60}$$

This construction gives rise to amplitudes in (super)gravity coupled to pure spontaneously-broken Yang-Mills theory. The supersymmetric $\mathcal{N}=4,2$ theories can be uplifted to $D=9,5$ dimensions without spoiling the construction, and similarly the bosonic theories can be considered in any dimension. The various supergravity theories constructed in this section are collected in table 2.

The spectra of the above supergravity theories share their bosonic part of the spectrum with the double-copy between the spontaneously-broken dimensionally-reduced YM theory and the YM theory coupled to ϕ^3 scalar theory with broken global symmetry. The result, $(\not{\text{YM}}_{\text{DR}}) \otimes (\text{YM} + \not{\phi}^3)$, is shown in table 3.

3 Spontaneously-broken Yang-Mills-Einstein supergravity theories

The double-copy construction described in the previous section should give amplitudes of large classes of Yang-Mills-Einstein theories, with or without supersymmetry, and with or without spontaneously-broken gauge symmetry. Given a procedure to compute all the tree-level scattering amplitudes of a field theory, it is in principle possible to reconstruct its

	$\mathcal{V}_R$	$V_R(m^2)$	$\overline{V}_R(m^2)$
$\mathcal{V}_L$	$h^{\mu\nu}, \phi, B^{\mu\nu}, A^{\mu a}, A^{\mu a'}, \phi^{aa'}$	$\emptyset$	$\emptyset$
$V_L(m^2)$	$\emptyset$	$W^{\mu}_{\alpha}, \varphi^{a'}_{\alpha}$	$\emptyset$
$\overline{V}_L(m^2)$	$\emptyset$	$\emptyset$	$\overline{W}^{\mu\alpha}, \overline{\varphi}^{a'\alpha}$

Table 3: The spectrum of the double-copy of $\cancel{\text{YM}}_{\text{DR}}$ and $\text{YM} + \cancel{\phi^3}$. The bosonic spectra of the $\mathcal{N} = 4, 2$ YMESG theories are similar. Here a, α are G_k indices and a' is an R-symmetry index.

Lagrangian order by order in the number of fields. However, since gravitational Lagrangians of the type discussed here involve quantities depending non-polynomially from the scalar fields, this procedure is in practice quite cumbersome.

However, there exist a very special class of $\mathcal{N} = 2$ supergravity theories in four and five dimensions for which the full non-polynomial Lagrangian can be reconstructed from the three-point interactions. Such theories provide the simplest examples of our construction and are reviewed in this section.

3.1 Higgs mechanism in five-dimensional $\mathcal{N} = 2$ YMESG theories

$\mathcal{N} = 2$ Maxwell-Einstein supergravity theories describe the coupling of an arbitrary number $\tilde{n}$ of vector multiplets to $\mathcal{N} = 2$ supergravity. An $\mathcal{N} = 2$ vector multiplet in five dimensions consists of a vector field A_μ, a symplectic Majorana spinor λ^i and a real scalar ϕ. The bosonic part of the $\mathcal{N} = 2$ MESG theory in five dimensions can be written in the form [77]

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F^I_{\mu\nu}F^{\mu\nu J} - \frac{1}{2}g_{xy}(\partial_\mu\phi^x)(\partial^\mu\phi^y) + \frac{e^{-1}}{6\sqrt{6}}C_{IJK}\varepsilon^{\mu\nu\rho\sigma\lambda}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda \ , \quad (3.1)$$

where A^I_μ $(I = 0, 1, ...\tilde{n})$ denote the vector fields of the theory including the bare graviphoton A^0_μ, and $F^I_{\mu\nu}$ are the corresponding abelian field strengths. The scalar fields are labeled as ϕ^x $(x, y, .. = 1, .., \tilde{n})$, and g_{xy} is the metric of the scalar manifold. The Lagrangian is completely determined by the constant symmetric tensor C_{IJK}. Using this tensor one defines a cubic form

$$\mathcal{V}(\xi) \equiv C_{IJK}\xi^I\xi^J\xi^K \ , \quad (3.2)$$

in the ambient space coordinates ξ^I. The $(\tilde{n}+1)$-dimensional ambient space spanned by the ξ^I has the metric

$$a_{IJ}(\xi) \equiv -\frac{1}{3}\frac{\partial}{\partial\xi^I}\frac{\partial}{\partial\xi^J}\ln\mathcal{V}(\xi) \, , \quad (3.3)$$

while the $\tilde{n}$-dimensional scalar manifold $\mathcal{M}_5$ is the co-dimension one hypersurface given by the condition [77]:

$$\mathcal{V}(h) = C_{IJK}h^Ih^Jh^K = 1 \qquad \text{with} \qquad h^I = \sqrt{\frac{2}{3}}\xi^I \quad (3.4)$$

and is parameterized by the coordinates φ^x. The metric g_{xy} is the induced metric on the hypersurface $\mathcal{M}_5$, whereas the "metric" $\overset{\circ}{a}_{IJ}(\phi)$ in the kinetic-energy term of the vector fields is given by the restriction of a_{IJ} to the hypersurface $\mathcal{M}_5$,

$$g_{xy}(\phi) = \frac{3}{2}\frac{\partial\xi^I}{\partial\phi^x}\frac{\partial\xi^J}{\partial\phi^y}a_{IJ}\bigg|_{\mathcal{V}=1} , \qquad \overset{\circ}{a}_{IJ}(\phi) = a_{IJ}|_{\mathcal{V}=1} . \tag{3.5}$$

The ambient space indices are lowered and raised with the metric $\overset{\circ}{a}_{IJ}(\phi)$ and its inverse.

Defining

$$h^I_x \equiv -\sqrt{\frac{3}{2}}\frac{\partial h^I}{\partial\phi^x} , \tag{3.6}$$

one finds the following identities:

$$h^I h_I = 1 , \tag{3.7}$$

$$h^I_x h_I = h_{Ix}h^I = 0 , \tag{3.8}$$

$$\overset{\circ}{a}_{IJ} = h_I h_J + h^x_I h^y_J g_{xy} . \tag{3.9}$$

Next, we consider a group $\mathcal{G}$ of symmetry transformations acting on the ambient space as

$$\delta_\alpha \xi^I = (M_r)^I{}_J \xi^J \alpha^r , \tag{3.10}$$

where M_r satisfy the commutation relations

$$[M_r, M_s] = f_{rs}{}^t M_t . \tag{3.11}$$

If $\mathcal{G}$ is a symmetry of the Lagrangian of the five-dimensional MESG theory, then its C-tensor is invariant under it, and it satisfies the relation

$$(M_r)_{(I}{}^L C_{JK)L} = 0 . \tag{3.12}$$

The vector fields of the theory transform linearly under the action of $\mathcal{G}$,

$$\delta_\alpha A^I_\mu = (M_r)^I{}_J A^J_\mu \alpha^r , \tag{3.13}$$

and $\mathcal{G}$ acts as isometries of the scalar manifold $\mathcal{M}_5$

$$\delta_\alpha \varphi^x = K^x_r \alpha^r , \tag{3.14}$$

where K^x_r is a Killing vector of $\mathcal{M}_5$ given by

$$K^x_r = -\sqrt{\frac{3}{2}}(M_r)^J{}_I h_J h^{Ix} . \tag{3.15}$$

The $h^I(\varphi^x)$ transform linearly under $\mathcal{G}$ just like the vector fields,

$$\delta_\alpha h^I(\varphi^x) = (M_r)^I{}_J h^J(\varphi^x)\alpha^r . \tag{3.16}$$

Spin-1/2 fields undergo rotations under the maximal compact subgroup of the global symmetry group $\mathcal{G}$,

$$\delta_\alpha \lambda^a_{\hat{i}} = L^{ab}_r \lambda^b_{\hat{i}} \alpha^r \ , \qquad \text{with} \qquad L^{ab}_r = (M_r)^J{}_I h^{[a|}_J h^{I|b]} - \Omega^{ab}_x K^x_r \ , \tag{3.17}$$

where Ω^{ab}_x is the spin connection of $\mathcal{M}_5$ and $a, b, .. = 1, 2, ..\tilde{n}$ denote the flat tangent space indices. The remaining fields (gravitinos and graviton) are inert under the action of $\mathcal{G}$.

We should note that using the identity (3.9) one can write the kinetic term of the vector fields as

$$e^{-1}\hat{\mathcal{L}}_{\text{vec}} = -\frac{1}{4} \stackrel{\circ}{a}_{IJ} \mathcal{F}^I_{\mu\nu}\mathcal{F}^{\mu\nu J} = -\frac{1}{4}\mathcal{F}^0_{\mu\nu}\mathcal{F}^{\mu\nu 0} - \frac{1}{4} g_{xy}\mathcal{F}^x_{\mu\nu}\mathcal{F}^{\mu\nu y} \ , \tag{3.18}$$

where

$$A^0_\mu \ \equiv \ h_I A^I_\mu, \qquad A^x_\mu \ \equiv \ h^x_I A^I_\mu \ . \tag{3.19}$$

Supersymmetry rotates A^0_μ into the gravitini and A^x_μ into gaugini. Therefore in a given background the physical graviphoton and the physical gaugini are given by the linear combinations $\langle h_I \rangle A^I_\mu$ and $\langle h^x_I \rangle A^I_\mu$, respectively.

Yang-Mills-Einstein supergravity theories are obtained by gauging a subgroup K of full global symmetry group $\mathcal{G}$ of the corresponding MESG theories [78, 79, 80]. A subset of the vector fields, denoted as A^r_μ must then transform in the adjoint representation of K. We consider only gaugings of compact groups K such that the other non-gauge vector fields are spectator fields, i.e. they are inert under K. In this case the non-zero entries of the matrices M_r are simply

$$(M_r)^s{}_t = f^{rst} \ . \tag{3.20}$$

Throughout the paper it will be convenient to formally introduce group structure constants in which the indices can assume values outside the range corresponding to the adjoint vectors A^r_μ, i.e. f^{IJK}. Such structure constants will always vanish if one or more of the indices correspond to a spectator vector field.

The bosonic sector of the $\mathcal{N} = 2$ YMESG theory in five dimensions has the Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L} \ = \ & -\frac{R}{2} - \frac{1}{4}\mathring{a}_{IJ}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J\mu\nu} - \frac{1}{2} g_{xy}\mathcal{D}_\mu\varphi^x\mathcal{D}^\mu\varphi^y + \frac{e^{-1}}{6\sqrt{6}} C_{IJK}\epsilon^{\mu\nu\rho\sigma\lambda}\Big\{ F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda \\ & + \frac{3}{2} g_s f^K{}_{J'K'} F^I_{\mu\nu} A^J_\rho A^{J'}_\sigma A^{K'}_\lambda + \frac{3}{5} g_s^2 A^I_\mu f^J{}_{I'J'} A^{I'}_\nu A^{J'}_\rho f^K{}_{K'L'} A^{K'}_\sigma A^{L'}_\lambda \Big\} \ , \end{aligned} \tag{3.21}$$

where

$$\mathcal{D}_\mu\varphi^x = \partial_\mu\varphi^x + g_s A^r_\mu K^x_r \ , \tag{3.22}$$

$$\mathcal{F}^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]} + g_s f^I{}_{JK} A^J_\mu A^K_\nu \ . \tag{3.23}$$

To preserve supersymmetry, gauging also requires the introduction of a Yukawa-like term

$$\mathcal{L}' = -\frac{i}{2} g_s \bar{\lambda}^{ia}\lambda^b_i K_{r[a} h^r_{b]} \ , \tag{3.24}$$

into the Lagrangian. However, in five dimensions, $\mathcal{N}=2$ YMESG theories without tensor fields do not have any scalar potential terms, and therefore all their vacua are Minkowskian.

One can break the non-abelian gauge symmetry to a subgroup by giving a VEV to some of the scalars while preserving full $\mathcal{N}=2$ supersymmetry. In this paper we study the double-copy construction of the amplitudes of spontaneously-broken YMESG theories obtained by gauging the compact isometries of the $\mathcal{N}=2$ MESG theories belonging to the generic Jordan family. Their cubic forms are of the form

$$N(\xi) = C_{IJK}\xi^I\xi^J\xi^K = \sqrt{2}\xi^0\left((\xi^1)^2 - (\xi^2)^2 - \cdots - (\xi^{\tilde{n}})^2\right) , \tag{3.25}$$

corresponding to the C-tensor

$$C_{011} = \frac{\sqrt{3}}{2}, \qquad C_{0rs} = -\frac{\sqrt{3}}{2}\delta_{rs}, \qquad r,s = 2,\cdots,\tilde{n}\,, \tag{3.26}$$

and the base point[20]

$$c^I = (\frac{1}{\sqrt{2}}, 1, 0, \cdots, 0) \ . \tag{3.27}$$

The global symmetry group of the MESG theories belonging to the generic Jordan family is $SO(1,1)\times SO(\tilde{n},1)$. Since one can embed the adjoint representation of any simple group into the fundamental representation of an orthogonal group, $SO(\tilde{n})$, one can obtain a YMESG theory with an arbitrary simple gauge group by gauging of the generic Jordan family of MESG theories. We should note however that in five dimensions these YMESG theories will have at least one spectator vector field in addition to the graviphoton.

Starting from a YMESG theory belonging to the generic Jordan family with gauge group K we will spontaneously break the gauge symmetry by giving a VEV to the scalar partner of a gauge field in the adjoint of K following [81, 82], where the breaking of $SU(2)$ gauge group down to its $U(1)$ subgroup was studied. This can be achieved by expanding the Lagrangian around the VEV shifted base point

$$c^I_{V_s} = \left(\frac{1}{\sqrt{2}}, 1, V_s, 0, 0\right), \tag{3.28}$$

corresponding to giving a VEV to h^2. The resulting theory describes YMESG theory coupled to some massive BPS vector multiplets. The vector fields acquire their masses, via the Higgs mechanism, from the term that is quadratic in the vector fields in the covariantized kinetic-energy term for the scalar fields,

$$-\frac{1}{2}g_{xy}\mathcal{D}_\mu\varphi^x\mathcal{D}^\mu\varphi^y = -\frac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y - g_s g_{xy}A^{\mu r}K^x_r\partial_\mu\phi^y - \frac{g_s^2}{2}g_{xy}K^x_rK^y_sA^r_\mu A^{\mu s} \ , \tag{3.29}$$

[20]The base point is the point where the scalar metric as well as the "metric" of the kinetic energy term of the vector fields become the Kronecker delta symbol.

and the gaugini acquire their masses through the Yukawa-like term (3.24). To preserve $\mathcal{N}=2$ Poincaré supersymmetry the masses of the gauge fields and gaugini must be equal. At first glance the mass terms appear different. However, as was pointed out in [81], the mass term for the gauge fields can be written in the form

$$\frac{g_s^2}{2} g_{xy} K_r^x K_s^y A_\mu^r A^{\mu s} = \frac{1}{2} A_\mu^x A^{\mu y} g_s W_{xz} g_s W_{wy} g^{wz} \ , \qquad W_{xy} = h^r_{[x} K_{ry]} \ . \tag{3.30}$$

Comparing this with the mass term for the gaugini,

$$\frac{i}{2}(\bar{\lambda}^{ix}\lambda_i^y)(g_s W_{xy}) \ , \tag{3.31}$$

one sees that they have the same mass as required by supersymmetry. Therefore under this Higgs phenomenon, the gauge field corresponding to each broken generator "eats" one scalar field, and we end up with a massive BPS vector supermultiplet consisting of a massive vector and two massive spinor fields.

3.2 Higgs mechanism in four-dimensional $\mathcal{N}=2$ YMESG theories

Dimensional reduction of the five-dimensional $\mathcal{N}=2$ YMESG theory of the previous subsection leads to a four-dimensional YMESG theory with an additional abelian spectator vector multiplet. Hence the spectrum of the resulting four-dimensional $\mathcal{N}=2$ Yang-Mills-Einstein supergravity with gauge group K includes one graviton multiplet and $\tilde{n}+1$ vector multiplets. Each four-dimensional vector multiplet consists of a vector A_μ^I, two spin-1/2 fields $\lambda^{I\hat{i}}$ and a complex scalar z^I. In addition to $\dim(K)$ vector multiplets in the adjoint representation of K, we have $(\tilde{n}-\dim(K)+1)$ spectator vector multiplets that do not partake in the gauging. As in the previous subsection the vectors furnishing the adjoint representation will be denoted as A_μ^r.

The bosonic part of the four-dimensional $\mathcal{N}=2$ Yang-Mills-Einstein Lagrangian can be written in the form [83, 84, 85, 86, 87, 88][21]

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - g_{I\bar{J}}\mathcal{D}_\mu z^I \mathcal{D}^\mu \bar{z}^J + \frac{1}{4}\mathrm{Im}\,\mathcal{N}_{AB}\mathcal{F}^A_{\mu\nu}\mathcal{F}^{B\mu\nu} - \frac{e^{-1}}{8}\epsilon^{\mu\nu\rho\sigma}\mathrm{Re}\,\mathcal{N}_{AB}\mathcal{F}^A_{\mu\nu}\mathcal{F}^B_{\rho\sigma} + g_s^2\mathcal{P}_4 \ , \tag{3.32}$$

where the gauge covariant derivatives and the four-dimensional potential term $\mathcal{P}_4$ are given by

$$\mathcal{D}_\mu z^I \equiv \partial_\mu z^I + g_s A_\mu^J f^I_{JK} z^K \ , \tag{3.33}$$
$$\mathcal{F}^I_{\mu\nu} \equiv 2\partial_{[\mu}A^I_{\nu]} + g_s f^I_{JK} A^J_\mu A^K_\nu \ , \tag{3.34}$$
$$\mathcal{P}_4 \equiv -\frac{1}{2}e^{\mathcal{K}} g_{IJ} f^{IKL} f^{JMN} z^K \bar{z}^L z^M \bar{z}^N \ . \tag{3.35}$$

[21] For further references on the subject we refer to the excellent book by Freedman and Van Proeyen [89].

In the symplectic formulation, the target-space metric $g_{I\bar{J}}$ and the period matrix $\mathcal{N}_{AB}$ are obtained from an holomorphic prepotential F, which depends on $\tilde{n}+2$ complex variables. For YMESG theories obtained by dimensional reduction, the prepotential is expressed in terms of the five-dimensional C-tensor as

$$F(Z^A) = -\frac{2}{3\sqrt{3}} C_{IJK} \frac{Z^I Z^J Z^K}{Z^{-1}} \ , \tag{3.36}$$

where $Z^{-1} \equiv Z^{A=-1}$. The construction goes as follows. The prepotential is associated to a (holomorphic) symplectic vector

$$v(z) = \left(\begin{array}{c} Z^A(z) \\ \dfrac{\partial F}{\partial Z^A}(z) \end{array} \right) \ , \tag{3.37}$$

where the $Z^A(z)$ are $\tilde{n}+2$ arbitrary holomorphic functions of $\tilde{n}+1$ complex variables z^I, which need to satisfy a non-degeneracy condition. The specific choice for such holomorphic functions is related to the choice of the physical scalars and will be discussed shortly. The symplectic vector $v(z)$ defines a Kähler potential $\mathcal{K}(z,\bar{z})$,

$$e^{-\mathcal{K}} = -i\langle v, \bar{v}\rangle = -i\Big(Z^A \frac{\partial \bar{F}}{\partial \bar{Z}^A} - \bar{Z}^A \frac{\partial F}{\partial Z^A}\Big) \ . \tag{3.38}$$

One then introduces a second (non-holomorphic) symplectic vector,

$$V(z,\bar{z}) = \left(\begin{array}{c} X^A \\ F_A \end{array} \right) = e^{\frac{\mathcal{K}}{2}} v(z) \ , \tag{3.39}$$

and its target-space covariant derivatives,

$$\begin{aligned} D_{\bar{I}} \bar{X}^A &= \partial_{\bar{I}} \bar{X}^I + \frac{1}{2}(\partial_{\bar{I}}\mathcal{K})\bar{X}^A \ , \\ D_{\bar{I}} \bar{F}_A &= \partial_{\bar{I}} \bar{F}_A + \frac{1}{2}(\partial_{\bar{I}}\mathcal{K})\bar{F}_A \ . \end{aligned} \tag{3.40}$$

The scalar metric and the period matrix are expressed in terms of the quantities above as

$$g_{I\bar{J}} = \partial_I \partial_{\bar{J}} \mathcal{K} \ , \tag{3.41}$$

$$\mathcal{N}_{AB} = \big(F_A \ D_{\bar{I}}\bar{F}_A\big)\big(X^B \ D_{\bar{I}}\bar{X}^B\big)^{-1} \ . \tag{3.42}$$

As in the previous subsection, we will focus on the generic Jordan family of Yang-Mills-Einstein supergravities whose C-tensor was given in eq. (3.26) and only consider compact gaugings of the isometry group $SO(1,1)\times SO(\tilde{n}-1,1)$. It is important to note that, thanks to their five-dimensional origin, the Lagrangians of the theories we are considering are uniquely specified by the choice of C-tensor and by the compact gauge group K that is a subgroup of the global symmetry of the five-dimensional theory. This fact allows us to identify a theory simply by its three-point interactions as both the C-tensor and the gauge group appear explicitly in the expressions for the three-point amplitudes.

The choice of five-dimensional base-point (3.28) is equivalent to specifying the set of non-degenerate functions entering the symplectic vector $Z^A(z)$ as follows

$$Z^A(z) = \left(1,\ \frac{i}{2} + z^0,\ \frac{i}{\sqrt{2}} + z^1,\ \frac{i}{\sqrt{2}} V_s + z^2,\ z^3,\ \ldots,\ z^{\tilde{n}}\right) , \tag{3.43}$$

with real V_s. We have chosen to label z^0 and z^1 the scalars belonging to the two universal spectator vector multiplets and $z^2, z^3, \ldots, z^{\dim(K)+1}$ the scalars transforming in the adjoint of compact gauge group K.

It should be noted that for theories in the generic Jordan family all base points can be brought into this form with a $SO(\dim(K))$ transformation.[22]

For $V_s = 0$ we obtain a Yang-Mills-Einstein supergravity with unbroken gauge group. In contrast, a non-zero V_s breaks the gauge symmetry group K down to an unbroken subgroup $\tilde{K}$. In general $\tilde{K}$ will have at least a $U(1)$ factor since the choice of base point corresponds to an adjoint scalar acquiring an expectation value, i.e. a non-zero value of V_s takes us on the Coulomb branch of the theory, similarly to the gauge theory case discussed in section 2.4. To write explicitly the Lagrangian of the spontaneously-broken theory we split the indices running over the vectors of the theory $A, B = -1, 1, \ldots, \tilde{n}$ as

$$A = \left(a, \alpha, \bar{\alpha}\right) , \tag{3.44}$$

so that the index a runs over the gluons of the unbroken gauge-group $\tilde{K}$ as well as the spectator vectors, while α and $\bar{\alpha}$ run over two conjugate representations of the unbroken gauge group. Consequently, the vector fields are written as[23]

$$A^A_\mu = \left(A^a_\mu,\ W_{\alpha\mu},\ \overline{W}{}^\alpha_\mu\right) . \tag{3.45}$$

In general, the unbroken gauge group will not necessarily be semisimple and the indices $\alpha, \bar{\alpha}$ may give a reducible representation. Similarly, the scalars $z^I = x^I + iy^I$ are split as[24]

$$x^I = \left(x^i,\ \varphi_{x\alpha},\ \overline{\varphi}{}^\alpha_x\right) , \qquad y^I = \left(y^i,\ \varphi_{y\alpha},\ \overline{\varphi}{}^\alpha_y\right) . \tag{3.46}$$

[22] This $SO(\dim(K))$ transformation will in general not belong to K and can be thought of as a redefinition of the Lie algebra generators.

[23] An alternative notation is to introduce projectors acting in the space spanned by the A, B indices and to define spectators, unbroken gauge fields and massive vectors accordingly as,

$$\tilde{A}^A_\mu = (\mathcal{P}_0)^A{}_B A^B_\mu , \qquad W^A_\mu = (\mathcal{P}_W)^A{}_B A^B_\mu , \qquad \overline{W}{}^A_\mu = (\mathcal{P}_{\overline{W}})^A{}_B A^B_\mu .$$

This approach is closer to the paper [19].

[24] Indices $x, y, ..$ in $\varphi_{x\alpha}$ etc. are not to be confused with the labels of $D = 5$ scalar fields.

Under this split, the only non-zero entries of the structure constants f^{ABC} are

$$f^{ABC} \to (f^{abc}, f^{a\bar{\alpha}\beta}, f^{\alpha\bar{\beta}\gamma}, f^{\bar{\alpha}\beta\bar{\gamma}}) , \tag{3.47}$$

and yield the structure constants of the unbroken gauge group, the representation matrices for the massive fields and tensors with three representation indices which will give multi-flavor couplings involving three massive fields.[25] It should be noted that, as in the case of the gauge theories of the previous section, these objects all obey Jacobi-like relations.

Among the vector multiplets providing the adjoint representation of the unbroken gauge group $\tilde{K}$, there is always a preferred multiplet. The abelian vector of this multiplet, denoted with A^2_μ, gives the $U(1)$ factor which is always part of the unbroken gauge group $\tilde{K}$. One of the scalar fields of the preferred multiplet, y^2, can be thought of as the Higgs field. Note that this field is the imaginary part of a four-dimensional complex scalar, $z^2 = x^2 + iy^2$, because the gauge symmetry breaking has a five-dimensional origin.

The next step is to rewrite the covariant derivatives appearing in the Lagrangian before symmetry breaking as

$$\mathcal{D}_\mu y^I = \begin{pmatrix} D_\mu y^i + g_s \overline{W}_\mu f^i \varphi_y - g_s \overline{\varphi}_y f^i W_\mu \\ D_\mu \varphi_{y\alpha} - i(\tilde{m} W_\mu)_\alpha + g_s y^i (f^i W_\mu)_\alpha + g_s \overline{W}_\mu f_\alpha \varphi_y - g_s \overline{\varphi}_y f_\alpha W_\mu + g_s \varphi_y f_\alpha W_\mu \\ D_\mu \overline{\varphi}^\alpha_y + i(\overline{W}_\mu \tilde{m})^\alpha - g_s y^i (\overline{W}_\mu f^i)_\alpha + g_s \overline{W}_\mu f^\alpha \varphi_y - g_s \overline{\varphi}_y f^\alpha W_\mu + g_s \overline{\varphi}_y f^\alpha \overline{W}_\mu \end{pmatrix} , \tag{3.48}$$

where D_μ is the covariant derivative for the unbroken gauge group. We have introduced the Hermitian matrix $\tilde{m}$, which is proportional to the mass matrix m for the massive fields, and is defined as follows,

$$\tilde{m}_\beta{}^\alpha = i g_s V_s (f^2{}_\beta{}^\alpha) = \sqrt{1 - V_s{}^2}\; m_\beta{}^\alpha \, . \tag{3.49}$$

Without any loss of generality we will take m to be block-diagonal. The derivative $\mathcal{D}_\mu x^I$ of the real part of z^I has an analogous expression with the terms proportional to m missing. The covariant field strengths are rewritten as

$$\mathcal{F}^A_{\mu\nu} = \begin{pmatrix} \mathcal{F}^a_{\mu\nu} + 2 g_s \overline{W}_{[\mu} f^a W_{\nu]} \\ 2 D_{[\mu} W_{\nu]\alpha} + 2 g_s \overline{W}_{[\mu} f_\alpha W_{\nu]} - g_s \overline{W}_\mu f_\alpha W_\nu \\ 2 D_{[\mu} \overline{W}^\alpha_{\nu]} + 2 g_s \overline{W}_{[\mu} f^\alpha W_{\nu]} - g_s \overline{W}_\mu f^\alpha \overline{W}_\nu \end{pmatrix} . \tag{3.50}$$

According to the above index decomposition, both period matrix and scalar metric are split into blocks as

$$\mathcal{N}_{AB} = \begin{pmatrix} \mathcal{N}_{ab} & \mathcal{N}_a{}^\beta & \mathcal{N}_{a\beta} \\ \mathcal{N}^\alpha_b & 0 & \mathcal{N}^\alpha_\beta \\ \mathcal{N}_{\alpha b} & \mathcal{N}_\alpha{}^\beta & 0 \end{pmatrix} , \qquad g_{I\bar{J}} = \begin{pmatrix} g_{ij} & g_i{}^\beta & g_{i\beta} \\ g^\alpha_j & 0 & g^\alpha_\beta \\ g_{\alpha j} & g_\alpha{}^\beta & 0 \end{pmatrix} . \tag{3.51}$$

[25] Such couplings can be non-zero only when the mass of one of the fields equals the sum of the other two.

The full four-dimensional bosonic Lagrangian after dimensional reduction can be obtained by plugging (3.48), (3.50) and (3.51) into (3.32), and plugging (3.46) and (3.47) into (3.35). In analogy with our previous paper [27], we then take the following steps after dimensional reduction to four dimensions:

1. We dualize the graviphoton field $F^{-1}_{\mu\nu}$. Since this field is a spectator (as long as we are not considering R-symmetry gaugings), this dualization does not interfere with the gauging procedure.

2. We employ a linear field redefinition to canonically normalize the bosonic Lagrangian at the base point and to render the supersymmetry transformations diagonal in the sense that the indices A, B of the fields are not mixed by supersymmetry. Such a redefinition involves only spectator fields together with the preferred abelian vector field A^2_μ and takes the following form,

$$\begin{aligned}
A^{-1}_\mu &= -\frac{\sqrt{1-{V_s}^2}}{4}\left(A^{-1\prime}_\mu + A^{0\prime}_\mu + \sqrt{2}A^{1\prime}_\mu\right) , \\
A^{0}_\mu &= \frac{1}{2\sqrt{1-{V_s}^2}}\left(A^{-1\prime}_\mu + A^{0\prime}_\mu - \sqrt{2}A^{1\prime}_\mu\right) , \\
A^{1}_\mu &= \frac{1}{\sqrt{2-2{V_s}^2}}\left(A^{-1\prime}_\mu - A^{0\prime}_\mu + \sqrt{2}V_s A^{2\prime}_\mu\right) , \\
A^{2}_\mu &= \frac{1}{\sqrt{2-2{V_s}^2}}\left(V_s A^{-1\prime}_\mu - V_s A^{0\prime}_\mu + \sqrt{2}A^{2\prime}_\mu\right) , \\
x^1 &= x^{1\prime} + V_s x^{2\prime} , \\
x^2 &= V_s x^{1\prime} + x^{2\prime} , \\
y^1 &= y^{1\prime} + V_s y^{2\prime} , \\
y^2 &= V_s y^{1\prime} + y^{2\prime} ,
\end{aligned}$$

$$\begin{aligned}
\varphi_{x\alpha} &= \sqrt{1+{V_s}^2}\varphi'_{x\alpha} , \qquad \varphi_{y\alpha} = \sqrt{1+{V_s}^2}\varphi'_{y\alpha} , \\
\overline{\varphi}^\alpha_x &= \sqrt{1+{V_s}^2}\overline{\varphi}'^\alpha_x , \qquad \overline{\varphi}^\alpha_y = \sqrt{1+{V_s}^2}\overline{\varphi}'^\alpha_y .
\end{aligned} \tag{3.52}$$

3. We pick the standard R_ξ gauge and introduce the gauge-fixing term

$$\mathcal{L}_{gf} = -\frac{1}{\xi}\overline{G}_\alpha G^\alpha , \qquad G_\alpha = D^\mu W'_{\alpha\mu} + i\xi(m\varphi'_y)_\alpha . \tag{3.53}$$

   If we choose the unitarity gauge, $\xi \to \infty$, the scalar field $\varphi'_{y\alpha}$ acquires an infinite mass and can be integrated out.

The final expansions for the scalar metric and period matrix which will be used in the Feynman-rule computation can be found in appendix C. For notational simplicity we do not put a prime on the fields which appear in the final Lagrangian.

4 Tree-level scattering amplitudes

4.1 Gauge theory amplitudes

In this subsection we evaluate three- and four-point amplitudes in the gauge theories discussed in section 2. Three-point amplitudes will be the building-blocks used to construct three-points supergravity amplitudes using the double-copy prescription and, in the $\mathcal{N}=2$ case, will lead to the identification of the complete supergravity Lagrangian. Four-point amplitudes will enable us to study the constraints imposed by color/kinematics duality. Amplitudes in this section will be written using a metric with mostly-minus signature.

4.1.1 Three points

The completely-massless three-point amplitudes that follow from the YM-scalar theory described in section 2.5 are, up to field redefinitions, the same as the ones already considered in [27]. We therefore focus on amplitudes with massive fields.

In the single-flavor case the only non-vanishing amplitudes have two massive and one massless external states; they are:[26]

$$\mathcal{A}_3\big(1\phi^{a\hat{a}},2\varphi^{\hat{\alpha}}_{\alpha},3\overline{\varphi}^{\beta}_{\hat{\beta}}\big) = -\frac{i}{2}g\lambda\tilde{F}^{b}{}_{\beta}{}^{\alpha}\Delta^{ab}\tilde{f}^{\hat{a}\hat{\beta}}{}_{\hat{\alpha}}\ , \tag{4.2}$$

$$\mathcal{A}_3\big(1A^{\hat{a}}_{\mu},2\varphi^{\hat{\alpha}}_{\alpha},3\overline{\varphi}^{\beta}_{\hat{\beta}}\big) = \sqrt{2}ig(k_2\cdot\epsilon_1)\delta^{\alpha}_{\beta}\tilde{f}^{\hat{a}\hat{\beta}}{}_{\hat{\alpha}}\ . \tag{4.3}$$

In the multi-flavor case, we also have a non-zero amplitude with three massive fields; it is:

$$\mathcal{A}_3\big(1\varphi^{\hat{\alpha}}_{\alpha},2\varphi^{\hat{\beta}}_{\beta},3\overline{\varphi}^{\gamma}_{\hat{\gamma}}\big) = -\frac{i}{2}\lambda g\tilde{F}^{\alpha}_{\ \gamma}{}^{\beta}\tilde{f}_{\hat{\alpha}}{}^{\hat{\gamma}}{}_{\hat{\beta}}\ . \tag{4.4}$$

Inspecting the Lagrangian of the spontaneously-broken YM-scalar theory described in section 2.4, it is easy to see that the three-point amplitudes are:

$$\mathcal{A}_3\big(1\phi^{\hat{a}a},2\varphi_{\hat{\alpha}},3\overline{\varphi}^{\hat{\beta}}\big) = -\sqrt{2}igm\delta^{a0}\tilde{f}^{\hat{a}}{}_{\hat{\beta}}{}^{\hat{\alpha}}\ , \tag{4.5}$$

$$\mathcal{A}_3\big(1A^{\hat{a}},2\varphi_{\hat{\alpha}},3\overline{\varphi}^{\hat{\beta}}\big) = \sqrt{2}ig(k_2\cdot\varepsilon_1)\tilde{f}^{\hat{a}}{}_{\hat{\beta}}{}^{\hat{\alpha}}\ , \tag{4.6}$$

$$\mathcal{A}_3\big(1\phi^{\hat{a}a},2W_{\hat{\alpha}},3\overline{W}^{\hat{\beta}}\big) = \sqrt{2}igm\delta^{a0}(\varepsilon_2\cdot\varepsilon_3)\tilde{f}^{\hat{a}}{}_{\hat{\beta}}{}^{\hat{\alpha}}\ , \tag{4.7}$$

$$\mathcal{A}_3\big(1A^{\hat{a}},2W_{\hat{\alpha}},3\overline{W}^{\hat{\beta}}\big) = -\sqrt{2}ig\Big((k_2\cdot\varepsilon_1)(\varepsilon_2\cdot\varepsilon_3)+(k_1\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_2)-(k_1\cdot\varepsilon_2)(\varepsilon_1\cdot\varepsilon_3)\Big)\tilde{f}^{\hat{a}}{}_{\hat{\beta}}{}^{\hat{\alpha}}\ . \tag{4.8}$$

[26] We use the following conversion between structure constants of different normalizations:

$$\tilde{f}^{\hat{a}\hat{b}\hat{c}} = \sqrt{2}if^{\hat{a}\hat{b}\hat{c}}\ ,\qquad \tilde{F}^{abc} = \sqrt{2}iF^{abc}\ . \tag{4.1}$$

As for the YM-scalar theory, when more than one flavor is present, there are two additional non-zero amplitudes,

$$\begin{aligned}\mathcal{A}_3\big(1W_{\hat\alpha}, 2W_{\hat\beta}, 3\overline{W}^{\hat\gamma}\big) &= -\sqrt{2}ig\Big((k_2\cdot\varepsilon_1)(\varepsilon_2\cdot\varepsilon_3)+(k_1\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_2)-(k_1\cdot\varepsilon_2)(\varepsilon_1\cdot\varepsilon_3)\Big)\tilde f^{\hat\alpha}{}_{\hat\gamma}{}^{\hat\beta}, \\ \mathcal{A}_3\big(1W_{\hat\alpha}, 2\varphi_{\hat\beta}, 3\overline{\varphi}^{\hat\gamma}\big) &= \sqrt{2}ig(k_2\cdot\varepsilon_1)\tilde f^{\hat\alpha}{}_{\hat\gamma}{}^{\hat\beta}\ . \end{aligned} \tag{4.9}$$

We will see that, in the $\mathcal{N}=2$ case, these building blocks will be sufficient to identify the supergravity obtained from the double-copy prescription.

4.1.2 Four points

Using the four-point amplitudes we can study the constraints imposed by color/kinematics duality on the theories constructed in section 2. We start from the YM-scalar theory with explicitly-broken global symmetry and Lagrangian given by (2.45) and compute first the amplitude between two massive and two massless scalars. To have a non-zero amplitude, the two masses must be equal:

$$\begin{aligned}&\mathcal{A}_4\big(1\phi^{a\hat a}, 2\phi^{b\hat b}, 3\varphi^{\hat\alpha}_{\alpha}, 4\overline{\varphi}^{\beta}_{\hat\beta}\big) = \\ &-\frac{i}{2}g^2\Bigg\{\tilde f^{\hat a}{}_{\hat\gamma}{}^{\hat\beta}\tilde f^{\hat b}{}_{\hat\alpha}{}^{\hat\gamma}\Big(\frac{\frac{\lambda^2}{2}\tilde F^c{}_\beta{}^\gamma\tilde F^d{}_\gamma{}^\alpha\Delta^{ac}\Delta^{bd}}{(k_1+k_4)^2-m^2}+\delta^\alpha_\beta\delta^{ab}\Big)+\tilde f^{\hat b\hat\beta}{}_{\hat\gamma}\tilde f^{\hat a\hat\gamma}{}_{\hat\alpha}\Big(\frac{\frac{\lambda^2}{2}\tilde F^d{}_\beta{}^\gamma\tilde F^c{}_\gamma{}^\alpha\Delta^{ac}\Delta^{bd}}{(k_1+k_3)^2-m^2}+\delta^\alpha_\beta\delta^{ab}\Big) \\ &+\frac{\tilde f^{\hat a\hat b\hat c}\tilde f^{\hat c\hat\beta}{}_{\hat\alpha}}{(k_1+k_2)^2}\Big(\frac{\lambda^2}{2}\tilde F^{abc}\Delta^{cd}\tilde F^d{}_\beta{}^\alpha+2(k_1\cdot k_3-k_1\cdot k_4)\delta^\alpha_\beta\delta^{ab}\Big)\Bigg\}\ .\end{aligned} \tag{4.10}$$

The numerator factors are naturally organized by the power of λ. The $\mathcal{O}(\lambda^0)$ parts of the numerator factors are the same as in the massless theory and obey the kinematic Jacobi relations. Imposing color/kinematics duality at $\mathcal{O}(\lambda^2)$ and taking Δ^{ab} to be invertible leads to the requirement

$$F^a{}_\beta{}^\gamma F^b{}_\gamma{}^\alpha - F^b{}_\beta{}^\gamma F^a{}_\gamma{}^\alpha + F^{abc}F^c{}_\beta{}^\alpha = 0\ , \tag{4.11}$$

i.e. the tensors F^{abc}, $F^a{}_\beta{}^\alpha$ can be seen as the structure constants and representation matrices of the unbroken global symmetry group, respectively. Similarly, imposing color/kinematics duality on the amplitude with one massless and three massive scalars leads to the identity,

$$F^a{}_\epsilon{}^\gamma F^\epsilon{}_\delta{}^\beta - F^a{}_\epsilon{}^\beta F^\epsilon{}_\delta{}^\gamma = F^a{}_\delta{}^\epsilon F^\gamma{}_\epsilon{}^\beta\ . \tag{4.12}$$

We next turn to amplitudes with four massive fields. The terms with four such fields in eq. (2.42) may appear mysterious; let us assume therefore a generic dependence on such fields (constrained by symmetries) and see that the coefficients are fixed by color/kinematics duality as stated in that equation. Thus, we assume that the Lagrangian contains the contact terms[27]

$$\frac{g^2}{2}\big(b_2\overline{\varphi}^{\alpha} f^{\hat{a}}\varphi_{\alpha}\overline{\varphi}^{\beta} f^{\hat{a}}\varphi_{\beta} + b_1\overline{\varphi}^{\alpha} f^{\hat{a}}\varphi_{\beta}\overline{\varphi}^{\beta} f^{\hat{a}}\varphi_{\alpha} + b_3 f^{\hat{\epsilon}}{}_{\hat{\alpha}\hat{\beta}}\varphi_{\alpha}{}^{\hat{\alpha}}\varphi_{\beta}{}^{\hat{\beta}} f_{\hat{\epsilon}}{}^{\hat{\gamma}\hat{\delta}}\overline{\varphi}^{\alpha}{}_{\hat{\gamma}}\overline{\varphi}^{\beta}{}_{\hat{\delta}} + b_4\overline{\varphi}^{\alpha} f^{\hat{\epsilon}}\varphi_{\beta}\overline{\varphi}^{\beta} f_{\hat{\epsilon}}\varphi_{\alpha}\big) \ . \tag{4.13}$$

The scattering amplitude of four massive scalars can be cast in the form

$$\mathcal{A}_4\big(1\varphi^{\hat{\alpha}}_{\alpha}, 2\varphi^{\hat{\beta}}_{\beta}, 3\overline{\varphi}^{\gamma}_{\hat{\gamma}}, 4\overline{\varphi}^{\delta}_{\hat{\delta}}\big) = -ig^2\left(\frac{n_1c_1}{D_1} + \frac{n_2c_2}{D_2} + \frac{n_3c_3}{D_3} + \frac{n_4c_4}{D_4} + \frac{n_5c_5}{D_5} + \frac{n_6c_6}{D_6} + \frac{n_7c_7}{D_7}\right) \ , \tag{4.14}$$

where the terms contributing to the amplitude are shown in figure 5; they correspond to decomposing each of the s, t and u channels following the representation of the intermediate state. When more than one mass is present, graphs with different internal mass are regarded as distinct. The color factors are given by

$$\begin{aligned}
&c_1 = f^{\hat{a}\hat{\delta}}{}_{\hat{\alpha}}\tilde{f}^{\hat{a}\hat{\gamma}}{}_{\hat{\beta}} \ , \qquad c_2 = \tilde{f}_{\hat{\epsilon}}{}^{\hat{\delta}}{}_{\hat{\alpha}}\tilde{f}^{\hat{\epsilon}\hat{\gamma}}{}_{\hat{\beta}} \ , \qquad c_3 = \tilde{f}^{\hat{\epsilon}\hat{\delta}}{}_{\hat{\alpha}}\tilde{f}_{\hat{\epsilon}}{}^{\hat{\gamma}}{}_{\hat{\beta}} \ , \qquad c_4 = \tilde{f}^{\hat{a}\hat{\gamma}}{}_{\hat{\alpha}}\tilde{f}^{\hat{a}\hat{\delta}}{}_{\hat{\beta}} \ , \\
&c_5 = \tilde{f}^{\hat{\epsilon}\hat{\gamma}}{}_{\hat{\alpha}}\tilde{f}_{\hat{\epsilon}}{}^{\hat{\delta}}{}_{\hat{\beta}} \ , \qquad c_6 = \tilde{f}_{\hat{\epsilon}}{}^{\hat{\gamma}}{}_{\hat{\alpha}}\tilde{f}^{\hat{\epsilon}\hat{\delta}}{}_{\hat{\beta}} \ , \qquad c_7 = f^{\hat{\gamma}\hat{\delta}}{}_{\hat{\epsilon}}\tilde{f}^{\hat{\epsilon}}{}_{\hat{\alpha}\hat{\beta}} \ ,
\end{aligned} \tag{4.15}$$

while the (massless and massive) inverse propagators are

$$\begin{aligned}
&D_1 = (k_1+k_4)^2 \, , \qquad D_2 = D_3 = (k_1+k_4)^2 - (m_1-m_4)^2 \, , \\
&D_4 = (k_1+k_3)^2 \, , \qquad D_5 = D_6 = (k_1+k_3)^2 - (m_1-m_3)^2 \, , \quad D_7 = (k_1+k_2)^2 - (m_1+m_2)^2 \, .
\end{aligned} \tag{4.16}$$

The numerator factors have the following expressions,

$$\begin{aligned}
n_1 &= (k_1\cdot k_3 - k_1\cdot k_2)\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} - \frac{i}{4}\lambda^2\tilde{F}^{b}{}_{\delta}{}^{\alpha}\tilde{F}^{c}{}_{\gamma}{}^{\beta}\Delta^{ba}\Delta^{ac} - \frac{i}{2}(k_1+k_4)^2\big(b_1\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + b_2\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}\big) \ , \\
n_2 &= \frac{1}{4}\lambda^2\tilde{F}^{\epsilon}{}_{\delta}{}^{\alpha}\tilde{F}_{\epsilon\gamma}{}^{\beta} - \frac{i}{2}b_4\big((k_1+k_4)^2 - (m_1-m_4)^2\big)\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} \ , \\
n_3 &= \frac{1}{4}\lambda^2\tilde{F}_{\epsilon\delta}{}^{\alpha}\tilde{F}^{\epsilon}{}_{\gamma}{}^{\beta} - \frac{i}{2}b_4\big((k_1+k_4)^2 - (m_1-m_4)^2\big)\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} \ , \\
n_4 &= (k_1\cdot k_4 - k_1\cdot k_2)\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \frac{i}{4}\lambda^2\tilde{F}^{b}{}_{\gamma}{}^{\alpha}\tilde{F}^{c}{}_{\delta}{}^{\beta}\Delta^{ba}\Delta^{ac} - \frac{i}{2}(k_1+k_3)^2\big(b_2\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + b_1\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}\big) \ , \\
n_5 &= \frac{1}{4}\lambda^2\tilde{F}_{\epsilon\gamma}{}^{\alpha}\tilde{F}^{\epsilon}{}_{\delta}{}^{\beta} - \frac{i}{2}b_4\big((k_1+k_3)^2 - (m_1-m_3)^2\big)\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} \ , \\
n_6 &= \frac{1}{4}\lambda^2\tilde{F}^{\epsilon}{}_{\gamma}{}^{\alpha}\tilde{F}_{\epsilon\delta}{}^{\beta} - \frac{i}{2}b_4\big((k_1+k_3)^2 - (m_1-m_3)^2\big)\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} \ , \\
n_7 &= \frac{1}{4}\lambda^2\tilde{F}^{\alpha\beta}{}_{\epsilon}\tilde{F}^{\epsilon}{}_{\gamma\delta} - \frac{i}{2}b_3\big((k_1+k_2)^2 - (m_1+m_2)^2\big)\big(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}\big) \ .
\end{aligned} \tag{4.17}$$

Note that this amplitude vanishes unless the masses of the external scalars are related as[28]

$$m_1 + m_2 = m_3 + m_4 \ . \tag{4.18}$$

[27]Additionally, we could consider a contact term of the form $\overline{\varphi}^{\alpha} f^{\hat{\epsilon}}\varphi_{\alpha}\overline{\varphi}^{\beta} f_{\hat{\epsilon}}\varphi_{\beta}$, but it is possible to show that it gives vanishing contribution to all amplitudes entering the double-copy construction in the next section.

[28]This relation holds because of our choice of masses and gauge-theory representations for the theory with explicitly-broken global symmetry.

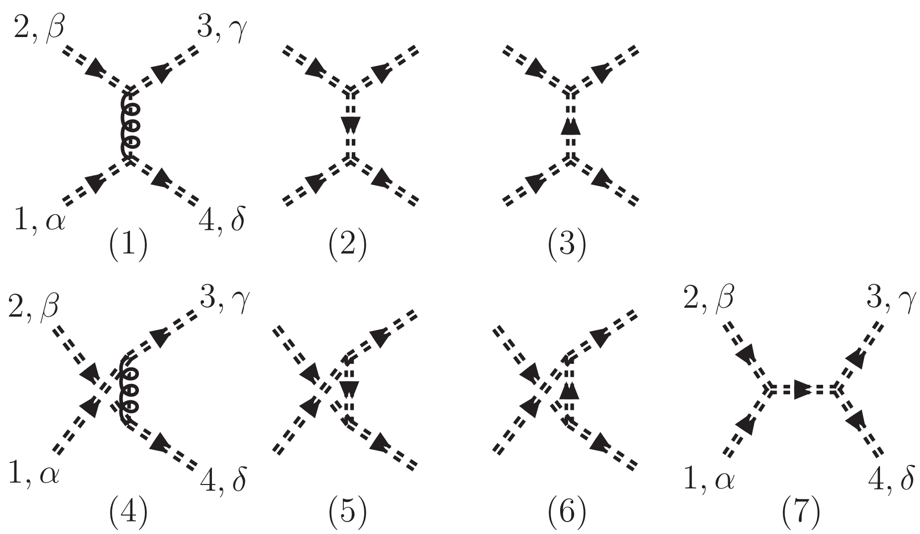


Figure 5: Seven separate contributions to tree amplitudes with four massive scalars. Dashed lines with arrows denote complex (massive) scalars. In diagram (1) and (4) the exchanged particle is a sum of a massless scalar and a gluon.

We start by looking at the kinematic counterpart of the color seven-term relation in eq. (2.26). As explained before, this color identity is to be thought of as a set of three-term identities. Consequently, different three-term numerator identities need to be imposed for the various possible choices of masses for the external particles.[29] We start by taking all masses to be equal. In this case, the color factors corresponding to massive $t-$ and $u-$channel exchanges vanish, and the seven terms relation collapses to

$$c_1 - c_4 + c_7 = 0 \ . \tag{4.19}$$

We proceed to impose the corresponding numerator relation

$$n_1 - n_4 + n_7 = 0 \ . \tag{4.20}$$

At the $\mathcal{O}(\lambda^0)$ order we obtain the condition

$$\Big\{2\big(1 - b_2 - b_1\big)(k_1 \cdot k_3) + \big(-1 - b_2 - b_3\big)(k_1 \cdot k_2)\Big\}\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} - (3 \leftrightarrow 4) = 0 \ , \tag{4.21}$$

which can be solved by

$$b_3 = -1 - b_2 \ , \qquad b_1 = 1 - b_2 \ . \tag{4.22}$$

[29]In principle, imposing several three-term relations on the numerator factors is different from imposing a single seven-term relation. The former choice is natural in our approach as the various graphs entering the amplitude presentation have a definite value of the mass for each internal or external line. Hence, graphs with different external masses are distinct and must be treated separately. Taking into account the possible values of the external mass, one finds that the color seven-term relation always reduces to three-term relations, and the corresponding three-term relations need to be imposed on the numerator factors.

The constraint at $\mathcal{O}(\lambda^2)$ is

$$\lambda^2\Big(2\tilde{F}^{b}{}_{[\delta}{}^{\alpha}\tilde{F}^{c}{}_{\gamma]}{}^{\beta}\Delta^{ba}\Delta^{ac}+\tilde{F}_{\epsilon}{}^{\alpha\beta}\tilde{F}^{\epsilon}{}_{\gamma\delta}\Big)=8m^2\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}-(3\leftrightarrow 4)\ . \tag{4.23}$$

The mass terms appear in this relation because the masses are chosen to be proportional to λ. This $\mathcal{O}(\lambda^2)$ identity forces us to pick one of the kinematic group generators, $F^{0}{}_{\alpha}{}^{\beta}$, to be proportional to the mass matrix; indeed, this has been our choice in section 2.5. In particular, the proportionality relation between $F^{0}{}_{\alpha}{}^{\beta}$ and the mass, together with the relations (4.12), implies

$$\begin{aligned} F^{\alpha}{}_{\beta}{}^{\gamma}\neq 0 \qquad &\Leftrightarrow \qquad m_{\alpha}+m_{\gamma}=m_{\beta}\ ,\\ F^{a}{}_{\beta}{}^{\gamma}\neq 0 \qquad &\Leftrightarrow \qquad m_{\gamma}=m_{\beta}\ . \end{aligned} \tag{4.24}$$

The right-hand side of (4.23) is cancelled by the terms depending on ρ in the matrix Δ on the left side and the remainder is just a part of the the last identity for the kinematic algebra in (2.44),

$$2\tilde{F}^{b}{}_{[\delta}{}^{\alpha}\tilde{F}^{c}{}_{\gamma]}{}^{\beta}\Delta^{ba}\Delta^{ac}+\tilde{F}_{\epsilon}{}^{\alpha\beta}\tilde{F}^{\epsilon}{}_{\gamma\delta}=0\ . \tag{4.25}$$

Next, we consider the cases in which masses are pairwise equal, $m_1=m_3$ and $m_2=m_4$ with $m_1\neq m_2$. In this case the seven-term identity reduces to[30]

$$c_2-c_4+c_7=0\ ,\qquad c_3-c_4+c_7=0\ . \tag{4.26}$$

Imposing the corresponding numerator identities and repeating the procedure above lead to the extra condition on the contact-term parameters

$$b_1=b_4\ , \tag{4.27}$$

together with the relations

$$\begin{aligned} -\tilde{F}^{b}{}_{\gamma}{}^{\alpha}\tilde{F}^{c}{}_{\delta}{}^{\beta}\Delta^{ba}\Delta^{ac}+\tilde{F}^{\epsilon}{}_{\delta}{}^{\alpha}\tilde{F}_{\epsilon\gamma}{}^{\beta}\ +\tilde{F}_{\epsilon}{}^{\alpha\beta}\tilde{F}^{\epsilon}{}_{\gamma\delta}=0\ ,\\ -\tilde{F}^{b}{}_{\gamma}{}^{\alpha}\tilde{F}^{c}{}_{\delta}{}^{\beta}\Delta^{ba}\Delta^{ac}+\tilde{F}^{\epsilon}{}_{\gamma}{}^{\beta}\tilde{F}_{\epsilon\delta}{}^{\alpha}\ +\tilde{F}_{\epsilon}{}^{\alpha\beta}\tilde{F}^{\epsilon}{}_{\gamma\delta}=0\ . \end{aligned} \tag{4.28}$$

In particular, all numerator relations can be satisfied if we choose $b_1=b_4=0$, $b_2=1$ and $b_3=-2$ as in the Lagrangian (2.45).

Finally, there are several three-term color relations corresponding to the case in which all external masses are different,

$$c_2-c_5+c_7=0\ ,\qquad c_3-c_5+c_7=0\ ,\qquad c_2-c_6+c_7=0\ ,\qquad c_3-c_6+c_7=0\ . \tag{4.29}$$

The corresponding numerator relations, combined with (4.25), (4.28) and (4.24), are equivalent to a seven-term relation for the global (kinematic) group structure constants,

$$2\tilde{F}^{b}{}_{[\delta}{}^{\alpha}\tilde{F}^{c}{}_{\gamma]}{}^{\beta}\Delta^{ba}\Delta^{ac}+4\tilde{F}^{\epsilon}{}_{[\delta}{}^{[\alpha}\tilde{F}_{\epsilon\gamma]}{}^{\beta]}+\tilde{F}_{\epsilon}{}^{\alpha\beta}\tilde{F}^{\epsilon}{}_{\gamma\delta}=0\ . \tag{4.30}$$

[30]There are two distinct cases, according to whether $m_1-m_4=+m_{\rm int}$ or $m_1-m_4=-m_{\rm int}$ for some possible mass of the t-channel particle.

There exist one more amplitude with four massive scalars,

$$\mathcal{A}_4\big(1\varphi^{\hat\alpha}_{\alpha}, 2\varphi^{\hat\beta}_{\beta}, 3\varphi^{\hat\gamma}_{\gamma}, 4\overline{\varphi}^{\delta}_{\hat\delta}\big) \ . \tag{4.31}$$

Its numerator factors obey an additional relation provided that

$$F^{\alpha}{}_{\epsilon}{}^{\gamma}F^{\epsilon}{}_{\delta}{}^{\beta} - F^{\alpha}{}_{\epsilon}{}^{\beta}F^{\epsilon}{}_{\delta}{}^{\gamma} = F^{\alpha}{}_{\delta}{}^{\epsilon}F^{\gamma}{}_{\epsilon}{}^{\beta} \ . \tag{4.32}$$

Equations (4.11), (4.12), (4.30) and (4.32) are equivalent to requiring that the F-tensors can be combined to give the structure constants of a larger global symmetry group, which is broken by the masses of some of the fields (and by the gauge-group representations). Indeed, this has been our approach in deriving the Lagrangian (2.42).

Next, we analyze scalar amplitudes in the spontaneously-broken YM-scalar theory reviewed in section 2.4. We specialize to the case in which the theory has only two real adjoint scalars (i.e. it can be seen as the bosonic part of the spontaneously-broken pure $\mathcal{N}=2$ SYM theory). There are two non-zero amplitudes with two massive and two massless scalars,

$$\begin{aligned}
&\mathcal{A}_4\big(1\phi^{0\hat a}, 2\phi^{0\hat b}, 3\varphi_{\hat\alpha}, 4\overline{\varphi}^{\hat\beta}\big) = \\
&\quad -ig^2\left\{\tilde f^{\hat a}{}_{\hat\beta}{}^{\hat\gamma}\tilde f^{\hat b}{}_{\hat\gamma}{}^{\hat\alpha}\frac{k_1\cdot k_4+2m^2}{(k_1+k_4)^2-m^2} + \tilde f^{\hat b}{}_{\hat\beta}{}^{\hat\gamma}\tilde f^{\hat a}{}_{\hat\gamma}{}^{\hat\alpha}\frac{k_1\cdot k_3+2m^2}{(k_1+k_3)^2-m^2} + \tilde f^{\hat a\hat b\hat c}\tilde f^{\hat c}{}_{\hat\beta}{}^{\hat\alpha}\frac{k_1\cdot k_3-k_1\cdot k_4}{(k_1+k_2)^2}\right\} , \\
&\mathcal{A}_4\big(1\phi^{1\hat a}, 2\phi^{1\hat b}, 3\varphi_{\hat\alpha}, 4\overline{\varphi}^{\hat\beta}\big) = \\
&\quad -ig^2\left\{\tilde f^{\hat a}{}_{\hat\beta}{}^{\hat\gamma}\tilde f^{\hat b}{}_{\hat\gamma}{}^{\hat\alpha}\frac{k_1\cdot k_4+2k_1\cdot k_2}{(k_1+k_4)^2-m^2} + \tilde f^{\hat b}{}_{\hat\beta}{}^{\hat\gamma}\tilde f^{\hat a}{}_{\hat\gamma}{}^{\hat\alpha}\frac{k_1\cdot k_3+2k_1\cdot k_2}{(k_1+k_3)^2-m^2} + \tilde f^{\hat a\hat b\hat c}\tilde f^{\hat c}{}_{\hat\beta}{}^{\hat\alpha}\frac{k_1\cdot k_3-k_1\cdot k_4}{(k_1+k_2)^2}\right\} ,
\end{aligned} \tag{4.33}$$

where ϕ^0 is the fluctuation of the field responsible for symmetry breaking (i.e. the fluctuations of the field which acquires a VEV), ϕ^1 is the other real scalar in the theory and the masses of the two massive scalars need to be equal to have a non-zero amplitude. These amplitudes manifestly display color/kinematics duality, as the numerator factors obey the same relations as the corresponding color factors.

Finally, we consider scalar amplitudes with four massive fields,

$$A\big(1\varphi_{\hat\alpha}, 2\varphi_{\hat\beta}, 3\overline{\varphi}^{\hat\gamma}, 4\overline{\varphi}^{\hat\delta}\big) \ . \tag{4.34}$$

The amplitude can be organized in the form (4.14) with inverse propagators (4.16), color factors

$$\begin{aligned}
&c_1 = f^{\hat a}{}_{\hat\delta}{}^{\hat\alpha}\tilde f^{\hat a}{}_{\hat\gamma}{}^{\hat\beta} \ , \qquad c_2 = \tilde f^{\hat\epsilon}{}_{\hat\delta}{}^{\hat\alpha}\tilde f_{\hat\epsilon\hat\gamma}{}^{\hat\beta} \ , \qquad c_3 = \tilde f_{\hat\epsilon\hat\delta}{}^{\hat\alpha}\tilde f^{\hat\epsilon}{}_{\hat\gamma}{}^{\hat\beta} \ , \qquad c_4 = \tilde f^{\hat a}{}_{\hat\gamma}{}^{\hat\alpha}\tilde f^{\hat a}{}_{\hat\delta}{}^{\hat\beta} \ , \\
&c_5 = \tilde f_{\hat\epsilon\hat\gamma}{}^{\hat\alpha}\tilde f^{\hat\epsilon}{}_{\hat\delta}{}^{\hat\beta} \ , \qquad c_6 = \tilde f^{\hat\epsilon}{}_{\hat\gamma}{}^{\hat\alpha}\tilde f_{\hat\epsilon\hat\delta}{}^{\hat\beta} \ , \qquad c_7 = f_{\hat\gamma\hat\delta}{}^{\hat\epsilon}\tilde f_{\hat\epsilon}{}^{\hat\alpha\hat\beta} \ ,
\end{aligned} \tag{4.35}$$

and numerator factors

$$\begin{aligned}
\tilde n_1 &= \tilde n_2 = \tilde n_3 = -\big(k_1\cdot k_2 - m_1m_2 - k_1\cdot k_3 - m_1m_3\big) \ , \\
\tilde n_4 &= \tilde n_5 = \tilde n_6 = -\big(2k_1\cdot k_2 - 2m_1m_2 + k_1\cdot k_3 + m_1m_3\big) \ , \\
\tilde n_7 &= \big(k_1\cdot k_2 + 2k_1\cdot k_3 - m_1m_2 + 2m_1m_3\big) \ .
\end{aligned} \tag{4.36}$$

It is immediate to verify that these numerators obey the same three-term relations as the corresponding color factors.

4.2 Supergravity amplitudes

In this section we compare, in explicit examples, the result of the double-copy construction described in section 2.6 with three- and four-point amplitudes computed from the expected supergravity Lagrangian derived in section 3 and find the map between the Lagrangian and double-copy fields.

One of the gauge-theory factors entering the construction is the spontaneously-broken $\mathcal{N}=2$ SYM theory. The bosonic part of the Lagrangian is shown in section 2.4. We list here the bosonic fields in four dimensions:

$$\left(A_\mu^{\hat{a}}\,, \phi^{\hat{a}a'}\,, W_{\hat{\alpha}\mu}\,, \varphi_{\hat{\alpha}}\,, \overline{W}_\mu^{\hat{\alpha}}\,, \overline{\varphi}^{\hat{\alpha}}\right)\,, \qquad a'=1,2\,. \tag{4.37}$$

The other gauge-theory factor is the YM-scalar theory discussed in section 2.5. Its field content is

$$\left(A_\mu^{\hat{a}}\,, \phi^{1\hat{a}}\,, \phi^{a\hat{a}}\,, \varphi_\alpha^{\hat{\alpha}}\,, \overline{\varphi}_{\hat{\alpha}}^{\alpha}\right)\,. \tag{4.38}$$

We will verify that the double-copy of these theories yields the spontaneously-broken generic Jordan family YMESG theory with general gauge group.

To identify the result of the double-copy construction as one of the supergravities discussed in section 3, we want the theory to have an uplift to five dimensions. To this end, we need to single out a particular adjoint scalar which does not enter the trilinear couplings in eq. (2.42) and hence can combine with the four-dimensional gluons to produce the gluons of the five-dimensional theory. We will denote this scalar as $\phi^{1\hat{a}}$. In contrast, the scalars corresponding to non-vanishing F^{abc} will be denoted as $\phi^{a\hat{a}}$, where the index a runs over the multiplets transforming in the adjoint representation of the unbroken gauge group $\tilde{K}$ ($a=2,3,\ldots,\dim(\tilde{K})+1$) and can also include extra spectator fields, when present. With a slight change of notation from section 2.5, the global-group generator proportional to the masses will be denoted as $F^2{}_\alpha{}^\beta$. The corresponding scalar field will be called ϕ^2, while $\phi^3,\phi^4,\ldots$ will be the other massless scalars partaking to the trilinear interactions controlled by the F^{abc} tensors. This shift of indices is necessary to "align" the gauge-theory global indices with the supergravity gauge adjoint indices, as the supergravity always has at least two spectator multiplets.

4.2.1 Three-point amplitudes and double-copy field map

We begin by finding the three-point amplitudes of two massive scalars and a massless non-spectator scalar in a spontaneously-broken generic Jordan family YMESG theory. There are two such amplitudes,

$$\begin{aligned}\mathcal{M}_3(1\varphi_\alpha, 2\overline{\varphi}^\beta, 3y^2) &= -i\Big(\frac{\kappa}{2}\Big)\frac{\sqrt{2}g_s m}{\sqrt{1-V_s^2}}\tilde{F}^2{}_\beta{}^\alpha \ , &(4.39)\\ \mathcal{M}_3(1\varphi_\alpha, 2\overline{\varphi}^\beta, 3y^a) &= -i\Big(\frac{\kappa}{2}\Big)\sqrt{2}g_s m\tilde{F}^a{}_\beta{}^\alpha \ . &(4.40)\end{aligned}$$

The first amplitude involves the scalar of the preferred vector multiplet which contains the fluctuations of the field acquiring a VEV, while the second involves the other massless scalars transforming in the adjoint representation of the unbroken gauge group. Note that we need both massive scalars to have the same mass in order for the amplitude to be non-zero.

It is natural to expect that these amplitudes are reproduced by the double copy

$$\begin{aligned}\mathcal{A}_3(1\varphi, 2\overline{\varphi}, 3\phi^0)\Big|_{\mathcal{N}=2} \otimes \mathcal{A}_3(1\varphi, 2\overline{\varphi}, 3\phi^2)\Big|_{\mathcal{N}=0} &= -\frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)\lambda m\sqrt{1+\rho^2}\tilde{F}^a{}_\beta{}^\alpha \ , &(4.41)\\ \mathcal{A}_3(1\varphi, 2\overline{\varphi}, 3\phi^0)\Big|_{\mathcal{N}=2} \otimes \mathcal{A}_3(1\varphi, 2\overline{\varphi}, 3\phi^a)\Big|_{\mathcal{N}=0} &= -\frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)\lambda m\tilde{F}^a{}_\beta{}^\alpha \ , &(4.42)\end{aligned}$$

where we have used eqs. (4.2) and (2.41). This double-copy can be constructed because eq. (2.46) guarantees that the massive fields have equal masses. The massless scalar ϕ^0 in the $\mathcal{N}=2$ theory is the fluctuation of the field that acquires the VEV, and the scalar ϕ^2 of the $\mathcal{N}=0$ theory is the scalar corresponding to the $U(1)$ generator related to the mass.

The amplitudes (4.39), (4.40) are equal to the amplitudes (4.41), (4.42) provided that we identify

$$\Big(\frac{\kappa}{2}\Big)\lambda = 2g_s \ , \qquad \rho = \frac{V_s}{\sqrt{1-V_s^2}} \ , \qquad f^{ABC}_{\text{sugra}} = F^{ABC} \ . \tag{4.43}$$

This identification, together with the relation between the gauge-theory mass and preferred $U(1)$ generator (2.40), leads precisely to the expression for the mass in the spontaneously-broken supergravity (3.49). The other supergravity amplitudes with two massive fields are:

$$\begin{aligned}\mathcal{M}_3(1\varphi_\alpha, 2\overline{\varphi}^\beta, 3y^0) &= \sqrt{2}i\Big(\frac{\kappa}{2}\Big)m^2\delta^\alpha_\beta \ ,\\ \mathcal{M}_3(1\varphi_\alpha, 2\overline{\varphi}^\beta, 3A^-) &= \frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)m\,\varepsilon_3\cdot(k_1-k_2)\delta^\alpha_\beta \ ,\\ \mathcal{M}_3(1\varphi_\alpha, 2\overline{\varphi}^\beta, 3A^0) &= -\frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)m\,\varepsilon_3\cdot(k_1-k_2)\delta^\alpha_\beta \ ,\end{aligned}$$

$$
\begin{aligned}
\mathcal{M}_3(1\varphi_\alpha,2\overline{\varphi}^\beta,3A^a) &= \frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)\lambda\,\varepsilon_3\cdot(k_1-k_2)\Delta^{ab}\tilde{F}^b{}_\beta{}^\alpha\ ,\\
\mathcal{M}_3(1\varphi_\alpha,2\overline{W}^\beta,3x^a) &= -\frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)\lambda\,\varepsilon_2\cdot(k_1-k_3)\Delta^{ab}\tilde{F}^b{}_\beta{}^\alpha\ ,\\
\mathcal{M}_3(1\overline{\varphi}^\alpha,2W_\beta,3x^a) &= -\frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)\lambda\,\varepsilon_2\cdot(k_1-k_3)\Delta^{ab}\tilde{F}^b{}_\alpha{}^\beta\ ,\\
\mathcal{M}_3(1W_\alpha,2\overline{W}^\beta,3A^-) &= -\frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)m\,\varepsilon_3\cdot(k_1-k_2)\,\varepsilon_1\cdot\varepsilon_2\delta^\alpha_\beta\ ,\\
\mathcal{M}_3(1W_\alpha,2\overline{W}^\beta,3A^0) &= \frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)m\,\varepsilon_3\cdot(k_1-k_2)\,\varepsilon_1\cdot\varepsilon_2\delta^\alpha_\beta\ ,\\
\mathcal{M}_3(1W_\alpha,2\overline{W}^\beta,3A^a) &= -\frac{i}{2\sqrt{2}}\Big(\frac{\kappa}{2}\Big)\lambda\big(\varepsilon_3\cdot(k_1-k_2)\varepsilon_1\cdot\varepsilon_2+\text{cyclic}\big)\Delta^{ab}\tilde{F}^b{}_\beta{}^\alpha\ ,\\
\mathcal{M}_3(1W_\alpha,2\overline{W}^\beta,3x^0) &= -\sqrt{2}i\Big(\frac{\kappa}{2}\Big)\,\epsilon(k_1,k_2,\varepsilon_1,\varepsilon_2)\delta^\alpha_\beta\ ,\\
\mathcal{M}_3(1W_\alpha,2\overline{W}^\beta,3y^0) &= -\sqrt{2}i\Big(\frac{\kappa}{2}\Big)(m^2\varepsilon_1\cdot\varepsilon_2+\varepsilon_1\cdot p_2\,\varepsilon_2\cdot p_1)\delta^\alpha_\beta\ ,\\
\mathcal{M}_3(1W_\alpha,2\overline{W}^\beta,3y^a) &= \frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)m\lambda\,\varepsilon_2\cdot\varepsilon_3\,\Delta^{ab}\tilde{F}^b{}_\beta{}^\alpha\ ,\\
\mathcal{M}_3(1\varphi_\alpha,2\overline{W}^\beta,3A^-) &= \sqrt{2}i\Big(\frac{\kappa}{2}\Big)\,\epsilon(k_2,k_3,\varepsilon_2,\varepsilon_3)\delta^\alpha_\beta\ ,\\
\mathcal{M}_3(1\varphi_\alpha,2\overline{W}^\beta,3A^0) &= \sqrt{2}i\Big(\frac{\kappa}{2}\Big)\,\epsilon(k_2,k_3,\varepsilon_2,\varepsilon_3)\delta^\alpha_\beta\ ,\\
\mathcal{M}_3(1\overline{\varphi}^\beta,2W_\alpha,3A^0) &= \sqrt{2}i\Big(\frac{\kappa}{2}\Big)\,\epsilon(k_2,k_3,\varepsilon_2,\varepsilon_3)\delta^\alpha_\beta\ ,
\end{aligned}
\tag{4.44}
$$

where the Levi-Civita tensor is normalized as $\epsilon^{0123}=-1$. These amplitudes can be reproduced by the double-copy prescription with the following field identification:

$$
\begin{aligned}
\tfrac{1}{\sqrt{2}}(A^{-1}_\pm-A^0_\pm) &= \phi^0\big|_{\mathcal{N}=2}\otimes A_\pm\big|_{\mathcal{N}=0}\,, & A^a_\pm &= A_\pm\big|_{\mathcal{N}=2}\otimes\phi^a\big|_{\mathcal{N}=0}\,,\\
\pm i\tfrac{1}{\sqrt{2}}(A^{-1}_\pm+A^0_\pm) &= \phi^1\big|_{\mathcal{N}=2}\otimes A_\pm\big|_{\mathcal{N}=0}\,, & \pm\alpha iA^1_\pm &= A_\pm\big|_{\mathcal{N}=2}\otimes\phi^1\big|_{\mathcal{N}=0}\,,\\
\tfrac{1}{\sqrt{2}}(y^0+ix^0) &= A_+\big|_{\mathcal{N}=2}\otimes A_-\big|_{\mathcal{N}=0}\,, & \tfrac{1}{\sqrt{2}}(y^0-ix^0) &= A_-\big|_{\mathcal{N}=2}\otimes A_+\big|_{\mathcal{N}=0}\,,\\
\varphi_\alpha &= \varphi\big|_{\mathcal{N}=2}\otimes\varphi_\alpha\big|_{\mathcal{N}=0}\,, & W_\alpha &= W\big|_{\mathcal{N}=2}\otimes\varphi_\alpha\big|_{\mathcal{N}=0}\,,\\
y^a &= \phi^0\big|_{\mathcal{N}=2}\otimes\phi^a\big|_{\mathcal{N}=0}\,, & x^a &= \phi^1\big|_{\mathcal{N}=2}\otimes\phi^a\big|_{\mathcal{N}=0}\,,\\
y^1 &= \phi^0\big|_{\mathcal{N}=2}\otimes\phi^1\big|_{\mathcal{N}=0}\,, & x^1 &= \phi^1\big|_{\mathcal{N}=2}\otimes\phi^1\big|_{\mathcal{N}=0}\,.
\end{aligned}
\tag{4.45}
$$

The field ϕ^1 is a distinguished spectator scalar in the YM-scalar theory which does not enter the trilinear couplings and is necessary for the theory to have a five-dimensional uplift, and the index a runs over the vector multiplets transforming in the adjoint representation of the unbroken gauge group plus extra spectator fields, when present. The free parameter $\alpha=\pm1$ reflects the symmetry $\phi^1\to-\phi^1$ of the $\mathcal{N}=0$ gauge-theory factor. Note that the spectator vectors $A_\mu^{-1,0,1}$ and spectator scalars $x^{0,1},y^{0,1}$ are always present due to the choice of compact gauging and to the requirement of a five-dimensional uplift.

In the case in which the supergravity has more than one flavor of massive vectors, additional multi-flavor amplitudes become possible,

$$\begin{aligned}\mathcal{M}_3(1W_\alpha, 2W_\beta, 3\overline{W}^\gamma) &= \frac{i}{2\sqrt{2}}\Big(\frac{\kappa}{2}\Big)\lambda\big(\varepsilon_3\cdot(k_1-k_2)\varepsilon_1\cdot\varepsilon_2+\text{cyclic}\big)\tilde{F}^{\alpha\beta}{}_{\gamma}\,,\\ \mathcal{M}_3(1W_\alpha, 2\varphi_\beta, 3\overline{\varphi}^\gamma) &= -\frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)\lambda\varepsilon_1\cdot k_2\tilde{F}^{\alpha\beta}{}_{\gamma}\,. \end{aligned} \tag{4.46}$$

They are reproduced by a double-copy construction with multi-flavor gauge theories.

4.2.2 Four-point amplitudes

To test the identification of parameters and fields constructed in section 4.2.1 we construct selected four-point amplitudes with two and four massive fields and compare them with the double-copy construction. We start with the supergravity amplitude with four massive scalars. Using the results from the previous sections, it can be expressed in the following form,

$$\mathcal{M}_4\big(1\varphi_\alpha, 2\varphi_\beta, 3\overline{\varphi}^\gamma, 4\overline{\varphi}^\delta\big) = -i\Big(\frac{\kappa}{2}\Big)^2\Big(\frac{n_1\tilde{n}_1}{D_1}+\frac{n_2\tilde{n}_2}{D_2}+\frac{n_3\tilde{n}_3}{D_3}+\frac{n_4\tilde{n}_4}{D_4}+\frac{n_5\tilde{n}_5}{D_5}+\frac{n_6\tilde{n}_6}{D_6}+\frac{n_7\tilde{n}_7}{D_7}\Big)\,, \tag{4.47}$$

where the numerators are given by (4.17) and (4.36). It is instructive to verify that all poles in the above amplitude correspond to the exchange of a particle of the theory. Specifically, for a given assignment of external masses, aside from two massless channels, there are three massive channels with square masses $(m_1+m_2)^2, (m_1-m_3)^2$ and $(m_1-m_4)^2$. One can see that, thanks to the relations (4.24), the numerators n_2, n_3, n_5, n_6, n_7 in (4.17) are either zero or proportional to inverse propagators when the mass of the intermediate channel is not one of the masses of the particles in the theory (i.e. one the eigenvalues of the matrix $m_\alpha{}^\beta$).

We have verified that the expression (4.47) reproduces the one from a Feynman-rule computation, once the field map (4.45) is employed. The expression for the general four massive scalar amplitude substantially simplifies in the simplest case in which the supergravity has a $SU(2)$ gauge group which is spontaneously-broken to its $U(1)$ subgroup. In this case only one flavor of massive fields is present, $\alpha, \beta \equiv 1$, and the structure constants become

$$F^{\alpha}{}_{\beta}{}^{\gamma} \equiv 0\,, \qquad F^{2}{}_{\alpha}{}^{\beta} \equiv i\,. \tag{4.48}$$

It is also convenient to absorb the ρ-dependent factor in the definition of λ,

$$\tilde{\lambda} = \sqrt{1+\rho^2}\lambda\,. \tag{4.49}$$

The amplitude has a simple expression,

$$\mathcal{M}_4(1\varphi, 2\varphi, 3\overline{\varphi}, 4\overline{\varphi}) = \frac{i}{2}\Big(\frac{\kappa}{2}\Big)^2(\tilde{\lambda}^2-4k_1\cdot k_2)\left[1+2\frac{k_1\cdot k_2-m^2}{(k_2+k_3)^2}+2\frac{k_1\cdot k_2-m^2}{(k_2+k_4)^2}\right]\,. \tag{4.50}$$

In this particular case, all non-zero amplitudes with two massive fields have simple expressions, and we list here some of them:

$$
\begin{aligned}
&\mathcal{M}_4(1x^0,2\overline{\varphi},3x^0,4\varphi) = i\Big(\frac{\kappa}{2}\Big)^2\left[-2m^2+4\frac{k_1\cdot k_4 k_3\cdot k_4}{(k_1+k_3)^2}\right]\ ,\\
&\mathcal{M}_4(1y^0,2\overline{\varphi},3y^0,4\varphi) = i\Big(\frac{\kappa}{2}\Big)^2\left[-2m^2-\frac{m^4}{k_2\cdot k_3}-\frac{m^4}{k_3\cdot k_4}+4\frac{k_2\cdot k_3 k_3\cdot k_4}{(k_2+k_4)^2}\right]\ ,\\
&\mathcal{M}_4(1x^1,2\overline{\varphi},3x^1,4\varphi) = i\Big(\frac{\kappa}{2}\Big)^2\left[-2m^2+4\frac{k_2\cdot k_3 k_3\cdot k_4}{(k_2+k_4)^2}\right]\ ,\\
&\mathcal{M}_4(1y^0,2\overline{\varphi},3y^2,4\varphi) = \frac{i}{\sqrt{2}}\Big(\frac{\kappa}{2}\Big)^2\tilde{\lambda}\,m\left[2+\frac{m^2}{k_2\cdot k_3}+\frac{m^2}{k_3\cdot k_4}\right]\ ,\\
&\mathcal{M}_4(1y^1,2\overline{\varphi},3y^1,4\varphi) = i\Big(\frac{\kappa}{2}\Big)^2\left[-(k_2+k_4)^2+4\frac{k_2\cdot k_3 k_3\cdot k_4}{(k_2+k_4)^2}\right]\ ,\\
&\mathcal{M}_4(1y^2,2\overline{\varphi},3y^2,4\varphi) = i\Big(\frac{\kappa}{2}\Big)^2\left[-\frac{\tilde{\lambda}^2}{2}-2m^2-\frac{m^2\tilde{\lambda}^2}{2k_2\cdot k_3}-\frac{m^2\tilde{\lambda}^2}{2k_3\cdot k_4}+4\frac{k_2\cdot k_3 k_3\cdot k_4}{(k_2+k_4)^2}\right]\ ,\\
&\mathcal{M}_4(1x^2,2\overline{\varphi},3x^2,4\varphi) = i\Big(\frac{\kappa}{2}\Big)^2\left(\frac{\tilde{\lambda}^2}{2}-(k_2+k_4)^2+\frac{\tilde{\lambda}^2}{2}\frac{k_3\cdot k_4}{k_2\cdot k_3}+\frac{\tilde{\lambda}^2}{2}\frac{k_2\cdot k_3}{k_3\cdot k_4}+4\frac{k_2\cdot k_3 k_3\cdot k_4}{(k_2+k_4)^2}\right).
\end{aligned}
\tag{4.51}
$$

There also exist amplitudes which vanish due to non-trivial cancellations. Among them there are

$$
\begin{aligned}
\mathcal{M}_4(1x^1,2\overline{\varphi},3x^2,4\varphi) &= 0\ ,\\
\mathcal{M}_4(1y^0,2\overline{\varphi},3y^1,4\varphi) &= 0\ ,\\
\mathcal{M}_4(1y^1,2\overline{\varphi},3y^2,4\varphi) &= 0\ .
\end{aligned}
\tag{4.52}
$$

We have explicitly checked that the result of the double-copy calculation for the amplitudes listed above matches the given expressions.

As an interesting aside, we note that the double copy of spontaneously-broken YM theory with itself, namely $\cancel{\text{YM}}\otimes\cancel{\text{YM}}$, is a valid construction in the current treatment. To understand what the result might be, let us take spontaneously-broken gauge theories with all masses equal and consider the scattering of four massive scalars. The kinematic numerator factor $\tilde{n}_7$ in eq. (4.36) is nonvanishing when $(k_1+k_2)^2=(2m)^2$ and thus the graph 7 in figure 5 exhibits a pole for such momentum configuration. This pole does not contribute to the gauge-theory amplitude due to the vanishing of the color factor c_7 in eq. (4.36). However, through the double copy, this pole features in the corresponding supergravity amplitude and signals the existence of a state of mass $2m$ in the spectrum. Such state is not part of the naive spectrum – the gauge invariant part of the tensor product of the two gauge-theory spectra; unitarity requires it to be included. The argument can be repeated starting from higher-point gauge-theory amplitudes and leads to the extension of the naive spectrum by an infinite number of states with equally spaced masses, $m_n = nm$ with n integer. These states also carry maximum spin two. Following from the discussion in appendix B.1 the amplitudes generated by the double-copy construction $\cancel{\text{SYM}}\otimes\cancel{\text{SYM}}$ should belong to $(D{+}1)$-dimensional Kaluza-Klein supergravity.

5 Loop amplitudes

Here we work out one of the simpler one-loop amplitudes in explicitly broken YM + ϕ^3 in a form that obeys color/kinematics duality. Then, using the corresponding amplitudes in spontaneously-broken SYM, the double copy gives the one-loop four-vector amplitude in spontaneously-broken YMESG.

5.1 One-loop massless-scalar amplitude in broken YM + ϕ^3 theory

Consider the one-loop amplitude for four massless external scalars in the explicitly broken YM + ϕ^3 theory. We write the complete amplitude in the cubic-diagram form (2.1), decomposed over the massless and massive internal modes,

$$\mathcal{A}_4^{\text{1-loop}} = g^4 \sum_{\mathcal{S}_4} \sum_{i\in\{\text{box,tri,bub}\}} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{S_i} \left(\frac{n_i c_i}{D_i} + \sum_\alpha \frac{n_{i,\alpha}\, c_i}{D_{i,\alpha}} + \sum_\alpha \frac{\overline{n}_{i,\alpha}\, c_i}{D_{i,\alpha}} \right), \tag{5.1}$$

where the first sum runs over the permutations $\mathcal{S}_4$ of all four external leg labels. The second sum runs over the three listed integral topologies, and the corresponding symmetry factors are $S_{\text{box}} = 8$, $S_{\text{tri}} = 4$ and $S_{\text{bub}} = 16$. The summation index α labels the massive modes, with mass $\pm m_\alpha$, inside the loop diagrams. The numerator without an α index corresponds to massless modes in the loop.

In the canonical order of the external legs, $(1, 2, 3, 4)$, the color factors are

$$\begin{aligned} c_{\text{box}} &= \tilde{f}^{\hat{b}\hat{a}_1\hat{c}}\,\tilde{f}^{\hat{c}\hat{a}_2\hat{d}}\,\tilde{f}^{\hat{d}\hat{a}_3\hat{e}}\,\tilde{f}^{\hat{e}\hat{a}_4\hat{b}}\,, \\ c_{\text{tri}} &= \tilde{f}^{\hat{a}_1\hat{a}_2\hat{c}}\,\tilde{f}^{\hat{b}\hat{c}\hat{d}}\,\tilde{f}^{\hat{d}\hat{a}_3\hat{e}}\,\tilde{f}^{\hat{e}\hat{a}_4\hat{b}}\,, \\ c_{\text{bub}} &= \tilde{f}^{\hat{a}_1\hat{a}_2\hat{c}}\,\tilde{f}^{\hat{b}\hat{c}\hat{d}}\,\tilde{f}^{\hat{d}\hat{e}\hat{b}}\,\tilde{f}^{\hat{e}\hat{a}_3\hat{a}_4}\,. \end{aligned} \tag{5.2}$$

The denominator factors in the canonical ordering are given by

$$\begin{aligned} &D_{\text{box}} = \ell_1^2\ell_2^2\ell_3^2\ell_4^2\,, \quad D_{\text{tri}} = s\ell_2^2\ell_3^2\ell_4^2\,, \quad D_{\text{bub}} = s^2\ell_2^2\ell_4^2\,, \\ &D_{\text{box},\alpha} = (\ell_1^2 - m_\alpha^2)(\ell_2^2 - m_\alpha^2)(\ell_3^2 - m_\alpha^2)(\ell_4^2 - m_\alpha^2)\,, \\ &D_{\text{tri},\alpha} = s(\ell_2^2 - m_\alpha^2)(\ell_3^2 - m_\alpha^2)(\ell_4^2 - m_\alpha^2)\,, \\ &D_{\text{bub},\alpha} = s^2(\ell_2^2 - m_\alpha^2)(\ell_4^2 - m_\alpha^2)\,, \end{aligned} \tag{5.3}$$

where $\ell_i = \ell - (k_1 + \ldots + k_i)$.

In ref. [27] the massless contributions to this amplitude were worked out using the unitarity method [90]; we quote the result again, in a slightly different form. We write the box numerator corresponding to massless fields, figure 6(a), as

$$n_{\rm box} = n^{(4)}_{\rm box} + n^{(2)}_{\rm box} + n^{(0)}_{\rm box}\,, \tag{5.4}$$

where the subscript denotes the order in λ. The $\mathcal{O}(\lambda^4)$ contribution of the box numerator, shown in figure 6(b), is entirely expressed in terms of the structure constants of the global group,

$$n^{(4)}_{\rm box}(1,2,3,4;\ell) = \frac{\lambda^4}{4} F^{ba_1c}F^{ca_2d}F^{da_3e}F^{ea_4b}\,. \tag{5.5}$$

The $\mathcal{O}(\lambda^2)$ numerator contributions, shown in figure 6(c), is given by

$$\begin{aligned} n^{(2)}_{\rm box}(1,2,3,4;\ell) \;=\; & \frac{\lambda^2}{24}\Big\{(N_\phi+D-2)\big(F^{a_1a_4b}F^{ba_3a_2}(\ell_2^2+\ell_4^2)+F^{a_1a_2b}F^{ba_3a_4}(\ell_1^2+\ell_3^2)\big)\\ & +24\big(sF^{a_1a_4b}F^{ba_3a_2}+tF^{a_1a_2b}F^{ba_3a_4}\big)+\delta^{a_3a_4}{\rm Tr}_{12}(6\ell_3^2-\ell_2^2-\ell_4^2)\\ & +\delta^{a_2a_3}{\rm Tr}_{14}(6\ell_2^2-\ell_1^2-\ell_3^2)+\delta^{a_1a_4}{\rm Tr}_{23}(6\ell_4^2-\ell_1^2-\ell_3^2)\\ & +\delta^{a_1a_2}{\rm Tr}_{34}(6\ell_1^2-\ell_2^2-\ell_4^2)+(\ell_1^2+\ell_2^2+\ell_3^2+\ell_4^2)(\delta^{a_2a_4}{\rm Tr}_{13}+\delta^{a_1a_3}{\rm Tr}_{24})\Big\}\,, \end{aligned} \tag{5.6}$$

and the $\mathcal{O}(\lambda^0)$ numerator contributions, shown in figure 6(d), is

$$\begin{aligned} n^{(0)}_{\rm box}(1,2,3,4;\ell) \;=\; & \frac{1}{24}\Big\{\delta^{a_1a_2}\delta^{a_3a_4}\big[24t(t-2\ell_1^2-2\ell_3^2)+2(N_\phi+D-2)(3\ell_1^2\ell_3^2-\ell_2^2\ell_4^2)\\ & \qquad +(N_\phi+D+14)\big(t(\ell_1^2+\ell_2^2+\ell_3^2+\ell_4^2)-u(\ell_1^2+\ell_3^2)\big)\big]\\ & +\delta^{a_2a_3}\delta^{a_1a_4}\big[24s(s-2\ell_2^2-2\ell_4^2)+2(N_\phi+D-2)(3\ell_2^2\ell_4^2-\ell_1^2\ell_3^2)\\ & \qquad +(N_\phi+D+14)\big(s(\ell_1^2+\ell_2^2+\ell_3^2+\ell_4^2)-u(\ell_2^2+\ell_4^2)\big)\big]\\ & +\delta^{a_1a_3}\delta^{a_2a_4}\big[2(N_\phi+D-2)(\ell_1^2\ell_3^2+\ell_2^2\ell_4^2)\\ & \qquad -(N_\phi+D+14)\big(s(\ell_1^2+\ell_3^2)+t(\ell_2^2+\ell_4^2)\big)\big]\Big\}\,. \end{aligned} \tag{5.7}$$

As before $\ell_i = \ell - (k_1 + \ldots + k_i)$ and in eq. (5.6) we use the shorthand notation ${\rm Tr}_{ij} = F^{ba_ic}F^{ca_jb}$. The parameter $N_\phi = \delta^{ab}\delta_{ab}$ is the number of massless scalars in the D-dimensional theory.

Finally, the numerator of the massive diagrams, figure 6(e) and figure 6(f), are conjugates of each other, and given by

$$\begin{aligned} n_{{\rm box},\alpha}(1,2,3,4;\ell) \;&=\; \frac{\lambda^4}{4}\widehat{F}^{a_1}{}_{\alpha}{}^{\beta}\widehat{F}^{a_2}{}_{\beta}{}^{\gamma}\widehat{F}^{a_3}{}_{\gamma}{}^{\delta}\widehat{F}^{a_4}{}_{\delta}{}^{\alpha}\,, \qquad (\text{no sum }\alpha)\\ \overline{n}_{{\rm box},\alpha}(1,2,3,4;\ell) \;&=\; \frac{\lambda^4}{4}\widehat{F}^{a_4}{}_{\alpha}{}^{\delta}\widehat{F}^{a_3}{}_{\beta}{}^{\gamma}\widehat{F}^{a_2}{}_{\gamma}{}^{\beta}\widehat{F}^{a_1}{}_{\beta}{}^{\alpha}\,, \qquad (\text{no sum }\alpha) \end{aligned} \tag{5.8}$$

where $\widehat{F}^{a}{}_{\alpha}{}^{\beta} \equiv \Delta^{ab}F^{b}{}_{\alpha}{}^{\beta}$. Since the mass depends on the index α we do not yet sum over the this index since the numerator has to first be combined with the correct denominator factor

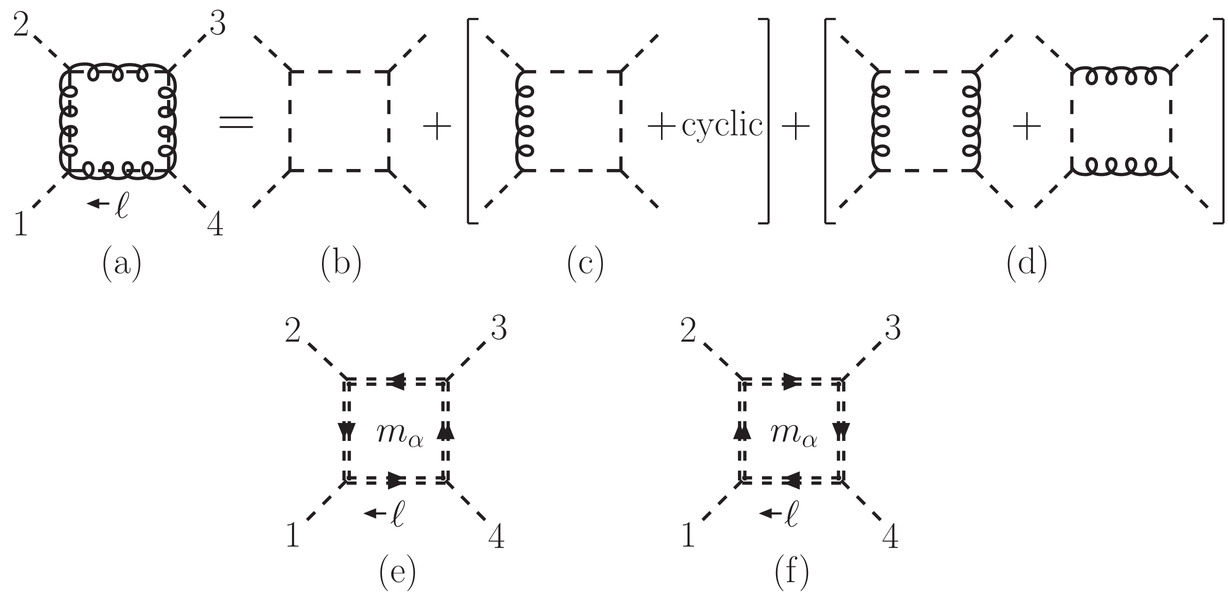


Figure 6: The different types of box diagrams that contribute to the one-loop amplitude with four massless scalars in explicitly broken YM + ϕ^3 theory. Diagram (a) denotes the sum of all box diagrams with massless internal states; the contribution (b) is of order λ^4, (c) is of order λ^2, and (d) is of order λ^0. Additionally, there are two conjugate diagrams (e), (f) with internal scalars of mass m_α. Dashed lines denote scalar fields, double lines of these corresponds to massive scalars, while curly lines denote vector fields. Note that quartic-scalar interactions are implicitly included in these diagrams, according to their power in the λ coupling.

(this is akin to not integrating over the loop momenta in numerators when they are not yet combined with their denominators).

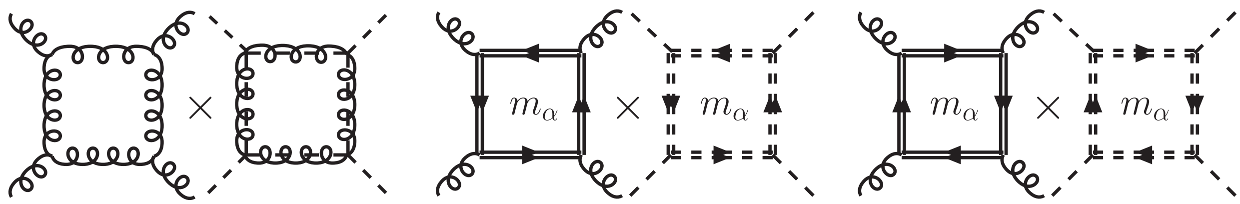
The box numerators have been constructed so to manifestly obey color/kinematics duality. In particular, the numerator factors for the remaining contributing diagrams, the triangles and bubbles, are given by the kinematic Lie algebra relations. For the massless numerators we have

$$\begin{aligned} n_{\text{tri}}(1,2,3,4;\ell) &= n_{\text{box}}(1,2,3,4;\ell) - n_{\text{box}}(2,1,3,4;\ell)\,, \\ n_{\text{bub}}(1,2,3,4;\ell) &= n_{\text{tri}}(1,2,3,4;\ell) - n_{\text{tri}}(1,2,4,3;\ell)\,, \end{aligned} \tag{5.9}$$

and for the massive ones, by virtue of the global-group Lie algebra, we have

$$\begin{aligned} n_{\text{tri},\alpha}(1,2,3,4;\ell) &= n_{\text{box},\alpha}(1,2,3,4;\ell) - n_{\text{box},\alpha}(2,1,3,4;\ell)\,, \\ n_{\text{bub},\alpha}(1,2,3,4;\ell) &= n_{\text{tri},\alpha}(1,2,3,4;\ell) - n_{\text{tri},\alpha}(1,2,4,3;\ell)\,, \end{aligned} \tag{5.10}$$

and similarly for the conjugate ones, $\overline{n}_{i,\alpha}$.



(a) (b) (c)

Figure 7: The three types of box diagrams in the one-loop four-vector spontaneously-broken YMESG amplitude. The contributions are, (a) the graviton and massless vector, (b) the massive W_α vectors, and (c) the massive $\overline{W}^\alpha$ vectors. These are given by double copies between spontaneously-broken SYM (left factors) and explicitly broken YM + ϕ^3 theory (right factors). The remaining triangle and bubble contributions are obtained from the boxes through the Jacobi and commutation relations of color/kinematics duality. See figure 6 for notation.

5.2 One-loop four-vector Yang-Mills-gravity amplitudes

The double-copy procedure, inherent in color/kinematics duality, provides a straightforward way to construct loop amplitudes in spontaneously-broken YMESG theory. For example, figure 7 illustrates how to obtain the different types of contributions – massless graviton and vector multiplets, massive W_α and $\overline{W}^\alpha$ multiplets – as double copies between spontaneously-broken SYM numerators and the explicitly broken YM + ϕ^3 numerators computed in the previous section.

The complete one-loop amplitude with four massless external vectors in spontaneously-broken YMESG theory, is given by the double-copy form (2.5),

$$\mathcal{M}_4^{\text{1-loop}} = \left(\frac{\kappa}{2}\right)^4 \sum_{\mathcal{S}_4} \sum_{i\in\{\text{box,tri,bub}\}} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{S_i} \left(\frac{n_i^{\text{SYM}}\, n_i}{D_i} + \sum_\alpha \frac{n_{i,\alpha}^{\text{SYM}}\, n_{i,\alpha}}{D_{i,\alpha}} + \sum_\alpha \frac{\overline{n}_{i,\alpha}^{\text{SYM}}\, \overline{n}_{i,\alpha}}{D_{i,\alpha}} \right), \tag{5.11}$$

where the sums, symmetry factors, and denominator factors, are the same as in eqs. (5.1) and (5.3). As before, the n_i are the numerators of explicitly broken YM + ϕ^3 theory given in section 5.1, and the n_i^{SYM} are numerators of spontaneously-broken SYM theory and we identify the combination $\kappa\lambda/2$ with the supergravity gauge coupling g.

For the one-loop amplitudes, the spontaneously-broken D-dimensional SYM numerators are given by $(D+1)$-dimensional SYM numerators with the last component of the loop momentum interpreted as mass: $\ell \to (\ell, \pm m_\alpha)$. We may write the numerators as

$$\begin{aligned} n_i^{\text{SYM}} &\equiv n_i^{\text{SYM}}(1,2,3,4;\ell;0)\,, \\ n_{i,\alpha}^{\text{SYM}} &\equiv n_i^{\text{SYM}}(1,2,3,4;\ell;m_\alpha)\,, \\ \overline{n}_{i,\alpha}^{\text{SYM}} &\equiv n_i^{\text{SYM}}(1,2,3,4;\ell;-m_\alpha)\,, \end{aligned} \tag{5.12}$$

meaning that the numerators for massive and massless internal states are described by the same function, only the value of m_α differs between them.[31]

Specifying to the maximally supersymmetric case, the box numerator of spontaneously-broken $\mathcal{N}=4$ SYM is given by

$$n_{\text{box}}^{\mathcal{N}=4\text{ SYM}}(1,2,3,4;\ell;m) = istA^{\text{tree}}(1,2,3,4) = \frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\delta^{(8)}\Big(\sum \eta_i^\alpha|i\rangle\Big)\,, \tag{5.13}$$

independent of the mass parameter m, whether zero or not, in agreement with ref. [76]. The corresponding triangle and bubble numerators vanish for this theory. Plugging this into (5.11) gives the four-vector amplitude in spontaneously-broken $\mathcal{N}=4$ YMESG theory.

The $\mathcal{N}=2$ SYM one-loop numerator factors may be written as the difference between $\mathcal{N}=4$ SYM and numerator factors for one adjoint $\mathcal{N}=2$ hypermultiplet running in the loop,

$$n_i^{\mathcal{N}=2\text{ SYM}}(1,2,3,4;\ell;m) = n_i^{\mathcal{N}=4\text{ SYM}}(1,2,3,4;\ell;m) - 2n_i^{\mathcal{N}=2,\text{mat.}}(1,2,3,4;\ell;m)\,. \tag{5.14}$$

Color/kinematics-satisfying one-loop numerator factors due to one adjoint hypermultiplet running in the loop may be found in refs. [15, 18, 16, 17]. A manifestly $\mathcal{N}=2$-supersymmetric box numerator valid for D-dimensional loop momenta was given in ref. [17]; introducing the mass-dependence we find

$$\begin{aligned}
n_{\text{box}}^{\mathcal{N}=2,\text{mat.}}(1,2,3,4,\ell;m) \;=\;& (\kappa_{12}+\kappa_{34})\frac{(s-\ell_s)^2}{2s^2} + (\kappa_{23}+\kappa_{14})\frac{\ell_t^2}{2t^2} + (\kappa_{13}+\kappa_{24})\frac{st+(s+\ell_u)^2}{2u^2}\\
&+ (\mu^2+m^2)\Big(\frac{\kappa_{12}+\kappa_{34}}{s} + \frac{\kappa_{23}+\kappa_{14}}{t} + \frac{\kappa_{13}+\kappa_{24}}{u}\Big)\\
&+ 2i\epsilon(k_1,k_2,k_3,\ell)\frac{\kappa_{13}-\kappa_{24}}{u^2}\,,
\end{aligned} \tag{5.15}$$

where $\ell_s = 2\ell\cdot(k_1+k_2)$, $\ell_t = 2\ell\cdot(k_2+k_3)$ and $\ell_u = 2\ell\cdot(k_1+k_3)$. The numerator factors of other box integrals are obtained by relabeling. The parameter μ is the component of the loop momenta that is orthogonal to four-dimensional spacetime, and $\epsilon(k_1,k_2,k_3,\ell) = k_1^\mu k_2^\nu k_3^\rho \ell^\lambda \epsilon_{\mu\nu\rho\lambda}$ is the Levi-Civita invariant. The external multiplet dependence is captured by the variables κ_{ij},

$$\kappa_{ij} = -\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle}\delta^{(4)}\Big(\sum \eta_i^\alpha|i\rangle\Big)\,\langle i\,j\rangle^2\,(\eta_i^3\eta_i^4)(\eta_j^3\eta_j^4)\,. \tag{5.16}$$

As before, the triangle and bubble numerators are given by the kinematic Jacobi relations,

$$\begin{aligned}
n_{\text{tri}}^{\mathcal{N}=2\text{ SYM}}(1,2,3,4;\ell;m) &= n_{\text{box}}^{\mathcal{N}=2\text{ SYM}}(1,2,3,4;\ell;m) - n_{\text{box}}^{\mathcal{N}=2\text{ SYM}}(2,1,3,4;\ell;m)\,,\\
n_{\text{bub}}^{\mathcal{N}=2\text{ SYM}}(1,2,3,4;\ell;m) &= n_{\text{tri}}^{\mathcal{N}=2\text{ SYM}}(1,2,3,4;\ell;m) - n_{\text{tri}}^{\mathcal{N}=2\text{ SYM}}(1,2,4,3;\ell;m)\,,
\end{aligned} \tag{5.17}$$

[31]Note that the numerators of spontaneously-broken SYM are thus straightforward to obtain from the massless theory in $(D+1)$ dimensions, in contrast to the numerators of explicitly broken YM + ϕ^3 theory which in general have no simple relation to the massless numerators.

which have no mass-dependence, since the mass term in eq. (5.15) is totally symmetric.

Plugging the $\mathcal{N}=2$ SYM numerators, together with the YM + ϕ^3 numerators (5.4), into equation (5.11) gives a four-vector amplitude in the spontaneously-broken $\mathcal{N}=2$ YMESG theory. The parameter N_ϕ in eq. (5.4) and (5.7) is identified with the number of massless vector multiplets (i.e. the number of massless vector fields excluding those in the graviton multiplet). One may verify the construction by observing that the unitarity cuts of these amplitudes match the direct evaluation of cuts in terms of tree diagrams.

It is not difficult to integrate the resulting expression and find the divergence of the four-vector amplitude. As usual, the masses do not enter the UV divergence, which is the same as that of the unbroken theory; it is naturally organized in the powers of λ.

- The $\mathcal{O}(\lambda^0)$ part of the amplitude is the same as in the MESG theory with the same field content. The four-vector amplitude diverges in four dimensions and, as a term in the effective action, the divergence is proportional to the square of the vector field stress tensor [91, 92].

- The $\mathcal{O}(\lambda^2)$ part of the amplitude is finite in four (and five) dimensions; it is given by a combination of the four- and six-dimensional box integrals with tree-level color structures.

- Since the $\mathcal{O}(\lambda^4)$ part of the YM + ϕ^3 numerators is momentum-independent, the divergence at this order is proportional to the divergence of the four-gluon amplitude in the $\mathcal{N}=2$ SYM theory. In the UV limit the masses drop out and the sum over the index α leads to a factor of the index of the adjoint representation, $T(A)\delta^{ab}=F^{acd}F^{bcd}$.

Next consider the maximally-helicity-violating (MHV) amplitude in $\mathrm{YM_{DR}}$ theory, which is the generalization of the bosonic part of SYM theories. For four-dimensional external states, the one-loop numerator factors may again be written as the difference between SYM numerators and numerator factors for scalar matter running in the loop,

$$\begin{aligned} n_i^{\mathrm{YM_{DR}}}(1,2,3,4;\ell;m) \;&=\; n_i^{\mathcal{N}=4\ \mathrm{SYM}}(1,2,3,4;\ell;m)-4n_i^{\mathcal{N}=2,\mathrm{mat.}}(1,2,3,4;\ell;m)\\ &\quad +(2+N'_\phi)\,n_{\mathrm{box}}^{\mathrm{YM_{DR},mat.}}(1,2,3,4;\ell;m)\,. \end{aligned} \tag{5.18}$$

where N'_ϕ is the number of real scalars in the loop (also counting the Goldstone boson). In the gravity theory this number gives the number of real vector fields in the graviton multiplet.

A box numerator for a four-vector amplitude with a single scalar running in the loop in the $\mathrm{YM_{DR}}$ theory, valid for D-dimensional loop momenta and four-dimensional external states, was given in ref. [17],

$$
\begin{aligned}
&n_{\rm box}^{\rm YM_{DR},mat.}(1,2,3,4,\ell;m) = \\
&\qquad -(\kappa_{12}+\kappa_{34})\Big(\frac{\ell_s^4}{4s^4}-\frac{\ell_s^2(2L+3\ell_s)}{4s^3}+\frac{2L\ell_s+\ell_s^2-2M^2}{2s^2}-\frac{2L-\ell_s+s}{4s}\Big)\\
&\qquad -(\kappa_{23}+\kappa_{14})\Big(\frac{\ell_t^4}{4t^4}-\frac{\ell_t^2(2L-\ell_s-\ell_u+t)}{4t^3}-\frac{M^2}{t^2}\Big)\\
&\qquad -(\kappa_{13}+\kappa_{24})\Big(\frac{\ell_u^3(\ell_u+3s)}{4u^4}-\frac{\ell_u(\ell_u(2L-\ell_s)-\ell_s^2+\ell_t^2+4s(L+\ell_u+2M))}{4u^3}\\
&\qquad\qquad -\frac{\ell_s^2-\ell_t^2+3\ell_u^2+4Lt+8M(\ell_u-s+M)}{8u^2}-\frac{\ell_s-s}{4u}\Big)\\
&\qquad -2i\epsilon(k_1,k_2,k_3,\ell)(\kappa_{13}-\kappa_{24})\frac{\ell_u^2-u\ell_u-2Mu}{u^4}\,.
\end{aligned}
\tag{5.19}
$$

with

$$
L=\ell^2-m^2 \qquad \text{and} \qquad M=\mu^2+m^2\,. \tag{5.20}
$$

In the above non-supersymmetric expressions it is understood that only the vector components of κ_{ij} should be kept; that is, in eq. (5.18) and eq. (5.19) we take

$$
\kappa_{ij}\to\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle}\,\langle i\,j\rangle^4\,(\eta_i^1\eta_i^2\eta_i^3\eta_i^4)(\eta_j^1\eta_j^2\eta_j^3\eta_j^4)\,. \tag{5.21}
$$

As before, the box numerator was constructed to obey color/kinematics duality, thus the triangle and bubble numerators are given by the kinematic Jacobi relations,

$$
\begin{aligned}
n_{\rm tri}^{\rm YM_{DR}}(1,2,3,4;\ell;m) &= n_{\rm box}^{\rm YM_{DR}}(1,2,3,4;\ell;m)-n_{\rm box}^{\rm YM_{DR}}(2,1,3,4;\ell;m)\,,\\
n_{\rm bub}^{\rm YM_{DR}}(1,2,3,4;\ell;m) &= n_{\rm tri}^{\rm YM_{DR}}(1,2,3,4;\ell;m)-n_{\rm tri}^{\rm YM_{DR}}(1,2,4,3;\ell;m)\,.
\end{aligned}
\tag{5.22}
$$

Plugging the $\mathrm{YM_{DR}}$ numerators together with the $\mathrm{YM}+\phi^3$ numerators (5.4), (5.9), (5.10) in eq. (5.11) gives the four-vector MHV amplitude in a spontaneously-broken $\mathrm{YM_{DR}}$-Einstein theory.

6 $\mathcal{N}=4$ supergravity theories

In this section we discuss the application of our results to construction of the amplitudes of $\mathcal{N}=4$ Maxwell-Einstein and Yang-Mills-Einstein supergravity theories. We begin with a review of the Lagrangians of these theories.[32]

[32]Note that the conventions for labeling various quantities used in this section are independent of the conventions used earlier for $\mathcal{N}=2$ supergravity theories.

6.1 $\mathcal{N}=4$ Maxwell-Einstein and Yang-Mills-Einstein supergravity theories

The $\mathcal{N}=4$ Maxwell-Einstein supergravity theories describe the coupling of $\mathcal{N}=4$ supergravity to $\mathcal{N}=4$ vector multiplets. Their construction and various gaugings in five dimensions were studied in [93, 94, 95, 96]. Our review will follow mainly [95].

The pure $\mathcal{N}=4$ supergravity in five dimensions contains one graviton $e_\mu{}^m$, four gravitini ψ^i_μ, six vector fields (A^{ij}_μ, a_μ), four spin 1/2 fermions χ^i and one real scalar field a:

$$\Big(e_\mu{}^m \,,\ \psi^i_\mu \,,\ A^{ij}_\mu \,,\ a_\mu \,,\ \chi^i \,,\ a \Big) . \tag{6.1}$$

Here, $\mu, \nu, \dots$ $(m, n, \dots)$ denote the five-dimensional curved (flat) indices and the $i, j = 1, \dots, 4$ are the indices of the fundamental representation of the R-symmetry group $USp(4)$. The vector field a_μ is a $USp(4)$ singlet and the vector fields A^{ij}_μ transform in the $\mathbf{5}$ of $USp(4)$, i.e.,

$$A^{ij}_\mu \ = \ -A^{ji}_\mu \,, \qquad A^{ij}_\mu \, \Omega_{ij} \ = \ 0 \,, \tag{6.2}$$

where Ω_{ij} is the symplectic metric of $USp(4) \cong SO(5)$. On the other hand an $\mathcal{N}=4$ vector multiplet contains the fields

$$\Big(A_\mu \,,\ \lambda^i \,,\ \phi^{ij} \Big), \tag{6.3}$$

where A_μ is a vector field, λ^i denotes four spin 1/2 fields, and the ϕ^{ij} are scalar fields in the $\mathbf{5}$ of $USp(4)$

$$\phi^{ij} = -\phi^{ji} \qquad \phi^{ij}\Omega_{ij} = 0 \ . \tag{6.4}$$

The total field content of the $\mathcal{N}=4$ MESG theory with n vector multiplets can be labelled as follows

$$\Big(e_\mu{}^m \,,\ \psi^i_\mu \,,\ A^{\tilde{I}}_\mu \,,\ a_\mu \,,\ \chi^i \,,\ \lambda^{ia} \,,\ a \,,\ \phi^x \Big) \,, \tag{6.5}$$

where the index $a = 1, \dots, n$ counts the number of $\mathcal{N}=4$ vector multiplets whereas the indices $\tilde{I}, \tilde{J}, ... = 1, \dots, (5+n)$ collectively denote the vector fields A^{ij}_μ of supergravity multiplet and the vector fields coming from the vector multiplets. The indices $x, y, .. = 1, \dots, 5n$ denotes the scalar fields in the n vector multiplets. The $USp(4)$ indices are raised and lowered with the symplectic metric Ω_{ij} and its inverse Ω^{ij}:

$$T^i \ = \ \Omega^{ij}\, T_j \ , \quad T_i \ = \ T^j\, \Omega_{ji} \,, \tag{6.6}$$

and the a, b indices are raised and lowered with δ^{ab}.

The scalar manifold spanned by the $(5n+1)$ scalar fields is [93]

$$\mathcal{M} \ = \ \frac{SO(5,n)}{SO(5) \times SO(n)} \times SO(1,1) \,, \tag{6.7}$$

where the $SO(1,1)$ factor corresponds to the $USp(4)$-singlet scalar field σ of the supergravity multiplet. The metric of the coset part $\mathcal{G}/\mathcal{H} = \frac{SO(5,n)}{SO(5)\times SO(n)}$ of the scalar manifold $\mathcal{M}$

parametrized by the $5n$ scalars is denoted as g_{xy} and the corresponding $SO(5) \times SO(n)$ "vielbeins" as f^a_{yij}

$$g_{xy} = \frac{1}{4} f^{ija}_x f^a_{yij} , \tag{6.8}$$

An equivalent description uses coset $\mathcal{G}/\mathcal{H}$ representatives $L_{\tilde{I}}{}^A$ where $\tilde{I}$ denotes a $\mathcal{G} = SO(5,n)$ index, and $A = (ij, a)$ is a $\mathcal{H} = SO(5) \times SO(n)$ index. Denoting the inverse of $L_{\tilde{I}}{}^A$ by $L_A{}^{\tilde{I}}$,

$$L_{\tilde{I}}{}^A \, L_B{}^{\tilde{I}} = \delta^A_B ,$$

one can define the vielbeins on $\mathcal{G}/\mathcal{H}$ and the composite $\mathcal{H}$-connections as follows:

$$L^{-1}\partial_\mu L = Q^{ab}_\mu \mathfrak{T}_{ab} + Q^{ij}_\mu \mathfrak{T}_{ij} + P^{aij}_\mu \mathfrak{T}_{aij}, \tag{6.9}$$

where $(\mathfrak{T}_{ab}, \mathfrak{T}_{ij})$ are the generators of the Lie algebra $\mathfrak{h}$ of $\mathcal{H}$, and $\mathfrak{T}_{aij}$ denotes the generators of the coset part of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. More explicitly the composite $SO(n)$ and $USp(4)$ connections are given by

$$Q^{ab}_\mu = L^{\tilde{I}a}\partial_\mu L_{\tilde{I}}{}^b = -Q^{ba}_\mu \qquad \text{and} \qquad Q^{ij}_\mu = L^{\tilde{I}ik}\partial_\mu L_{\tilde{I}k}{}^j = Q^{ji}_\mu . \tag{6.10}$$

Furthermore we have

$$P^{aij}_\mu = L^{\tilde{I}a}\partial_\mu L_{\tilde{I}}{}^{ij} = -\frac{1}{2} f^{aij}_x \, \partial_\mu \phi^x . \tag{6.11}$$

The Lagrangian of the five-dimensional $\mathcal{N} = 4$ MESG theory is reproduced in appendix D following [93, 95]. Its bosonic part can be written as:

$$\begin{aligned} e^{-1} \mathcal{L}_{Bosonic} = & -\frac{1}{2}R - \frac{1}{4}\Sigma^2 a_{\tilde{I}\tilde{J}} F^{\tilde{I}}_{\mu\nu} F^{\mu\nu\tilde{J}} - \frac{1}{4}\Sigma^{-4} G_{\mu\nu}G^{\mu\nu} \\ & -\frac{1}{2}(\partial_\mu a)^2 - \frac{1}{2} g_{xy}\partial_\mu \phi^x \partial^\mu \phi^y + \frac{\sqrt{2}}{8} e^{-1} C_{\tilde{I}\tilde{J}} \epsilon^{\mu\nu\rho a\lambda} F^{\tilde{I}}_{\mu\nu} F^{\tilde{J}}_{\rho a} a_\lambda , \end{aligned} \tag{6.12}$$

where

$$\Sigma = e^{\frac{1}{\sqrt{3}}a}. \tag{6.13}$$

and the abelian field strengths of vector fields are defined as

$$F^{\tilde{I}}_{\mu\nu} = (\partial_\mu A^{\tilde{I}}_\nu - \partial_\nu A^{\tilde{I}}_\mu) , \quad G_{\mu\nu} = (\partial_\mu a_\nu - \partial_\nu a_\mu) , \tag{6.14}$$

The main constraints imposed by supersymmetry are [33]

$$a_{\tilde{I}\tilde{J}} = L^{ij}_{\tilde{I}} L_{\tilde{J}ij} + L^a_{\tilde{I}} L^a_{\tilde{J}} , \quad C_{\tilde{I}\tilde{J}} = L^{ij}_{\tilde{I}} L_{\tilde{J}ij} - L^a_{\tilde{I}} L^a_{\tilde{J}} , \tag{6.15}$$

where $C_{\tilde{I}\tilde{J}}$ is the constant $SO(5,n)$ invariant metric.

Five-dimensional $\mathcal{N} = 4$ MESG theories can be truncated to $\mathcal{N} = 2$ MESG theories with or without $\mathcal{N} = 2$ hypermultiplets. To understand the structure of truncations we note that

[33] The indices $\tilde{I}, \tilde{J}, \ldots$ are raised and lowered by $a_{\tilde{I}\tilde{J}}$ and its inverse, e.g $L^A_{\tilde{I}} = a_{\tilde{I}\tilde{J}} L^{\tilde{J}A}$.

the pure $\mathcal{N}=4$ supergravity theory can be truncated to $\mathcal{N}=2$ supergravity coupled to a single vector multiplet by discarding two of the $\mathcal{N}=2$ gravitino supermultiplets where each gravitino multiplet contains a gravitino, two vectors and one spin 1/2 field. The remaining vector multiplet involves the $SO(n,5)$ singlet vector a_μ. On the other hand an $\mathcal{N}=4$ vector multiplet decomposes into an $\mathcal{N}=2$ vector multiplet plus an $\mathcal{N}=2$ hypermultiplet which has four scalars. One can discard either the $\mathcal{N}=2$ hypermultiplet or the $\mathcal{N}=2$ vector multiplet in truncation. If one throws away the $\mathcal{N}=2$ hypers from all the $\mathcal{N}=4$ multiplets the resulting theory is an $\mathcal{N}=2$ MESG theory belonging to the generic Jordan family with the scalar manifold

$$\mathcal{M}_{V_{(n+1)}} = \frac{SO(1,1)\times SO(n,1)}{SO(n)} \,,$$

which is unique modulo the embedding of $\mathcal{N}=2$ R-symmetry group $SU(2)$ inside $USp(4)$. On the other hand if one throws away m of the $\mathcal{N}=2$ vector multiplets and keeps the corresponding hypermultiplets the resulting theory is an $\mathcal{N}=2$ MESG theory coupled to m hypermultiplets with the moduli space:

$$\mathcal{M}_{V_{(n-m+1)}} \times \mathcal{V}_{H_m} = \frac{SO(1,1)\times SO(n-m,1)}{SO(n-m)} \times \frac{SO(m,4)}{SO(m)\times SO(4)} \,.$$

The $\mathcal{N}=2$ MESG theory sector of all these truncations is of the generic Jordan type. The $F\wedge F\wedge A$ term

$$\frac{\sqrt{2}}{8} e^{-1}\, C_{\tilde{I}\tilde{J}}\, \epsilon^{\mu\nu\rho a\lambda}\, F^{\tilde{I}}_{\mu\nu} F^{\tilde{J}}_{\rho a}\, a_\lambda \,, \tag{6.16}$$

of the $\mathcal{N}=4$ MESG theory reduces to

$$\frac{\sqrt{2}}{8} e^{-1}\, C_{RS}\, \epsilon^{\mu\nu\rho a\lambda}\, F^{R}_{\mu\nu} F^{S}_{\rho a}\, a_\lambda \,, \tag{6.17}$$

where $R,S,..=1,2,..(n-m+1)$. If we denote the singlet vector a_μ as A^0_μ this implies that the C-tensor of the $\mathcal{N}=2$ MESG theory is simply given by

$$C_{0RS} = \frac{\sqrt{3}}{2} C_{RS} \,,$$

where C_{RS} is proportional to the constant $SO(n-m,1)$ invariant metric, namely

$$\begin{aligned} C_{011} &= \frac{\sqrt{3}}{2}\,, \\ C_{0rs} &= -\frac{\sqrt{3}}{2}\delta_{rs}\,, \quad r,s,..=1,2,...(n-m)\,. \end{aligned} \tag{6.18}$$

Four-dimensional $\mathcal{N}=4$ MESG theories and their gaugings were first studied in [97, 98]. Their most general gaugings both in four and five dimensions using the embedding tensor formalism was given more recently [96]. Under dimensional reduction the five-dimensional

$\mathcal{N}=4$ MESG theory with n vector multiplets leads to the four-dimensional MESG theory with $(n+1)$ vector multiplets and the scalar manifold

$$\mathcal{M}_4 = \frac{SO(6,n+1)}{SO(6)\times SO(n+1)} \times \frac{SU(1,1)}{U(1)}\,. \tag{6.19}$$

The $SU(1,1)$ symmetry acts via electric and magnetic dualities and in the symplectic section that descends directly from five dimensions via dimensional reduction the Lagrangian is invariant under the five-dimensional U-duality group. These $\mathcal{N}=4$ MESG theories in four dimensions can be truncated to $\mathcal{N}=2$ MESG theories belonging to the generic Jordan family with or without hypermultiplets. Truncation to maximal $\mathcal{N}=2$ MESG theory with $(n+1)$ vector multiplets without hypers is unique modulo the embedding of the $\mathcal{N}=2$ R-symmetry group $U(2)$ inside $\mathcal{N}=4$ R-symmetry group $SO(6)=SU(4)$. The resulting theory has the scalar manifold

$$\frac{SO(n+1,2)}{SO(n+1)\times SO(2)} \times \frac{SU(1,1)}{U(1)}\,. \tag{6.20}$$

If one retains m, $\mathcal{N}=2$ hypermultiplets and $(n+1-m)$ vector multiplets in the truncation the resulting theory is a MESG theory coupled to m hypermultiplets with the scalar manifold

$$\frac{SO(n+1-m,2)}{SO(n+1-m)\times SO(2)} \times \frac{SU(1,1)}{U(1)} \times \frac{SO(m,4)}{SO(m)\times SO(4)}\,. \tag{6.21}$$

Most general gaugings of $\mathcal{N}=4$ supergravity theories coupled to $\mathcal{N}=4$ vector multiplets were studied in [96] using the embedding tensor formalism. In this paper we will only focus on gaugings that lead to $\mathcal{N}=4$ supergravity coupled to Yang-Mills gauge theories with a compact gauge group that allow Minkowski vacua only. For this we will follow the work of [95] on gaugings of five-dimensional $\mathcal{N}=4$ MESG theories. As was shown by the authors of [95] gauging with tensors requires an abelian gauge group whose gauge field is the singlet vector a_μ. Furthermore gauging a semisimple subgroup of the global symmetry group $SO(5,n)$ by itself does not require coupling to any tensors and allows Minkowski vacua only. To gauge a semisimple subgroup K_S of $SO(5,n)$ one identifies the subset of vector fields A^I_μ that transform in the adjoint representation of K_S with the remaining vector fields being spectators. Since such gaugings do not have tensors we can formally use the same index $\tilde{I},\tilde{J},..=1,2,...,n+5$ to collectively denote the K_S gauge fields plus the spectators with the understanding that the structure constants $f_{\tilde{I}\tilde{J}}{}^{\tilde{K}}$ of the gauge group vanishes when any one of the indices corresponds to the spectator vector fields. In this paper we restrict ourselves to gaugings of a compact subgroup K of $SO(n)$ global symmetry which do not involve any tensor fields and will use this formal trick to simplify the formulas.

The gauging of a subgroup K requires that all derivatives acting on fields that transform non-trivially under K be covariantized. This is implemented by the following substitutions in the Lagrangian $A_\mu^{\tilde I}$ in the standard way:

$$\begin{aligned} F^{\tilde I}_{\mu\nu} &\longrightarrow \mathcal{F}^{\tilde I}_{\mu\nu} = F^{\tilde I}_{\mu\nu} + g_S\, A^{\tilde J}_\mu f^{I}_{\tilde J \tilde K} A^{\tilde K}_\nu\,, \\ \partial_\mu L^{\tilde I}_A &\longrightarrow \mathfrak{D}_\mu L^{\tilde I}_A = \partial_\mu L^{\tilde I}_A + g_S A^{\tilde J}_\mu f^{I}_{\tilde J\tilde K} L^{\tilde K}_A\,. \end{aligned} \tag{6.22}$$

The composite $USp(4)$ and $SO(n)$ connections, the vielbein $P^a_{\mu ij}$ as well as the derivatives $\mathcal{D}_\mu$ acting on fermions are also modified by the new g_S dependent contributions as reviewed in appendix D where we also reproduce the Lagrangian of the $\mathcal{N}=4$ YMESG theory in five dimensions following [95]. The bosonic part of the Lagrangian of the YMESG theory has the form [95] :

$$\begin{aligned} e^{-1}\,\mathcal{L}_{\text{YMESG}} = & -\frac{1}{2}R - \frac{1}{4}\Sigma^2\, a_{\tilde I\tilde J}\,\mathcal{F}^{\tilde I}_{\mu\nu}\mathcal{F}^{\mu\nu\tilde J} - \frac{1}{4}\Sigma^{-4}\,G_{\mu\nu}G^{\mu\nu} \\ & -\frac{1}{2}\,(\partial_\mu a)^2 - \frac{1}{2}\,\mathcal{P}^{aij}_\mu \mathcal{P}^\mu_{aij} + \frac{\sqrt{2}}{8} e^{-1}\, C_{\tilde I\tilde J}\,\epsilon^{\mu\nu\rho a\lambda}\,\mathcal{F}^{\tilde I}_{\mu\nu}\mathcal{F}^{\tilde J}_{\rho a}\,a_\lambda \\ & - g_S^2\left(-\frac{9}{2}\,S_{ij}\Delta^{ij} + \frac{1}{2}\,T^a_{ij}\,T^{aij}\right)\,, \end{aligned} \tag{6.23}$$

where

$$S_{ij} = -\frac{2}{9}\Sigma^{-1}L^{\tilde J}_{(i|k|}f^{\tilde K}_{\tilde J\tilde I}L^{kl}_{\tilde K}L^{\tilde I}_{|l|j)}, \tag{6.24}$$

$$T^a_{ij} = -\Sigma^{-1}L^{\tilde J a}L^{\tilde K k}_{(i}f^{\tilde I}_{\tilde J\tilde K}L_{\tilde I|k|j)}, \tag{6.25}$$

$$\mathcal{P}^a_{\mu ij} = P^a_{\mu ij} - g_S A^{\tilde J}_\mu L^{\tilde K}_{ij} f^{\tilde I}_{\tilde J\tilde K} L^a_{\tilde I}\,. \tag{6.26}$$

The $\mathcal{N}=4$ Yang-Mills-Einstein supergravity with a compact gauge group K that is a subgroup of $SO(n)$ can be truncated to $\mathcal{N}=2$ Yang-Mills-Einstein supergravity with the same gauge group that belongs to the generic Jordan family discussed in section 3. This truncation is unique for a given compact gauge group K, modulo the equivalence class of embeddings of K in $SO(n)$ and R symmetry group $SU(2)$ inside $USp(4)$, and assuming that the number of spectator vector multiplets is the same in the truncated theory as the original $\mathcal{N}=4$ theory. Conversely one can extend a YMESG theory belonging to the generic Jordan family to an $\mathcal{N}=4$ YMESG theory with the same gauge group. These results hold true also for the corresponding YMESG theories in four dimensions so long as one works in the symplectic section that descends directly from five dimensions. The four-dimensional YMESG theories have one extra spectator vector multiplet coming from the supergravity multiplet in five dimensions.

The spontaneous symmetry breaking mechanism of $\mathcal{N}=2$ YMESG theories induced by giving a VEV to some of the scalars in the vectors multiplets can be extended to the $\mathcal{N}=4$ YMESG theories for compact gauge groups K that are subgroups of $SO(n)$ both in five as well as in four dimensions. For example the $\mathcal{N}=4$ supersymmetric Yang-Mills theory with gauge group $SU(2)$ spontaneously-broken down to $U(1)$ subgroup by giving a VEV to one of

$\mathcal{V}_R \backslash \mathcal{V}_L$	A_μ	λ^i	$\phi^{[ij]}$
A_ν	$h_{\mu\nu} \oplus \sigma \oplus \phi$	$\Psi^i_\mu \oplus \psi^i$	$A_\nu^{[ij]}$
ϕ^c	A^c_μ	$\psi^{i,c}$	$\phi^{[ij],c}$

Table 4: The spectrum of the $D = 4$, $\mathcal{N} = 4$ Maxwell-Einstein and Yang-Mills-Einstein supergravity theories from the double-copy construction: one $\mathcal{N} = 4$ supergravity multiplet given by the second, third and fourth entries of the second line and as many vector multiplets as the range of the index c of the scalar fields in the non-supersymmetric gauge-theory factor given in the third line.

the scalars leads to a massless gauge multiplet and two massive BPS vector multiplets which can be written as complex fields carrying opposite $U(1)$ charges. In four dimensions these charged BPS vector multiplets have 5 massive complex scalars and four massive fermions [99]. A massive $\mathcal{N} = 4$ BPS vector multiplet decomposes into a massive BPS $\mathcal{N} = 2$ vector multiplet plus a massive $\mathcal{N} = 2$ BPS hypermultiplet. Therefore a spontaneously-broken $\mathcal{N} = 4$ YMESG theory can be truncated to a spontaneously-broken $\mathcal{N} = 2$ YMESG theory by throwing away the massive hypermultiplets. The spontaneous symmetry breaking by giving a VEV to one of the scalars in a gauge vector multiplet breaks the R-symmetry from $SO(6)$ down to $SO(5) = USp(4)$ in four dimensions and from $USp(4)$ down to $SO(4)$ in five dimensions.

6.2 More on double copies with $\mathcal{N} = 4$ supersymmetry

In the double-copy construction of the amplitudes of $\mathcal{N} = 2$ MESG theories one gauge theory copy is $\mathcal{N} = 2$ supersymmetric and the other copy has no supersymmetry. If one replaces the $\mathcal{N} = 2$ gauge-theory factor with an $\mathcal{N} = 4$ supersymmetric theory one obtains the amplitudes of an $\mathcal{N} = 4$ MESG theory both in five as well as in four dimensions. The fields of four-dimensional $\mathcal{N} = 4$ MESG theory and YMESG theory in terms of those of $\mathcal{N} = 4$ SYM and of the pure YM theory coupled to scalars in a specific way can be obtained by restricting to the product $\mathcal{V}_L \otimes \mathcal{V}_R$ in section 2.6.2 which we give in Table 4.

The double-copy construction yields the superamplitudes of $\mathcal{N} = 4$ MESG theory in terms of the $\mathcal{N} = 4$ SYM superamplitudes and the amplitudes of the dimensional reduction of pure YM theory from $D = 4 + n_V$, where n_V is the number of vector multiplets. In the same sense as from a Lagrangian point of view we can truncate these $\mathcal{N} = 4$ supergravity superamplitudes to a combination of $\mathcal{N} = 2$ superamplitudes corresponding to vector and hypermultiplets that describe the amplitudes of $\mathcal{N} = 2$ MESG theory coupled to hypermultiplets corresponding to the quaternionic manifold $\frac{SO(m,4)}{SO(m)\times SO(4)}$. Special cases of such amplitudes that arise in $\mathcal{N} = 2$ MESG theories which are orbifolds of $\mathcal{N} = 8$ supergravity were discussed in [15, 27].

Similarly, the superamplitudes of $\mathcal{N} = 4$ YMESG theories can be obtained as double copies by replacing the $\mathcal{N} = 2$ supersymmetric gauge-theory factor by an $\mathcal{N} = 4$ supersymmetric gauge-theory factor while keeping the $\mathcal{N} = 0$ gauge copy as in section 2.3, with only the ϕ fields but not the φ fields. (Keeping both the fields ϕ and φ leads to a theory that does not obey color/kinematics duality.)

Similarly to the unbroken symmetry case, scattering amplitudes with manifest $\mathcal{N} = 4$ supersymmetry and spontaneously-broken gauge symmetry can be constructed rather straightforwardly by replacing the spontaneously-broken $\mathcal{N} = 4$ SYM theory in place of the spontaneously-broken $\mathcal{N} = 2$ SYM theory in section 2.6. These amplitudes are expected to describe spontaneously-broken YMESG theories that preserve all $\mathcal{N} = 4$ supersymmetries. Some of them, such as the anomalous amplitudes discussed in [100] for MESG theories, are particularly easy to find from the expressions found in sec. 5. Apart from the rational contribution present in the MESG and unbroken YMESG theories, they will also acquire nontrivial dependence on the non-zero masses. However, spontaneous partial supersymmetry breaking is possible in $\mathcal{N} = 4$ YMESG theories [98, 101, 102]. In particular the work of [102] studies in depth the breaking of $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 2$ supersymmetry in $\mathcal{N} = 4$ supergravity theories. Detailed study of the double-copy construction of the amplitudes of spontaneously broken $\mathcal{N} = 4$ YMESG theories with or without partial supersymmetry breaking is beyond the scope of this paper. It will be studied in a separate work where we will discuss explicitly these amplitudes and their comparison with a direct Feynman graph-based calculations.

7 Conclusions and outlook

In this paper we have extended color/kinematics duality and the double-copy construction to gauge and gravity theories that are spontaneously broken by an adjoint scalar VEV.

As demonstrated in earlier work [27], abelian and non-abelian gauge theories that couple to (super)gravity provide a rich class of theories for which both spectra and interactions appear to exhibit a double-copy structure. The tree-level S matrices and the loop-level integrands of these YMESG theories can be constructed in terms of the tree-level S matrices and the loop-level integrands of particular matter-coupled YM theories. Color/kinematics duality is the main agent behind the consistency of this construction.

In the presence of a non-abelian gauge symmetry it is particularly natural to consider spontaneous symmetry breaking. We observe that the gravity double-copy structure is present at the level of the spectrum of spontaneously-broken YMESG theories. The YMESG spectra can be expressed as the tensor product of the spectrum of two types of gauge theories: a spontaneously-broken YM theory and a YM theory coupled to massive scalars charged under a global symmetry. In the latter YM + ϕ^3 theory, the scalar fields have acquired mass as a consequence of an explicit breaking of the global symmetry.

The double-copy construction is shown to work for the interacting fields given that the two gauge-theory factors obey color/kinematics duality of a form specific to broken gauge theories. As discussed in section 2.4, in addition to the Jacobi relation and commutation relation, there are new types of color-factor relations in a gauge theory spontaneously broken by an adjoint scalar VEV. Color/kinematics duality then requires that corresponding kinematic identities are satisfied by the kinematic numerators of the diagrammatic expansion of an amplitude. With the appropriate definition of the numerator factors, the spontaneously-broken YM theory inherits color/kinematics duality from the corresponding unbroken $(D+1)$-dimensional theory. For the explicitly-broken YM + ϕ^3 theory, color/kinematics duality acts as a highly non-trivial constraint on the terms in the Lagrangian that are introduced to break the global symmetry. These terms exhibit certain similarities with terms appearing in spontaneously-broken gauge theories, but the details differ significantly. While we do not discuss it in the current work, it should be interesting to understand these terms as originating from some limiting case (perhaps a double-scaling limit) of a spontaneously-broken gauge theory.

Using the above gauge-theory ingredients, and building on our earlier work [27], we discussed in detail the $\mathcal{N}=2$ generic Jordan family YMESG theories with spontaneously-broken gauge symmetry and showed that they continue to exhibit a double-copy structure on the Coulomb branch. By computing three-point and four-point scattering amplitudes we identified the map relating the double-copy asymptotic states and the asymptotic states of the supergravity Lagrangian. Similar to the orbifold constructions of ref. [15], the supergravity fields are related to bilinears of the gauge theory fields which are neutral under an appropriately-identified global symmetry. The double-copy construction of the asymptotic states also follows closely the approach taken in ref. [17]. Similar to the unbroken case [27], upon comparing the scattering amplitudes we identify the parameters of the supergravity Lagrangian in terms of the parameters of the two gauge-theory factors. This gives non-trivial relations between the dimensionful and dimensionless couplings of the various theories.

The details of the double-copy construction extend to YMESG theories with $\mathcal{N}\leq 4$ supersymmetry with little change. In this paper, the $\mathcal{N}<2$ YMESG theories have only been considered as obtained through the double copy, without detailing their Lagrangian formulation. Nevertheless, as pointed out in ref. [27], the formalism described there should extend to unbroken $\mathcal{N}=1$ supersymmetric and non-supersymmetric theories. Since $\mathcal{N}=1$ YMESG theories have no adjoint scalars they cannot be considered on the Coulomb branch. For non-supersymmetric YME theories with adjoint scalars, the spontaneously-broken phase is straightforwardly obtained through the double copy.

We addressed with more details the case of $\mathcal{N}=4$ MESG and YMESG theories. The $\mathcal{N}=4$ MESG theories are obtained as a double-copy of $\mathcal{N}=4$ SYM theory with the dimensional reduction of some higher-dimensional pure YM theory [27]. The corresponding $\mathcal{N}=4$ YMESG theories are obtained by gauging a subgroup of the global symmetry group, which in terms of the double-copy construction amounts to adding a ϕ^3 term to the non-supersymmetric gauge-theory factor [27]. In analogy with the double-copy construction for $\mathcal{N} = 2$ theories, amplitudes in the spontaneously-broken $\mathcal{N} = 4$ theory should be obtained from a double copy between spontaneously-broken $\mathcal{N} = 4$ SYM and explicitly broken YM + ϕ^3 theory. We leave for future work a thorough understanding of the $\mathcal{N} = 4$ YMESG theories from a Lagrangian perspective, as well as a comparison of the resulting scattering amplitudes with the results of the double-copy construction outlined here.

To illustrate the power of the double-copy construction we presented several one-loop four-point amplitudes. For the broken YM + ϕ^3 theory, we considered one-loop diagrams with external massless scalars, and internal massless vectors and massive scalars. After double-copying this theory with the corresponding spontaneously-broken $\mathcal{N} = 4, 2$ SYM numerators, we obtained amplitudes in spontaneously broken $\mathcal{N} = 4, 2$ YMESG theories. Corresponding one-loop amplitudes with no supersymmetry were also presented.

A future relevant study would be the case of $\mathcal{N} = 2$ YMESG theories with hypermultiplets in the fundamental representation. An Higgs mechanism with fields in representations different from the adjoint gives distinct scenarios for breaking the gauge group. It would be interesting to explore whether gauge theories with a fundamental scalar VEV exhibit color/kinematics duality, and similarly to check the result of the double-copy construction against scattering amplitudes evaluated with the corresponding supergravity Lagrangian as a starting point.

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A Summary of index notation

Here we give a brief summary of the various indices used in the paper, with the exception of section 6, which follows a different notation consistent with the relevant supergravity literature. The types of indices are:

$A, B, C = -1, 0, \ldots, \tilde{n}$	index running over all sugra vectors in $4D$, global gauge-theory index (before symmetry breaking),
$I, J, K = 0, \ldots, \tilde{n}$	index running over matter vectors in $4D$; index running over all vector fields in $5D$,
$x, y = 1, 2, \ldots, \tilde{n}$	curved target space indices in $5D$,
a, b, c	index running over massless vectors, in the Higgsed supergravity; global index in gauge-theory,
i, j, k	index running over massless scalars, in the Higgsed supergravity ; fundamental global indices in gauge theory,
$\hat{\imath}, \hat{\jmath}, \hat{k}$	fundamental rep. indices in gauge theory,
α, β, γ	index running over massive fields,
$\hat{a}, \hat{b}, \hat{c}$	gauge-theory adjoint indices,
$\hat{\alpha}, \hat{\beta}, \hat{\gamma}$	gauge-theory matter-representation indices,
m, n, o	flavor indices.

With this notation we have

$$A = \big(-1,\ I\big) = \big(a, \alpha, \bar{\alpha}\big) = \big(-1, i, \alpha, \bar{\alpha}\big)\ . \tag{A.1}$$

B Symmetry breaking vs. dimensional compactification

B.1 Spontaneously broken SYM

In this appendix we show that SYM spontaneously-broken by an adjoint scalar VEV is equivalent to a dimensional compactification $(D+1) \to D$ of SYM such that for each field the extra-dimensional momentum becomes a mass that is proportional to a U(1) charge.

Consider that the gluons and scalars fields in the higher-dimensional theory satisfy the following differential equation with respect to a derivative in the internal direction $(D+1)$:

$$\partial_{D+1} \begin{pmatrix} A^{\mu\hat{A}} \\ \phi^{a\hat{A}} \end{pmatrix} = -gV f^{0\hat{A}\hat{B}} \begin{pmatrix} A^{\mu\hat{B}} \\ \phi^{a\hat{B}} \end{pmatrix} \equiv i\, m^{\hat{A}\hat{B}} \begin{pmatrix} A^{\mu\hat{B}} \\ \phi^{a\hat{B}} \end{pmatrix}\ . \tag{B.1}$$

This means that some fields have a momentum turned on in the internal direction $(D+1)$, corresponding to the eigenvalues of $-gVf^{0\hat{A}\hat{B}} \equiv i\,m^{\hat{A}\hat{B}}$. Fields that commute with the generator t^0 will not have a mass since that implies that $f^{0\hat{A}\hat{B}}$ vanish. If needed one can decompose this equation into massive and massless field, with the corresponding renaming and complexification as in section 2.4, giving

$$\begin{aligned}
\partial_{D+1}\begin{pmatrix} A^{\mu\hat{a}} \\ \phi^{a\hat{a}} \end{pmatrix} &= 0\,, \\
\partial_{D+1}\begin{pmatrix} W^{\mu}_{\hat{\alpha}} \\ \varphi^{a}_{\hat{\alpha}} \end{pmatrix} &= -gVf^{0}{}_{\hat{\alpha}}{}^{\hat{\beta}}\begin{pmatrix} W^{\mu}_{\hat{\beta}} \\ \varphi^{a}_{\hat{\beta}} \end{pmatrix} \equiv i\,m_{\hat{\alpha}}{}^{\hat{\beta}}\begin{pmatrix} W^{\mu}_{\hat{\beta}} \\ \varphi^{a}_{\hat{\beta}} \end{pmatrix}.
\end{aligned} \tag{B.2}$$

However, for the exercise in this appendix it is more convenient to work with the real fields and mass matrix in eq. (B.1).

The kinetic term of the scalars $\phi^{a>0}$ in $(D+1)$ dimensions can now be shown to be identical to a kinetic term in D dimensions plus a ϕ^4-term containing a VEV:

$$\begin{aligned}
&\frac{1}{2}\big(\mathcal{D}_\mu\phi^{a\hat{A}}\big)^2 - \frac{1}{2}\big(\partial_{D+1}\phi^{a\hat{A}} + gf^{A\hat{B}\hat{C}}A^{\hat{B}}_{D+1}\phi^{a\hat{C}}\big)^2 \\
&= \frac{1}{2}\big(\mathcal{D}_\mu\phi^{a\hat{A}}\big)^2 - \frac{1}{2}\big(i\,m^{\hat{A}\hat{B}}\phi^{a\hat{B}} + gf^{ABC}\phi^{0\hat{B}}\phi^{a\hat{C}}\big)^2 \\
&= \frac{1}{2}\big(\mathcal{D}_\mu\phi^{a\hat{A}}\big)^2 + \frac{g^2}{2}\mathrm{tr}\big([Vt^0+\phi^0,\phi^a]^2\big)\,,
\end{aligned} \tag{B.3}$$

where the second term on the first row corresponds to the $(D+1)$ component of the kinetic term, similarly $A^{\hat{A}}_{D+1}$ is the gauge field in that direction. The latter is renamed to $\phi^{0\hat{A}}$ on the second line. The full expression for $\mathcal{D}_\mu\phi^{a\hat{A}}$ can be found in eq. (2.23), remembering that the global index a does not yet include $a=0$.

To get the kinetic term for the ϕ^0 field we need to look at the $(D+1)$-dimensional vector-field kinetic term. It is straightforward to see that it is identical to the D-dimensional vector-field kinetic term plus the kinetic term of ϕ^0, including a VEV for the latter,

$$\begin{aligned}
&-\frac{1}{4}\big(\mathcal{F}^{\hat{A}}_{\mu\nu}\big)^2 + \frac{1}{2}\big(\partial_\mu A^{\hat{A}}_{D+1} - \partial_{D+1}A^{\hat{A}}_\mu + gf^{\hat{A}\hat{B}\hat{C}}A^{\hat{B}}_\mu A^{\hat{C}}_{D+1}\big)^2 \\
&= -\frac{1}{4}\big(\mathcal{F}^{\hat{A}}_{\mu\nu}\big)^2 + \frac{1}{2}\big(\partial_\mu\phi^{0\hat{A}} - i\,m^{\hat{A}\hat{B}}A^{\hat{B}}_\mu + gf^{\hat{A}\hat{B}\hat{C}}A^{\hat{B}}_\mu\phi^{0\hat{C}}\big)^2 \\
&= -\frac{1}{4}\big(\mathcal{F}^{\hat{A}}_{\mu\nu}\big)^2 + \frac{1}{2}\big((\mathcal{D}_\mu\phi^0)^{\hat{A}} - i\,m^{\hat{A}\hat{B}}A^{\hat{B}}_\mu\big)^2 \\
&= -\frac{1}{4}\big(\mathcal{F}^{\hat{A}}_{\mu\nu}\big)^2 + \frac{1}{2}\big((\mathcal{D}_\mu\phi^0 + \mathcal{D}_\mu\langle\phi^0\rangle)^{\hat{A}}\big)^2\,,
\end{aligned} \tag{B.4}$$

where $\langle\phi^0\rangle = Vt^0$. Similar to before, the second term on the first line is the contribution of the field-strength in the $\mu\otimes(D+1)$ direction, on the second line $A^{\hat{A}}_{D+1}$ is renamed to $\phi^{0\hat{A}}$, and on the third and fourth lines terms are reassembled into covariant derivatives. Again, the full expressions for $\mathcal{F}^{\hat{A}}_{\mu\nu}$ and $\mathcal{D}_\mu\phi^{0\hat{A}}$ can be found in eq. (2.23).

Finally, including the quartic terms for the $\phi^{a>0}$ scalars, the $(D+1)$-dimensional (massless/unbroken) SYM Lagrangian has become a spontaneously-broken D-dimensional SYM Lagrangian,

$$\mathcal{L}_{\rm SYM} = -\frac{1}{4}\big(\mathcal{F}^{\hat{A}}_{\mu\nu}\big)^2 + \frac{1}{2}\big((\mathcal{D}_\mu\phi^a + \mathcal{D}_\mu\langle\phi^a\rangle)^{\hat{A}}\big)^2 + \frac{g^2}{4}{\rm tr}\big([\phi^a + \langle\phi^a\rangle, \phi^b + \langle\phi^b\rangle]^2\big) + \text{fermions}\,, \tag{B.5}$$

where $\langle\phi^a\rangle = \delta^{a0}Vt^0$.

A practical implication of this identification is that scattering amplitudes for SYM spontaneously-broken by a adjoint scalar VEV can be computed from unbroken $(D+1)$-dimensional SYM given that for each field there is a relation between the internal-space momentum and a U(1) charge. This relation can be stated as an operator equation, simply by rewriting the differential equation (B.1) as follows:

$$i(p_{D+1} - gVq)\begin{pmatrix} A^{\mu\hat{A}} \\ \phi^{a\hat{A}} \end{pmatrix} = 0\,, \tag{B.6}$$

where p_{D+1} is the momentum operator pointing in the $(D{+}1)$ direction, and q is a U(1)-charge operator that acts as $q\Phi = [t^0, \Phi]$, for some field Φ. Similarly, we have that $p_{D+1}\Phi = m\Phi$, where m is the mass of Φ, implying that a (massive) field carries the U(1) charge $m/(gV)$.

For example, for tree amplitudes it is sufficient to impose the constraint (B.6) on the external states, then the internal states will automatically have this satisfied by virtue of charge/momentum conservation. Similarly, for loop amplitudes, it is sufficient to have this constraint imposed once for each independent loop momenta.

B.2 Explicitly broken YM + ϕ^3

Here we re-derive the Lagrangian (2.42) for explicitly broken YM + ϕ^3, without explicitly using color/kinematics duality. Similar to the derivation in section B.1, it is given by a dimensional compactification $(D+1) \to D$ of the corresponding unbroken theory, after a proper identification of the extra-dimensional momentum and the U(1) charge of each field. Although, the details are strikingly different compared to the SYM case.

Consider that the scalars fields in the higher-dimensional theory satisfy the following differential equation with respect to a derivative in the internal direction $(D+1)$:

$$\partial_{D+1}\phi^{A\hat{a}} = -\frac{1}{2}\rho\lambda F^{0AB}\phi^{B\hat{a}} \equiv i\, m^{AB}\phi^{B\hat{a}}\,. \tag{B.7}$$

Similar to before we let $\widetilde{\phi}^{0\hat{a}} = A^{\hat{a}}_{D+1}$ represent the gluon that is converted to a scalar upon dimensional reduction. The covariant derivative, applied in the $(D+1)$ direction, of the other scalars is then given by

$$
\begin{aligned}
-\frac{1}{2}\big((\mathcal{D}_{D+1}\phi^A)^{\hat a}\big)^2 =& -\frac{1}{2}\big(\partial_{D+1}\phi^{A\hat a}+gf^{\hat a\hat b\hat c}A^{\hat b}_{D+1}\phi^{A\hat c}\big)^2\\
=& -\frac{1}{2}\Big(-\frac{1}{2}\rho\lambda F^{0AB}\phi^{B\hat a}+gf^{\hat a\hat b\hat c}\widetilde{\phi}^{0\hat b}\phi^{A\hat c}\Big)^2\\
=& -\frac{1}{2}m^{AC}m^{CB}\phi^{A\hat a}\phi^{B\hat a}+\frac{1}{2}g\rho\lambda F^{0AB}f^{\hat a\hat b\hat c}\widetilde{\phi}^{0\hat a}\phi^{A\hat b}\phi^{B\hat c}-\frac{g^2}{2}f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\widetilde{\phi}^{0\hat a}\phi^{A\hat b}\widetilde{\phi}^{0\hat c}\phi^{A\hat d}\\
=& -(m^2)_\alpha{}^\beta\overline{\varphi}^{\alpha\hat a}\varphi_\beta{}^{\hat a}+g\rho\lambda F^0_\alpha{}^\beta f^{\hat a\hat b\hat c}\widetilde{\phi}^{0\hat a}\overline{\varphi}^{\alpha\hat b}\varphi_\beta{}^{\hat c}-g^2f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\widetilde{\phi}^{0\hat a}\widetilde{\phi}^{0\hat c}\overline{\varphi}^{\alpha\hat b}\varphi_\alpha{}^{\hat d}\\
&-\frac{g^2}{2}f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\widetilde{\phi}^{0\hat a}\phi^{a\hat b}\widetilde{\phi}^{0\hat c}\phi^{a\hat d}\,.
\end{aligned}
\tag{B.8}
$$

On the last line the proper massive fields have been identified (and complexified), and $m_\alpha{}^\beta$ is the proper mass matrix corresponding to these fields, similar to the presentation in section 2.5.

An important difference from the derivation in section B.1 is that the extra-dimensional gluon is also charged under the global group, since $F^{0AB}\neq 0$ is assumed in order to have a mass term. However, in the derivation in eq. (B.8) this field is a U(1) singlet in the $\rho\to 0$ limit, which appears to be inconsistent with this assumption. To ensure the existence of a non-singlet scalar in this limit we demand that the true ϕ^0 scalar is a linear combination of $\widetilde{\phi}^0$ and a scalar $\widehat{\phi}^0$ that was present already before the dimensional compactification. The non-kinetic terms in the Lagrangian containing $\widehat{\phi}^0$ is then,

$$
g\lambda F^0_\alpha{}^\beta f^{\hat a\hat b\hat c}\widehat{\phi}^{0\hat a}\overline{\varphi}^{\alpha\hat b}\varphi_\beta{}^{\hat c}-g^2f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\widehat{\phi}^{0\hat a}\widehat{\phi}^{0\hat c}\overline{\varphi}^{\alpha\hat b}\varphi_\alpha{}^{\hat d}-\frac{1}{2}g^2f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\widehat{\phi}^{0\hat a}\phi^{a\hat b}\widehat{\phi}^{0\hat c}\phi^{a\hat d}\,. \tag{B.9}
$$

Indeed, if we add the terms in eq. (B.8) and eq. (B.9) and do the unitary rotation

$$
\begin{pmatrix}\widehat{\phi}^0\\ \widetilde{\phi}^0\end{pmatrix}=\frac{1}{\sqrt{1+\rho^2}}\begin{pmatrix}1 & -\rho\\ \rho & 1\end{pmatrix}\begin{pmatrix}\phi^0\\ \phi'^0\end{pmatrix}, \tag{B.10}
$$

of the two scalars, then only the field ϕ^0 has a cubic interaction, and ϕ'^0 becomes a U(1) singlet of the global group. We may drop the latter field since it can be absorbed into the freedom of redefining the global group, e.g. $G_k\times U(1)\to G_k$. We then get the following modification of the covariant derivative considered in eq. (B.8):

$$
\begin{aligned}
-\frac{1}{2}\big((\mathcal{D}_{D+1}\phi^A)^{\hat a}\big)^2\to& -(m^2)_\alpha{}^\beta\overline{\varphi}^{\alpha\hat a}\varphi_\beta{}^{\hat a}+g\lambda\sqrt{1+\rho^2}F^0_\alpha{}^\beta f^{\hat a\hat b\hat c}\phi^{0\hat a}\overline{\varphi}^{\alpha\hat b}\varphi_\beta{}^{\hat c}-g^2f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\phi^{0\hat a}\phi^{0\hat c}\overline{\varphi}^{\alpha\hat b}\varphi_\alpha{}^{\hat d}\\
&-\frac{1}{2}g^2f^{\hat a\hat b\hat e}f^{\hat e\hat c\hat d}\phi^{0\hat a}\phi^{a\hat b}\phi^{0\hat c}\phi^{a\hat d}\,.
\end{aligned}
\tag{B.11}
$$

Compared to a massless unbroken theory, the only new terms in this expression are the two first ones. It is not surprising that a quadratic mass term appears, but that the cubic term corresponding to the global-group coupling gets modified by a square-root function is striking. The remaining two terms are simply a group decomposition of certain quartic scalar terms already present in the original unbroken Lagrangian (2.37). Ignoring these, we get the explicitly broken YM + ϕ^3 by adding the above mass-term to the Lagrangian (2.37) and at the same time swapping the cubic λ-dependent term as

$$g\lambda F^{a}_{\alpha}{}^{\beta} f^{\hat{a}\hat{b}\hat{c}} \phi^{a\hat{a}} \overline{\varphi}^{\alpha\hat{b}} \varphi_{\beta}{}^{\hat{c}} \to g\lambda \Delta^{ab} F^{b}_{\alpha}{}^{\beta} f^{\hat{a}\hat{b}\hat{c}} \phi^{a\hat{a}} \overline{\varphi}^{\alpha\hat{b}} \varphi_{\beta}{}^{\hat{c}} , \tag{B.12}$$

with $\Delta^{ab} = \delta^{ab} + \big(\sqrt{1+\rho^2} - 1\big)\delta^{a0}\delta^{0b}$. Finally, carrying out the full decomposition of $\phi^{A\hat{a}}$ into real and complex massive fields, we obtain precisely the Lagrangian in eq. (2.42).

Even though the terms in the Lagrangian can be obtained as a dimensional compactification of the unbroken $(D+1)$ dimensional theory, the amplitudes do not enjoy the same straightforward relation. The reason is that in the $(D+1)$-dimensional theory the (massive) scalars can source W-bosons that are not part of the explicitly broken YM + ϕ^3 theory. Without some highly special treatment of amplitudes in the $(D+1)$ theory, such "illegal" particles will appear as intermediate states. An example of such a special treatment would be to impose the operator equation (B.6), with the gauge-group generator replaced by the global-group generator $t^0 \to T^0$, on all external and all internal states in the amplitude. However, in practice, that approach may have no significant advantage compared to computing the amplitudes from scratch using the Lagrangian or using some recursive method.

In fact, it is no surprise that explicitly broken YM + ϕ^3 theory cannot be a straightforward dimensional compactification. If it were then, through the double-copy construction, the spontaneously broken YMESG would inherit this property. This is impossible, spontaneously broken YMESG is clearly not a straightforward dimensional compactification of a $(D+1)$-dimensional theory; for example, it does not have massive modes of gravitons.

C Expansions for the supergravity Lagrangian

In this appendix we list expansions for the period matrix and scalar metric entering the supergravity Higgs Lagrangian after the field redefinition (3.52) and up to terms linear in the physical scalar fields. The non-zero entries of the period matrix are the following,

$$\begin{aligned}
\mathcal{N}_{-1-1} &= -i + O(\phi^2) , & \mathcal{N}_{-1a} &= 2z^a + O(\phi^2) , \\
\mathcal{N}_{-10} &= 2z^0 + O(\phi^2) , & \mathcal{N}_{0a} &= 2\bar{z}^a + O(\phi^2) , \\
\mathcal{N}_{-11} &= 2z^1 + O(\phi^2) , & \mathcal{N}_{-1}{}^{\beta} &= \sqrt{2}(\overline{\varphi}^{\beta}_x + i\overline{\varphi}^{\beta}_y) + O(\phi^2) , \\
\mathcal{N}_{00} &= -i + O(\phi^2) , & \mathcal{N}_{-1\beta} &= \sqrt{2}(\varphi_{x\beta} + i\varphi_{y\beta}) + O(\phi^2) , \\
\mathcal{N}_{01} &= -2\bar{z}^1 + O(\phi^2) , & \mathcal{N}_{0}{}^{\beta} &= \sqrt{2}(\overline{\varphi}^{\beta}_x - i\overline{\varphi}^{\beta}_y) + O(\phi^2) , \\
\mathcal{N}_{11} &= -i + 2\bar{z}^0 + O(\phi^2) , & \mathcal{N}_{0\beta} &= \sqrt{2}(\varphi_{x\beta} - i\varphi_{y\beta}) + O(\phi^2) , \\
& & \mathcal{N}_{ab} &= (-i + 2\bar{z}^0)\delta^{ab} + O(\phi^2) , \\
& & \mathcal{N}^{\alpha}{}_{\beta} &= (-i + 2\bar{z}^0)\delta^{\alpha}_{\beta} + O(\phi^2) ,
\end{aligned} \tag{C.1}$$

where the indices a, b run over fields transforming in the adjoint of the unbroken gauge group and additional (non universal) spectators. Similarly, the scalar metric has non-zero entries,

$$
\begin{aligned}
g_{00} &= 1 - 2\sqrt{2}y^0 + O(\phi^2) \ , & g_{1a} &= -y^a + O(\phi^2) \ , & g_{ab} &= (1-2y^1)\delta^{ab} + O(\phi^2) \ , \\
g_{11} &= 1 - 2y^1 + O(\phi^2) \ , & g_1{}^{\beta} &= -\overline{\varphi}_y^{\beta} + O(\phi^2) \ , & g^{\alpha}{}_{\beta} &= (1-2y^1)\delta^{\alpha}_{\beta} + O(\phi^2) \ . \\
& & g_{1\beta} &= -\overline{\varphi}_{y\beta} + O(\phi^2) \ , & &
\end{aligned}
\tag{C.2}
$$

Note that the differences between these expansions and the ones in the appendix of [27] arise because the vector field A^1_μ has not been dualized. These expansions are sufficient for calculating the three-point amplitudes presented in section 4.2.

D Lagrangians of $\mathcal{N} = 4$ MESG and YMESG theories in five dimensions

The Lagrangian of the five-dimensional $\mathcal{N} = 4$ MESG theory is given by [93, 95]:

$$
\begin{aligned}
e^{-1}\mathcal{L} \;=\; & -\frac{1}{2}R - \frac{1}{2}\,\bar{\psi}^i_\mu\,\Gamma^{\mu\nu\rho}\,D_\nu\psi_{\rho i} - \frac{1}{4}\Sigma^2\,a_{\tilde{I}\tilde{J}}\,F^{\tilde{I}}_{\mu\nu}F^{\mu\nu\tilde{J}} - \frac{1}{4}\Sigma^{-4}\,G_{\mu\nu}G^{\mu\nu} \\
& - \frac{1}{2}\,(\partial_\mu a)^2 - \frac{1}{2}\,\bar{\chi}^i \not{D}\chi_i - \frac{1}{2}\bar{\lambda}^{ia}\not{D}\lambda^a_i - \frac{1}{2}\,P^{aij}_\mu P^\mu_{aij} \\
& - \frac{i}{2}\bar{\chi}^i\Gamma^\mu\Gamma^\nu\psi_{\mu i}\,\partial_\nu a + i\bar{\lambda}^{ia}\,\Gamma^\mu\Gamma^\nu\psi^j_\mu\,P_{\nu ij}{}^a \\
& + \frac{\sqrt{3}}{6}\Sigma L^{ij}_{\tilde{I}}\,F^{\tilde{I}}_{\rho a}\bar{\chi}_i\,\Gamma^\mu\Gamma^{\rho a}\psi_{\mu j} - \frac{1}{4}\Sigma L^a_{\tilde{I}}\,F^{\tilde{I}}_{\rho a}\bar{\lambda}^{ai}\,\Gamma^\mu\Gamma^{\rho a}\psi_{\mu i} \\
& - \frac{1}{2\sqrt{6}}\Sigma^{-2}\,\bar{\chi}^i\,\Gamma^\mu\Gamma^{\rho a}\psi_{\mu i}G_{\rho a} + \frac{5i}{24\sqrt{2}}\Sigma^{-2}\bar{\chi}^i\Gamma^{\rho a}\chi_i G_{\rho a} \\
& - \frac{i}{12}\Sigma L^{ij}_{\tilde{I}}F^{\tilde{I}}_{\rho a}\bar{\chi}_i\,\Gamma^{\rho a}\chi_j - \frac{i}{2\sqrt{3}}\Sigma L^a_{\tilde{I}}\,F^{\tilde{I}}_{\rho a}\bar{\lambda}^{ia}\Gamma^{\rho a}\chi_i - \frac{i}{8\sqrt{2}}\Sigma^{-2}G_{\rho a}\bar{\lambda}^{ia}\Gamma^{\rho a}\lambda^a_i \\
& + \frac{i}{4}\Sigma L^{ij}_{\tilde{I}}F^{\tilde{I}}_{\rho a}\bar{\lambda}^a_i\Gamma^{\rho a}\lambda^a_j - \frac{i}{4}\Sigma L^{ij}_{\tilde{I}}\,F^{\tilde{I}}_{\rho a}\,[\bar{\psi}_{\mu i}\Gamma^{\mu\nu\rho a}\psi_{\nu j} + 2\,\bar{\psi}^\rho_i\,\psi^a_j] \\
& - \frac{i}{8\sqrt{2}}\,\Sigma^{-2}G_{\rho a}\,[\bar{\psi}^i_\mu\Gamma^{\mu\nu\rho a}\psi_{\nu i} + 2\bar{\psi}^{\rho i}\,\psi^a_i] \\
& + \frac{\sqrt{2}}{8}e^{-1}\,C_{\tilde{I}\tilde{J}}\,\epsilon^{\mu\nu\rho a\lambda}\,F^{\tilde{I}}_{\mu\nu}F^{\tilde{J}}_{\rho a}\,a_\lambda + e^{-1}\mathcal{L}_{4f} \ ,
\end{aligned}
\tag{D.1}
$$

where $\mathcal{L}_{4f}$ denotes the four fermion terms in the Lagrangian.The supersymmetry transformation laws are given by[34]

$$
\delta e^m_\mu \;=\; \frac{1}{2}\bar{\varepsilon}^i\Gamma^m\psi_{\mu i} \ ,
$$

[34]All symmetrizations (ij) and anti-symmetrizations $[ij]$ are of weight one.

$$
\begin{aligned}
\delta\psi_{\mu i} &= D_\mu\varepsilon_i + \frac{i}{6}\Sigma L_{\tilde{I}ij}F^{\tilde{I}}_{\rho\sigma}(\Gamma_\mu^{\ \rho\sigma} - 4\delta^\rho_\mu\Gamma^\sigma)\varepsilon^j\,, \\
&\quad + \frac{i}{12\sqrt{2}}\Sigma^{-2}G_{\rho\sigma}(\Gamma_\mu^{\ \rho\sigma} - 4\delta^\rho_\mu\Gamma^\sigma)\varepsilon_i + \text{3-fermion terms}\,, \\
\delta\chi_i &= -\frac{i}{2}\partial\!\!\!/\sigma\varepsilon_i + \frac{\sqrt{3}}{6}\Sigma L_{\tilde{I}ij}F^{\tilde{I}}_{\rho\sigma}\Gamma^{\rho\sigma}\varepsilon^j - \frac{1}{2\sqrt{6}}\Sigma^{-2}G_{\rho\sigma}\Gamma^{\rho\sigma}\varepsilon_i\,, \\
\delta\lambda^a_i &= iP^a_{\mu ij}\Gamma^\mu\varepsilon^j - \frac{1}{4}\Sigma\, L^a_{\tilde{I}}F^{\tilde{I}}_{\rho\sigma}\Gamma^{\rho\sigma}\varepsilon_i + \text{3-fermion terms}\,, \\
\delta A^{\tilde{I}}_\mu &= \vartheta^{\tilde{I}}_\mu\,, \\
\delta a_\mu &= \frac{1}{\sqrt{6}}\Sigma^2\bar{\varepsilon}^i\Gamma_\mu\chi_i - \frac{i}{2\sqrt{2}}\Sigma^2\bar{\varepsilon}^i\psi_{\mu i}\,, \\
\delta\sigma &= \frac{i}{2}\bar{\varepsilon}^i\chi_i\,, \\
\delta L^{ij}_{\tilde{I}} &= -iL^a_{\tilde{I}}(\delta^{[i}_k\delta^{j]}_l - \frac{1}{4}\Omega^{ij}\Omega_{kl})\bar{\varepsilon}^k\lambda^{la}\,, \\
\delta L^a_{\tilde{I}} &= -iL_{\tilde{I}ij}\bar{\varepsilon}^i\lambda^{ja}\,,
\end{aligned}
\tag{D.2}
$$

where

$$
\vartheta^{\tilde{I}}_\mu \equiv -\frac{1}{\sqrt{3}}\Sigma^{-1}L^{\tilde{I}}_{ij}\bar{\varepsilon}^i\Gamma_\mu\chi^j - i\Sigma^{-1}L^{\tilde{I}}_{ij}\bar{\varepsilon}^i\psi^j_\mu + \frac{1}{2}L^{\tilde{I}}_a\Sigma^{-1}\bar{\varepsilon}^i\Gamma_\mu\lambda^a_i\,. \tag{D.3}
$$

and

$$
\Sigma = e^{\frac{1}{\sqrt{3}}a}. \tag{D.4}
$$

The abelian field strengths of vector fields are defined as

$$
F^{\tilde{I}}_{\mu\nu} = (\partial_\mu A^{\tilde{I}}_\nu - \partial_\nu A^{\tilde{I}}_\mu)\,, \quad G_{\mu\nu} = (\partial_\mu a_\nu - \partial_\nu a_\mu)\,, \tag{D.5}
$$

and the covariant derivative, D_μ involves the composite connections:

$$
D_\mu\lambda^a_i = \nabla_\mu\lambda^a_i + Q_{\mu i}{}^j\lambda^a_j + Q^{ab}_\mu\lambda^b_i\,, \tag{D.6}
$$

where ∇_μ is the Lorentz- and spacetime covariant derivative.

To construct an $\mathcal{N}=4$ YMESG theory with a semisimple subgroup K_S of the global symmetry group $SO(5,n)$ as the non-abelian gauge symmetry one replaces all derivatives acting on fields that transform non-trivially under K_S with K_S gauge covariant derivatives [93, 95]. As explained in section 6 this is implemented by the following substitutions in the Lagrangian:

$$
\begin{aligned}
F^{\tilde{I}}_{\mu\nu} &\longrightarrow \mathcal{F}^{\tilde{I}}_{\mu\nu} = F^{\tilde{I}}_{\mu\nu} + g_S\, A^{\tilde{J}}_\mu f^{\tilde{I}}_{\tilde{J}\tilde{K}}A^{\tilde{K}}_\nu\,, \\
\partial_\mu L^{\tilde{I}}_A &\longrightarrow \mathfrak{D}_\mu L^{\tilde{I}}_A = \partial_\mu L^{\tilde{I}}_A + g_S A^{\tilde{J}}_\mu f^{\tilde{I}}_{\tilde{J}\tilde{K}}L^{\tilde{K}}_A.
\end{aligned}
\tag{D.7}
$$

where $f_{\tilde{I}\tilde{J}}{}^{\tilde{K}}$ are non-vanishing only when the indices take values in the adjoint representation of the semisimple gauge group K_S and vanish whenever any one of the indices labels the spectator vector fields. The $USp(4)$ and $SO(n)$ connections, as well as the vielbein $P^a_{\mu ij}$ are also modified by the new g_S dependent contributions, i.e.,

$$
\begin{aligned}
\mathcal{Q}_{\mu i}{}^{j} &= Q_{\mu i}{}^{j} + g_S A^{\tilde{J}}_{\mu} L^{\tilde{K}}_{ik} f^{\tilde{I}}_{\tilde{J}\tilde{K}} L^{kj}_{\tilde{I}} \,, && \text{(D.8)}\\
\mathcal{Q}_{\mu a}{}^{b} &= Q_{\mu a}{}^{b} - g_S A^{\tilde{J}}_{\mu} L^{\tilde{K}}_{a} f^{\tilde{I}}_{\tilde{J}\tilde{K}} L^{b}_{\tilde{I}} \,, && \text{(D.9)}\\
\mathcal{P}^a_{\mu ij} &= P^a_{\mu ij} - g_S A^{\tilde{J}}_{\mu} L^{\tilde{K}}_{ij} f^{\tilde{I}}_{\tilde{J}\tilde{K}} L^{a}_{\tilde{I}} \,. && \text{(D.10)}
\end{aligned}
$$

The derivatives acting on the fermions get modified accordingly

$$
\mathcal{D}_\mu \lambda^a_i = \nabla_\mu \lambda^a_i + \mathcal{Q}_{\mu i}{}^{j} \lambda^a_j + \mathcal{Q}^{ab}_{\mu} \lambda^b_i \,, \tag{D.11}
$$

where $\mathcal{Q}_{\mu i}{}^{j}$ and $\mathcal{Q}_{\mu a}{}^{b}$ now include the g_S dependent terms.

To restore supersymmetry with the above covariantizations one adds to the Lagrangian following Yukawa couplings as well as potential terms[93, 95]

$$
\Delta\mathcal{L} = \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{Potential}} \,, \tag{D.12}
$$

where

$$
\begin{aligned}
\mathcal{L}_{\text{Yukawa}} \;=\;& \frac{3i}{2} g_S S_{ij}\, \bar{\psi}^i_\mu \Gamma^{\mu\nu} \psi^j_\nu + i g_S I_{ijab}\, \bar{\lambda}^{ia} \lambda^{jb} + \frac{i}{2} g_S S_{ij}\, \bar{\chi}^i \chi^j + g_S T^a_{ij}\, \bar{\psi}^i_\mu \Gamma^\mu \lambda^{ja} \,,\\
& + \sqrt{3} g_S S_{ij}\, \bar{\psi}^i_\mu \Gamma^\mu \chi^j - \frac{2i}{\sqrt{3}} g_S T^a_{ij}\, \bar{\chi}^i \lambda^{ja} \,,
\end{aligned} \tag{D.13}
$$

and

$$
\mathcal{L}_{\text{Potential}} = -g_S^2 \mathcal{V}^{(S)} = -g_S^2 \left(-\frac{9}{2}\, S_{ij} \Delta^{ij} + \frac{1}{2}\, T^a_{ij}\, T^{aij} \right) . \tag{D.14}
$$

Various scalar field dependent quantities above are defined as follows:

$$
\begin{aligned}
S_{ij} &= -\frac{2}{9} \Sigma^{-1} L^{\tilde{J}}_{(i|k|} f^{\tilde{K}}_{\tilde{J}\tilde{I}} L^{kl}_{\tilde{K}} L^{\tilde{I}}_{|l|j)} \,, && \text{(D.15)}\\
T^a_{ij} &= -\Sigma^{-1} L^{\tilde{J}a} L^{\tilde{K}}_{(i}{}^{k} f^{\tilde{I}}_{\tilde{J}\tilde{K}} L_{\tilde{I}|k|j)} \,, && \text{(D.16)}\\
I_{ijab} &= -\frac{3}{2} S_{ij} \delta_{ab} - \Sigma^{-1} L^{\tilde{J}a} L^{\tilde{K}}_{ij} f^{\tilde{I}}_{\tilde{J}\tilde{K}} L^{b}_{\tilde{I}} \,. && \text{(D.17)}
\end{aligned}
$$

Furthermore one needs to modify the transformation rules of the fermions as follows:

$$
\begin{aligned}
\delta_{\text{new}} \psi_{\mu i} &= i g_S S_{ij} \Gamma_\mu \varepsilon^j, && \text{(D.18)}\\
\delta_{\text{new}} \lambda^a_i &= g_S T^a_{ij} \varepsilon^j, && \text{(D.19)}\\
\delta_{\text{new}} \chi_i &= g_S \sqrt{3}\, S_{ij} \varepsilon^j. && \text{(D.20)}
\end{aligned}
$$

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