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Berends–Giele recursions and the BCJ duality in superspace and components

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The recursive method of Berends and Giele to compute tree-level gluon amplitudes is revisited using the framework of ten-dimensional super Yang–Mills. First we prove that the pure spinor formula to compute SYM tree amplitudes derived in 2010 reduces to the standard Berends–Giele formula from the 80s when restricted to gluon amplitudes and additionally determine the fermionic completion. Second, using BRST cohomology manipulations in superspace, alternative representations of the component amplitudes are explored and the Bern–Carrasco–Johansson relations among partial tree amplitudes are derived in a novel way. Finally, it is shown how the supersymmetric components of manifestly local BCJ-satisfying tree-level numerators can be computed in a recursive fashion.

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Contents

1. Introduction

Ten-dimensional super Yang–Mills (SYM) provides a simplified description of maximally supersymmetric gauge theories [1]. On the one hand, its spectrum comprises just a gluon and a gluino which automatically cover the scalars in lower-dimensional formulations [2]. On the other hand, pure spinors allow to formulate the on-shell conditions as a cohomology problem [3], and the BRST operator in the associated pure spinor superspace powerfully embodies gauge invariance and supersymmetry [4]. This framework naturally appears in the manifestly super Poincaré-covariant quantization of the superstring [4].

Using a confluence of string-theory techniques and field-theory intuition, scattering amplitudes in ten-dimensional SYM have been compactly represented in pure spinor super-space [5,6,7]. This construction crucially rests on the notion of multiparticle superfields [8] which were motivated by superstring computations [9–13]. Multiparticle superfields collect the contributions of tree-level subdiagrams at arbitrary multiplicity and can be flexibly attached to multiloop diagrams, see [7] for a two-loop application.

In a companion paper [14], the construction of multiparticle superfields and their expansion in the Grassmann variable θ^α of pure spinor superspace have been tremendously simplified. In the following, we will revisit tree-level amplitudes in the light of the new theta-expansions and in particular:

- recover and supersymmetrize the Berends–Giele recursion for gluonic tree amplitudes
- present a simplified component realization of the BCJ color-kinematics duality, along with a new superspace proof for the closely related BCJ relations.

1.1. Summary of results on the Berends–Giele recursion

The theta-expansions of ten-dimensional multiparticle superfields have recently [14] been simplified using supersymmetric Berends–Giele currents which generalize the gluonic currents defined by Berends and Giele [15]. Using these simplified expansions, the pure spinor superspace formula to compute ten-dimensional color-ordered SYM amplitudes at tree level [5],

$$A^{\text{SYM}}(1, 2, \ldots, p, p+1) = \langle E_{12\ldots p} M_{p+1} \rangle \,, \tag{1.1}$$

will be explicitly evaluated in components and shown to be

$$A^{\text{SYM}}(1, 2, \ldots, p, p+1) = s_{12\ldots p}(\mathfrak{e}_{12\ldots p} \cdot \mathfrak{e}_{p+1}) + k_{12\ldots p}^m (\mathcal{X}_{12\ldots p} \gamma_m \mathcal{X}_{p+1}) \,. \tag{1.2}$$

The superfields $E_{12...p}$ and M_{p+1} will be introduced in section 2.3, and the square of the region momentum $k^m_{12...p} \equiv k^m_1 + k^m_2 + \ldots + k^m_p$ is denoted by $s_{12...p}$. Moreover, $\mathfrak{e}^m_{12...p}$ and $\mathcal{X}^\alpha_{12...p}$ in (1.2) denote the component Berends–Giele currents which depend on the gluon and gluino polarizations e^m_i, χ^α_i as well as light-like momenta k^m_i associated with legs $i = 1, 2, \ldots, p$, where $m = 0, \ldots, 9$ and $\alpha = 1, \ldots, 16$ are vector and Weyl-spinor indices of the Lorentz group $SO(1,9)$.

After setting the fermionic fields to zero, the first term in (1.2) will be shown to reproduce the gluonic Berends–Giele formula [15],

$$A^{\mathrm{YM}}(1, 2, \ldots, p+1) = s_{12...p}(J_{12...p} \cdot J_{p+1}) \,, \tag{1.3}$$

making (1.2) its supersymmetric generalization for ten-dimensional SYM.

Furthermore, the same Berends–Giele currents $\mathfrak{e}^m_{12...p}$ and $\mathcal{X}^\alpha_{12...p}$ together with a field-strength companion $\mathfrak{f}^{mn}_{12...p}$ will be shown to yield economic and manifestly cyclic representations of SYM amplitudes such as

$$
\begin{aligned}
A^{\mathrm{SYM}}(1, 2, 3, 4, 5) = &\frac{1}{2}(\mathfrak{e}^m_{12}\mathfrak{f}^{mn}_{34}\mathfrak{e}^n_5 + \mathfrak{e}^m_{34}\mathfrak{f}^{mn}_5\mathfrak{e}^n_{12} + \mathfrak{e}^m_5\mathfrak{f}^{mn}_{12}\mathfrak{e}^n_{34}) \\
&+ (\mathcal{X}_{12}\gamma_m\mathcal{X}_5)\mathfrak{e}^m_{34} + (\mathcal{X}_{34}\gamma_m\mathcal{X}_{12})\mathfrak{e}^m_5 + (\mathcal{X}_5\gamma_m\mathcal{X}_{34})\mathfrak{e}^m_{12} + \mathrm{cyclic}(12345) \,,
\end{aligned}
\tag{1.4}
$$

streamlining the earlier approach in [16] based on the above $J^m_{12...p}$.

Using the generating series of supersymmetric Berends–Giele currents discussed in [17,14], it will be shown that the generating series of ten-dimensional SYM tree-level amplitudes takes a very simple form,

$$\mathrm{Tr}\Big(\frac{1}{4}\mathbb{F}_{mn}\mathbb{F}^{mn} + (\mathbb{W}\gamma^m\nabla_m\mathbb{W})\Big)\Big|_{\theta=0} = \sum_{n=3}^{\infty} \frac{n-2}{n} \sum_{i_1, i_2, \ldots, i_n} \mathrm{Tr}(t^{i_1}t^{i_2}\ldots t^{i_n})A^{\mathrm{SYM}}(i_1, i_2, \ldots, i_n)\,. \tag{1.5}$$

Note that the left-hand side of (1.5) matches the ten-dimensional SYM Lagrangian evaluated on the generating series $\mathbb{F}^{mn}(x, \theta = 0)$ and $\mathbb{W}^\alpha(x, \theta = 0)$ defined below.

1.2. Summary of results on the BCJ duality

The virtue of the simplified theta-expansions in [14] can be reconciled with a manifestation of the duality between color and kinematics due to Bern, Carrasco and Johansson (BCJ) [18] (see [19] for a review). A concrete tree-level realization of the BCJ duality was given in [20] at any multiplicity, based on local numerators in pure spinor superspace. The

components are accessible through the zero-mode treatment in [21], but we will present a significantly accelerated approach where the zero-mode manipulations are trivialized.

The BCJ duality immediately led to the powerful prediction that only $(n-3)!$ permutations of SYM tree-level subamplitudes (1.2) are linearly independent [18]. This basis dimension was later derived from the monodromy properties of the string worldsheet [22,23], by the field-theory limit of the n-point superstring disk amplitude [12,24] and by BCFW on-shell recursions in field theory [25]. In addition to these proofs, the following explicit BCJ relations among color-ordered amplitudes will be obtained from pure spinor cohomology arguments,

$$\sum_{i=1}^{|A|}\sum_{j=1}^{|B|}(-1)^{i-j}s_{a_ib_j}A^{\mathrm{SYM}}((a_1\ldots a_{i-1}\sqcup\!\sqcup a_{|A|}\ldots a_{i+1}),a_i,b_j,(b_{j-1}\ldots b_1\sqcup\!\sqcup b_{j+1}\ldots b_{|B|}),n)=0\,,$$

$$(1.6)$$

where the words $A=a_1a_2\ldots a_{|A|}$ and $B=b_1b_2\ldots b_{|B|}$ have total length $|A|+|B|=n-1$. The shuffle product $\sqcup\!\sqcup$ is defined recursively as

$$\emptyset\sqcup\!\sqcup A=A\sqcup\!\sqcup\emptyset=A,\qquad A\sqcup\!\sqcup B\equiv a_1(a_2\ldots a_{|A|}\sqcup\!\sqcup B)+b_1(b_2\ldots b_{|B|}\sqcup\!\sqcup A)\,,\qquad(1.7)$$

where $\emptyset$ denotes the case when no "letter" is present.

2. Review

2.1. Berends–Giele recursion relations

In the 80s, Berends and Giele proposed a recursive method to compute color-ordered gluon amplitudes at tree-level using multiparticle currents J_P^m defined[1] as [15]

$$J_i^m\equiv e_i^m\,,\qquad s_PJ_P^m\equiv\sum_{XY=P}[J_X,J_Y]^m+\sum_{XYZ=P}\{J_X,J_Y,J_Z\}^m\,,\qquad(2.1)$$

where e_i^m denotes the polarization vector of a single-particle gluon, $P=12\ldots p$ encompasses several external particles, and the Mandelstam invariants are

$$s_P\equiv\frac{1}{2}k_P^2\,,\qquad k_P^m\equiv k_1^m+k_2^m+\cdots+k_p^m\,.\qquad(2.2)$$

---

[1] The original definition of J_P^m in [15] contains the factor $1/k_P^2$ instead of $1/s_P$ as adopted here. An overall factor of $\frac{1}{2}$ in (2.3) and (2.4) compensates this difference.

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The notation $\sum_{XY=P}$ in (2.1) instructs to deconcatenate $P = 12\ldots p$ into non-empty words $X = 12\ldots j$ and $Y = j+1\ldots p$ with $j = 1, 2, \ldots, p-1$ and the obvious generalization to $\sum_{XYZ=P}$. The brackets $[\cdot, \cdot]^m$ and $\{\cdot, \cdot, \cdot\}^m$ are given by stripping off one gluon field (with vector index m) from the cubic and quartic vertices of the Yang–Mills Lagrangian,

$$[J_X, J_Y]^m \equiv (k_Y \cdot J_X)J_Y^m + \frac{1}{2}k_X^m(J_X \cdot J_Y) - (X \leftrightarrow Y) \tag{2.3}$$

$$\{J_X, J_Y, J_Z\}^m \equiv (J_X \cdot J_Z)J_Y^m - \frac{1}{2}(J_X \cdot J_Y)J_Z^m - \frac{1}{2}(J_Y \cdot J_Z)J_X^m. \tag{2.4}$$

The *Berends–Giele* currents J_P^m are conserved [15] and satisfy certain symmetries [26],

$$k_P^m J_P^m = 0, \qquad J_{A\sqcup B}^m = 0, \quad \forall A, B \neq \emptyset. \tag{2.5}$$

The purely gluonic amplitudes are then computed as [15]

$$A^{\mathrm{YM}}(1, 2, \ldots, p, p+1) = s_{12\ldots p} J_{12\ldots p}^m J_{p+1}^m. \tag{2.6}$$

For example, the Berends–Giele current of multiplicity two following from (2.1) is

$$s_{12}J_{12}^m = e_2^m(e_1 \cdot k_2) - e_1^m(e_2 \cdot k_1) + \frac{1}{2}(k_1^m - k_2^m)(e_1 \cdot e_2) \tag{2.7}$$

and leads to the well-known three-point amplitude

$$A^{\mathrm{YM}}(1, 2, 3) = s_{12}J_{12}^m J_3^m = (e_1 \cdot e_2)(k_1 \cdot e_3) + \mathrm{cyclic}(123). \tag{2.8}$$

Note that the Berends–Giele formula (2.6) as presented in [15] is not supersymmetric, it computes purely gluonic amplitudes.

2.2. Super Yang–Mills superfields in ten dimensions

SYM in ten dimensions admits a super-Poincare-invariant description in terms of four types of superfields: the spinor potential $\mathbb{A}_\alpha(x, \theta)$, the vector potential $\mathbb{A}^m(x, \theta)$ and their associated field-strengths $\mathbb{W}^\alpha(x, \theta)$, $\mathbb{F}^{mn}(x, \theta)$. They satisfy the following non-linear field equations[2] [1],

$$\{D_{(\alpha}, \mathbb{A}_{\beta)}\} = \gamma_{\alpha\beta}^m \mathbb{A}_m + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\} \tag{2.9}$$

$$[D_\alpha, \mathbb{A}_m] = [\partial_m, \mathbb{A}_\alpha] + (\gamma_m \mathbb{W})_\alpha + [\mathbb{A}_\alpha, \mathbb{A}_m]$$

$$\{D_\alpha, \mathbb{W}^\beta\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn} + \{\mathbb{A}_\alpha, \mathbb{W}^\beta\}$$

$$[D_\alpha, \mathbb{F}^{mn}] = [\partial^{[m}, (\mathbb{W}\gamma^{n]})_\alpha] - [\mathbb{A}^{[m}, (\mathbb{W}\gamma^{n]})_\alpha] + [\mathbb{A}_\alpha, \mathbb{F}^{mn}].$$

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[2] Our convention for (anti)symmetrizing indices does not include $\frac{1}{2}$, e.g. $\partial^{[m}\gamma^{n]} = \partial^m\gamma^n - \partial^n\gamma^m$.

For later convenience, we use the notation where $\mathbb{K}$ refers to any element of the set containing these superfields,

$$\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}_m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}. \tag{2.10}$$

In the context of scattering amplitudes or vertex operators of the superstring [4], one discards the quadratic terms from (2.9) to obtain the linearized superfields of ten-dimensional SYM $K_i \in \{A_\alpha^i, A_m^i, W_i^\alpha, F_i^{mn}\}$ satisfying

$$\{D_{(\alpha}, A_{\beta)}^i\} = \gamma_{\alpha\beta}^m A_m^i, \qquad \{D_\alpha, W_i^\beta\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}^i$$
$$[D_\alpha, A_m^i] = (\gamma_m W_i)_\alpha + [\partial_m, A_\alpha^i], \qquad [D_\alpha, F_{mn}^i] = [\partial_{[m}, (\gamma_{n]} W_i)_\alpha]. \tag{2.11}$$

They describe a single gluon and/or gluino which furnishes the ith leg in the amplitude.

In pursuing compact expressions for superstring scattering amplitudes one is led to a natural multiparticle generalization of the above description, where the single-particle labels are replaced by "words" $P = 123\ldots p$. In particular, amplitudes can be compactly written in terms of non-local[3] superfields called Berends–Giele currents $\mathcal{K}_P \in \{\mathcal{A}_\alpha^P, \mathcal{A}_m^P, \mathcal{W}_P^\alpha, \mathcal{F}_P^{mn}\}$ encompassing several legs $1, 2, \ldots, p$ in an amplitude. They are recursively constructed from linearized superfields in (2.11), and the original expressions in [8] are related to simplified representations in [14] via non-linear gauge transformations. This gauge freedom affects the generating series $\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}_m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$ of Berends–Giele currents

$$\mathbb{K} = \sum_i \mathcal{K}_i t^i + \sum_{i,j} \mathcal{K}_{ij} t^i t^j + \sum_{i,j,k} \mathcal{K}_{ijk} t^i t^j t^k + \cdots, \tag{2.12}$$

where t^i are generators of a non-abelian gauge group. The generating series in (2.12) were shown in [17] to solve the non-linear field equations[4] (2.9) by the properties of the constituent Berends–Giele currents $\mathcal{K}_P \in \{\mathcal{A}_\alpha^P, \mathcal{A}_m^P, \mathcal{W}_P^\alpha, \mathcal{F}_P^{mn}\}$.

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[3] A discussion of *local* multiparticle superfields K_P can be found in [14,8].

[4] It should be pointed out that the notion of a generating series which solves the field equations and gives rise to tree amplitudes corresponds to the "perturbiner" formalism [27]. This approach has been applied to the self-dual sector of Yang–Mills theory and led to a generating series of MHV amplitudes, see [28] for a supersymmetric extension. However, the generic Yang–Mills amplitudes have never been obtained this way (see also [29]). We thank Nima Arkani-Hamed for pointing out these references.

2.2.1. Simplifying component expansions with superfield gauge transformations

The aforementioned gauge freedom of the generating series (2.12) allows to tune the theta-expansion of the multiparticle supersymmetric Berends–Giele currents such that [14]

$$\mathcal{A}_\alpha^P(x,\theta) = \left(\frac{1}{2}(\theta\gamma_m)_\alpha \mathfrak{e}_P^m + \frac{1}{3}(\theta\gamma^m)_\alpha(\theta\gamma_m\mathcal{X}_P) - \frac{1}{32}(\gamma^P\theta)_\alpha(\theta\gamma_{mnp}\theta)\mathfrak{f}_P^{mn} + \dots \right) e^{k_P \cdot x} \quad (2.13)$$

takes the same form as the linearized superfield A_α^i subject to (2.11) [30,31],

$$A_\alpha^i(x,\theta) = \left(\frac{1}{2}(\theta\gamma_m)_\alpha e_i^m + \frac{1}{3}(\theta\gamma^m)_\alpha(\theta\gamma_m\chi_i) - \frac{1}{32}(\gamma^P\theta)_\alpha(\theta\gamma_{mnp}\theta)f_i^{mn} + \dots \right) e^{k_i \cdot x} . \quad (2.14)$$

The components $\mathfrak{e}_P^m, \mathcal{X}_P^\alpha, \mathfrak{f}_P^{mn}$ depend on the momenta k_i^m, polarizations e_i^m and wavefunctions χ_i^α of the gluons and gluinos encompassed in the multiparticle label $P = 12\dots p$ and can be obtained from the recursions [14]

$$\mathfrak{e}_P^m = \frac{1}{s_P} \sum_{XY=P} \mathfrak{e}_{[X,Y]}^m , \quad \mathcal{X}_P^\alpha = \frac{1}{s_P} \sum_{XY=P} \mathcal{X}_{[X,Y]}^\alpha , \quad (2.15)$$

where $\mathfrak{e}_i^m \equiv e_i^m$ and $\mathcal{X}_i^\alpha \equiv \chi_i^\alpha$ for a single-particle label as well as

$$\mathfrak{e}_{[X,Y]}^m = -\frac{1}{2} \left[\mathfrak{e}_X^m(k^X \cdot \mathfrak{e}^Y) + \mathfrak{e}_n^X \mathfrak{f}_Y^{mn} - (\mathcal{X}^X \gamma^m \mathcal{X}^Y) - (X \leftrightarrow Y) \right] \quad (2.16)$$

$$\mathcal{X}_{[X,Y]}^\alpha = \frac{1}{2}(k_X^p + k_Y^p)\gamma_p^{\alpha\beta} \left[\mathfrak{e}_X^m(\gamma_m \mathcal{X}_Y)_\beta - \mathfrak{e}_Y^m(\gamma_m \mathcal{X}_X)_\beta \right] . \quad (2.17)$$

The non-linear component field-strength is given by

$$\mathfrak{f}_P^{mn} = k_P^m \mathfrak{e}_P^n - k_P^n \mathfrak{e}_P^m - \sum_{XY=P} \left(\mathfrak{e}_X^m \mathfrak{e}_Y^n - \mathfrak{e}_X^n \mathfrak{e}_Y^m \right) \quad (2.18)$$

and generalizes the single-particle instance $\mathfrak{f}_i^{mn} \equiv f_i^{mn} = k_i^m e_i^n - k_i^n e_i^m$ in (2.14).

The expressions in (2.16), (2.17) and (2.18) are obtained from the theta-independent terms of the superfields $\mathcal{A}_P^m, \mathcal{W}_P^\alpha, \mathcal{F}_P^{mn}$ evaluated at $x = 0$ [14],

$$\mathfrak{e}_P^m \equiv \mathcal{A}_P^m(0,0) , \qquad \mathcal{X}_P^\alpha \equiv \mathcal{W}_P^\alpha(0,0) , \qquad \mathfrak{f}_P^{mn} \equiv \mathcal{F}_P^{mn}(0,0) , \quad (2.19)$$

in the same way as e_i^m, χ_i^α and f_i^{mn} stem from the linearized superfields $A_i^m, W_i^\alpha, F_i^{mn}$. Accordingly, the recursions in (2.15) to (2.17) for $\mathfrak{e}_P^m$ and $\mathcal{X}_P^\alpha$ descend from the recursive construction of superspace Berends–Giele currents $\mathcal{A}_P^m, \mathcal{W}_P^\alpha, \mathcal{F}_P^{mn}$ described in [14].

Note that the transversality of the gluon and the Dirac equation of the gluino propagate as follows to the multiparticle level,

$$(k_P \cdot \mathfrak{e}_P) = 0 , \qquad k_m^P(\gamma^m \mathcal{X}_P)_\alpha = \sum_{XY=P} \left[\mathfrak{e}_X^m(\gamma_m \mathcal{X}_Y)_\alpha - \mathfrak{e}_Y^m(\gamma_m \mathcal{X}_X)_\alpha \right], \quad (2.20)$$

where transversality of $\mathfrak{e}_P^m$ is a peculiarity of the Lorentz gauge chosen in the derivation of the corresponding superspace Berends–Giele current $\mathcal{A}_P^m(x,\theta)$ [14].

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2.3. The pure spinor superspace formula for SYM tree amplitudes

Tree-level amplitudes in ten-dimensional SYM have been constructed in [5] from cohomology methods in pure spinor superspace [4]. Inspired by OPEs in string theory, the BRST-invariant superspace expression

$$A^{\mathrm{SYM}}(1,2,\ldots,p,p+1) = \langle E_{12\ldots p} M_{p+1} \rangle \equiv \sum_{XY=12\ldots p} \langle M_X M_Y M_{p+1} \rangle \qquad (2.21)$$

with the pole structure of a color-ordered $(p+1)$-point amplitude has been proposed and shown to reproduce known component expressions for various combinations of gluons and gluinos. BRST invariance of the superfields implies gauge-invariant and supersymmetric components. In (2.21) the bracket $\langle\ldots\rangle$ instructs to pick up terms of order $\lambda^3\theta^5$ of the enclosed superfields [4] and the following shorthand has been used

$$M_P \equiv \lambda^\alpha \mathcal{A}_\alpha^P(x,\theta). \qquad (2.22)$$

At this point, we make use of the gauge choice in [14] where the theta-expansion (2.13) of the multiparticle superfield mimics the single-particle counterpart (2.14). In this way, the same $\lambda^3\theta^5$ correlators listed on appendix A of [32] govern both the three-point amplitude

$$A^{\mathrm{SYM}}(1,2,3) = \langle M_1 M_2 M_3 \rangle = \frac{1}{2}\mathfrak{e}_1^m \mathfrak{f}_2^{mn} \mathfrak{e}_3^n + (\mathcal{X}_1 \gamma_m \mathcal{X}_2)\mathfrak{e}_3^m + \mathrm{cyclic}(123) \qquad (2.23)$$

and a generic multiparticle constituent of the n-point amplitudes (2.21),

$$\langle M_X M_Y M_Z \rangle = \frac{1}{2}\mathfrak{e}_X^m \mathfrak{f}_Y^{mn} \mathfrak{e}_Z^n + (\mathcal{X}_X \gamma_m \mathcal{X}_Y)\mathfrak{e}_Z^m + \mathrm{cyclic}(XYZ) \equiv \mathfrak{M}_{X,Y,Z}. \qquad (2.24)$$

This makes the gluon and gluino components of an arbitrary n-point tree amplitude easily accessible through the recursion (2.15) to (2.18) for the components $\mathfrak{e}_P^m, \mathcal{X}_P^\alpha$ and $\mathfrak{f}_P^{mn}$. Using the component field-strength (2.18), it follows that the gluonic three-point amplitudes of the Berends–Giele and pure spinor formulæ match. In the following section, we will demonstrate that the same is true for an arbitrary number of external legs.

3. The supersymmetric completion of the Berends–Giele formula

In this section, the pure spinor superspace formula for ten-dimensional SYM tree amplitudes (2.21) will be shown to reduce *ipsis litteris* to the Berends–Giele formula (2.6) when restricted to its gluonic expansion. Given the supersymmetry of the pure spinor approach, we will use it to derive the supersymmetric completion of the Berends–Giele formula.

3.1. Bosonic Berends–Giele current from superfields

In a first step, the lowest components $\mathfrak{e}_P^m$ in the superfield (2.13) are demonstrated to reproduce the bosonic Berends–Giele currents in (2.1) once the fermions are decoupled, i.e.

$$\mathfrak{e}_P^m \big|_{\chi_j=0} = J_P^m \ . \tag{3.1}$$

Plugging the field-strength $\mathfrak{f}_P^{mn}$ (2.18) into the recursive definition of $\mathfrak{e}_P^m$ (2.15) leads to

$$2s_P \mathfrak{e}_P^m = - \sum_{XY=P} \left[2\mathfrak{e}_X^m (k_X \cdot \mathfrak{e}_Y) + k_Y^m (\mathfrak{e}_X \cdot \mathfrak{e}_Y) - (\mathcal{X}_X \gamma^m \mathcal{X}_Y) - (X \leftrightarrow Y) \right]$$
$$+ \sum_{XYZ=P} \left[2(\mathfrak{e}_X \cdot \mathfrak{e}_Z) \mathfrak{e}_Y^m - (\mathfrak{e}_X \cdot \mathfrak{e}_Y) \mathfrak{e}_Z^m - (\mathfrak{e}_Y \cdot \mathfrak{e}_Z) \mathfrak{e}_X^m \right] \ . \tag{3.2}$$

In absence of fermions, $\chi_j^\alpha = 0$, the first line (3.2) yields the contribution of the cubic vertex (2.3) to the Berends–Giele current, and the second line due to the non-linear part of the field-strength $\mathfrak{f}_P^{mn}$ reproduces the quartic vertex (2.4). This is natural since the quartic interaction in the YM Lagrangian arises from the non-linear part of the field-strength. Together with the single-particle case $\mathfrak{e}_i^m = J_i^m = e_i^m$, the matching of (3.2) at $\chi_j^\alpha = 0$ with the Berends–Giele recursion (2.1) completes the inductive proof of (3.1).

Also note that the recursion (2.17) for $\mathcal{X}_P^\alpha$ amounts to a resummation of Feynman diagrams incorporating both the fermion propagator $k_m \gamma_{\alpha\beta}^m / k^2$ and the cubic coupling of two fermions with a boson, in accordance with the Berends–Giele method [15] applied to ten-dimensional SYM theory.

3.2. Supersymmetric Berends–Giele amplitude from the pure spinor formula

The relation (3.1) between the ten-dimensional Berends–Giele current $\mathfrak{e}_P^m$ in superspace and its purely gluonic counterpart J_P^m is now extended to their corresponding tree-level amplitudes: the pure spinor formula (2.21) versus the Berends–Giele formula (2.6).

To see the relation, note that (2.24) can be rewritten as

$$\langle M_X M_Y M_Z \rangle = (\mathfrak{e}_{[X,Y]} \cdot \mathfrak{e}_Z) + \mathfrak{e}_X^m (\mathcal{X}_Y \gamma_m \mathcal{X}_Z) - \mathfrak{e}_Y^m (\mathcal{X}_X \gamma_m \mathcal{X}_Z) \tag{3.3}$$
$$+ \frac{1}{2} \sum_{RS=Z} \left[(\mathfrak{e}_R \cdot \mathfrak{e}_X)(\mathfrak{e}_S \cdot \mathfrak{e}_Y) - (\mathfrak{e}_R \cdot \mathfrak{e}_Y)(\mathfrak{e}_S \cdot \mathfrak{e}_X) \right] \ ,$$

provided that transversality (2.20) and momentum conservation holds, $k_X^m + k_Y^m + k_Z^m = 0$. In particular, when $Z \to p+1$ is a single-particle label associated with the $(p+1)^{\text{th}}$ massless leg, the deconcatenation terms in the second line of (3.3) vanish:

$$\langle M_X M_Y M_{p+1} \rangle = (\mathfrak{e}_{[X,Y]} \cdot \mathfrak{e}_{p+1}) + \mathfrak{e}_X^m (\mathcal{X}_Y \gamma_m \mathcal{X}_{p+1}) - \mathfrak{e}_Y^m (\mathcal{X}_X \gamma_m \mathcal{X}_{p+1}) . \tag{3.4}$$

Plugging the correlator (3.4) into the pure spinor superspace formula for tree-level SYM amplitudes (2.21) yields

$$A^{\mathrm{SYM}}(1,2,\ldots p,p{+}1) = \sum_{XY=12\ldots p}\left[(\mathfrak{e}_{[X,Y]}\cdot\mathfrak{e}_{p+1})+\mathfrak{e}_X^m(\mathcal{X}_Y\gamma_m\mathcal{X}_{p+1})-\mathfrak{e}_Y^m(\mathcal{X}_X\gamma_m\mathcal{X}_{p+1})\right].\quad(3.5)$$

Alternatively, using (2.15) and (2.20) to identify $\mathfrak{e}_{12\ldots p}^m$ and $\mathcal{X}_{12\ldots p}^\alpha$, this can be written as

$$A^{\mathrm{SYM}}(1,2,\ldots p,p+1) = s_{12\ldots p}(\mathfrak{e}_{12\ldots p}\cdot\mathfrak{e}_{p+1}) + k_{12\ldots p}^m(\mathcal{X}_{12\ldots p}\gamma_m\mathcal{X}_{p+1})\,.\quad(3.6)$$

In view of (3.1), the expression (3.6) reproduces the gluonic Berends–Giele formula [15] in absence of fermions,

$$A^{\mathrm{SYM}}(1,2\ldots,p,p+1)\big|_{\chi_j=0} = s_{12\ldots p}(J_{12\ldots p}\cdot J_{p+1}) = A^{\mathrm{YM}}(1,2\ldots,p,p+1)\,,\quad(3.7)$$

and additionally provides its supersymmetric completion. Note that the bosonic currents $\mathfrak{e}_P^m$ contain even powers of gluino wavefunctions χ_i^α from the last term in (2.16) such as $s_{12}\mathfrak{e}_{12}^m = s_{12}J_{12}^m + (\chi_1\gamma^m\chi_2)$. Hence, both classes of terms on the right hand side of (3.6) contribute to fermionic amplitudes.

3.3. Divergent propagators and their cancellation

3.3.1. In components

From the definition (2.15) it follows that both of $\mathfrak{e}_P^m$ and $\mathcal{X}_P^\alpha$ in (3.6) are proportional to a divergent propagator since $s_P = 0$ for a massless $(p+1)$-point amplitude. As well known from the Berends–Giele formula for gluons [15], this is compensated by the formally vanishing numerator containing $s_P = 0$ in (2.6). The same is true for its supersymmetric completion derived in (3.6) since $k_P^m(\gamma_m\mathcal{X}_{p+1})_\alpha = 0$ using $k_P^m = -k_{p+1}^m$ and the massless Dirac equation. The interpretation is also the same; s_P is the inverse of the bosonic propagator $1/\partial^2$ while $k_m^P\gamma_{\alpha\beta}^m$ is the inverse of the fermion propagator $\partial_m\gamma_{\alpha\beta}^m/\partial^2$.

3.3.2. In pure spinor superspace

The supersymmetric way to cancel a divergent propagator relies on the action of the pure spinor BRST charge $Q \equiv \lambda^\alpha D_\alpha$ [4] on the currents M_P [5],

$$E_P \equiv QM_P = \sum_{XY=P} M_X M_Y\,.\quad(3.8)$$

10

The integration of schematic form $\langle \lambda^3 \theta^5 \rangle = 1$ annihilates BRST-exact expressions [4]. Because the single-particle superfield M_{p+1} is BRST closed, $QM_{p+1} = 0$, the superspace representation of tree-level amplitudes in (2.21) would be BRST exact $Q(M_P M_{p+1})$ if the current M_P was well defined in the phase space of $p+1$ massless particles [5]. However, $M_P \sim 1/s_P$ and therefore the vanishing of s_P prevents the amplitude from being BRST exact. Just like (3.5), the expression $\langle \sum_{XY=P} M_X M_Y M_{p+1} \rangle$ does not contain any divergent propagator.

The assessment of BRST-exactness for a given superfield will play an important role in the derivation of BCJ relations in section 4.2.

3.4. Short representations and BRST integration by parts

At first sight the Berends–Giele formula (2.6) requires the p-current $J^m_{12...p}$ in the computation of the $(p+1)$-gluon amplitude. However, a diagrammatic method has been used by Berends and Giele in [16] to obtain "short" representations of bosonic amplitudes up to eight points which required no more than the four-current and led to manifestly cyclic formulæ for $A^{\mathrm{YM}}(1, 2, \ldots, p+1)$. For example, the six-point amplitude was found to be

$$A^{\mathrm{YM}}(1, 2, \ldots, 6) = \frac{1}{2} s_{123} J^m_{123} J^m_{456} + \frac{1}{3} [J_{12}, J_{34}]^m J^m_{56} \tag{3.9}$$
$$+ \frac{1}{2} \{J_1, J_{23}, J_4\}^m J^m_{56} + \{J_1, J_2, J_{34}\}^m J^m_{56} + \mathrm{cyclic}(123456) \, ,$$

and similar expressions were written for the seven- and eight-point amplitudes [16].

In the framework of pure spinor superspace, the multiplicity of currents can be shortened using integration by parts of the BRST charge. By (3.8), this amounts to

$$\sum_{XY=P} \langle M_X M_Y M_Q \rangle = \sum_{XY=Q} \langle M_P M_X M_Y \rangle \, , \tag{3.10}$$

which has been used in [5] to cast the superspace formula (2.21) for n-point trees into a manifestly cyclic form without any current of multiplicity higher than $\frac{n}{2}$, e.g.

$$A^{\mathrm{SYM}}(1, 2, \ldots, 6) = \frac{1}{3} \langle M_{12} M_{34} M_{56} \rangle + \frac{1}{2} \langle M_{123}(M_{45} M_6 + M_4 M_{56}) \rangle + \mathrm{cyclic}(123456) \, . \tag{3.11}$$

In terms of the components $\mathfrak{M}_{X,Y,Z}$ from the evaluation (2.24) of pure spinor superspace expressions, the component expressions for amplitudes of multiplicity ≤ 8 are given by

$$A^{\text{SYM}}(1,2,\ldots,4) = \frac{1}{2}\mathfrak{M}_{12,3,4} + \text{cyclic}(12\ldots 4) \tag{3.12}$$

$$A^{\text{SYM}}(1,2,\ldots,5) = \mathfrak{M}_{12,3,45} + \text{cyclic}(12\ldots 5)$$

$$A^{\text{SYM}}(1,2,\ldots,6) = \frac{1}{3}\mathfrak{M}_{12,34,56} + \frac{1}{2}(\mathfrak{M}_{123,45,6} + \mathfrak{M}_{123,4,56}) + \text{cyclic}(12\ldots 6)$$

$$A^{\text{SYM}}(1,2,\ldots,7) = \mathfrak{M}_{123,45,67} + \mathfrak{M}_{1,234,567} + \text{cyclic}(12\ldots 7)$$

$$A^{\text{SYM}}(1,2,\ldots,8) = \frac{1}{2}(\mathfrak{M}_{1234,567,8} + \mathfrak{M}_{1234,56,78} + \mathfrak{M}_{1234,5,678})$$

$$+ \mathfrak{M}_{123,456,78} + \text{cyclic}(12\ldots 8) \ ,$$

see [5] for the nine- and ten-point analogues. Given the recursive nature of the definitions of $\mathfrak{e}_P^m$, $\mathfrak{f}_P^{mn}$ and $\mathcal{X}_P^\alpha$, the full component expansion of the above amplitudes is readily available and reproduce the results available in the website [33].

Note that the manipulations leading to (3.4) rely on a single-particle current M_{p+1} and therefore do not apply to the $\mathfrak{M}_{X,Y,Z}$ in (3.12).

3.5. The generating series of tree-level amplitudes

The way how component amplitudes (3.6) of SYM descend from the pure spinor superspace expression (2.21) can be phrased in the language of generating series. The solution

$$\mathbb{V} \equiv \lambda^\alpha \mathbb{A}_\alpha = \sum_i M_i t^i + \sum_{i,j} M_{ij} t^i t^j + \sum_{i,j,k} M_{ijk} t^i t^j t^k + \cdots \tag{3.13}$$

of the non-linear SYM equations (2.9) generates color-dressed SYM amplitudes via[5] [17]

$$\frac{1}{3}\text{Tr}\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle = \sum_{n=3}^\infty \frac{n-2}{n} \sum_{i_1,i_2,\ldots,i_n} \text{Tr}(t^{i_1} t^{i_2} \ldots t^{i_n}) A^{\text{SYM}}(i_1, i_2, \ldots, i_n) \ . \tag{3.14}$$

Note from (2.19) that $\mathfrak{e}_P^m$, $\mathcal{X}_P^\alpha$ and $\mathfrak{f}_P^{mn}$ are just the $\theta = 0$ components of the corresponding generating series $\mathbb{A}^m$, $\mathbb{W}^\alpha$ and $\mathbb{F}^{mn}$. Therefore (2.24) implies that

$$\frac{1}{3}\text{Tr}\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle = \frac{1}{4}\text{Tr}([\mathbb{A}_m, \mathbb{A}_n]\mathbb{F}^{mn}) + \text{Tr}(\mathbb{W}\gamma^m \mathbb{A}_m \mathbb{W})\Big|_{\theta=0}$$

$$= \text{Tr}\Big(\frac{1}{4}\mathbb{F}_{mn}\mathbb{F}^{mn} + (\mathbb{W}\gamma^m \nabla_m \mathbb{W})\Big)\Big|_{\theta=0}. \tag{3.15}$$

---

[5] The representations of SYM amplitudes generated by $\text{Tr}\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle$ are related to (2.21) by BRST integration by parts (3.10).

In passing to the second line of (3.15), we have used the massless Dirac equation $\nabla_m \gamma_{\alpha\beta}^m \mathbb{W}^\beta = 0$ as well as the field equation $\partial_m \mathbb{F}^{mn} = [\mathbb{A}_m, \mathbb{F}^{mn}] + \gamma_{\alpha\beta}^n \{\mathbb{W}^\alpha, \mathbb{W}^\beta\}$ and discarded a total derivative to rewrite $(\partial_m \mathbb{A}_n)\mathbb{F}^{mn} = -\mathbb{A}_n \big([\mathbb{A}_m, \mathbb{F}^{mn}] + \gamma_{\alpha\beta}^n \{\mathbb{W}^\alpha, \mathbb{W}^\beta\}\big)$. The factor $1/3$ on the left-hand side of (3.15) offsets the sum over three terms that results from the cyclic symmetry of the trace.

It is interesting to observe that the generating series of tree-level amplitudes (3.15) matches the ten-dimensional SYM Lagrangian evaluated on the generating series of (non-local) Berends–Giele currents in superspace: $\mathbb{F}^{mn}(x,0)$ and $\mathbb{W}^\alpha(x,0)$.

4. BCJ relations from the cohomology of pure spinor superspace

In this section, we prove that the BCJ relations [18] among partial SYM amplitudes follow from the vanishing of certain BRST-exact expressions in pure spinor superspace and find a closed formula for them. A closely related property of tree amplitudes is the possibility to express the complete kinematic dependence in terms of $(n-2)!$ master numerators through a sequence of Jacobi-like relations [18]. A superspace representation of such master numerators was given in [20], and we will provide a compact component evaluation along the lines of the previous section.

4.1. Kleiss–Kuijf relations from symmetries of Berends–Giele currents

For completeness, we start by revisiting from a superspace perspective the Kleiss–Kuijf (KK) relations among color-ordered amplitudes [34], firstly proven in [35].

The KK relations are conveniently described in the Berends–Giele framework. To see this, recall that the superspace currents $\mathcal{K}_P \in \{\mathcal{A}_\alpha^P, \mathcal{A}_P^m, \mathcal{W}_P^\alpha, \mathcal{F}_P^{mn}\}$ satisfy the symmetry property [8]

$$\mathcal{K}_{A \shuffle B} = 0, \quad \forall A, B \neq \emptyset , \tag{4.1}$$

see appendix B of [14] for a proof. The symmetry (4.1) of course also holds for theta-independent components $\{\mathfrak{e}_P^m, \mathcal{X}_P^\alpha, \mathfrak{f}_P^{mn}\}$ of $\mathcal{K}_P$, see (2.19). Since the currents $\mathfrak{e}_P^m$ reduce to J_P^m via (3.1), this is consistent with the symmetry $J_{A \shuffle B}^m = 0, \forall A, B \neq \emptyset$ derived by Berends and Giele in [26]. The symmetry (4.1) together with the identity[6]

$$\mathcal{K}_{B1A} - (-1)^{|B|} \mathcal{K}_{1(A \shuffle B^T)} = - \sum_{XY=B} (-1)^{|X|} \mathcal{K}_{X^T \shuffle (Y1A)} , \tag{4.2}$$

---

[6] Incidentally, the identity (4.2) shows the equivalence between the statements given in equation (2) of [36] and Theorem 2.2 of [37].

where B^T denotes the reversal of the word B, lead to an alternative form of (4.1),

$$\mathcal{K}_{B1A} - (-1)^{|B|}\mathcal{K}_{1(A\sqcup B^T)} = 0 \, . \tag{4.3}$$

Since $E_P \equiv QM_P$ generalizes (4.3) to $\mathcal{K}_P \to E_P$, the tree-level amplitude representation[7] (3.6) $A_{12\ldots n} = \langle E_{12\ldots n-1}M_n \rangle$ immediately yields the Kleiss–Kuijf relations

$$A_{C1Bn} - (-1)^{|C|}A_{1(B\sqcup C^T)n} = \langle (E_{C1B} - (-1)^{|C|}E_{1(B\sqcup C^T)})M_n \rangle = 0 \, , \tag{4.4}$$

which reduce the number of independent color-ordered amplitudes to $(n-2)!$ [34].

4.2. BCJ relations from the BRST cohomology

4.2.1. Berends–Giele currents in BCJ gauge

There is a method to construct Berends–Giele currents from quotients of local superfields $K_{[P,Q]}$ by Mandelstam invariants whose precise form follows from an intuitive mapping with cubic graphs (or planar binary trees) [8,14]. For example, the Berends–Giele currents associated with the local superfield $V_{[P,Q]} \equiv \lambda^\alpha A_\alpha^{[P,Q]}$ up to multiplicity four are given by

$$M_{12}^{\mathrm{BCJ}} = \frac{V_{[1,2]}}{s_{12}}, \qquad M_{123}^{\mathrm{BCJ}} = \frac{V_{[[1,2],3]}}{s_{12}s_{123}} + \frac{V_{[1,[2,3]]}}{s_{23}s_{123}}, \tag{4.5}$$

$$M_{1234}^{\mathrm{BCJ}} = \frac{1}{s_{1234}}\left(\frac{V_{[[[1,2],3],4]}}{s_{12}s_{123}} + \frac{V_{[[1,[2,3]],4]}}{s_{23}s_{123}} + \frac{V_{[[1,2],[3,4]]}}{s_{12}s_{34}} + \frac{V_{[1,[2,[3,4]]]}}{s_{34}s_{234}} + \frac{V_{[1,[[2,3],4]]}}{s_{23}s_{234}} \right).$$

As discussed in a companion paper [14], one can perform a multiparticle gauge transformation (denoted BCJ gauge) which enforces the superfields

$$V_{123\ldots p} \equiv V_{[[\ldots[[1,2],3],\ldots],p]} \tag{4.6}$$

in (4.5) with diagrammatic interpretation shown in fig. 1 to satisfy the Lie symmetries of nested commutators $[[\ldots[[t^1,t^2],t^3],\ldots],t^p]$, e.g.

$$V_{12} + V_{21} = 0, \quad V_{123} + V_{231} + V_{312} = 0 \, . \tag{4.7}$$

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[7] We omit the superscript from A^{SYM} and write the labels as a subscript to avoid cluttering.
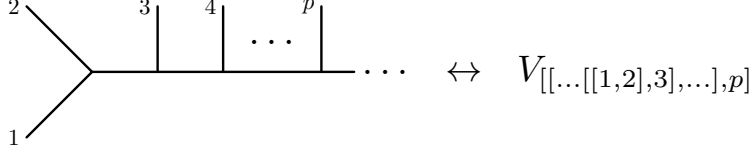
Fig. 1 The tree diagram with an off-shell leg is represented by the local superfield (4.6).

Moreover, BCJ gauge allows to reduce any other topology of bracketings to the master topology (4.6) by a sequence of Jacobi-like identities

$$V_{\ldots[[P,Q],R]\ldots} + V_{\ldots[[Q,R],P]\ldots} + V_{\ldots[[R,P],Q]\ldots} = 0\,, \quad \text{e.g. } V_{[[1,2],[3,4]]} = V_{1234} - V_{1243}\,. \quad (4.8)$$

Hence, the Berends–Giele current $M^{\mathrm{BCJ}}_{12\ldots p}$ can be expanded in terms of the $(p-1)!$ independent permutations of $V_{12\ldots p}$. This is the same number of independent components as left by the Berends–Giele symmetry (4.1) (here for $\mathcal{K}_{12\ldots p} \to M^{\mathrm{BCJ}}_{12\ldots p}$). As a crucial feature of Berends–Giele currents in BCJ gauge, there is an invertible mapping between the local superfields $V_{12\ldots p}$ and $M^{\mathrm{BCJ}}_{12\ldots p}$. More explicitly, for multiplicity $p \leq 4$ one can use (4.6) and (4.8) to invert (4.5) and obtain

$$V_{12} = s_{12}M^{\mathrm{BCJ}}_{12}, \qquad V_{123} = s_{12}(s_{23}M^{\mathrm{BCJ}}_{123} - s_{13}M^{\mathrm{BCJ}}_{213})\,, \quad (4.9)$$

$$V_{1234} = s_{12}\big[s_{23}s_{34}M^{\mathrm{BCJ}}_{1234} - s_{13}s_{34}M^{\mathrm{BCJ}}_{2134} + s_{14}s_{23}M^{\mathrm{BCJ}}_{3214} - s_{13}s_{24}M^{\mathrm{BCJ}}_{3124}$$
$$+ s_{23}s_{24}(M^{\mathrm{BCJ}}_{1234} + M^{\mathrm{BCJ}}_{1243}) - s_{13}s_{14}(M^{\mathrm{BCJ}}_{2134} + M^{\mathrm{BCJ}}_{2143})\big]\ .$$

The generalization to arbitrary rank can be read off from the formula [12]

$$\frac{V_{12\ldots p}}{z_{12}z_{23}\cdots z_{p-1,p}} + \mathrm{perm}(2,\ldots,p) = \prod_{k=2}^{p}\sum_{m=1}^{k-1}\frac{s_{mk}}{z_{mk}}M^{\mathrm{BCJ}}_{12\ldots p} + \mathrm{perm}(2,\ldots,p)\,, \quad (4.10)$$

using partial fraction relations[8] among the denominators made of $z_{ij} \equiv z_i - z_j$.

It is important to stress that the left-hand sides in (4.9) are *local* expressions; all the kinematic poles in Mandelstam invariants cancel out from the linear combinations of currents on the right-hand side. The poles cancel only when the superfields are in the BCJ gauge. As we will see below, this fact can be exploited to derive the BCJ relations [18] among color-ordered amplitudes.

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[8] Note that $Z_{12\ldots p-1,p} \equiv 1/(z_{12}z_{23}\ldots z_{p-1,p})$ satisfies $Z_{A \sqcup\!\sqcup B} = 0, \forall A, B \neq \emptyset$.

4.2.2. Four- and five-point BCJ relations

We shall now connect superfields in BCJ gauge with BCJ relations among partial SYM amplitudes. At the four- and five-point level, one multiplies the local expressions in (4.9) by a single-particle V_n (which is BRST closed) and uses the vanishing of BRST-exact expressions under the pure spinor bracket prescription $\langle \ldots \rangle$ [4]. For example,

$$\frac{V_{123}}{s_{12}} = s_{23}M_{123}^{\text{BCJ}} - s_{13}M_{213}^{\text{BCJ}} \quad \Rightarrow \quad 0 = \left\langle Q\left(\frac{V_{123}}{s_{12}}V_4\right)\right\rangle = \langle (s_{23}E_{123}^{\text{BCJ}} - s_{13}E_{213}^{\text{BCJ}})V_4\rangle \quad (4.11)$$

with $E_P^{\text{BCJ}} \equiv QM_P^{\text{BCJ}}$ corresponds to the four-point[9] BCJ relation [18] by (2.21),

$$0 = s_{23}A^{\text{SYM}}(1,2,3,4) - s_{13}A^{\text{SYM}}(2,1,3,4) \ . \tag{4.12}$$

Note that the BCJ gauge for the local superfields is a crucial requirement in this derivation: In a generic gauge, $s_{23}M_{123} - s_{13}M_{213}$ would be an ill-defined expression containing divergent propagators of the form $1/s_{123}$, and BRST-exactness would be obstructed as argued in subsection 3.3 for individual SYM subamplitudes.

Similarly, the identities

$$\frac{V_{1234}}{s_{12}s_{123}} + \frac{V_{3214}}{s_{23}s_{123}} = s_{34}M_{1234}^{\text{BCJ}} + s_{14}M_{3214}^{\text{BCJ}} - s_{24}(M_{1324}^{\text{BCJ}} + M_{3124}^{\text{BCJ}}) \tag{4.13}$$

$$\frac{V_{1234} - V_{1243}}{s_{12}s_{34}} = s_{23}M_{1234}^{\text{BCJ}} - s_{13}M_{2134}^{\text{BCJ}} - s_{24}M_{1243}^{\text{BCJ}} + s_{14}M_{2143}^{\text{BCJ}}$$

derived from (4.9) with manifestly well-defined left-hand side imply the BCJ relations [18]

$$0 = \left\langle Q\left(\frac{V_{1234}}{s_{12}s_{123}} + \frac{V_{3214}}{s_{23}s_{123}}\right)V_5\right\rangle = s_{34}A^{\text{SYM}}(1,2,3,4,5) + s_{14}A^{\text{SYM}}(3,2,1,4,5) \quad (4.14)$$

$$- s_{24}\left[A^{\text{SYM}}(1,3,2,4,5) + A^{\text{SYM}}(3,1,2,4,5)\right]$$

$$0 = \left\langle Q\left(\frac{V_{1234} - V_{1243}}{s_{12}s_{34}}\right)V_5\right\rangle = s_{23}A^{\text{SYM}}(1,2,3,4,5) - s_{13}A^{\text{SYM}}(2,1,3,4,5)$$

$$- s_{24}A^{\text{SYM}}(1,2,4,3,5) + s_{14}A^{\text{SYM}}(2,1,4,3,5) \ .$$

Even though the above derivation relies on the choice of BCJ gauge, the subamplitudes in the resulting BCJ relations are independent on the multiparticle gauge for the currents M_P. This can be seen from the non-linear gauge invariance in the generating series (3.14) of the amplitude formula (2.21).

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[9] The three-point BCJ-relation $0 = s_{12}A^{\text{SYM}}(1,2,3)$ following from $s_{12} = 0$ can be formally derived via $0 = \langle QV_{12}V_3\rangle = s_{12}\langle V_1V_2V_3\rangle$.

4.2.3. Higher-point BCJ relations

Along the same lines, one can verify in a basis of V_P that the expression [8]

$$M_{S[A,B]}^{\mathrm{BCJ}} \equiv \sum_{i=1}^{|A|} \sum_{j=1}^{|B|} (-1)^{i-j+|A|-1} s_{a_i b_j} M_{(a_1 a_2 \ldots a_{i-1} \sqcup a_{|A|} a_{|A|-1} \ldots a_{i+1}) a_i b_j (b_{j-1} \ldots b_2 b_1 \sqcup b_{j+1} \ldots b_{|B|})}^{\mathrm{BCJ}}$$

(4.15)

with $A = a_1 a_2 \ldots a_{|A|}$ and $B = b_1 b_2 \ldots b_{|B|}$ does not have any pole in s_{AB}. One can therefore identify the following BRST-exact combinations of $(|A| + |B| + 1)$-point amplitudes,

$$0 = (-1)^{|A|-1} \langle Q(M_{S[A,B]}^{\mathrm{BCJ}} M_n) \rangle \qquad (4.16)$$

$$= \sum_{i=1}^{|A|} \sum_{j=1}^{|B|} (-1)^{i-j} s_{a_i b_j} \langle E_{(a_1 a_2 \ldots a_{i-1} \sqcup a_{|A|} a_{|A|-1} \ldots a_{i+1}) a_i b_j (b_{j-1} \ldots b_2 b_1 \sqcup b_{j+1} \ldots b_{|B|})}^{\mathrm{BCJ}} M_n \rangle$$

$$= \sum_{i=1}^{|A|} \sum_{j=1}^{|B|} (-1)^{i-j} s_{a_i b_j} A^{\mathrm{SYM}}((a_1 \ldots a_{i-1} \sqcup a_{|A|} \ldots a_{i+1}), a_i, b_j, (b_{j-1} \ldots b_1 \sqcup b_{j+1} \ldots b_{|B|}), n) ,$$

which all boil down to BCJ relations in some representation [18,22,23,25]. For the single-particle choice $A = 1$ along with $B = 2, n, (n-1), \ldots, 4, 3$, (4.16) reduces to the fundamental BCJ relations

$$0 = (-1)^n \langle Q(M_{S[1,2n(n-1)\ldots54]}^{\mathrm{BCJ}} M_3) \rangle \qquad (4.17)$$
$$= s_{12} A^{\mathrm{SYM}}(2, 1, 3, \ldots, n) + (s_{12} + s_{13}) A^{\mathrm{SYM}}(2, 3, 1, 4, \ldots, n)$$
$$+ \cdots + (s_{12} + s_{13} + \ldots + s_{1,n-1}) A^{\mathrm{SYM}}(2, 3, \ldots, n-1, 1, n) ,$$

which are well-known to leave $(n-3)!$ independent subamplitudes [18,22,23,25].

4.3. Component form of BCJ numerators

The initial derivation of BCJ relations in [18] relied on the duality between color and kinematics, i.e. the existence of particular representations of tree amplitudes. The functions of polarizations and momenta associated with the cubic graphs in such a "BCJ representation" are assumed to obey the same Jacobi identities as the color factors made of structure constants f^{abc} of the gauge group. As a consequence, the complete information on polarizations and momenta reside in $(n-2)!$ master graphs which can be chosen to be the half-ladder diagrams with fixed endpoints 1 and $n-1$ as depicted in fig. 2 and arbitrary permutations of the remaining legs $2, 3, \ldots, n-2$ and n.

An explicit realization of the BCJ duality for tree-level amplitudes was given in [20] based on the tree amplitudes of the pure spinor superstring. The master graphs in the figure were associated with local kinematic numerators[10] $\langle V_{12\ldots j} V_{n-1,n-2\ldots j+1} V_n \rangle$ labeled

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[10] Note that the precursors of $V_{12\ldots p}$ were denoted by $T_{12\ldots p}$ in [20].
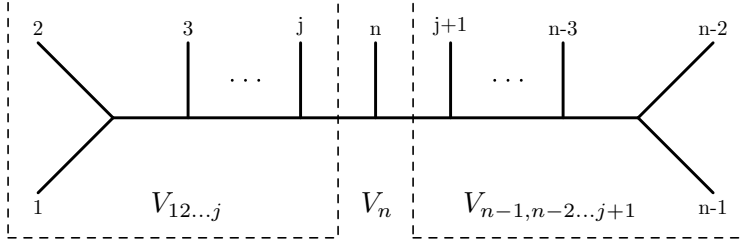
Fig. 2 The $(n-2)!$ half-ladder diagrams with legs 1 and $n-1$ attached to opposite endpoints encode the complete kinematic dependence in a BCJ representation.

by $j = 1, 2, \ldots, n-2$ along with the $(n-3)!$ permutations in the legs $2, 3, \ldots, n-2$. The kinematic factors for any other graph can be reached by a sequence of Jacobi relations, and this representation agrees with the field-theory limit of the open superstring amplitude, i.e. yields the right SYM amplitude.

The techniques of [14] (in particular the discussion of BCJ/HS gauge) give rise to a compact formula for their components,

$$\langle V_A V_B V_C \rangle = \frac{1}{2} e_A^m f_B^{mn} e_C^n + (\chi_A \gamma_m \chi_B) e_C^m + \text{cyclic}(ABC) \,, \tag{4.18}$$

whose form is completely analogous to (2.24). The constituents e_A^m, f_A^{mn} and χ_A^α of (4.18) are *local* multiparticle polarizations and will be explained below.

4.3.1. Local multiparticle polarizations

The discussion of recursion relations for local superfields given in [14] has a direct counterpart for their multiparticle polarizations e_A^m, f_A^{mn} and χ_A^α which constitute their theta-independent terms. The setup starts with a recursive definition for local multiparticle polarizations $\hat{e}_A^m$, $\hat{f}_A^{mn}$ and $\hat{\chi}_A^\alpha$ whose labels do not satisfy the symmetries of a Lie algebra, for example $\hat{e}_{123}^m + \hat{e}_{231}^m + \hat{e}_{312}^m \neq 0$ (their hatted notation is a reminder of this symmetry failure). However, non-linear gauge variations of their multiparticle superfields can be exploited to find a gauge where the symmetries are indeed satisfied.

The recursive definition of the hatted components is given by

$$\hat{e}_{12\ldots p}^m = -\frac{1}{2}\left[\hat{e}_{12\ldots p-1}^m (k_{12\ldots p-1} \cdot \hat{e}_p) + \hat{e}_{12\ldots p-1}^n \hat{f}_p^{mn} - (\hat{\chi}_{12\ldots p-1} \gamma^m \hat{\chi}_p) - (12\ldots p-1 \leftrightarrow p)\right]$$

$$\hat{\chi}_{12\ldots p}^\alpha = \frac{1}{2} k_{12\ldots p}^n \gamma_n^{\alpha\beta} \left[\hat{e}_{12\ldots p-1}^m (\gamma_m \hat{\chi}_p)_\beta - (12\ldots p-1 \leftrightarrow p)\right] \,, \tag{4.19}$$

and it starts with $\hat{e}_i^m = e_i^m$ and $\hat{\chi}_i^\alpha = \chi_i^\alpha$. The local field-strength is defined by

$$\hat{f}_{mn}^{12\ldots p} \equiv k_m^{12\ldots p} \hat{e}_n^{12\ldots p} - k_n^{12\ldots p} \hat{e}_m^{12\ldots p} + \sum_{j=2}^{p} \sum_{\delta \in P(\beta_j)} (k_{12\ldots j-1} \cdot k_j) \, \hat{e}_{[n}^{12\ldots j-1,\{\delta\}} \, \hat{e}_{m]}^{j,\{\beta_j \backslash \delta\}} \,, \tag{4.20}$$

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with shorthand $\beta_j = \{j+1, j+2, \ldots, p\}$ and $P(\beta_j)$ denoting the power set of β_j, e.g.

$$\hat{f}_1^{mn} = f_1^{mn} = k_1^m e_1^n - k_1^n e_1^m \ , \qquad \hat{f}_{12}^{mn} = f_{12}^{mn} = k_{12}^m e_{12}^n - k_{12}^n e_{12}^m - s_{12} e_1^{[m} e_2^{n]}$$

$$\hat{f}_{123}^{mn} = k_{123}^m \hat{e}_{123}^n - k_{123}^n \hat{e}_{123}^m - (s_{13} + s_{23}) e_{12}^{[n} e_3^{m]} - s_{12}(e_1^{[n} e_{23}^{m]} - e_2^{[n} e_{13}^{m]}) \ . \qquad (4.21)$$

Up to and including multiplicity $p = 2$, the multiparticle polarizations in the BCJ numerators (4.18) agree with their hatted counterparts in (4.19),

$$e_{12}^m = \hat{e}_{12}^m = e_2^m(e_1 \cdot k_2) - e_1^m(e_2 \cdot k_1) + \frac{1}{2}(k_1^m - k_2^m)(e_1 \cdot e_2) + (\chi_1 \gamma^m \chi_2) \qquad (4.22)$$

$$\chi_{12}^\alpha = \hat{\chi}_{12}^\alpha = \frac{1}{2} k_{12}^p \gamma_p^{\alpha\beta} \left[e_1^m (\gamma_m \chi_2)_\beta - e_2^m (\gamma_m \chi_1)_\beta \right] \ ,$$

while multiplicities $p \geq 3$ require redefinitions $\hat{h}_{12\ldots p}$ starting with

$$e_{123}^m = \hat{e}_{123}^m - k_{123}^m \hat{h}_{123} \ , \qquad \chi_{123}^\alpha = \hat{\chi}_{123}^\alpha \ . \qquad (4.23)$$

The redefinition of $\hat{e}_{123}^m$ in (4.23) ensures the Lie symmetry $e_{123}^m + e_{231}^m + e_{312}^m = 0$. At multiplicity $p = 4$, we have

$$e_{1234}^m = \hat{e}_{1234}^m + (k_{123}\cdot k_4)\hat{h}_{123}e_4^m - (k_{12}\cdot k_3)\hat{h}_{124}e_3^m - (k_1\cdot k_2)(\hat{h}_{134}e_2^m - \hat{h}_{234}e_1^m) - k_{1234}^m \hat{h}_{1234}$$

$$\chi_{1234}^\alpha = \hat{\chi}_{1234}^\alpha + (k_{123}\cdot k_4)\hat{h}_{123}\chi_4^\alpha - (k_{12}\cdot k_3)\hat{h}_{124}\chi_3^\alpha - (k_1\cdot k_2)(\hat{h}_{134}\chi_2^\alpha - \hat{h}_{234}\chi_1^\alpha) \ , \qquad (4.24)$$

and the rank-five example can be extracted from [14] as will be explained shortly. The scalar correction terms $\hat{h}_{12\ldots p}$ in (4.23) and (4.24) can be reduced to building blocks

$$h_{A,B,C} \equiv \frac{1}{4} e_A^m f_B^{mn} e_C^n + \frac{1}{2}(\chi_A \gamma_m \chi_B) e_C^m + \text{cyclic}(ABC) \qquad (4.25)$$

made of multiparticle polarizations of lower multiplicity $\leq p - 2$ via

$$3\hat{h}_{123} \equiv h_{1,2,3}$$

$$4\hat{h}_{1234} \equiv h_{12,3,4} + h_{34,1,2} - \frac{1}{2}h_{1,2,3}(k_{123} \cdot e_4) \qquad (4.26)$$

$$+ \frac{1}{6}\left[h_{1,3,4}(k_{134} \cdot e_2) - h_{2,3,4}(k_{234} \cdot e_1) - h_{124}(k_{124} \cdot e_3) \right] \ .$$

Once the redefinition $e_{12\ldots p}^m = \hat{e}_{12\ldots p}^m + \ldots$ for the multiparticle polarization has been performed, the corresponding "unhatted" field-strength relevant for the BCJ numerators in (4.18) is obtained completely analogously to (4.20),

$$f_{mn}^{12\ldots p} \equiv k_m^{12\ldots p} e_n^{12\ldots p} - k_n^{12\ldots p} e_m^{12\ldots p} + \sum_{j=2}^{p} \sum_{\delta \in P(\beta_j)} (k_{12\ldots j-1} \cdot k_j)\, e_{[n}^{12\ldots j-1,\{\delta\}}\, e_{m]}^{j,\{\beta_j \backslash \delta\}} \ . \qquad (4.27)$$

4.3.2. Higher multiplicity

As already mentioned, the above redefinitions of $\hat{e}^m_{12\ldots p}$, $\hat{\chi}^\alpha_{12\ldots p}$ and $\hat{f}^{mn}_{12\ldots p}$ descend from the superspace discussion in section 3 of [14]. In particular, the corrections $h_{A,B,C}$ defined in (4.25) are the $\theta = 0$ component of a local superfield $H_{A,B,C}(x,\theta)$ which was completely specified up to multiplicity five in [14]. So the full expressions of e^m_{12345}, χ^α_{12345} and f^{mn}_{12345} are readily available.

At the same time, there is no obstruction to pushing these recursive constructions even further, leading to local multiparticle polarizations e^m_P, χ^α_P and f^{mn}_P of higher multiplicity. Therefore, together with the central formula (4.18) for local components, the discussion in this section provides access to the supersymmetric components of the *local* BCJ-satisfying numerators of [20] in a recursive fashion.

5. Conclusion and outlook

In this work, we have extracted and streamlined component information from tree-level scattering amplitudes in pure spinor superspace. The results are based on simplified theta-expansions for multiparticle superfields of ten-dimensional SYM which are derived from non-linear gauge transformations in a companion paper [14]. More specifically:

- The n-point tree-level amplitude derived in [5] from locality, supersymmetry and gauge invariance is shown to reproduce the Berends–Giele formula, and the supersymmetrization by fermionic component amplitudes is worked out.
- BCJ relations are derived from the decoupling of BRST-exact expressions in pure spinor superspace.
- Kinematic tree-level numerators satisfying the BCJ duality between color and kinematics from [20] are translated into components.

The resulting ten-dimensional component amplitudes together with their BCJ representations and dimensional reductions will have a broad range of applications. With appropriate truncations of the gluon and gluino components, they are suitable to determine D-dimensional unitarity cuts in a variety of theories including QCD, see e.g. [38,39,40] and references therein.

It would be interesting to relate the multiparticle polarizations in the component form of the BCJ numerators to the approach of [41]. In that reference, formally vanishing non-local terms are added to the Yang–Mills Lagrangian to automatically produce BCJ

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numerators. The interplay between Lagrangians and generating series of kinematic factors might shed further light on the superfield redefinitions in [14] underlying our BCJ numerators.

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