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Non-linear gauge transformations in $D = 10$ SYM theory and the BCJ duality

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Recent progress on scattering amplitudes in super Yang–Mills and superstring theory benefitted from the use of multiparticle superfields. They universally capture tree-level subdiagrams, and their generating series solve the non-linear equations of ten-dimensional super Yang–Mills. We provide simplified recursions for multiparticle superfields and relate them to earlier representations through non-linear gauge transformations of their generating series. In this work we discuss the gauge transformations which enforce their Lie symmetries as suggested by the Bern–Carrasco–Johansson duality between color and kinematics. Another gauge transformation due to Harnad and Shnider is shown to streamline the theta-expansion of multiparticle superfields, bypassing the need to use their recursion relations beyond the lowest components. The findings of this work tremendously simplify the component extraction from kinematic factors in pure spinor superspace.

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1. Introduction

In recent years, the super-Poincaré covariant description [1] of ten-dimensional super Yang–Mills theory (SYM) has been extensively used to compute scattering amplitudes in string and field theory. This description features the ten-dimensional superfields,

$$\mathbb{A}_\alpha(x,\theta),\ \ \mathbb{A}^m(x,\theta),\ \ \mathbb{W}^\alpha(x,\theta),\ \ \mathbb{F}^{mn}(x,\theta)\ , \tag{1.1}$$

where $\mathbb{A}_\alpha, \mathbb{A}_m$ are the spinor and vector potentials and $\mathbb{W}^\alpha, \mathbb{F}^{mn}$ their associated field-strengths. They satisfy certain non-linear field equations to be reviewed below.

The appearance of the linearized versions $A_\alpha(x,\theta)$, $A^m(x,\theta)$, $W^\alpha(x,\theta)$ and $F^{mn}(x,\theta)$ of (1.1) in the massless vertex operators of the pure spinor superstring [2] have brought these superfields to the forefront of perturbation theory: They compactly encode the kinematical factors of scattering amplitudes in string and field theory.

Following the standard CFT prescription for scattering amplitudes in the pure spinor superstring, it soon became clear that the *linearized* superfields repeatedly appeared in the same meaningful combinations. The study of short-distance singularities among massless vertex operators gave rise to the notion of multiparticle superfields,

$$K_P \in \{A^P_\alpha(x,\theta),\ A^m_P(x,\theta),\ W^\alpha_P(x,\theta),\ F^{mn}_P(x,\theta)\}\ .$$

We gather the labels of several particles in $P = 12\ldots p$ and collectively refer to the four types of superfields via K_P to avoid the cluttering of Lorentz indices.

In the last years, two distinct ways of obtaining the explicit expressions of multiparticle superfields have been proposed. In 2011 and 2012 [3,4], their construction closely followed the (lengthy) OPE calculations in superstring tree amplitudes, leading to expressions for K_P which satisfy the Lie symmetries of nested commutators $[\ldots[[t^1,t^2],t^3],\ldots,t^p]$ under permutations of the labels in $P = 12\ldots p$. In 2014 [5], an efficient recursive definition of multiparticle superfields was given in terms of a cubic-vertex prescription $K_{[P,Q]}$, bypassing the need to perform OPEs beyond multiplicity $p = 2$. A chain of redefinitions was supplemented in order to recover the same Lie symmetries as in the previous approach.

In addition to the (local) multiparticle superfields, the superstring amplitude calculations also suggested natural definitions of their non-local counterparts, called *Berends–Giele currents* and represented by calligraphic letters,

$$\mathcal{K}_P \in \{\mathcal{A}_\alpha(x,\theta),\ \mathcal{A}_m(x,\theta),\ \mathcal{W}^\alpha(x,\theta),\ \mathcal{F}^{mn}(x,\theta)\}\ . \tag{1.2}$$

As described in [3,5], the precise definition of $\mathcal{K}_P$ used an intuitive mapping between planar binary trees (or cubic graphs) and Lie symmetry-satisfying multiparticle superfields, dressed with the propagators of the graph. These Berends–Giele currents elegantly capture kinematic factors of multiparticle amplitudes in both string and field-theory.

As one of the main result of this article, we provide an alternative definition of Berends–Giele currents which tremendously simplifies the construction of earlier work [5] while preserving their equations of motion.

1.1. Generating series and non-linear gauge transformations

A new perspective on multiparticle superfields K_P and their associated Berends–Giele currents $\mathcal{K}_P$ is provided by the generating series of Berends–Giele currents. These generating series are an expansion in terms of Lie-algebra generators t^i with multiparticle Berends–Giele currents as coefficients [6],

$$\mathbb{K} \equiv \sum_{p=1}^{\infty} \sum_{i_1,i_2,\ldots,i_p} \mathcal{K}_{i_1 i_2 \ldots i_p} t^{i_1} t^{i_2} \ldots t^{i_p} \ . \tag{1.3}$$

As a key feature of these generating series $\mathbb{K} \in \{\mathbb{A}_\alpha(x,\theta), \mathbb{A}^m(x,\theta), \mathbb{W}^\alpha(x,\theta), \mathbb{F}^{mn}(x,\theta)\}$, they are Lie algebra-valued and solve the non-linear field equations of ten-dimensional SYM theory. These equations are invariant under non-linear gauge transformations [1],

$$\begin{aligned} \delta_{\mathbf{\Omega}} \mathbb{A}_\alpha &= \big[D_\alpha, \mathbf{\Omega}\big] - \big[\mathbb{A}_\alpha, \mathbf{\Omega}\big] \, , \qquad \delta_{\mathbf{\Omega}} \mathbb{W}^\alpha = \big[\mathbf{\Omega}, \mathbb{W}^\alpha\big] \, , \\ \delta_{\mathbf{\Omega}} \mathbb{A}_m &= \big[\partial_m, \mathbf{\Omega}\big] - \big[\mathbb{A}_m, \mathbf{\Omega}\big] \, , \quad \delta_{\mathbf{\Omega}} \mathbb{F}^{mn} = \big[\mathbf{\Omega}, \mathbb{F}^{mn}\big] \, , \end{aligned} \tag{1.4}$$

where $\mathbf{\Omega}(x,\theta)$ is a generating series of multiparticle gauge parameters Ω_P. This non-linear gauge invariance will be the main topic of this work. It underpins the earlier constructions of multiparticle superfields and provides a surprising link between the Bern–Carrasco–Johansson (BCJ) duality [7,8,9] and multiparticle gauge transformations.

1.2. Non-linear gauge transformations and the BCJ duality

As will be shown in this paper, the cubic-vertex prescription $K_{[P,Q]}$ appearing in the earlier construction of multiparticle superfields turns out to have a direct non-local counterpart for Berends–Giele currents

$$\mathcal{K}_P \equiv \frac{1}{s_P} \sum_{XY=P} \mathcal{K}_{[X,Y]} \tag{1.5}$$

with the same functional form for the currents $\mathcal{K}_{[X,Y]}$ as seen for the local fields $K_{[X,Y]}$. The recursive definition (1.5) yields a particular gauge where $k_m^P \mathcal{A}_P^m(x,\theta) = 0$, in other words, the generating series $\mathbb{A}_m^{\rm L}$ of the currents in (1.5) realizes *Lorentz gauge*.

The redefinitions required by imposing the Lie symmetries on the multiparticle superfields in the previous constructions [3,5] are now understood as a change of gauge. Starting from the definitions in the Lorentz gauge as above, the superfield redefinitions discussed in [3,5] amount to enforcing the *BCJ gauge*, e.g.,

$$\mathbb{A}_m^{\rm BCJ} = \mathbb{A}_m^{\rm L} + \big[\partial_m, \mathbf{\Omega}^{\rm BCJ}\big] - \big[\mathbb{A}_m^{\rm L}, \mathbf{\Omega}^{\rm BCJ}\big] \,, \tag{1.6}$$

where the superscripts $^{\rm BCJ}$ and $^{\rm L}$ refer to the redefined superfields of [3,5] and the new recursive constructions discussed in this paper. The gauge parameter[1] $\mathbf{\Omega}^{\rm BCJ}$ in the sense of (1.4) will be described in section 3, with complete expressions up to the fifth order in the multiparticle expansion.

The terminology "BCJ gauge" for the above transformations is motivated by the BCJ conjecture [7] on a duality between color and kinematics: The kinematic factors N_i of scattering amplitudes can be arranged to satisfy the same Jacobi identity as their associated color factors C_i, see [8] for the striking impact on gravity amplitudes, [9] for the loop-level formulation of the conjecture and [10] for a review. Incidentally, the family $K_P^{\rm BCJ}$ of multiparticle superfields in the BCJ gauge satisfy the same "generalized Lie symmetries" [11] as a string of structure constants in $[t^a, t^b] = f^{abc}t^c$,

$$\text{"kinematics"} \ K_{12\ldots p}^{\rm BCJ} \ \longleftrightarrow \ f^{12a_3}f^{a_3 3 a_4}f^{a_4 4 a_5}\ldots f^{a_p p a_{p+1}} \ \text{"color"} \,. \tag{1.7}$$

The relation between the tree-level BCJ duality and the superfields in the BCJ gauge can be seen from the tree-level amplitudes computed with the pure spinor superstring.

At tree level, the numerators N_i are assembled from cubic expressions $A_\alpha^P A_\beta^Q A_\gamma^R$ where the particular linear combinations of multiparticle labels P, Q, R follow from the field-theory limit of the superstring amplitude, see fig. 1. As shown in [12], the numerators resulting from this procedure obey the color-kinematics duality for any number of external particles. The superfields in the "BCJ gauge" were an essential requirement in the derivation of BCJ-satisfying numerators from the pure spinor superstring[2]. Non-linear gauge transformations of the generating series (1.3) of multiparticle superfields reparametrize the SYM amplitudes by moving terms between different cubic diagrams. They can therefore be viewed as an example of the "generalized gauge freedom" of [7,8,9].

[1] For historical reasons, $\mathbf{\Omega}^{\rm BCJ}$ will be denoted by $-\mathbb{H}$ in section 3.

[2] In eliminating spurious double poles from the string computation, BCJ gauge of the multiparticle superfields is automatically attained [3].

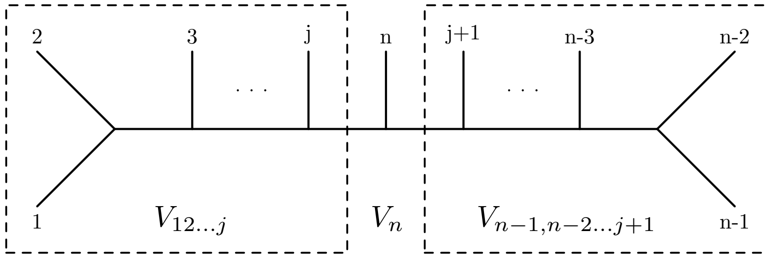


Fig. 1 The basis of half-ladder diagrams with legs 1 and $n-1$ attached to opposite endpoints furnish the manifestly-local pure spinor representation of tree-level numerators $V_{12\dots j}V_nV_{n-1,n-2,\dots j+1}$ built from SYM superfields in the BCJ gauge.

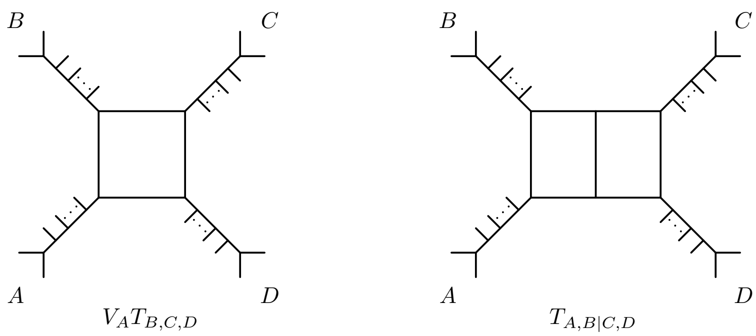


Fig. 2 The pure spinor expressions of arbitrary box and double-box numerators are given by certain multiparticle building blocks $V_A T_{B,C,D}$ [13] and $T_{A,B|C,D}$ [14]. They furnish a manifestly local representation that satisfies the BCJ identities within each external tree subdiagram when the SYM superfields are in the BCJ gauge.

At loop level, BCJ-satisfying five-point integrands at both one- and two-loops were recently derived using multiparticle superfields in the BCJ gauge [13,14][3]. At any multiloop order, kinematic Jacobi identities within tree-level subdiagram are manifestly satisfied if they are represented by multiparticle superfields in BCJ gauge. This for instance applies to the general box and double-box diagram displayed in fig. 2 where the multiparticle labels A, B, C and D refer to appropriate superfields with the symmetry (1.7). The ubiquitous appearance of multiparticle superfields calls for an efficient handle on their components, i.e. their dependence on the Grassmann-odd superspace coordinates θ^α.

1.3. Theta-expansions in Harnad–Shnider gauge

In the same way as the Lie symmetries required by the BCJ duality could be attained through a non-linear gauge transformation (1.6), we will simplify the theta-expansion of

[3] It should be pointed out that the straightforward derivation of the six-point integrand at one-loop does not satisfy the BCJ duality [13]. Although not conclusive, the failure seems to be related to the well-known six-point gauge anomaly and deserves further investigation.

Berends–Giele currents through a convenient choice of multiparticle gauge parameters. The underlying gauge condition $\theta^\alpha \mathbb{A}_\alpha^{\rm HS} = 0$ goes back to Harnad and Shnider (HS) [15] and has been further studied in the context of linearized superfields [16]. We apply this line of thoughts to the multiparticle level and obtain economic theta-expansions for Berends–Giele currents $\mathcal{K}_P$ which largely resemble the linearized counterparts. Non-linear deviations at higher powers of theta are controlled by superfields of higher mass dimension [6].

The theta-expansions of HS gauge significantly alleviate the conversion of kinematic factors in pure spinor superspace to their components involving gluons and gluinos. The computational effort caused by large numbers of external states [17] can be tremendously reduced, and the resulting structural insights into the tree-level components are discussed in a companion paper [18]. A huge long-term benefit for higher orders in perturbation theory is expected from the quick access to the component information on multiloop kinematic factors.

1.4. Outline

This paper is organized as follows. In section 2, the field equations of ten-dimensional SYM are reviewed and exploited to construct Berends–Giele currents in Lorentz gauge. Their gauge equivalence to the earlier construction of [5] in BCJ gauge is clarified in section 3. In section 4, the key ideas of HS gauge are reviewed and applied to streamline the theta-expansions of Berends–Giele currents, starting from either Lorentz gauge or BCJ gauge. Finally, we conclude in section 5 with applications of the improved theta-expansions to scattering amplitudes in pure spinor superspace.

2. Super-Poincare description of ten-dimensional super Yang–Mills

2.1. Non-linear super Yang–Mills

Ten-dimensional SYM can be described in a super-Poincaré covariant manner using superspace coordinates (x^m, θ^α) where $m, n = 0, \ldots, 9$ and $\alpha, \beta = 1, \ldots, 16$ denote vector and spinor indices of the Lorentz group. Using Lie algebra-valued connections $\mathbb{A}_\alpha = \mathbb{A}_\alpha(x, \theta)$ and $\mathbb{A}_m = \mathbb{A}_m(x, \theta)$, one defines supercovariant derivatives [19,1],

$$\nabla_\alpha \equiv D_\alpha - \mathbb{A}_\alpha \, , \qquad \nabla_m \equiv \partial_m - \mathbb{A}_m \ . \tag{2.1}$$

The fermionic differential operators

$$D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m\,, \qquad \{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m \tag{2.2}$$

involve the 16×16 Pauli matrices $\gamma^m_{\alpha\beta} = \gamma^m_{\beta\alpha}$ subject to the Clifford algebra $\gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma} = 2\eta^{mn}\delta^\gamma_\alpha$, and the convention for (anti)symmetrizing indices does not include $\frac{1}{2}$. The constraint equation $\big\{\nabla_\alpha, \nabla_\beta\big\} = \gamma^m_{\alpha\beta}\nabla_m$ together with Bianchi identities then lead to the non-linear equations of motion [1],

$$\begin{aligned} &\big\{\nabla_\alpha, \nabla_\beta\big\} = \gamma^m_{\alpha\beta}\nabla_m\,, &&\big\{\nabla_\alpha, \mathbb{W}^\beta\big\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn}\,,\\ &\big[\nabla_\alpha, \nabla_m\big] = -(\gamma_m\mathbb{W})_\alpha\,, &&\big[\nabla_\alpha, \mathbb{F}^{mn}\big] = (\mathbb{W}^{[m}\gamma^{n]})_\alpha\,, \end{aligned} \tag{2.3}$$

where

$$\mathbb{F}_{mn} \equiv -\big[\nabla_m, \nabla_n\big]\,, \qquad \mathbb{W}^\alpha_m \equiv \big[\nabla_m, \mathbb{W}^\alpha\big]\,. \tag{2.4}$$

Equivalently, using the definitions (2.1) the equations of motion (2.3) become

$$\begin{aligned} &\{D_{(\alpha}, \mathbb{A}_{\beta)}\} = \gamma^m_{\alpha\beta}\mathbb{A}_m + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\}\,, &&\{D_\alpha, \mathbb{W}^\beta\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn} + \{\mathbb{A}_\alpha, \mathbb{W}^\beta\}\\ &[D_\alpha, \mathbb{A}_m] = [\partial_m, \mathbb{A}_\alpha] + (\gamma_m\mathbb{W})_\alpha + [\mathbb{A}_\alpha, \mathbb{A}_m]\,, &&[D_\alpha, \mathbb{F}^{mn}] = (\mathbb{W}^{[m}\gamma^{n]})_\alpha + [\mathbb{A}_\alpha, \mathbb{F}^{mn}]\,. \end{aligned} \tag{2.5}$$

It is straightforward to check that (2.3) or (2.5) are preserved by the non-linear gauge transformations,

$$\begin{aligned} &\delta_{\mathbf{\Omega}}\mathbb{A}_\alpha = \big[\nabla_\alpha, \mathbf{\Omega}\big], \qquad \delta_{\mathbf{\Omega}}\mathbb{A}_m = \big[\nabla_m, \mathbf{\Omega}\big]\\ &\delta_{\mathbf{\Omega}}\mathbb{W}^\alpha = \big[\mathbf{\Omega}, \mathbb{W}^\alpha\big], \quad \delta_{\mathbf{\Omega}}\mathbb{F}^{mn} = \big[\mathbf{\Omega}, \mathbb{F}^{mn}\big], \quad \delta_{\mathbf{\Omega}}\mathbb{W}^{m\alpha} = \big[\mathbf{\Omega}, \mathbb{W}^{m\alpha}\big]\,, \end{aligned} \tag{2.6}$$

with Lie algebra-valued gauge parameter $\mathbf{\Omega} = \mathbf{\Omega}(x,\theta)$.

2.1.1. Linearized super Yang–Mills

Discarding the quadratic terms in the superfields from the equations of motion (2.5) yields the field equations of linearized SYM,

$$\begin{aligned} &\{D_{(\alpha}, A_{\beta)}\} = \gamma^m_{\alpha\beta}A_m, &&\{D_\alpha, W^\beta\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}\\ &[D_\alpha, A_m] = (\gamma_m W)_\alpha + [\partial_m, A_\alpha] &&[D_\alpha, F_{mn}] = [\partial_{[m}, (\gamma_{n]}W)_\alpha]\,. \end{aligned} \tag{2.7}$$

They are invariant under the linearized gauge transformations,

$$\delta_\Omega A_\alpha = \big[D_\alpha, \Omega\big]\,, \quad \delta_\Omega A_m = \big[\partial_m, \Omega\big]\,, \quad \delta_\Omega W^\alpha = 0\,, \quad \delta_\Omega F^{mn} = 0\,. \tag{2.8}$$

Note that the massless vertex operators in the open pure spinor superstring [2] are given in terms of these linearized superfields and the equations of motion (2.7) imply their BRST invariance [20].

2.2. Supersymmetric Berends–Giele currents in Lorentz gauge

For a multiparticle label $P \equiv i_1 i_2 i_3 \ldots i_p$ with each i_j referring to an external SYM state, we define a set of multiparticle *Berends–Giele currents*

$$\mathcal{K}_P \in \{\mathcal{A}_\alpha^P, \mathcal{A}_P^m, \mathcal{W}_P^\alpha, \mathcal{F}_P^{mn}\} \tag{2.9}$$

as follows. The single-particle currents $\mathcal{K}_i$ are given by the linearized superfields, $\mathcal{K}_i \in \{A_\alpha^i, A_i^m, W_i^\alpha, F_i^{mn}\}$, while multiparticle instances follow from the recursion[4]

$$\mathcal{K}_P \equiv \frac{1}{s_P} \sum_{XY=P} \mathcal{K}_{[X,Y]} \,, \tag{2.10}$$

where

$$\mathcal{A}_\alpha^{[P,Q]} = -\frac{1}{2}\big[\mathcal{A}_\alpha^P (k^P \cdot \mathcal{A}^Q) + \mathcal{A}_m^P (\gamma^m \mathcal{W}^Q)_\alpha - (P \leftrightarrow Q)\big] \tag{2.11}$$

$$\mathcal{A}_m^{[P,Q]} = -\frac{1}{2}\big[\mathcal{A}_m^P (k^P \cdot \mathcal{A}^Q) + \mathcal{A}_n^P \mathcal{F}_{mn}^Q - (\mathcal{W}^P \gamma_m \mathcal{W}^Q) - (P \leftrightarrow Q)\big] \tag{2.12}$$

$$\mathcal{W}_{[P,Q]}^\alpha = \frac{1}{2}(k_P^m + k_Q^m)\gamma_m^{\alpha\beta}\big[\mathcal{A}_P^n (\gamma_n \mathcal{W}_Q)_\beta - (P \leftrightarrow Q)\big] \tag{2.13}$$

$$\mathcal{F}_P^{mn} = k_P^m \mathcal{A}_P^n - k_P^n \mathcal{A}_P^m - \sum_{XY=P} \big(\mathcal{A}_X^m \mathcal{A}_Y^n - \mathcal{A}_X^n \mathcal{A}_Y^m\big) \,. \tag{2.14}$$

Multiparticle momenta as well as their associated Mandelstam invariants are defined by

$$k_P^m \equiv k_{i_1}^m + k_{i_2}^m + \cdots + k_{i_p}^m \,, \qquad s_P \equiv \frac{1}{2} k_P^2 \,, \tag{2.15}$$

and the sum over multiparticle labels $XY = P$ in (2.10) and (2.14) instructs to deconcatenate $P = i_1 i_2 i_3 \ldots i_p$ into non-empty words $X = i_1 i_2 \ldots i_j$ and $Y = i_{j+1} \ldots i_p$ with $j = 1, 2, \ldots, p-1$. Alternative recursive formulæ for $\mathcal{W}_P^\alpha$ and $\mathcal{F}_P^{mn}$ read[5]

$$\mathcal{W}_{[P,Q]}^\alpha = -\frac{1}{2}\big[\mathcal{W}_P^\alpha (k_P \cdot \mathcal{A}_Q) + \mathcal{W}_P^{m\alpha} \mathcal{A}_Q^m + \frac{1}{2}(\gamma_{rs}\mathcal{W}_P)^\alpha \mathcal{F}_Q^{rs} - (P \leftrightarrow Q)\big] \tag{2.16}$$

$$\mathcal{F}_{[P,Q]}^{mn} = -\frac{1}{2}\big[\mathcal{F}_P^{mn}(k_P \cdot \mathcal{A}_Q) + \mathcal{F}_P^{p|mn}\mathcal{A}_p^Q + 2\mathcal{F}_P^{mp}\mathcal{F}_{Q\,p}^{n} + 4\gamma_{\alpha\beta}^{[m}\mathcal{W}_P^{n]\alpha}\mathcal{W}_Q^\beta - (P \leftrightarrow Q)\big] \,,$$

[4] This definition of the supersymmetric Berends–Giele currents closely generalizes the standard Berends–Giele currents J_P^m of [21]. When the fermions are set to zero, J_P^m can be identified as the theta-independent component of $\mathcal{A}_P^m(x,\theta)$. Furthermore, the quartic-vertex interaction $\{J_X, J_Y, J_Z\}$ of [21] is automatically included in the cubic-vertex prescription $\mathcal{K}_{[X,Y]}$ [18].

[5] The recursion for Berends–Giele currents $\mathcal{W}_P^\alpha$ and $\mathcal{F}_P^{mn}$ based on (2.16) is actually closer to the original string-inspired construction of multiparticle superfields where the key input stems from the short-distance behaviour of integrated vertex operators [5].

with superfields $\mathcal{W}_P^{m\alpha}, \mathcal{F}_P^{m|pq}$ of higher mass dimension,

$$\mathcal{W}_P^{m\alpha} \equiv k_P^m \mathcal{W}_P^\alpha + \sum_{XY=P} \big(\mathcal{W}_X^\alpha \mathcal{A}_Y^m - \mathcal{W}_Y^\alpha \mathcal{A}_X^m \big) \tag{2.17}$$

$$\mathcal{F}_P^{m|pq} \equiv k_P^m \mathcal{F}_P^{pq} + \sum_{XY=P} \big(\mathcal{F}_X^{pq} \mathcal{A}_Y^m - \mathcal{F}_Y^{pq} \mathcal{A}_X^m \big) \ . \tag{2.18}$$

One can show by induction that the Berends–Giele currents defined in (2.11) to (2.14) obey the equations of motion

$$\begin{aligned}
D_{(\alpha} \mathcal{A}^P_{\beta)} &= \gamma^m_{\alpha\beta} \mathcal{A}^P_m + \sum_{XY=P} \big(\mathcal{A}^X_\alpha \mathcal{A}^Y_\beta - \mathcal{A}^Y_\alpha \mathcal{A}^X_\beta \big) \\
D_\alpha \mathcal{A}^P_m &= k^P_m \mathcal{A}^P_\alpha + (\gamma_m \mathcal{W}_P)_\alpha + \sum_{XY=P} \big(\mathcal{A}^X_\alpha \mathcal{A}^Y_m - \mathcal{A}^Y_\alpha \mathcal{A}^X_m \big) \\
D_\alpha \mathcal{W}^\beta_P &= \frac{1}{4} (\gamma^{mn})_\alpha{}^\beta \mathcal{F}^P_{mn} + \sum_{XY=P} \big(\mathcal{A}^X_\alpha \mathcal{W}^\beta_Y - \mathcal{A}^Y_\alpha \mathcal{W}^\beta_X \big) \\
D_\alpha \mathcal{F}^{mn}_P &= (\mathcal{W}^{[m}_P \gamma^{n]})_\alpha + \sum_{XY=P} \big(\mathcal{A}^X_\alpha \mathcal{F}^{mn}_Y - \mathcal{A}^Y_\alpha \mathcal{F}^{mn}_X \big) \ .
\end{aligned} \tag{2.19}$$

Apart from the terms along with the deconcatenation sum $\sum_{XY=P}$, these multiparticle equations of motion have the same form as the linearized ones (2.7). They play a key role for the BRST invariance of scattering amplitudes in string and field theory, see [22,23,3] for examples at tree-level and [4,24,13,14] at loop-level. The need for such objects was also observed in the worldline version of the pure spinor formalism [25,26].

In addition, one can also show by induction that the currents defined in (2.10) satisfy,

$$k^P_m \mathcal{A}^m_P = 0 \tag{2.20}$$

$$k^P_m (\gamma^m \mathcal{W}_P)_\alpha = \sum_{XY=P} \big[\mathcal{A}^X_m (\gamma^m \mathcal{W}_Y)_\alpha - \mathcal{A}^Y_m (\gamma^m \mathcal{W}_X)_\alpha \big] \tag{2.21}$$

$$k^P_m \mathcal{F}^{mn}_P = \sum_{XY=P} \big[2(\mathcal{W}_X \gamma^n \mathcal{W}_Y) + \mathcal{A}^X_m \mathcal{F}^{mn}_Y - \mathcal{A}^Y_m \mathcal{F}^{mn}_X \big] \ . \tag{2.22}$$

As we will see, (2.20) implies that the generating series of Berends–Giele currents (2.10) is in *Lorentz gauge*.

2.2.1. Symmetries of supersymmetric Berends–Giele currents

The currents $\mathcal{K}_P(x,\theta)$ defined above satisfy the following symmetry proven in appendix A,

$$\mathcal{K}_{A \sqcup\!\sqcup B} = 0, \quad \forall A, B \neq \emptyset \ , \tag{2.23}$$

where the shuffle product $\shuffle$ between the words $A = a_1 a_2 \ldots a_{|A|}$ and $B = b_1 b_2 \ldots b_{|B|}$ is defined recursively by

$$\emptyset \shuffle A = A \shuffle \emptyset = A, \qquad A \shuffle B \equiv a_1 (a_2 \ldots a_{|A|} \shuffle B) + b_1 (b_2 \ldots b_{|B|} \shuffle A) \,, \tag{2.24}$$

and $\emptyset$ denotes the empty word.

As elaborated in a companion paper [18], setting the fermions to zero reduces the θ^α-independent component of $\mathcal{A}^m_P(x,\theta)$ to the gluonic current J^m_P defined by Berends and Giele [21], thus (2.23) implies the symmetry $J^m_{A \shuffle B} = 0$ derived in [27]. These facts explain why $\mathcal{K}_P(x,\theta)$ are called supersymmetric Berends–Giele currents.

2.3. Generating series of Berends–Giele currents

The generating series of multiparticle Berends–Giele currents $\mathcal{K}_P \in \{\mathcal{A}^P_\alpha, \mathcal{A}^m_P, \mathcal{W}^\alpha_P, \ldots\}$

$$\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}^m, \mathbb{W}^\alpha, \ldots\} \tag{2.25}$$

is an expansion in terms of Lie-algebra generators t^{i_j} [6]

$$\begin{aligned} \mathbb{K} &\equiv \sum_{p=1}^{\infty} \sum_{i_1, i_2, \ldots, i_p} \mathcal{K}_{i_1 i_2 \ldots i_p} t^{i_1} t^{i_2} \ldots t^{i_p} \\ &= \sum_{p=1}^{\infty} \sum_{i_1, i_2, \ldots, i_p} \frac{1}{p} \, \mathcal{K}_{i_1 i_2 \ldots i_p} [t^{i_1}, [t^{i_2}, \ldots, [t^{i_{p-1}}, t^{i_p}]] \ldots] \,. \end{aligned} \tag{2.26}$$

The second line follows from the Berends–Giele symmetry (2.23) and guarantees that $\mathbb{K}$ is Lie algebra valued, see [28] for a proof. The equations of motion (2.19) satisfied by the Berends–Giele currents imply that $\mathbb{K}$ satisfies the non-linear field equations (2.5) [6][6].

Given that the Mandelstam invariant s_P in (2.15) arises from half the d'Alembertian

$$\Box \mathbb{K} \equiv \big[\partial^m, [\partial_m, \mathbb{K}]\big] \,, \tag{2.27}$$

[6] The notion of a generating series which solves the field equations and gives rise to tree amplitudes is also central to the "perturbiner" formalism [29]. This approach has been used to derive a generating series of Yang–Mills MHV amplitudes, see [30] for a supersymmetric extension. However, the generic Yang–Mills amplitudes have never been obtained (see also [31]). We thank Nima Arkani-Hamed for pointing out these references.

the recursive prescriptions (2.11) to (2.13) for $\mathcal{A}^P_\alpha, \mathcal{A}^P_m, \mathcal{W}^\alpha_P$ can be reexpressed at the level of the generating series as

$$\Box\mathbb{A}_\alpha = \big[\mathbb{A}_m, [\partial^m, \mathbb{A}_\alpha]\big] + \big[(\gamma^m\mathbb{W})_\alpha, \mathbb{A}_m\big] \tag{2.28}$$

$$\Box\mathbb{A}_m = \big[\mathbb{A}_p, [\partial^p, \mathbb{A}^m]\big] + \big[\mathbb{F}^{mp}, \mathbb{A}_p\big] + \gamma^m_{\alpha\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\} \tag{2.29}$$

$$\Box\mathbb{W}^\alpha = \big[\partial_m, [\mathbb{A}_n, (\gamma^m\gamma^n\mathbb{W})^\alpha]\big] \,. \tag{2.30}$$

As detailed in the following subsection, these second-order differential equations can be verified from (2.5) and (2.4), provided that Lorentz gauge is imposed,

$$[\partial_m, \mathbb{A}^m] = 0 \,. \tag{2.31}$$

Similar manipulations lead to the generating-series representation of (2.16),

$$\Box\mathbb{W}^\alpha = \big[\mathbb{A}_m, [\partial^m, \mathbb{W}^\alpha]\big] + \big[\mathbb{A}^m, \mathbb{W}^\alpha_m\big] + \frac{1}{2}\big[\mathbb{F}_{mn}, (\gamma^{mn}\mathbb{W})^\alpha\big] \tag{2.32}$$

$$\Box\mathbb{F}^{mn} = [\mathbb{A}_p, [\partial^p, \mathbb{F}^{mn}]] + [\mathbb{A}_p, \mathbb{F}^{p|mn}] + 2[\mathbb{F}^{mp}, \mathbb{F}_p{}^n] + 4\{(\mathbb{W}^{[m}\gamma^{n]})_\alpha, \mathbb{W}^\alpha\} \,, \tag{2.33}$$

where $\mathbb{F}^{p|mn}$ denotes the generating series of (2.18). Equivalence of (2.32) and (2.30) follows from the Dirac equation,

$$\nabla_m(\gamma^m\mathbb{W})_\alpha = 0 \,, \tag{2.34}$$

i.e. the generating series of (2.21). In summary, the recursive prescriptions (2.11) to (2.13) for multiparticle superfields yield a solution of the SYM equations in Lorentz gauge.

2.3.1. Deriving non-linear wave equations

We shall now derive the non-linear wave equations (2.28), (2.29), (2.32) and (2.33) for the non-linear superfields $\mathbb{K}$ in Lorentz gauge. By Jacobi identities and repeated use of $\partial^m = \nabla^m + \mathbb{A}^m$, we have

$$\begin{aligned} \Box\mathbb{K} &= [\nabla^m + \mathbb{A}^m, [\partial_m, \mathbb{K}]] \\ &= [[\nabla^m, \partial_m], \mathbb{K}] + [\mathbb{A}^m, [\partial_m, \mathbb{K}]] + [\mathbb{A}^m, [\nabla_m, \mathbb{K}]] + [\nabla^m, [\nabla_m, \mathbb{K}]] \,. \end{aligned} \tag{2.35}$$

The first term in the second line vanishes in Lorentz gauge (2.31) by $[\partial_m, \nabla^m] = -[\partial_m, \mathbb{A}^m]$. For any of the gauge-covariant choices $\mathbb{K} \rightarrow \{\nabla_\alpha, \nabla_m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$, the last term of (2.35) can be converted to quadratic expressions in the non-linear fields using (2.34) and

$$\begin{aligned} [\nabla_m, \mathbb{F}^{mp}] &= \gamma^p_{\alpha\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\} \\ [\nabla_m, \mathbb{W}^{m\alpha}] &= \frac{1}{2}[\mathbb{F}_{mn}, (\gamma^{mn}\mathbb{W})^\alpha] \\ [\nabla_m, \mathbb{F}^{m|pq}] &= 2[\mathbb{F}^{pn}, \mathbb{F}_n{}^q] + 4\{(\mathbb{W}^{[m}\gamma^{n]})_\alpha, \mathbb{W}^\alpha\} \,. \end{aligned} \tag{2.36}$$

Upon inserting (2.36) into (2.35), one can reproduce the wave equations (2.28), (2.29), (2.32) and (2.33) from $\mathbb{K} \rightarrow \{\nabla_\alpha, \nabla_m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$.

2.4. Generating series of gauge transformations

In general, the non-linear gauge transformations (2.6) are a symmetry of the non-linear SYM equations of motion (2.5) for any Lie algebra-valued gauge parameter $\mathbf{\Omega}$ with generating series,

$$\mathbf{\Omega} = \sum_{p=1}^{\infty} \sum_{i_1,i_2,\ldots,i_p} \Omega_{i_1 i_2 \ldots i_p} t^{i_1} t^{i_2} \ldots t^{i_p} \, , \qquad \Omega_{A \sqcup\!\sqcup B} = 0 \;\forall A, B \neq \emptyset \, . \tag{2.37}$$

In the remainder of this work we will exploit the effects of different gauge parameters Ω_P. One particular choice to be discussed in the next subsection efficiently encodes the multiparticle response to linearized gauge variations (2.8), possibly for several external legs. But more importantly, the multiparticle gauge freedom parameterized by Ω_P can be exploited as a tool to:

1. Find a representation of multiparticle superfields which manifestly obey generalized Lie symmetries, so-called *BCJ representations* discussed in section 3.
2. Considerably simplify the theta-expansions of multiparticle superfields as discussed in section 4.
3. Find a multiparticle representation which combines both features above.

The benefits for scattering amplitudes are sketched in section 5, and the tree-level applications are deepened in [18].

2.4.1. Generating series of polarization shifts

Standard linearized gauge variations of the form $\delta_{\mathcal{G}} A^i_m = k^i_m \mathcal{G}_i$ with scalar parameter $\mathcal{G}_i = e^{k_i x}$ induce multiparticle transformations of the Berends–Giele currents by their recursive construction, see (2.10) to (2.14). They effectively shift gluon polarizations e^i_m by k^i_m and do not affect the transversality $(k_i \cdot A_i) = 0$, hence, they cannot alter the property $k^m_P \mathcal{A}^P_m = 0$ at any multiparticle level and preserve Lorentz gauge (2.31). The resulting condition $[\partial^m, \delta_{\mathcal{G}} \mathbb{A}_m] = 0$ applied to $\delta_{\mathcal{G}} \mathbb{A}_m = [\partial_m, \mathbb{G}] - [\mathbb{A}_m, \mathbb{G}]$ (see (2.6)) yields a recursion for the multiparticle components of $\mathbb{G}$,

$$\Box \mathbb{G} = \big[\mathbb{A}_m, [\partial^m, \mathbb{G}]\big] \, , \qquad \mathcal{G}_P = -\frac{1}{2s_P} \sum_{XY=P} \big[\mathcal{G}_X (k^X \cdot \mathcal{A}^Y) - (X \leftrightarrow Y)\big] \, . \tag{2.38}$$

This formula generalizes the transformations of multiparticle fields discussed in [32]. In that reference the single-particle initial conditions for the recursion in (2.38) were specialized to $\mathcal{G}_i = \delta_{i,1} e^{k_1 x}$; only the gluon polarization of particle $i = 1$ is shifted. Note that (2.38) with several non-vanishing $\mathcal{G}_i$ in the initial conditions allows to address simultaneous shifts of multiple polarization vectors e_i^m by the corresponding k_i^m. One can show that (2.38) is the supersymmetric generalization of the complicated-looking formula (2.24) of [27], highlighting the benefits of the superspace approach to the Berends–Giele currents adopted here.

3. Non-linear superfields and Berends–Giele currents in BCJ gauge

In a previous paper, supersymmetric Berends–Giele currents were constructed in a totally different fashion [5]. Starting with a *local* representation of multiparticle superfields

$$K_P \in \{A_\alpha^P, A_m^P, W_P^\alpha, F_{mn}^P\}\,, \tag{3.1}$$

redefinitions were employed in order to enforce the symmetries of nested commutators $[[t^1, t^2], t^3]$ in a Lie algebra such as $K_{123} + K_{231} + K_{312} = 0$. Their Berends–Giele currents were constructed by adjoining propagators, i.e. inverse Mandelstam invariants (2.15), to Lie symmetry-satisfying numerators, following an intuitive mapping to cubic graphs compatible with the ordering of the external legs. Despite their different construction, the Berends–Giele currents $\mathcal{K}_P^{\rm BCJ}$ of [5] or those in the *Lorentz gauge* $\mathcal{K}_P^{\rm L} \equiv \mathcal{K}_P$ constructed in the previous section give rise to identical tree-level amplitudes. As verified below up to multiplicity five, these different currents are in fact related by a non-linear gauge transformation and are therefore equivalent. As indicated by the superscript in $\mathcal{K}_P^{\rm BCJ}$, the constituents $K_{12\ldots p}$ of the Berends–Giele currents in [5] have the symmetries suggested by the BCJ duality between color and kinematics [7]. Accordingly, the currents $\mathcal{K}_P^{\rm BCJ}$ are said to be in *BCJ gauge*.

3.1. Recursive definition of local superfields in Lorentz gauge

The definition of local superfields $\hat{K}_{[P,Q]}$ in Lorentz gauge[7] is given by

$$\hat{A}_\alpha^{[P,Q]} = -\frac{1}{2}\big[\hat{A}_\alpha^P (k^P \cdot \hat{A}^Q) + \hat{A}_m^P (\gamma^m \hat{W}^Q)_\alpha - (P \leftrightarrow Q)\big] \tag{3.2}$$

$$\hat{A}_m^{[P,Q]} = -\frac{1}{2}\big[\hat{A}_m^P (k^P \cdot \hat{A}^Q) + \hat{A}_n^P \hat{F}_{mn}^Q - (\hat{W}^P \gamma_m \hat{W}^Q) - (P \leftrightarrow Q)\big] \tag{3.3}$$

$$\hat{W}^\alpha_{[P,Q]} = \frac{1}{2}(k_P^m + k_Q^m)\gamma_m^{\alpha\beta}\big[\hat{A}_P^n (\gamma_n \hat{W}_Q)_\beta - (P \leftrightarrow Q)\big]\,, \tag{3.4}$$

[7] Starting from rank four, the superfields denoted by $\{\hat{A}_\alpha^P, \hat{A}_m^P, \hat{W}_P^\alpha, \hat{F}_P^{mn}\}$ in this work and [5] do not match.

it amounts to picking up the numerator on top of various inverse s_X in the recursions (2.11) to (2.13) for Berends–Giele currents. We will often use a simplified notation for brackets $[P, Q]$ when one of P, Q is of single-particle type,

$$\hat{K}_{12\ldots p} \equiv \hat{K}_{[12\ldots p-1,p]} \ . \tag{3.5}$$

In this topology, the field-strength[8] appearing above is given by

$$\hat{F}^{12\ldots p}_{mn} \equiv k^{12\ldots p}_m \hat{A}^{12\ldots p}_n - k^{12\ldots p}_n \hat{A}^{12\ldots p}_m + \sum_{j=2}^{p} \sum_{\delta \in P(\beta_j)} (k_{12\ldots j-1} \cdot k_j)\, \hat{A}^{12\ldots j-1,\{\delta\}}_{[n}\, \hat{A}^{j,\{\beta_j \setminus \delta\}}_{m]} \,, \tag{3.6}$$

where $\beta_j \equiv \{j+1, j+2, \ldots, p\}$ and $P(\beta_j)$ denotes its power set.

3.2. Review of generalized Lie symmetries for multiparticle superfields

The approach of [5] to Berends–Giele currents in BCJ gauge $\mathcal{K}^{\rm BCJ}_P$ is based on local superfields $K_{12\ldots p}$ satisfying all generalized Lie symmetries $\pounds_k$ up to $k = p$,

$$\pounds_k \circ K_{12\ldots p} = 0, \quad k = 2, \ldots, p \tag{3.7}$$

$$\pounds_{k=2n+1}\!: \quad K_{12\ldots n+1[n+2[\ldots[2n-1[2n,2n+1]]\ldots]]} - K_{2n+1\ldots n+2[n+1[\ldots[3[21]]\ldots]]} = 0$$

$$\pounds_{k=2n}\!: \quad K_{12\ldots n[n+1[\ldots[2n-2[2n-1,2n]]\ldots]]} + K_{2n\ldots n+1[n[\ldots[3[21]]\ldots]]} = 0 \ .$$

For example,

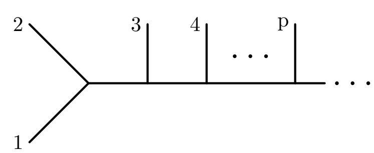
$$\pounds_2 \circ K_{12} = K_{12} + K_{21} = 0\,, \qquad \pounds_3 \circ K_{123} = K_{123} + K_{231} + K_{321} = 0 \tag{3.8}$$

$$\pounds_4 \circ K_{1234} = K_{1234} - K_{1243} + K_{3412} - K_{3421} = 0\,,$$

and so forth. These symmetries leave $(p-1)!$ independent permutations of $K_{12\ldots p}$ and are also obeyed by nested commutators $[\ldots[[t^1, t^2], t^3], \ldots, t^p]$ and the color factors in

$$K_{12\ldots p} \ \longleftrightarrow\ f^{12a_3} f^{a_3 3 a_4} f^{a_4 4 a_5} \ldots f^{a_p p a_{p+1}} \ . \tag{3.9}$$

Therefore the local superfields K_P admit the following diagrammatic interpretation:



$$\leftrightarrow \quad K_{123\ldots p}$$

[8] Field-strenghts $\hat{F}^{mn}_{[P,Q]}$ of more general topologies beyond (3.5) such as $\hat{F}^{mn}_{[12,34]}$ can be addressed along the lines of (2.16).

3.3. Recursive definition of local superfields in BCJ gauge

The recursively defined superfields $\hat{K}_{12\ldots p}$ in (3.2) to (3.6) do not yet satisfy the Lie symmetries (3.7). However, this can be compensated by redefinitions $K_{12\ldots p} = \hat{K}_{12\ldots p} + \ldots$ via superfields $\hat{H}_{12\ldots p} \equiv \hat{H}_{[12\ldots p-1,p]}$ which amount to a non-linear gauge transformation of their corresponding generating series. Starting from $\hat{H}_i = \hat{H}_{ij} = 0$, the superfields $\hat{H}_{12\ldots p}$ at multiplicity p enter through the following recursive system of equations [5]

$$K_{[12\ldots p-1,p]} \equiv \hat{K}_{[12\ldots p-1,p]} - \sum_{j=2}^{p} \sum_{\delta \in P(\beta_j)} (k^{1\ldots j-1} \cdot k^j) \big[\hat{H}_{1\ldots j-1,\{\delta\}} \ \hat{K}_{j,\{\beta_j \setminus \delta\}} - (1\ldots j-1 \leftrightarrow j) \big]$$
$$- \begin{cases} D_\alpha \hat{H}_{[12\ldots p-1,p]} & : \ K = A_\alpha \\ k^m_{12\ldots p} \hat{H}_{[12\ldots p-1,p]} & : \ K = A^m \\ 0 & : \ K = W^\alpha \end{cases} \tag{3.10}$$

and will be introduced separately in the next subsection.

The redefinitions in (3.10) have been originally designed in a two-step procedure which yields the expressions for $\hat{H}_{12\ldots p}$ in a constructive manner[9] [5]. As a result, the superfields $K_{12\ldots p}$ defined by (3.10) as well as

$$F^{12\ldots p}_{mn} \equiv k^{12\ldots p}_m A^{12\ldots p}_n - k^{12\ldots p}_n A^{12\ldots p}_m + \sum_{j=2}^{p} \sum_{\delta \in P(\beta_j)} (k_{12\ldots j-1} \cdot k_j) \, A^{12\ldots j-1,\{\delta\}}_{[n} \ A^{j,\{\beta_j \setminus \delta\}}_{m]} \tag{3.11}$$

satisfy all the Lie symmetries $\mathcal{L}_2, \mathcal{L}_3, \ldots$ in (3.7) up to and including $\mathcal{L}_p$. For example, since $\hat{H}_i = \hat{H}_{ij} = 0$, the definitions in (3.10) yield

$$K_1 = \hat{K}_1 \ , \qquad K_{12} = \hat{K}_{12} \ , \qquad \forall \ K \in \{A_\alpha, A^m, W^\alpha, F^{mn}\} \ , \tag{3.12}$$

and the first non-trivial redefinition occurs at multiplicity three with

$$A^{123}_\alpha = \hat{A}^{[12,3]}_\alpha - D_\alpha \hat{H}_{[12,3]} \ , \qquad A^m_{123} = \hat{A}^m_{[12,3]} - k^m_{123} \hat{H}_{[12,3]} \ , \qquad W^\alpha_{123} = \hat{W}^\alpha_{[12,3]} \ . \tag{3.13}$$

A rank-four sample of the redefinitions (3.10) is provided by

$$A^m_{1234} = \hat{A}^m_{[123,4]} - (k^{123} \cdot k^4) \hat{H}_{[12,3]} A^m_4 - (k^{12} \cdot k^3) \hat{H}_{[12,4]} A^m_3$$
$$- (k^1 \cdot k^2) \big(\hat{H}_{[13,4]} A^m_2 - \hat{H}_{[23,4]} A^m_1 \big) - k^m_{1234} \hat{H}_{[123,4]} \ . \tag{3.14}$$

[9] As discussed in [5], an intermediate step of the redefinition procedure gives rise to redefined superfields $A'^m_{12\ldots p}$ which determine the definition of $H_{[12\ldots p-1,p]}$ via $\mathcal{L}_p \circ A'^m_{[12\ldots p-1,p]} \equiv p \, k^m_{12\ldots p} H_{[12\ldots p-1,p]}$. For this definition to work, the overall momentum $k^m_{12\ldots p}$ must factorize in the sum dictated by $\mathcal{L}_p \circ A'^m_{[12\ldots p-1,p]}$, providing a strong consistency check of the setup. The relation between $H_{12\ldots p}$ and $\hat{H}_{12\ldots p}$ will be given in (3.15).

3.4. Explicit form of the redefinitions $\hat{H}$

One can show that expressions for $\hat{H}_{[12\ldots p-1,p]}$ can be conveniently summarized by

$$\hat{H}_{[A,B]} \equiv H_{[A,B]} - \frac{1}{2}\big[\hat{H}_A(k_A\cdot A_B) - (A\leftrightarrow B)\big] \tag{3.15}$$

$$H'_{A,B,C} \equiv H_{A,B,C} + \frac{1}{2}\big[H_{[A,B]}(k_{AB}\cdot A_C) + \text{cyclic}(A,B,C)\big]\,, \tag{3.16}$$

with the central building block

$$H_{A,B,C} \equiv -\frac{1}{4}A_A^m A_B^n F_C^{mn} + \frac{1}{2}(W_A\gamma_m W_B)A_C^m + \text{cyclic}(A,B,C)\,. \tag{3.17}$$

In particular, the redefinitions up to multiplicity five are captured by

$$\begin{aligned} H_{[12,3]} &= \frac{1}{3}H_{1,2,3}\\ H_{[123,4]} &= \frac{1}{4}\big(H'_{12,3,4} + H'_{34,1,2}\big)\\ H_{[12,34]} &= \frac{1}{4}\big(-2H'_{12,3,4} + 2H'_{34,1,2}\big)\\ H_{[1234,5]} &= \frac{1}{5}\big(H'_{123,4,5} - H'_{543,2,1} + H'_{12,3,45}\big)\\ H_{[123,45]} &= \frac{1}{5}\big(-3H'_{123,4,5} - 2H'_{543,2,1} + 2H'_{12,3,45}\big)\;. \end{aligned} \tag{3.18}$$

The treatment and significance of the additional topologies $H_{[12,34]}$ and $H_{[123,45]}$ is explained around (3.30) and in appendix B. Higher-rank versions of H_P are under investigation, and it would be interesting to extend the simple expressions in (3.18) to arbitrary multiplicity[10]. The expressions above are sufficient to identify the redefinitions up to and including multiplicity five as originating from a non-linear gauge transformation.

It is worth mentioning a remarkable feature of $H_{A,B,C}$ in (3.17): Upgrading the polarization vectors and spinors in the color-ordered SYM three-point amplitude at tree level,

$$A^{\text{SYM}}(1,2,3) = -\frac{1}{2}e_1^m e_2^n f_3^{mn} + (\chi_1\gamma_m\chi_2)e_3^m + \text{cyclic}(1,2,3)\,. \tag{3.19}$$

to superfields according to $e_i^m \to A_i^m(\theta)$, $\chi_i^\alpha \to W_i^\alpha(\theta)$ and $f_i^{mn} = k_i^{[m}e_i^{n]} \to F_i^{mn}(\theta)$, the amplitude (3.19) can be rewritten as

$$A^{\text{SYM}}(1,2,3) = 2H_{1,2,3}(\theta=0)\,. \tag{3.20}$$

[10] Noting that $H_{[12\ldots p-1,p]}$ here corresponds to $H_{12\ldots p}$ from [5], the expression of $H_{[123,4]}$ presented in (3.18) considerably simplifies the expression of H_{1234} given in the appendix C of [5].

3.5. Supersymmetric Berends–Giele currents in BCJ gauge

In this section, we will justify the terminology of Lorentz and BCJ gauge for the representations $\mathcal{K}_P^{\rm L}$ and $\mathcal{K}_P^{\rm BCJ}$ of Berends–Giele currents. It will be verified up to multiplicity five that they are indeed related by a non-linear gauge transformation, e.g.

$$\mathbb{A}_m^{\rm BCJ} = \mathbb{A}_m^{\rm L} - [\partial_m, \mathbb{H}] + [\mathbb{A}_m^{\rm L}, \mathbb{H}] \ , \tag{3.21}$$

translating into

$$\mathcal{A}_P^{m,\rm BCJ} = \mathcal{A}_P^{m,\rm L} - k_P^m \mathcal{H}_P + \sum_{XY=P} (\mathcal{A}_X^{m,\rm L} \mathcal{H}_Y - \mathcal{A}_Y^{m,\rm L} \mathcal{H}_X) \ . \tag{3.22}$$

Clearly, (3.21) is a special case of a non-linear gauge transformation (2.6) with $\mathbf{\Omega} \to -\mathbb{H}$. The generating series of gauge parameters

$$\mathbb{H} \equiv \sum_{i_1,i_2,i_3} \mathcal{H}_{i_1 i_2 i_3} t^{i_1} t^{i_2} t^{i_3} + \sum_{i_1,i_2,i_3,i_4} \mathcal{H}_{i_1 i_2 i_3 i_4} t^{i_1} t^{i_2} t^{i_3} t^{i_4} + \cdots \tag{3.23}$$

is built from Berends–Giele currents $\mathcal{H}_P$ of the superfields $\hat{H}_{[A,B]}$. As before, the Berends–Giele symmetry $\mathcal{H}_{A \sqcup\!\sqcup B} = 0$ implies Lie algebra-valuedness of the series (3.23) [28]. Of course, the same $\mathbb{H}$ describes the transformation of the remaining series $\mathbb{A}_\alpha$, $\mathbb{W}^\alpha$, $\mathbb{F}^{mn}$, see (2.6). We will focus on the transformation between the currents $\mathcal{A}_P^{m,\rm BCJ}$ and $\mathcal{A}_P^{m,\rm L}$ of the vector potential since the remaining superfields follow the same or simpler lines.

In the following discussion we will construct Berends–Giele currents up to rank four using the mapping between planar binary trees and nested brackets [5], see appendix B for rank five. By (3.12), the two gauge choices are identical at multiplicities one and two,

$$\mathcal{K}_1^{\rm BCJ} = \mathcal{K}_1^{\rm L} \ , \qquad \mathcal{K}_{12}^{\rm BCJ} = \mathcal{K}_{12}^{\rm L} \, , \tag{3.24}$$

reflecting the vanishing of the simplest redefinitions,

$$\hat{H}_1 = \hat{H}_{12} = 0 \quad \Rightarrow \quad \mathcal{H}_1 = \mathcal{H}_{12} = 0 \ , \tag{3.25}$$

and justifying the absence of single-particle and two-particle contributions in the series (3.23).

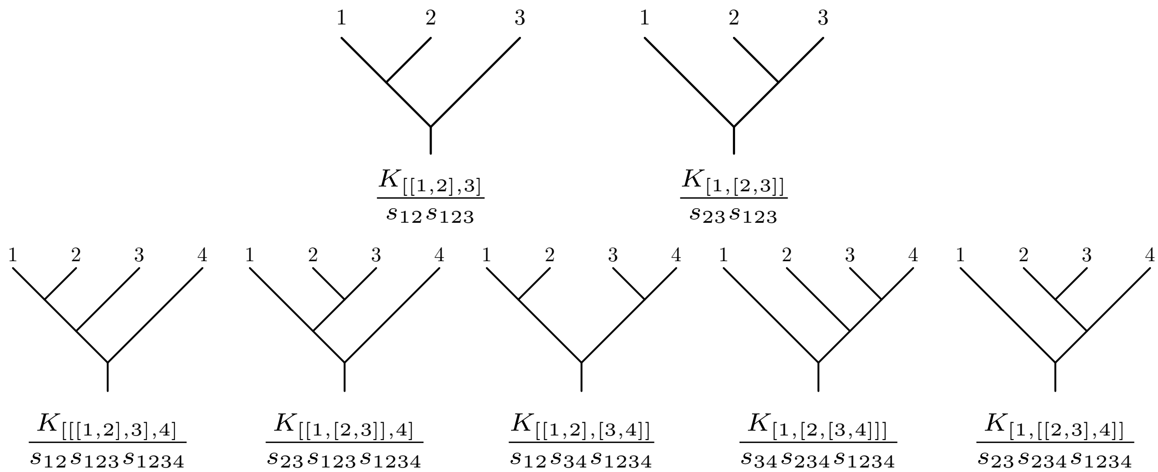


Fig. 3 The planar binary trees used to define $\mathcal{K}_{123}$ and $\mathcal{K}_{1234}$.

3.5.1. Rank three

At multiplicity three, the two binary trees displayed in fig. 3 lead to

$$\mathcal{K}_{123}^{\mathrm{BCJ}} = \frac{K_{[12,3]}}{s_{12}s_{123}} + \frac{K_{[1,23]}}{s_{23}s_{123}}, \qquad \mathcal{K}_{123}^{\mathrm{L}} = \frac{\hat{K}_{[12,3]}}{s_{12}s_{123}} + \frac{\hat{K}_{[1,23]}}{s_{23}s_{123}}\,, \tag{3.26}$$

with $\hat{K}_{[P,Q]} = -\hat{K}_{[Q,P]}$ from (3.2) to (3.4). Hence, the relation (3.11) between the local superfields in the two gauges is sufficient to determine the corresponding relation between their Berends–Giele currents. For example, $A^m_{[12,3]} = \hat{A}^m_{[12,3]} - k^m_{123}\hat{H}_{[12,3]}$ implies that

$$\mathcal{A}_{123}^{m,\mathrm{BCJ}} = \mathcal{A}_{123}^{m,\mathrm{L}} - k^m_{123}\mathcal{H}_{123}, \qquad \mathcal{H}_{123} = \frac{\hat{H}_{[12,3]}}{s_{12}s_{123}} + \frac{\hat{H}_{[1,23]}}{s_{23}s_{123}}\,, \tag{3.27}$$

where (3.25) allows to restore a vanishing deconcatenation term $0 = \mathcal{A}_1^{m,\mathrm{L}}\mathcal{H}_{23} + \mathcal{A}_{12}^{m,\mathrm{L}}\mathcal{H}_3 - \mathcal{A}_{23}^{m,\mathrm{L}}\mathcal{H}_1 - \mathcal{A}_3^{m,\mathrm{L}}\mathcal{H}_{12}$ and to verify (3.22) at $P = 123$.

3.5.2. Rank four

Similar calculations at multiplicity four lead to the relation

$$\mathcal{A}_{1234}^{m,\mathrm{BCJ}} = \mathcal{A}_{1234}^{m,\mathrm{L}} - k^m_{1234}\mathcal{H}_{1234} + \mathcal{A}_1^m\mathcal{H}_{234} - \mathcal{A}_4^m\mathcal{H}_{123} \tag{3.28}$$

required by (3.22), where (3.25) identifies the last two terms on the right-hand side as a perfect deconcatenation $\sum_{XY=1234}(\mathcal{A}_X^{m,\mathrm{L}}\mathcal{H}_Y - \mathcal{A}_Y^{m,\mathrm{L}}\mathcal{H}_X)$. The Berends–Giele currents comprise the five binary trees depicted in fig. 3,

$$\begin{aligned}
\mathcal{A}^{m,\mathrm{BCJ}}_{1234} &= \frac{1}{s_{1234}}\Big(\frac{A^m_{[123,4]}}{s_{12}s_{123}}+\frac{A^m_{[321,4]}}{s_{23}s_{123}}+\frac{A^m_{[12,34]}}{s_{12}s_{34}}+\frac{A^m_{[342,1]}}{s_{34}s_{234}}+\frac{A^m_{[324,1]}}{s_{23}s_{234}}\Big)\\
\mathcal{A}^{m,\mathrm{L}}_{1234} &= \frac{1}{s_{1234}}\Big(\frac{\hat{A}^m_{[123,4]}}{s_{12}s_{123}}+\frac{\hat{A}^m_{[321,4]}}{s_{23}s_{123}}+\frac{\hat{A}^m_{[12,34]}}{s_{12}s_{34}}+\frac{\hat{A}^m_{[342,1]}}{s_{34}s_{234}}+\frac{\hat{A}^m_{[324,1]}}{s_{23}s_{234}}\Big) \qquad (3.29)\\
\mathcal{H}_{1234} &= \frac{1}{s_{1234}}\Big(\frac{\hat{H}_{[123,4]}}{s_{12}s_{123}}+\frac{\hat{H}_{[321,4]}}{s_{23}s_{123}}+\frac{\hat{H}_{[12,34]}}{s_{12}s_{34}}+\frac{\hat{H}_{[342,1]}}{s_{34}s_{234}}+\frac{\hat{H}_{[324,1]}}{s_{23}s_{234}}\Big)\,,
\end{aligned}$$

where four of the five numerators in (3.29) belong to the topology of (3.14). However, the third term representing the middle diagram in fig. 3 follows the separate conversion rule

$$\begin{aligned}
A^m_{[12,34]} &= \hat{A}^m_{[12,34]} - k^m_{1234}\hat{H}_{[12,34]} \qquad (3.30)\\
&\quad + (k^1\cdot k^2)\big(\hat{H}_{[13,4]}A^m_2 - \hat{H}_{[23,4]}A^m_1\big) + (k^3\cdot k^4)\big(\hat{H}_{[12,4]}A^m_3 - \hat{H}_{[12,3]}A^m_4\big)
\end{aligned}$$

between Lorentz gauge and BCJ gauge. As a defining property of BCJ gauge, the left-hand side can be expressed in terms of the basic topology (3.10) via $A^m_{[12,34]} = A^m_{1234} - A^m_{1243}$. The new topology $\hat{H}_{[12,34]}$ of redefining fields (see [32]) is determined by (3.30) whose solution can be found in (3.18).

Upon insertion into (3.29), contributions of the form $\hat{H}_{[12,3]}A^m_4$ in (3.14) and (3.30) conspire to the desired deconcatenation term in (3.28), verifying the mediation of a non-linear gauge transformation between $\mathcal{A}^{m,\mathrm{BCJ}}_{1234}$ and $\mathcal{A}^{m,\mathrm{L}}_{1234}$. The analogous analysis of the gauge transformation at multiplicity five is relegated to appendix B.

4. Theta-expansions in Harnad–Shnider gauge

In the last section we have identified a particular gauge transformation $\mathbb{H}$ which relates the Berends–Giele currents in the BCJ gauge to their counterparts in the Lorentz gauge. Similarly, we will now construct another gauge transformation

$$\mathbb{L} \equiv \sum_{i_1,i_2}\mathcal{L}_{i_1 i_2}t^{i_1}t^{i_2} + \sum_{i_1,i_2,i_3}\mathcal{L}_{i_1 i_2 i_3}t^{i_1}t^{i_2}t^{i_3} + \cdots \qquad (4.1)$$

whose expansion starts at multiplicity two and is designed to simplify the theta-expansions of the multiparticle superfields.

4.1. Generating series of Harnad–Shnider gauge variations

A convenient gauge choice to expand the superfields of ten-dimensional SYM in θ^α is the Harnad–Shnider (HS) gauge [15],

$$\theta^\alpha \mathbb{A}_\alpha^{\rm HS} = 0 \,. \tag{4.2}$$

At the linearized level, the gauge $\theta^\alpha A_\alpha^i = 0$ has been used in [16] to obtain the theta-expansions of the single-particle superfields to arbitrary order. However, the recursive definition (2.11) of multiparticle Berends–Giele currents $\mathcal{A}_\alpha^P$ in Lorentz gauge does not preserve linearized HS gauge, e.g.

$$\theta^\alpha A_\alpha^i = 0 \quad \Rightarrow \quad \theta^\alpha \mathcal{A}_\alpha^{12} = \frac{1}{2s_{12}} \big[A_m^2 (\theta\gamma^m W_1) - (1 \leftrightarrow 2) \big] \neq 0 \ . \tag{4.3}$$

Still, there is a non-linear gauge transformation $\mathbb{L}$ which brings the currents from Lorentz gauge into HS gauge via

$$\mathbb{A}_\alpha^{\rm HS} = \mathbb{A}_\alpha^{\rm L} - [D_\alpha, \mathbb{L}] + [\mathbb{A}_\alpha^{\rm L}, \mathbb{L}] \,. \tag{4.4}$$

It can determined recursively by contracting with θ^α:

$$[\mathcal{D}, \mathbb{L}] = \theta^\alpha \mathbb{A}_\alpha^{\rm L} + [\theta^\alpha \mathbb{A}_\alpha^{\rm L}, \mathbb{L}] \,, \tag{4.5}$$

where the Euler operator

$$\mathcal{D} \equiv \theta^\alpha D_\alpha = \theta^\alpha \frac{\partial}{\partial \theta^\alpha} \tag{4.6}$$

weights the $k^{\rm th}$ order in θ by a factor of k. At the level of multiparticle components in (4.1), this translates into

$$\mathcal{D}\mathcal{L}_P = \theta^\alpha \mathcal{A}_\alpha^P + \sum_{XY=P} \left(\theta^\alpha \mathcal{A}_\alpha^X \mathcal{L}_Y - \theta^\alpha \mathcal{A}_\alpha^Y \mathcal{L}_X \right) \,, \tag{4.7}$$

where the Berends–Giele currents $\mathcal{L}_X, \mathcal{L}_Y$ on the right hand side have lower multiplicity than $\mathcal{L}_P$ on the left hand side. Hence, (4.7) is a recursion w.r.t. multiplicity in its Lie-series expansion (4.1). The currents $\mathcal{A}_\alpha^P$ are understood to follow the Lorentz-gauge setup in (2.10) to (2.14). Using $\theta^\alpha A_\alpha^i = \mathcal{L}_i = 0$ at the linear level, we have for instance

$$\mathcal{D}\mathcal{L}_{12} = \theta^\alpha \mathcal{A}_\alpha^{12} \,, \quad \mathcal{D}\mathcal{L}_{123} = \theta^\alpha \mathcal{A}_\alpha^{123} \,, \quad \mathcal{D}\mathcal{L}_{1234} = \theta^\alpha \mathcal{A}_\alpha^{1234} + \theta^\alpha \mathcal{A}_\alpha^{12} \mathcal{L}_{34} - \theta^\alpha \mathcal{A}_\alpha^{34} \mathcal{L}_{12} \,. \tag{4.8}$$

By imposing $\mathbb{L}(\theta = 0) = 0$, we arrive at explicit theta-expansions such as

$$\begin{aligned} \mathcal{L}_{12} = \frac{1}{2s_{12}} \Big(&(\theta\gamma_m \chi_1) e_2^m + \frac{1}{8} (\theta\gamma_{mnp}\theta) e_1^m f_2^{np} \\ &+ \frac{1}{12} (\theta\gamma_{mnp}\theta)(\theta\gamma^m \chi_1) k_{12}^n e_2^p - (1 \leftrightarrow 2) + \cdots \Big) e^{k_{12}x} \,, \end{aligned} \tag{4.9}$$

with terms of order $\theta^{\geq 4}$ in the ellipsis and analogous expressions for $\mathcal{L}_{12\ldots p}$ at $p \geq 3$.

4.2. Multiparticle theta-expansions in Harnad–Shnider gauge

The theta-expansion of non-linear fields in HS gauge (4.2) can be elegantly captured by means of higher mass dimension superfields [6],

$$\mathbb{W}^{m_1\ldots m_k\alpha} \equiv [\nabla^{m_1}, \mathbb{W}^{m_2\ldots m_k\alpha}] , \qquad \mathbb{F}^{m_1\ldots m_k|pq} \equiv [\nabla^{m_1}, \mathbb{F}^{m_2\ldots m_k|pq}] , \tag{4.10}$$

subject to non-linear gauge transformations [6]

$$\delta_{\mathbf{\Omega}}\mathbb{W}^{m_1\ldots m_k\alpha} = \big[\mathbf{\Omega}, \mathbb{W}^{m_1\ldots m_k\alpha}\big], \quad \delta_{\mathbf{\Omega}}\mathbb{F}^{m_1\ldots m_k|pq} = \big[\mathbf{\Omega}, \mathbb{F}^{m_1\ldots m_k|pq}\big] . \tag{4.11}$$

In the subsequent, we assume that the superfields have been brought to HS gauge via (2.6) through the transformation $\mathbf{\Omega} \to \mathbb{L}$ constructed from (4.7). For ease of notation, the accompanying $^{\rm HS}$ superscripts as in (4.4) will henceforth be suppressed. Contracting the non-linear equations of motion (2.5) with θ^α yields [15]

$$\begin{aligned} \big(\mathcal{D}+1\big)\mathbb{A}_\beta &= (\theta\gamma^m)_\beta \mathbb{A}_m , \qquad & \mathcal{D}\mathbb{A}_m &= (\theta\gamma_m \mathbb{W}) \\ \mathcal{D}\mathbb{W}^\beta &= \frac{1}{4}(\theta\gamma^{mn})^\beta \mathbb{F}_{mn} , \qquad & \mathcal{D}\mathbb{F}^{mn} &= -(\mathbb{W}^{[m}\gamma^{n]}\theta) \end{aligned} \tag{4.12}$$

by virtue of HS gauge. This can be used to reconstruct the entire theta-expansion of any superfield from their zero'th orders $\mathbb{K}(\theta=0)$ [15],

$$\begin{aligned} [\mathbb{A}_\alpha]_k &= \frac{1}{k+1}(\theta\gamma^m)_\alpha [\mathbb{A}_m]_{k-1} , \qquad & [\mathbb{A}_m]_k &= \frac{1}{k}(\theta\gamma_m [\mathbb{W}]_{k-1}) \\ [\mathbb{W}^\alpha]_k &= \frac{1}{4k}(\theta\gamma^{mn})^\alpha [\mathbb{F}_{mn}]_{k-1} , \qquad & [\mathbb{F}^{mn}]_k &= -\frac{1}{k}([\mathbb{W}^{[m}]_{k-1}\gamma^{n]}\theta) , \end{aligned} \tag{4.13}$$

where the notation $[\ldots]_k$ instructs to only keep terms of order $(\theta)^k$ of the enclosed superfields. The analogous expressions for superfields at higher mass dimensions are

$$\begin{aligned} [\mathbb{W}^\alpha_m]_k &= \frac{1}{k}\bigg\{\frac{1}{4}(\theta\gamma^{pq})^\alpha [\mathbb{F}_{m|pq}]_{k-1} - (\theta\gamma_m)_\beta \sum_{l=0}^{k-1} \{[\mathbb{W}^\beta]_l, [\mathbb{W}^\alpha]_{k-l-1}\}\bigg\} \\ [\mathbb{F}^{m|pq}]_k &= -\frac{1}{k}\bigg\{([\mathbb{W}^{m[p}]_{k-1}\gamma^{q]}\theta) + (\theta\gamma^m)_\alpha \sum_{l=0}^{k-1} [[\mathbb{W}^\alpha]_l, [\mathbb{F}^{pq}]_{k-l-1}]\bigg\} \\ [\mathbb{W}^\alpha_{mn}]_k &= \frac{1}{k}\bigg\{\frac{1}{4}(\theta\gamma^{pq})^\alpha [\mathbb{F}_{mn|pq}]_{k-1} + (\theta\gamma_m)_\beta \sum_{l=0}^{k-1} \{[\mathbb{W}^\beta]_l, [\mathbb{W}^\alpha_n]_{k-l-1}\} \\ &\quad + (\theta\gamma_n)_\beta \sum_{l=0}^{k-1} \Big(\{[\mathbb{W}^\beta_m]_l, [\mathbb{W}^\alpha]_{k-l-1}\} + \{[\mathbb{W}^\beta]_l, [\mathbb{W}^\alpha_m]_{k-l-1}\}\Big)\bigg\} , \end{aligned} \tag{4.14}$$

see [6] for the underlying equations of motion and (C.8) for generalizations to higher mass dimension.

4.2.1. The component wavefunctions

The θ-independent terms $[\mathbb{K}]_0$ initiate the above recursions in the order of θ, and their multiparticle components $[\mathcal{K}_P]_0$ at lowest mass dimensions

$$[\mathcal{A}^m_P]_0 \equiv \mathfrak{e}^m_P e^{k_P x} \ , \qquad [\mathcal{W}^\alpha_P]_0 \equiv \mathcal{X}^\alpha_P e^{k_P x} \tag{4.15}$$

are shown in [18] to supersymmetrize the Berends–Giele currents in [21], e.g.

$$\begin{aligned} s_{12}\mathfrak{e}^m_{12} &= e^m_2(k_2\cdot e_1) - e^m_1(k_1\cdot e_2) + \frac{1}{2}(k^m_1 - k^m_2)(e_1\cdot e_2) + (\chi_1\gamma^m\chi_2) \\ s_{12}\mathcal{X}^\alpha_{12} &= \frac{1}{2}k^m_{12}\gamma^{\alpha\beta}_m\big[e^n_1(\gamma_n\chi_2)_\beta - e^n_2(\gamma_n\chi_1)_\beta\big] \ . \end{aligned} \tag{4.16}$$

Note that Lorentz gauge for the superfields $\mathcal{A}^m_P$ propagates to the currents $\mathfrak{e}^m_P$,

$$(k_P\cdot\mathfrak{e}_P) = (k_P\cdot[\mathcal{A}_P]_0) = 0\,, \tag{4.17}$$

since the transformation towards HS gauge in (4.5) is chosen with $\mathbb{L}(\theta=0)=0$.

At higher mass dimensions, the wavefunctions in

$$[\mathcal{W}^{m_1\ldots m_k\alpha}_P]_0 \equiv \mathcal{X}^{m_1\ldots m_k\alpha}_P e^{k_P x} \ , \qquad [\mathcal{F}^{m_1\ldots m_k|pq}_P]_0 \equiv \mathfrak{f}^{m_1\ldots m_k|pq}_P e^{k_P x} \tag{4.18}$$

with $k=0,1,2,\ldots$ inherit the recursive expressions from (4.10) such that

$$\begin{aligned} \mathfrak{f}^{mn}_P &\equiv k^m_P\mathfrak{e}^n_P - k^n_P\mathfrak{e}^m_P - \sum_{XY=P}(\mathfrak{e}^m_X\mathfrak{e}^n_Y - \mathfrak{e}^n_X\mathfrak{e}^m_Y) \\ \mathcal{X}^{m_1\ldots m_k\alpha}_P &\equiv k^{m_1}_P\mathcal{X}^{m_2\ldots m_k|pq}_P - \sum_{XY=P}(\mathfrak{e}^{m_1}_X\mathcal{X}^{m_2\ldots m_k\alpha}_Y - \mathcal{X}^{m_2\ldots m_k\alpha}_X\mathfrak{e}^{m_1}_Y)\ , \qquad k=1,2,\ldots \\ \mathfrak{f}^{m_1\ldots m_k|pq}_P &\equiv k^{m_1}_P\mathfrak{f}^{m_2\ldots m_k|pq}_P - \sum_{XY=P}(\mathfrak{e}^{m_1}_X\mathfrak{f}^{m_2\ldots m_k|pq}_Y - \mathfrak{f}^{m_2\ldots m_k|pq}_X\mathfrak{e}^{m_1}_Y)\ , \qquad k=1,2,\ldots\ . \end{aligned} \tag{4.19}$$

4.2.2. The theta-expansion

Using the notation $\mathcal{K}_P(x,\theta)\equiv\mathcal{K}_P(\theta)e^{k_P\cdot x}$ one can show that the recursions (4.13) and (4.14) lead to the following multiparticle theta-expansions,

$$\begin{aligned} \mathcal{A}^P_\alpha(\theta) &= \frac{1}{2}(\theta\gamma_m)_\alpha\mathfrak{e}^m_P + \frac{1}{3}(\theta\gamma_m)_\alpha(\theta\gamma^m\mathcal{X}_P) - \frac{1}{32}(\theta\gamma_m)^\alpha(\theta\gamma^{mnp}\theta)\mathfrak{f}^P_{np} \\ &+ \frac{1}{60}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\mathcal{X}^P_n\gamma_p\theta) + \frac{1}{1152}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\theta\gamma^{pqr}\theta)\mathfrak{f}^{n|qr}_P \\ &+ \sum_{XY=P}[\mathcal{A}^{X,Y}_\alpha]_5 + \ldots \end{aligned} \tag{4.20}$$

$$
\begin{aligned}
\mathcal{A}_P^m(\theta) &= \mathfrak{e}_P^m + (\theta\gamma^m \mathcal{X}_P) - \frac{1}{8}(\theta\gamma^{mpq}\theta)\mathfrak{f}_P^{pq} + \frac{1}{12}(\theta\gamma^{mnp}\theta)(\mathcal{X}_P^n\gamma^p\theta) \\
&+ \frac{1}{192}(\theta\gamma^m{}_{nr}\theta)(\theta\gamma^r{}_{pq}\theta)\mathfrak{f}_P^{n|pq} - \frac{1}{480}(\theta\gamma^m{}_{nr}\theta)(\theta\gamma^r{}_{pq}\theta)(\mathcal{X}_P^{np}\gamma^q\theta) \\
&+ \sum_{XY=P}\Big([\mathcal{A}^m_{X,Y}]_4 + [\mathcal{A}^m_{X,Y}]_5\Big) + \ldots \\
\mathcal{W}_P^\alpha(\theta) &= \mathcal{X}_P^\alpha + \frac{1}{4}(\theta\gamma^{mn})^\alpha \mathfrak{f}^P_{mn} - \frac{1}{4}(\theta\gamma_{mn})^\alpha(\mathcal{X}_P^m\gamma^n\theta) - \frac{1}{48}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)\mathfrak{f}_P^{m|np} \\
&+ \frac{1}{96}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)(\mathcal{X}_P^{mn}\gamma^p\theta) - \frac{1}{1920}(\theta\gamma_m{}^r)^\alpha(\theta\gamma_{nr}{}^s\theta)(\theta\gamma_{spq}\theta)\mathfrak{f}_P^{mn|pq} \\
&+ \sum_{XY=P}\Big([\mathcal{W}^\alpha_{X,Y}]_3 + [\mathcal{W}^\alpha_{X,Y}]_4 + [\mathcal{W}^\alpha_{X,Y}]_5\Big) + \ldots \\
\mathcal{F}_P^{mn}(\theta) &= \mathfrak{f}_P^{mn} - (\mathcal{X}_P^{[m}\gamma^{n]}\theta) + \frac{1}{8}(\theta\gamma_{pq}{}^{[m}\theta)\mathfrak{f}_P^{n]|pq} - \frac{1}{12}(\theta\gamma_{pq}{}^{[m}\theta)(\mathcal{X}_P^{n]p}\gamma^q\theta) \\
&- \frac{1}{192}(\theta\gamma_{ps}{}^{[m}\theta)\mathfrak{f}_P^{n]p|qr}(\theta\gamma^s{}_{qr}\theta) + \frac{1}{480}(\theta\gamma^{[m}{}_{ps}\theta)(\mathcal{X}_P^{n]pq}\gamma^r\theta)(\theta\gamma^s{}_{qr}\theta) \\
&+ \sum_{XY=P}\Big([\mathcal{F}^{mn}_{X,Y}]_2 + [\mathcal{F}^{mn}_{X,Y}]_3 + [\mathcal{F}^{mn}_{X,Y}]_4 + [\mathcal{F}^{mn}_{X,Y}]_5\Big) + \sum_{XYZ}[\mathcal{F}^{mn}_{X,Y,Z}]_5 + \ldots
\end{aligned}
$$

with terms of order $\theta^{\geq 6}$ in the ellipsis. The non-linearities of the form $\sum_{XY=P}[\mathcal{K}_{X,Y}]_l$ can be traced back to the quadratic expressions in (4.14), e.g.

$$
\begin{aligned}
[\mathcal{A}_\alpha^{X,Y}]_5 &= \frac{1}{144}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\mathcal{X}^X\gamma_n\theta)(\mathcal{X}^Y\gamma_p\theta) \\
[\mathcal{A}^m_{X,Y}]_4 &= \frac{1}{24}(\theta\gamma^m{}_{np}\theta)(\mathcal{X}^X\gamma^n\theta)(\mathcal{X}^Y\gamma^p\theta) \\
[\mathcal{W}^\alpha_{X,Y}]_3 &= -\frac{1}{6}(\theta\gamma_{mn})^\alpha(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}_Y\gamma^n\theta) \\
[\mathcal{F}^{mn}_{X,Y}]_2 &= -(\mathcal{X}_X\gamma^{[m}\theta)(\mathcal{X}_Y\gamma^{n]}\theta) \ ,
\end{aligned} \tag{4.21}
$$

and further instances as to make the complete orders $\theta^{\leq 5}$ available are spelt out in appendix C. It is easy to see that these non-linear terms vanish in the single-particle case, and one recovers the linearized expansions of [16].

Analogous theta-expansions for superfields (4.10) of higher mass dimensions start with

$$
\begin{aligned}
\mathcal{W}_P^{m\alpha}(x,\theta) &= e^{k_P x}\Big(\mathcal{X}_P^{m\alpha} + \frac{1}{4}(\theta\gamma_{np})^\alpha\mathfrak{f}_P^{m|np} + \sum_{XY=P}\big[(\mathcal{X}_X\gamma^m\theta)\mathcal{X}_Y^\alpha - (X\leftrightarrow Y)\big] + \ldots\Big) \\
\mathcal{F}_P^{m|pq}(x,\theta) &= e^{k_P x}\Big(\mathfrak{f}_P^{m|pq} - (\mathcal{X}_P^{m[p}\gamma^{q]}\theta) + \sum_{XY=P}\big[(\mathcal{X}_X\gamma^m\theta)\mathfrak{f}_Y^{pq} - (X\leftrightarrow Y)\big] + \ldots\Big) \ ,
\end{aligned} \tag{4.22}
$$

where the lowest two orders $\sim \theta^2, \theta^3$ in the ellipsis along with generalizations to higher mass dimensions are spelt out in appendix C.

4.3. Combining HS gauge with BCJ gauge

The steps in (4.4) and (4.5) towards HS gauge can be literally repeated when starting with BCJ gauge:

$$\begin{aligned} \mathbb{A}_\alpha^{\rm BCJ-HS} &= \mathbb{A}_\alpha^{\rm BCJ} - [D_\alpha, \mathbb{L}'] + [\mathbb{A}_\alpha^{\rm BCJ}, \mathbb{L}'] \\ [\mathcal{D}, \mathbb{L}'] &= \theta^\alpha \mathbb{A}_\alpha^{\rm BCJ} + [\theta^\alpha \mathbb{A}_\alpha^{\rm BCJ}, \mathbb{L}'] \ . \end{aligned} \tag{4.23}$$

The multiparticle expansion of the gauge parameter $\mathbb{L}'$ can be constructed along the lines of (4.7), where we again set $\mathbb{L}'(\theta=0)=0$. The resulting gauge combines the benefits of a simplified theta-expansion due to

$$\theta^\alpha \mathbb{A}_\alpha^{\rm BCJ-HS} = 0 \tag{4.24}$$

with a manifestation of the BCJ duality in cubic-diagram numerators subject to Lie symmetries. The arguments of subsection 4.2 give rise to theta-expansions completely analogous to HS gauge, see (4.20) and appendix C. The only difference is a redefinition of the component Berends–Giele currents according to

$$\begin{aligned} \mathfrak{e}_P^m \rightarrow \mathcal{A}_P^{m,\rm BCJ}(\theta=0) &= \mathfrak{e}_P^m + \sum_{XY=P} (\mathfrak{e}_X^m \mathfrak{h}_Y - \mathfrak{e}_Y^m \mathfrak{h}_X) - k_P^m \mathfrak{h}_P \\ \mathcal{X}_P^\alpha \rightarrow \mathcal{W}_P^{\alpha,\rm BCJ}(\theta=0) &= \mathcal{X}_P^\alpha + \sum_{XY=P} (\mathcal{X}_X^\alpha \mathfrak{h}_Y - \mathcal{X}_Y^\alpha \mathfrak{h}_X) \ , \end{aligned} \tag{4.25}$$

where the multiparticle gauge parameters contribute through their $\theta=0$ order,

$$\mathfrak{h}_P \equiv \mathcal{H}_P(\theta=0) \ . \tag{4.26}$$

The redefinitions in (4.25) propagate to their counterparts at higher mass dimension via (4.19). Since BCJ gauge already violates the Lorentz-gauge condition at the three-particle level, e.g. $k_m^{123} \mathcal{A}_{123}^{m,\rm BCJ} = -2 s_{123} \mathcal{H}_{123}$, transversality (4.17) of the modified current $\mathfrak{e}_P^m \rightarrow \mathcal{A}_P^{m,\rm BCJ}(\theta=0)$ will no longer hold.

Similarly, the theta-expansions of higher-mass dimension Berends–Giele currents given in (4.22) and appendix C preserve their structure after the replacements in (4.25). As mentioned earlier, the BCJ gauge appears naturally in the context of string amplitudes due to the redefinitions induced by the double poles in OPE contractions. Hence, BCJ-HS gauge is particularly convenient for an accelerated approach to component amplitudes of the pure spinor superstring.

5. Application of Berends–Giele currents in Harnad–Shnider gauge

In this subsection, we sketch applications of multiparticle superfields in HS gauge to scattering amplitudes in pure spinor superspace, relevant to both string and field theories. The identification of gluon and gluino components in supersymmetric kinematic factors is shown to simplify enormously in HS gauge, in particular for large numbers of external legs.

5.1. Pure spinor superspace

Pure spinor superspace is obtained by supplementing ten-dimensional superspace $\{x^m, \theta^\alpha\}$ with a bosonic Weyl spinor λ^α subject to the pure spinor constraint

$$(\lambda\gamma^m\lambda) = 0 \ . \tag{5.1}$$

Physical components in pure spinor superspace reside at the order $\lambda^3\theta^5$ [2],

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 2880 \ , \tag{5.2}$$

and group theory fixes any other tensor structure in terms of the above scalar [33]. The prescription (5.2) guarantees that kinematic factors $S(\theta,\lambda)$ in the cohomology of the BRST operator

$$Q \equiv \lambda^\alpha D_\alpha \tag{5.3}$$

are supersymmetric and gauge invariant [2]. On these grounds, various scattering amplitudes in ten-dimensional SYM have been proposed by constructing BRST-invariant expressions with the required propagator structure [22,23,13,14]. Also, cohomology arguments have given constructive input to the computation of superstring amplitudes [3,4,24].

Up to now, in order to extract the kinematic components from scattering amplitudes in pure spinor superspace, the theta-expansions of the linearized superfields are inserted into the recursive definitions of multiparticle superfields, leaving a huge number of tensor contractions of $\lambda^3\theta^5$ for a computer-based evaluation [17]. Many kinematic factors obtained from this procedure have been gathered on the website [34]. HS gauge, on the other hand, drastically reduces the number of different $\lambda^3\theta^5$ contractions. This makes kinematic factors with an arbitrary number of external legs tractable for manual evaluation.

5.2. Applications at tree level

Tree-level kinematics of both the open superstring [3] and ten-dimensional SYM [23] can be expressed in terms of the building block

$$\langle M_A M_B M_C \rangle \ , \qquad M_A \equiv \lambda^\alpha \mathcal{A}^A_\alpha \ . \tag{5.4}$$

BRST-invariant combinations of the building block (5.4) descend from a generating series of color-dressed tree-level amplitudes $\mathcal{M}^{\rm SYM}(1,2,\ldots,n)$ [6],

$$\frac{1}{3}{\rm Tr}\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle = \sum_{n=3}^{\infty}(n-2)\sum_{i_1<i_2<\ldots<i_n} \mathcal{M}^{\rm SYM}(i_1,i_2,\ldots,i_n) \ , \qquad \mathbb{V}\equiv \lambda^\alpha \mathbb{A}_\alpha \ . \tag{5.5}$$

Since (5.5) is also invariant under non-linear gauge transformations, the components of (5.4) can be equivalently evaluated in HS gauge for arbitrary multiplicity,

$$\langle M^{\rm HS}_A M^{\rm HS}_B M^{\rm HS}_C \rangle = \frac{1}{2}\mathfrak{e}^m_A \mathfrak{e}^n_B \mathfrak{f}^C_{mn} + (\mathcal{X}_A\gamma_m \mathcal{X}_B)\mathfrak{e}^m_C + {\rm cyc}(A,B,C) \ . \tag{5.6}$$

The component currents $\mathfrak{e}^m_A, \mathcal{X}^\alpha_A$ and $\mathfrak{f}^{mn}_A$ defined in (4.15) and (4.18) can be obtained by truncating the superspace recursion (2.10) to (2.14) to $\theta = 0$. By the theta-expansions in (4.20), this component extraction involves no tensor structures $\sim \lambda^3\theta^5$ other than

$$\begin{aligned} \langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma_r\theta)(\theta\gamma^{pqr}\theta)\rangle &= 32(\delta^{mp}\delta^{nq}-\delta^{mq}\delta^{np}) \\ \langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\gamma_n\theta)_\alpha(\gamma_p\theta)_\beta\rangle &= -18\gamma^m_{\alpha\beta} \ , \end{aligned} \tag{5.7}$$

and elegantly settles the building blocks for components of tree-level amplitudes. In a companion paper [18], it will be demonstrated that (5.6) reproduces the Berends–Giele formula for bosonic tree amplitudes [21] along with its supersymmetric completion from the pure spinor superspace formula [23].

The generating series (5.5) found appearance in [35] as a superspace action for ten-dimensional SYM. The component evaluation in (5.6) is compatible with the component action of SYM in the sense that

$$\frac{1}{3}{\rm Tr}\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle = {\rm Tr}\Big(\frac{1}{4}\mathbb{F}_{mn}\mathbb{F}^{mn} + (\mathbb{W}\gamma^m\nabla_m\mathbb{W})\Big)\Big|_{\theta=0} \ . \tag{5.8}$$

The fermionic coupling vanishes on-shell by the Dirac equation (2.34) and a total derivative ∂_m has been discarded to relate

$$(\partial_m\mathbb{A}_n)\mathbb{F}^{mn} = \partial_m(\mathbb{A}_n\mathbb{F}^{mn}) - \mathbb{A}_n\big([\mathbb{A}_m,\mathbb{F}^{mn}] + \gamma^n_{\alpha\beta}\{\mathbb{W}^\alpha,\mathbb{W}^\beta\}\big) \tag{5.9}$$

through the expression for $\partial_m\mathbb{F}^{mn}$ in (2.36).

5.3. Applications at loop level

In the same way as the building block (5.4) is specific to tree amplitudes, any loop order singles out specific scalar combinations of multiparticle superfields which are BRST invariant at the linearized level, e.g.

$$\begin{aligned} M_A(\lambda\gamma_m\mathcal{W}_B)(\lambda\gamma_n\mathcal{W}_C)\mathcal{F}_D^{mn} \ &\leftrightarrow \ 1-\text{loop } [36,4,5] \\ (\lambda\gamma_{mnpqr}\lambda)(\lambda\gamma_s\mathcal{W}_A)\mathcal{F}_B^{mn}\mathcal{F}_C^{pq}\mathcal{F}_D^{rs} \ &\leftrightarrow \ 2-\text{loop } [37,14] \\ (\lambda\gamma_m\mathcal{W}_A^n)(\lambda\gamma_n\mathcal{W}_B^p)(\lambda\gamma_p\mathcal{W}_C^m) \ &\leftrightarrow \ 3-\text{loop } [6] \ . \end{aligned} \tag{5.10}$$

They describe the low-energy limit in string theory and are motivated by the zero-mode saturation rules of the pure spinor formalism [2,36]. Moreover, they are believed to represent box, double-box and Mercedes-star diagrams in SYM amplitudes to arbitrary multiplicity, see [13,14]. Again, HS gauge as well as the theta-expansions in (4.20), (4.22) and appendix C greatly simplify their component evaluation via (5.2).

In contrast to tree-level, loop amplitudes in SYM and superstring theory additionally involve tensorial building blocks contracting the loop momenta where HS gauge yields comparable benefits in the component evaluation. One-loop kinematic factors generalizing (5.10) to arbitrary tensor rank have been constructed in [32], and some of them have been defined in terms of the superfields $H_{12\ldots p}$ from the transformation to BCJ gauge. As will be described elsewhere, kinematic factors with explicit reference to gauge parameters will require extra care when adapted to different non-linear gauges. At any rate, HS gauge for Berends–Giele currents sets new scales for the computational effort in component evaluations.

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Appendix A. Proof of the Berends–Giele symmetries

In this appendix, the symmetry property (2.23) of Berends–Giele currents will be proven from their recursive definition (2.10). The idea is to regard the bracketing operation in

$$s_{AB}\mathcal{K}_{A\sqcup\!\sqcup B} = \sum_{XY=A\sqcup\!\sqcup B} \mathcal{K}_{[X,Y]} \tag{A.1}$$

as a linear and antisymmetric map $\mathcal{B}$ acting on a tensor product of words $X \otimes Y$,

$$\mathcal{B}:\ X\otimes Y \to \mathcal{K}_{[X,Y]}\ , \qquad \mathcal{B}(X\otimes Y) = -\mathcal{B}(Y\otimes X)\ . \tag{A.2}$$

We will then show by induction that

$$s_{AB}\mathcal{K}_{A\sqcup\!\sqcup B} = \sum_{XY=A\sqcup\!\sqcup B} \mathcal{B}(X\otimes Y) = 0\ , \tag{A.3}$$

starting with $0 = \mathcal{K}_{1\sqcup\!\sqcup 2} = \mathcal{K}_{12} + \mathcal{K}_{21}$ by antisymmetry of the bracket.

As pointed out below (2.10), the convention for deconcatenation sums $\sum_{XY=P}$ is to exclude the empty words $X=\emptyset$ and $Y=\emptyset$. Hence, they have to be subtracted in relating (A.3) to the deconcatenation coproduct for (possibly empty) words P,

$$\Delta(P) \equiv 1\otimes P + P\otimes 1 + \sum_{XY=P} X\otimes Y\ . \tag{A.4}$$

This coproduct is known to be compatible with the shuffle product in the sense that

$$\begin{aligned}\Delta(A\sqcup\!\sqcup B) &= \Delta(A)\sqcup\!\sqcup\Delta(B) \\ &= 1\otimes(A\sqcup\!\sqcup B) + (A\sqcup\!\sqcup B)\otimes 1 + A\otimes B + B\otimes A + \sum_{PQ=A}\sum_{RS=B}(P\sqcup\!\sqcup R)\otimes(Q\sqcup\!\sqcup S) \\ &\quad + \sum_{RS=B}\big(R\otimes(A\sqcup\!\sqcup S) + (A\sqcup\!\sqcup R)\otimes S\big) + \sum_{PQ=A}\big(P\otimes(Q\sqcup\!\sqcup B) + (P\sqcup\!\sqcup B)\otimes Q\big)\ ,\end{aligned} \tag{A.5}$$

see e.g. section 1.5 in [11]. The tensor product in (A.3) can then be written as

$$\begin{aligned}\sum_{XY=A\sqcup\!\sqcup B} X\otimes Y &= A\otimes B + B\otimes A + \sum_{PQ=A}\sum_{RS=B}(P\sqcup\!\sqcup R)\otimes(Q\sqcup\!\sqcup S) \\ &+ \sum_{RS=B}\big(R\otimes(A\sqcup\!\sqcup S) + (A\sqcup\!\sqcup R)\otimes S\big) + \sum_{PQ=A}\big(P\otimes(Q\sqcup\!\sqcup B) + (P\sqcup\!\sqcup B)\otimes Q\big)\ .\end{aligned} \tag{A.6}$$

In turns out that the right hand side is annihilated by $\mathcal{B}$ in (A.2) since the first two terms $A\otimes B + B\otimes A$ drop out by antisymmetry of $\mathcal{B}$ and the remaining terms are mapped to the

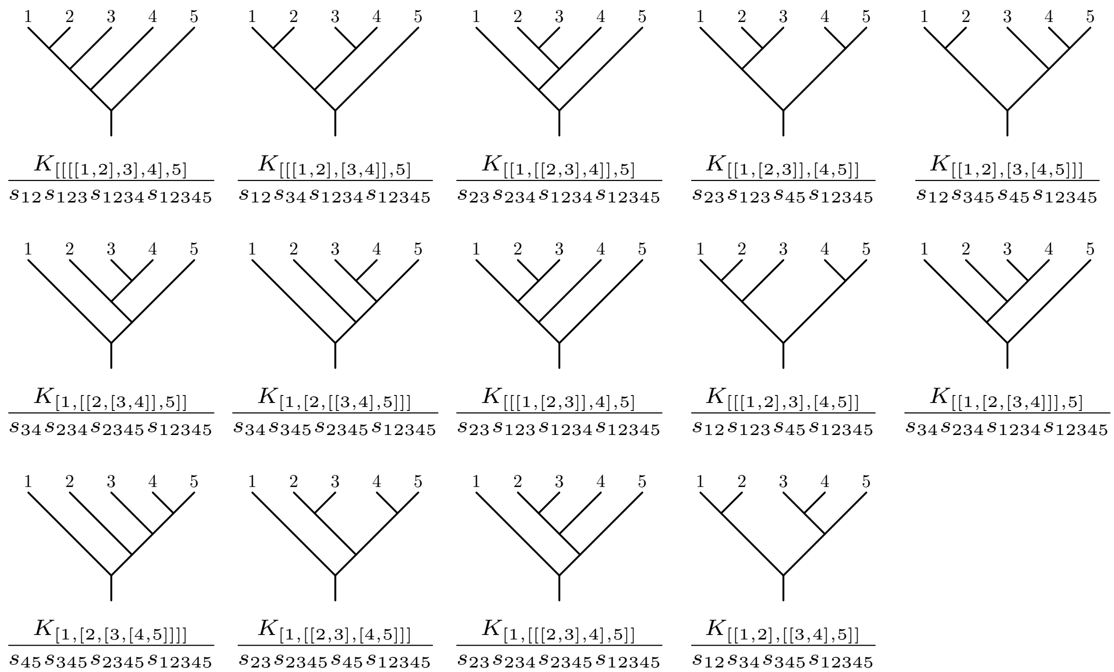

Fig. 4 The fourteen binary trees used in the definition of $\mathcal{K}_{12345}$.

schematic form $\mathcal{K}_{[X\sqcup Y,Z]}$ under $\mathcal{B}$ with all of $X,Y,Z\neq\emptyset$. By the bracketing rules (2.11) to (2.13), the latter yields antisymmetric combinations of $\mathcal{K}_{X\sqcup Y}$ and $\mathcal{K}_Z$ with $X\sqcup Y$ of multiplicity smaller than $A\sqcup B$. Hence, we can set $\mathcal{K}_{X\sqcup Y}=0$ by the inductive assumption which concludes the proof of (A.3).

The proof can be easily extended to $\mathcal{F}_P^{mn}$ and higher-mass dimension superfields with recursive definition in (2.14) and (4.10): The deconcatenation sums along with the non-linearities can be treated using the same arguments as above, and the linear contributions from superfields of the same multiplicity inherit the shuffle property of lower-mass dimension superfields.

Appendix B. BCJ gauge versus Lorentz gauge at rank five

In this appendix, we verify that the supersymmetric Berends–Giele currents at rank five in BCJ gauge and Lorentz gauge are related by a non-linear gauge transformation as in (3.22). Straightforward but tedious calculations lead to the following translation between local superfields in BCJ and Lorentz gauge,

$$\begin{aligned} A^m_{[1234,5]} &= \hat{A}^m_{[1234,5]} - k^m_{12345}\hat{H}_{[1234,5]} \\ &\quad - (k^1\cdot k^2)(\hat{H}_{[134,5]}A^m_2 + \hat{H}_{[14,5]}A^m_{23} + \hat{H}_{[13,5]}A^m_{24} + \hat{H}_{[13,4]}A^m_{25} - (1\leftrightarrow 2)) \end{aligned} \tag{B.1}$$

$$
\begin{aligned}
&- (k^{12}\cdot k^3)(\hat H_{[124,5]}A^m_3 + \hat H_{[12,5]}A^m_{34} + \hat H_{[12,4]}A^m_{35} - \hat H_{[34,5]}A^m_{12})\\
&- (k^{123}\cdot k^4)(\hat H_{[123,5]}A^m_4 + \hat H_{[12,3]}A^m_{45})\\
&- (k^{1234}\cdot k^5)(\hat H_{[123,4]}A^m_5)\\
A^m_{[123,45]} &= \hat A^m_{[123,45]} - k^m_{12345}\hat H_{[123,45]}\\
&- (k^1\cdot k^2)(\hat H_{[13,45]}A^m_2 + \hat H_{[45,2]}A^m_{13} - (1\leftrightarrow 2)) \qquad \text{(B.2)}\\
&- (k^{12}\cdot k^3)(\hat H_{[12,45]}A^m_3 + \hat H_{[45,3]}A^m_{12})\\
&- (k^{123}\cdot k^{45})(\hat H_{[12,3]}A^m_{45})\\
&- (k^4\cdot k^5)(\hat H_{[123,4]}A^m_5 - \hat H_{[123,5]}A^m_4)\\
A^m_{[[12,34],5]} &= \hat A^m_{[[12,34],5]} - k^m_{12345}\hat H_{[[12,34],5]} \qquad \text{(B.3)}\\
&- (k^1\cdot k^2)(\hat H_{[34,2]}A^m_{15} - \hat H_{[34,1]}A^m_{25} + \hat H_{[342,5]}A^m_1 - \hat H_{[341,5]}A^m_2)\\
&- (k^3\cdot k^4)(\hat H_{[12,3]}A^m_{45} - \hat H_{[12,4]}A^m_{35} + \hat H_{[123,5]}A^m_4 - \hat H_{[124,5]}A^m_3)\\
&- (k^{12}\cdot k^{34})(\hat H_{[12,5]}A^m_{34} - \hat H_{[34,5]}A^m_{12})\\
&- (k^{1234}\cdot k^5)(\hat H_{[12,34]}A^m_5)\ ,
\end{aligned}
$$

where the second and third equations can be regarded as the definitions of $\hat H_{[123,45]}$ and $\hat H_{[[12,34],5]}$. The solution of the former is given in (3.18) and (3.15) while the latter is

$$
\hat H_{[[12,34],5]} = H_{[1234,5]} - H_{[1243,5]} - \frac{1}{2}H_{[12,34]}(k_{1234}\cdot A_5)\ . \qquad \text{(B.4)}
$$

Plugging the above equations into the generic definition of the rank-five Berends–Giele current as displayed in fig. 4, namely,

$$
\begin{aligned}
s_{12345}\mathcal{K}_{12345} = {}& \frac{K_{[1,4532]}}{s_{2345}s_{345}s_{45}} - \frac{K_{[1,3452]}}{s_{2345}s_{345}s_{34}} - \frac{K_{[1,3425]}}{s_{2345}s_{234}s_{34}} + \frac{K_{[1,2345]}}{s_{2345}s_{234}s_{23}} - \frac{K_{[12,453]}}{s_{345}s_{12}s_{45}}\\
&+ \frac{K_{[12,345]}}{s_{345}s_{12}s_{34}} + \frac{K_{[45,231]}}{s_{123}s_{23}s_{45}} - \frac{K_{[45,123]}}{s_{123}s_{12}s_{45}} + \frac{K_{[3421,5]}}{s_{1234}s_{234}s_{34}} - \frac{K_{[2341,5]}}{s_{1234}s_{234}s_{23}}\\
&- \frac{K_{[2314,5]}}{s_{1234}s_{123}s_{23}} + \frac{K_{[1234,5]}}{s_{1234}s_{123}s_{12}} + \frac{K_{[1,[23,45]]}}{s_{2345}s_{23}s_{45}} - \frac{K_{[5,[12,34]]}}{s_{1234}s_{12}s_{34}}\ , \qquad \text{(B.5)}
\end{aligned}
$$

leads to

$$
\mathcal{A}^{m,\mathrm{BCJ}}_{12345} = \mathcal{A}^{m,\mathrm{L}}_{12345} - k^m_{12345}\mathcal{H}_{12345} + \mathcal{A}^m_1\mathcal{H}_{2345} + \mathcal{A}^m_{12}\mathcal{H}_{345} - \mathcal{A}^m_5\mathcal{H}_{1234} - \mathcal{A}^m_{45}\mathcal{H}_{123}\ . \qquad \text{(B.6)}
$$

By the vanishing of $\mathcal{H}_i$ and $\mathcal{H}_{ij}$, this reproduces the non-linear gauge transformation (3.22) at multiplicity five.

Appendix C. Theta-expansions in Harnad–Shnider gauge

C.1. Theta-expansions of $\mathcal{A}^\alpha_P, \mathcal{A}^m_P, \mathcal{W}^\alpha_P, \mathcal{F}^{mn}_P$

The component prescription (5.2) in pure spinor superspace requires the theta-expansion of the enclosed superfields up to the order θ^5. The expansions up to θ^5 of the Berends–Giele currents $\mathcal{A}^\alpha_P, \mathcal{A}^m_P, \mathcal{W}^\alpha_P, \mathcal{F}^{mn}_P$ in HS gauge can be found in (4.20) up to deconcatenation terms. These are now spelt out:

$$[\mathcal{A}^m_{X,Y}]_5 = \frac{1}{320}(\theta\gamma^{mnr}\theta)(\theta\gamma_{rpq}\theta)(\mathcal{X}_X\gamma_n\theta)\mathfrak{f}^{pq}_Y - (X\leftrightarrow Y) \tag{C.1}$$

$$[\mathcal{W}^\alpha_{X,Y}]_4 = -\frac{1}{64}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)(\mathcal{X}_X\gamma^m\theta)\mathfrak{f}^{np}_Y - (X\leftrightarrow Y) \tag{C.2}$$

$$\begin{aligned}[\mathcal{W}^\alpha_{X,Y}]_5 = {}& \frac{1}{120}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{npq}\theta)(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}^n_Y\gamma^p\theta)\\ &+\frac{1}{240}(\theta\gamma_n{}^q)^\alpha(\theta\gamma_{mpq}\theta)(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}^n_Y\gamma^p\theta)\\ &-\frac{1}{1280}(\theta\gamma^{rs})^\alpha(\theta\gamma_{mnr}\theta)(\theta\gamma_{pqs}\theta)\mathfrak{f}^{mn}_X\mathfrak{f}^{pq}_Y - (X\leftrightarrow Y)\end{aligned} \tag{C.3}$$

$$[\mathcal{F}^{mn}_{X,Y}]_3 = \frac{1}{8}\big[(\theta\gamma_{pq}{}^{[m}\theta)(\mathcal{X}_X\gamma^{n]}\theta)\mathfrak{f}^{pq}_Y - (X\leftrightarrow Y)\big] \tag{C.4}$$

$$\begin{aligned}[\mathcal{F}^{mn}_{X,Y}]_4 = {}& -\frac{1}{12}(\theta\gamma_{pq}{}^{[m}\theta)(\mathcal{X}_X\gamma^{n]}\theta)(\mathcal{X}^p_Y\gamma^q\theta)\\ &-\frac{1}{24}(\theta\gamma^{pq[m}\theta)(\mathcal{X}_X\gamma_p\theta)(\mathcal{X}^{n]}_Y\gamma_q\theta)\\ &-\frac{1}{128}(\theta\gamma^{[m}{}_{pq}\theta)(\theta\gamma^{n]}{}_{rs}\theta)\mathfrak{f}^{pq}_X\mathfrak{f}^{rs}_Y - (X\leftrightarrow Y)\end{aligned} \tag{C.5}$$

$$\begin{aligned}[\mathcal{F}^{mn}_{X,Y}]_5 = {}& -\frac{1}{192}(\theta\gamma^{[m}{}_{ps}\theta)(\mathcal{X}_X\gamma^{n]}\theta)\mathfrak{f}^{p|qr}_Y(\theta\gamma^s{}_{qr}\theta)\\ &-\frac{1}{320}(\mathcal{X}_X\gamma^p\theta)(\theta\gamma_{ps}{}^{[m}\theta)\mathfrak{f}^{n]|qr}_Y(\theta\gamma^s{}_{qr}\theta)\\ &-\frac{1}{320}(\theta\gamma_{ps}{}^{[m}\theta)(\mathcal{X}^{n]}_X\gamma^p\theta)\mathfrak{f}^{qr}_Y(\theta\gamma^s{}_{qr}\theta)\\ &+\frac{1}{96}(\theta\gamma^{[m}{}_{pq}\theta)(\theta\gamma^{n]}{}_{rs}\theta)(\mathcal{X}^p_X\gamma^q\theta)\mathfrak{f}^{rs}_Y - (X\leftrightarrow Y)\end{aligned}$$

$$[\mathcal{F}^{mn}_{X,Y,Z}]_5 = -\frac{1}{24}(\theta\gamma_{pq}{}^{[m}\theta)(\mathcal{X}_X\gamma^{n]}\theta)(\mathcal{X}_Y\gamma^p\theta)(\mathcal{X}_Z\gamma^q\theta) + (X\leftrightarrow Z) \tag{C.6}$$

C.2. Theta-expansions of the simplest higher mass dimension superfields

For the simplest superfields of higher mass dimension, the theta-expansion in HS gauge that starts as in (4.22) and has the following second and third order:

$$[\mathcal{W}^{m\alpha}_P]_2 = -\frac{1}{4}(\theta\gamma_{np})^\alpha(\mathcal{X}^{mn}_P\gamma^p\theta) + \sum_{XY=P}\big[\frac{1}{4}(\theta\gamma_{np})^\alpha(\mathcal{X}_X\gamma^m\theta)\mathfrak{f}^{np}_Y$$

$$
- \frac{1}{8}(\theta\gamma^m{}_{np}\theta)\mathcal{X}^\alpha_X \mathfrak{f}^{np}_Y - (X \leftrightarrow Y)\big]
$$

$$
[\mathcal{W}^{m\alpha}_P]_3 = -\frac{1}{48}(\theta\gamma_n{}^r)^\alpha(\theta\gamma_{rpq}\theta)\mathfrak{f}^{mn|pq}_P + \sum_{XY=P}\big[-\frac{1}{4}(\theta\gamma_{np})^\alpha(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}^n_Y\gamma^p\theta)
$$

$$
- \frac{1}{6}(\theta\gamma_{np})^\alpha(\mathcal{X}_X\gamma^n\theta)(\mathcal{X}^m_Y\gamma^p\theta) - \frac{1}{12}(\theta\gamma^m{}_{np}\theta)(\mathcal{X}^n_X\gamma^p\theta)\mathcal{X}^\alpha_Y
$$

$$
- \frac{1}{32}(\theta\gamma_{np})^\alpha(\theta\gamma^m{}_{qr}\theta)\mathfrak{f}^{np}_X\mathfrak{f}^{qr}_Y - (X \leftrightarrow Y)\big]
$$

$$
[\mathcal{F}^{m|pq}_P]_2 = -\frac{1}{8}\mathfrak{f}^{m[p}{}_{|nr}(\theta\gamma^{q]nr}\theta) - \sum_{XY=P}\big[(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}^{[p}_Y\gamma^{q]}\theta) \tag{C.7}
$$

$$
+ (\mathcal{X}^m_X\gamma^{[p}\theta)(\mathcal{X}_Y\gamma^{q]}\theta) + \frac{1}{8}(\theta\gamma^m{}_{nr}\theta)\mathfrak{f}^{pq}_X\mathfrak{f}^{nr}_Y - (X \leftrightarrow Y)\big]
$$

$$
[\mathcal{F}^{m|pq}_P]_3 = \frac{1}{12}(\mathcal{X}^{m[p}_B{}_n\gamma_r\theta)(\theta\gamma^{q]nr}\theta) + \sum_{XY=P}\big[\frac{1}{8}(\mathcal{X}_X\gamma^m\theta)(\theta\gamma^{[p}{}_{nr}\theta)\mathfrak{f}^{q]|nr}
$$

$$
+ \frac{1}{8}(\theta\gamma_{nr}{}^{[p}\theta)(\mathcal{X}_X\gamma^{q]}\theta)\mathfrak{f}^{m|nr}_Y - \frac{1}{8}(\mathcal{X}^m_X\gamma^{[p}\theta)(\theta\gamma^{q]}{}_{nr}\theta)\mathfrak{f}^{nr}_Y
$$

$$
+ \frac{1}{8}(\theta\gamma^m{}_{nr}\theta)(\mathcal{X}^{[p}_X\gamma^{q]}\theta)\mathfrak{f}^{nr}_Y - \frac{1}{12}(\theta\gamma^m{}_{nr}\theta)(\mathcal{X}^n_X\gamma^r\theta)\mathfrak{f}^{pq}_Y - (X \leftrightarrow Y)\big]
$$

$$
+ \sum_{XYZ=P}\big[(\mathcal{X}_X\gamma^{[p}\theta)(\mathcal{X}_Y\gamma^{q]}\theta)(\mathcal{X}_Z\gamma^m\theta) + (X \leftrightarrow Z)\big] \ .
$$

C.3. Theta-expansions of generic higher mass dimension superfields

For superfields of higher mass dimension as defined in (4.10), the theta-expansion in HS gauge is governed by the recursion

$$
[\mathbb{W}^{N\alpha}]_k = \frac{1}{k}\bigg\{\frac{1}{4}(\theta\gamma_{pq})^\alpha[\mathbb{F}^{N|pq}]_{k-1} + \sum_{\substack{M\in P(N)\\ M\neq 0}}\sum_{l=0}^{k-1}\big[\,([\mathbb{W}]_l\gamma\theta)^M\,,\,[\mathbb{W}^{(N\setminus M)\alpha}]_{k-l-1}\,\big]\bigg\}
$$

$$
[\mathbb{F}^{N|pq}]_k = -\frac{1}{k}\bigg\{([\mathbb{W}^{N[p}]_{k-1}\gamma^{q]}\theta) - \sum_{\substack{M\in P(N)\\ M\neq 0}}\sum_{l=0}^{k-1}\big[\,([\mathbb{W}]_l\gamma\theta)^M\,,\,[\mathbb{F}^{(N\setminus M)|pq}]_{k-l-1}\,\big]\bigg\} \ . \tag{C.8}
$$

We are using multi-index notation $N \equiv n_1 n_2 \ldots n_k$ where the power set $P(N)$ consists of the 2^k ordered subsets, and $(\mathbb{W}\gamma)^N \equiv (\mathbb{W}^{n_1\ldots n_{k-1}}\gamma^{n_k})$. Their resulting theta-expansion to subleading order is given by

$$
\mathcal{W}^{N\alpha}_P(\theta) = \mathcal{X}^{N\alpha}_P + \frac{1}{4}(\theta\gamma_{pq})^\alpha\mathfrak{f}^{N|pq}_P
$$

$$
+ \sum_{XY=P}\sum_{\substack{M\in P(N)\\ M\neq 0}}\big[(\mathcal{X}_X\gamma\theta)^M\mathcal{X}^{(N\setminus M)\alpha}_Y - (\mathcal{X}_Y\gamma\theta)^M\mathcal{X}^{(N\setminus M)\alpha}_X\big] + \ldots
$$

$$
\mathcal{F}^{N|pq}_P(\theta) = \mathfrak{f}^{N|pq}_P - (\mathcal{X}^{N[p}\gamma^{q]}\theta) \tag{C.9}
$$

$$
+ \sum_{XY=P}\sum_{\substack{M\in P(N)\\ M\neq 0}}\big[(\mathcal{X}_X\gamma\theta)^M\mathfrak{f}^{(N\setminus M)|pq}_Y - (\mathcal{X}_Y\gamma\theta)^M\mathfrak{f}^{(N\setminus M)|pq}_X\big] + \ldots \ .
$$

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