Ricci-flat metrics on cones over toric complex surfaces

Dmitri Bykov*

• Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut,
Am Mühlenberg 1, D-14476 Potsdam-Golm, Germany
• Steklov Mathematical Institute of Russ. Acad. Sci., Gubkina str. 8, 119991 Moscow, Russia

Abstract. The goal of the talk was to describe recent results in constructing Ricci-flat metrics on complex cones over positively-curved complex surfaces.

There exist many explicitly known Ricci-flat metrics on *noncompact* Calabi-Yau manifolds (the first examples being [?], [?], [?]). The reason is that these latter metrics possess sufficiently many isometries. The role of these metrics is that they describe the geometry of the compact Calabi-Yau manifold in the vicinity of a singularity, after it has been resolved. One particular type of singularity that can occur for a complex Calabi-Yau threefold is that of a cone over a complex surface. From the algebraic perspective, such complex surfaces are characterized by the fact that they have an ample anticanonical bundle. They have been classified: such surface is either $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$ or the blow-up of $\mathbb{CP}^2$ in no more than eight sufficiently generic points. The latter are called *del Pezzo surfaces*. The goal of the talk was to provide a framework for constructing the most general Ricci-flat metric (with the relevant isometries) on the anticanonical cone over the del Pezzo surface of rank one — the blow-up of $\mathbb{CP}^2$ at one point. The metric of [?], which can be found by the so-called orthotoric ansatz of [?], fits in our construction as a particular case. The results reported in the talk were obtained in [?].

1 The del Pezzo surface and the cone: geometry

We will be interested in the del Pezzo surface of rank one, further denoted by $\mathbf{dP}_1$. The del Pezzo surface $\mathbf{dP}_1$ is a compact simply-connected Kähler manifold of complex dimension 2, such that $H^2(\mathbf{dP}_1, \mathbb{Z}) = \mathbb{Z}^2$, and the intersection pairing on $H^2(\mathbf{dP}_1, \mathbb{Z})$ has the form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

1.1 The differential-geometric model

The Kähler metric on the cone Y over the del Pezzo surface of rank one has the isometry group

$$\mathrm{Isom}(Y) = U(2) \times U(1) \tag{1}$$

In more practical terms, we will introduce three complex coordinates z_1, z_2, u on Y and, due to the $U(2) \times U(1)$ isometry, we will assume that the Kähler potential depends on the two combinations of them:

$$K = K(|z_1|^2 + |z_2|^2, |u|^2) := K(e^t, e^s) \tag{2}$$

*Emails: dmitri.bykov@aei.mpg.de, dbykov@mi.ras.ru

It turns out useful to perform a Legendre transform, passing from the variables $\{t, s\}$ to the new independent variables

$$\mu=\frac{\partial K}{\partial t}\,,\qquad \nu=\frac{\partial K}{\partial s} \tag{3}$$

and from the Kähler potential $K(t,s)$ to the dual potential $G(\mu,\nu)$:

$$G=\mu\, t+\nu\, s-K \tag{4}$$

The usefulness of the new variables (3) to a large extent relies on the fact that they are the moment maps for the following two $U(1)$ actions on Y:

$$U(1)_\mu:\quad (\,z_1\to e^{i\alpha}\,z_1,\quad z_2\to e^{i\alpha}\,z_2\,)\qquad\qquad U(1)_\nu:\quad u\to e^{i\beta}\,u \tag{5}$$

The Ricci-flatness equation assumes the following form:

$$e^{\frac{\partial G}{\partial \mu}+\frac{\partial G}{\partial \nu}}\left(\frac{\partial^2 G}{\partial\mu^2}\frac{\partial^2 G}{\partial\nu^2}-\left(\frac{\partial^2 G}{\partial\mu\partial\nu}\right)^2\right)=\tilde{a}\,\mu \tag{6}$$

Denoting (μ,ν) by (μ_1,μ_2), we can recover the metric from the dual potential G [?] using the formula

$$ds^2=\mu\, g_{\mathbb{CP}^1}+\sum_{i,j=1}^{2}\frac{\partial^2 G}{\partial\mu_i\partial\mu_j}\,d\mu_i\,d\mu_j+\sum_{i,j=1}^{2}\left(\frac{\partial^2 G}{\partial\mu^2}\right)^{-1}_{ij}(d\phi_i-A_i)\,(d\phi_j-A_j)\,, \tag{7}$$

where $g_{\mathbb{CP}^1}$ is the standard round metric on $\mathbb{CP}^1$, $A_2=0$ and A_1 is the 'Kähler current' of $\mathbb{CP}^1$, i.e. a connection, whose curvature is the Fubini-Study form of $\mathbb{CP}^1$: $dA_1=\omega_{\mathbb{CP}^1}$.

1.2 The moment 'biangle'

Since (μ,ν) are moment maps for the $U(1)^2$ action, the domain on which the potential $G(\mu,\nu)$ is defined is the moment polygon for this $U(1)^2$ action. In this case it is an unbounded domain with two vertices. Hence we may call it a 'biangle', and it is depicted in Fig. 1.2.

From the perspective of the equation (6), it is the singularities of the function G that determine the polytope. It is known [?] that in the simplest case of a (generally non-Ricci-flat) metric induced by a Kähler quotient of flat space with respect to an action of a complex torus, the potential G takes the form of a superposition of 'hyperplanes':

$$G_{\text{toric}}=\sum_{i=1}^{M}L_i\,(\log L_i-1)\quad\text{with}\quad L_i=\alpha_i\mu+\beta_i\nu+\gamma_i\,. \tag{8}$$

In general, a potential G satisfying (6) will not have this form. However, we will assume that it has the corresponding *asymptotic* behavior at the faces of the moment polytope. More exactly, when we approach an arbitrary face L_i, i.e. when $L_i\to 0$, we impose the asymptotic condition

$$G=L_i\,(\log L_i-1)+\ldots\quad\text{as}\quad L_i\to 0, \tag{9}$$

where the ellipsis indicates terms regular at $L_i\to 0$. Despite being subleading, they are important for the equation (6) to be consistent even in the limit $L_i\to 0$. Consistency of the equation as well requires that $\alpha_i+\beta_i=1$.

The fiber over a generic point of the moment polytope shown in Fig. 1.2 is $\mathbb{CP}^1\times\mathbb{T}^2$. We will now demonstrate how the angles of the moment polytope are detemined by the normal bundles to the two $\mathbb{CP}^1$'s 'located' in the corners.

Figure 1: The trapezium – the moment polygon of $\mathbf{dP}_1$ – and the (μ,ν) plane section of the moment polytope for the cone over $\mathbf{dP}_1$.

A corner of the moment polytope may be given by the equations

$$\lambda_i = \alpha_i \mu + \beta_i \nu + \gamma_i = 0, \quad i = 1, 2\,. \tag{10}$$

Moreover, according to the discussion above we assume that the behavior of the potential G near the corner is as follows:

$$G = \lambda_1(\log\lambda_1 - 1) + \lambda_2(\log\lambda_2 - 1) + \ldots, \tag{11}$$

where $\ldots$ denotes less singular terms. Compatibility with the Ricci-flatness condition (6) implies

$$\alpha_i + \beta_i = 1, \quad i = 1, 2 \tag{12}$$

We wish to determine what the behavior (11) implies for the *metric* near a given embedded $\mathbb{CP}^1$. The Kähler potential corresponding to the asymptotic behavior (11) looks as follows:

$$K = \kappa\ \log\left(|z_1|^2 + |z_2|^2\right) + \left(|z_1|^2 + |z_2|^2\right)^n\ |u|^{n'} + \left(|z_1|^2 + |z_2|^2\right)^m\ |u|^{m'} + \ldots, \tag{13}$$

where $\kappa = 2\,\frac{\gamma_2\beta_1 - \gamma_1\beta_2}{\beta_2 - \beta_1}$ and

$$n = \frac{2\beta_2}{\beta_2 - \beta_1}, \quad m = -\frac{2\beta_1}{\beta_2 - \beta_1}, \tag{14}$$
$$n' = -\frac{2(1-\beta_2)}{\beta_2 - \beta_1}, \quad m' = \frac{2(1-\beta_1)}{\beta_2 - \beta_1}$$

Upon changing the complex coordinates we can bring the Kähler potential to the form

$$K = \kappa\ \log\left(1 + |w|^2\right) + \left(1 + |w|^2\right)^n\ |x|^2 + \left(1 + |w|^2\right)^m\ |y|^2 + \ldots, \tag{15}$$

For $\kappa > 0$ this implies that the normal bundle $N_{\mathbb{CP}^1}$ to the $\mathbb{CP}^1$ parametrized by the inhomogeneous coordinate w and located in a given corner of the moment polytope is[1]

$$N_{\mathbb{CP}^1} = \mathcal{O}(-n)\ \oplus\ \mathcal{O}(-m), \qquad n + m = 2 \tag{16}$$

Note that $n + m = 2$, as required by the Calabi-Yau condition.

In the del Pezzo cone case the two corners of the moment biangle in the (μ,ν)-plane correspond to the two bases of the trapezium representing the moment polytope of the del Pezzo surface itself. This is emphasized in Fig. 1.2. These two bases of the trapezium correspond to the two $\mathbb{CP}^1$'s embedded in the del Pezzo surface:

- One $\mathbb{CP}^1$ is inherited from $\mathbb{CP}^2$, hence the normal bundle inside $\mathbf{dP}_1$ is $N = \mathcal{O}(1)$. This implies that the normal bundle inside the *cone over* $\mathbf{dP}_1$ is $N = \mathcal{O}(1) \oplus \mathcal{O}(-3)$

- The second $\mathbb{CP}^1$ is the exceptional divisor of the blow-up and is embedded with normal bundle $N = \mathcal{O}(-1)$. The normal bundle inside the *cone over* $\mathbf{dP}_1$ is therefore $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

These two spheres generate the second homology group of the del Pezzo surface, and their intersection matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

[1] See [?] for a detailed discussion of how the Kähler potential encodes the normal bundle to a $\mathbb{CP}^1$ in the analogous situation, when the $\mathbb{CP}^1$ is embedded in a complex surface.

2 An expansion away from the vertex of the cone

We aim at building an expansion of the metric at 'infinity', i.e. far from the 'vertex'. For this purpose, instead of the $\{\mu, \nu\}$ variables, we will use a 'radial' variable ν and an angular variable ξ (by introducing the constant μ_0 we shift the origin to the intersection point of the two outer lines of the moment 'biangle'): $\{\mu, \nu\} \to \{\nu,\ \xi = \frac{\mu-\mu_0}{\nu}\}$. We propose the following expansion for the potential G at $\nu \to \infty$ (b is a constant):

$$G = 3\nu(\log\nu - 1) + \nu\, P_0(\xi) + b\log\nu + \sum_{k=0}^{\infty} \nu^{-k}\; P_{k+1}(\xi) \tag{17}$$

Substituting this expansion in the Monge-Ampere equation, we obtain a 'master' equation, which can then be expanded in powers of $\frac{1}{\nu}$ and solved iteratively for the functions $P_k(\xi)$:

$$\begin{aligned} \sum_{k=0}^{\infty} \nu^{-k}\, P_k''(\xi) \;\times\; \left(3 - \frac{b}{\nu} + \sum_{k=2}^{\infty} k(k-1) P_k(\xi)\nu^{-k}\right) - \left(\sum_{k=1}^{\infty} k P_k'(\xi)\,\nu^{-k}\right)^2 &= \\ = a\left(\xi + \frac{\mu_0}{\nu}\right)\, e^{-\frac{b}{\nu} + \sum\limits_{k=0}^{\infty}\left((\xi-1)\,P_k' + (k-1)\,P_k\right)\nu^{-k}} \end{aligned} \tag{18}$$

2.1 Leading order

The first equation is obtained from (18) in the limit $\nu \to \infty$ and has the solution

$$P_0(\xi) = \log\left(-\frac{a}{9}\right) - \sum_{i=0}^{2} \frac{\xi - \xi_i}{\xi_i - 1}\; \log\left(\xi - \xi_i\right), \tag{19}$$

where ξ_i are the roots of the polynomial

$$Q(\xi) = \xi^3 - \frac{3}{2}\xi^2 + d, \tag{20}$$

and d is a constant of integration, which plays a crucial geometric role that we will now reveal. We will assume that $d \neq \frac{1}{2}$.

The function $P_0(\xi)$ determines the metric at infinity by means of the formulas (17) and (7). Tthe 'radial' part of the metric looks as follows ($r = 2\sqrt{3\nu}$):

$$\left[ds^2\right]_\mu := \frac{\partial^2 G}{\partial\mu_i \partial\mu_j}\, d\mu_i d\mu_j = 3\frac{d\nu^2}{\nu} + \nu\, P_0''(\xi)\, d\xi^2 = dr^2 + r^2\, \frac{P_0''}{12}\, d\xi^2 \tag{21}$$

In particular, we see that positivity of the metric requires $P_0'' > 0$.

The potential corresponding to this metric may be written in the (μ, ν) variables as follows:

$$G_0 = \sum_{i=0}^{2} \frac{\mu - \xi_i\,\nu}{1 - \xi_i}\; \left(\log\left(\mu - \xi_i\,\nu\right) - 1\right) \tag{22}$$

The slopes of the three lines involved are defined by the roots ξ_i:

$$\mathrm{Slope}_i = \left(\frac{\mu}{\nu}\right)_i = \xi_i\,. \tag{23}$$

In the notations (10) of the moment polytope, which we used before, one has

$$\xi_1 = -\frac{\beta_1}{1-\beta_1} \quad \text{and} \quad \xi_2 = -\frac{\beta_2}{1-\beta_2} \tag{24}$$

On the other hand, from the normal bundle formulas (14) and Fig. 1.2 it follows that

$$1-\frac{\beta_2}{\beta_3}=-2, \quad 1-\frac{\beta_1}{\beta_3}=2 \tag{25}$$

Hence $\frac{\beta_2}{\beta_1}=-3$. This implies the following relation for ξ_1, ξ_2:

$$-\frac{\xi_2}{1-\xi_2}=\frac{3\xi_1}{1-\xi_1} \tag{26}$$

One can show that it has two solutions: $(\xi_1^{(1)}, \xi_2^{(1)})$, $(\xi_1^{(2)}, \xi_2^{(2)})$. However, for $\xi \in (\xi_1^{(2)}, \xi_2^{(2)})$ one has $P_0'' < 0$ and for $\xi \in (\xi_1^{(1)}, \xi_2^{(1)})$ one has $P_0'' > 0$, so the positivity of the metric requires that we choose the first solution. It corresponds to

$$d=\frac{16+\sqrt{13}}{64}\,. \tag{27}$$

The third root of $Q(\xi)=0$, which we will denote ξ_0, is smaller than the two other roots.

2.2 Regularity requirement

Recall that we required that near each edge $L_i=0$ of the moment polytope the function G should behave as in (9):

$$G=L_i\,(\log L_i-1)+\ldots \quad \text{as} \quad L_i\to 0 \tag{28}$$

By placing the origin at the intersection point of the lines 1, 2 of Fig. 1.2, we make sure that the equations of these lines have the form

$$\text{Line 1}: \quad \mu-\mu_0=\xi_1\,\nu, \qquad \text{Line 2}: \quad \mu-\mu_0=\xi_2\,\nu, \tag{29}$$

to *all orders* of perturbation theory. Indeed, the lines clearly cannot change their slopes, and ξ_1, ξ_2 are their slopes at infinity. The only thing that could happen in higher orders of perturbation theory is that the lines could shift and no longer pass through the origin $\mu=\nu=0$. Precisely to account for this modification we shift the origin to the new intersection point of the two lines. To summarize, G can be written as

$$G=\frac{\mu-\mu_0-\xi_1\,\nu}{1-\xi_1}\,(\log{(\mu-\mu_0-\xi_1\,\nu)}-1)+\frac{\mu-\mu_0-\xi_2\,\nu}{1-\xi_2}\,(\log{(\mu-\mu_0-\xi_2\,\nu)}-1)+\Delta, \tag{30}$$

where Δ is a function regular at $\mu-\mu_0=\xi_1\nu$ and $\mu-\mu_0=\xi_2\nu$. In terms of the (ν,ξ) variables the statement is that $\Delta(\nu,\xi)$ is regular at $\xi=\xi_1,\xi_2$ for any fixed ν. In the forthcoming analysis of the higher orders of perturbation theory around infinity we will make the crucial *assumption* that each term of the expansion of $\Delta(\nu,\xi)$ in powers of $\frac{1}{\nu}$ is a function of ξ, regular at the two points $\xi=\xi_1,\xi_2$.

2.3 Arbitrary order

It will be explained in the following sections that the function G satisfying eq. (6) has the following structure:

$$G=3\nu\,(\log\nu-1)+\nu\,P_0(\xi)+b\log{(\nu(\xi-\xi_0))}+\sum_{k=1}^{\infty}\nu^{-k}\,P_{k+1}(\xi) \tag{31}$$

$$\text{with} \quad P_k(\xi)=b^k\left(\frac{(-1)^k}{k(k-1)}\left(\frac{1-\xi_0}{\xi-\xi_0}\right)^{k-1}+\text{Polyn}_{k-3}(\xi)\right), \quad k\geq 2 \tag{32}$$

As it should be clear from the notation, $\mathrm{Polyn}_{k-3}(\xi)$ is a polynomial of degree $k-3$ for $k \geq 3$ (and is zero for $k < 3$).

The terms in (31)-(32) singular in $\xi - \xi_0$ can be easily summed to produce the following:

$$G = \frac{\tilde{\mu} - \xi_1 \nu}{1 - \xi_1} \left(\log \left(\frac{\tilde{\mu} - \xi_1 \nu}{1 - \xi_1} \right) - 1 \right) + \frac{\tilde{\mu} - \xi_2 \nu}{1 - \xi_2} \left(\log \left(\frac{\tilde{\mu} - \xi_2 \nu}{1 - \xi_2} \right) - 1 \right) + \tag{33}$$
$$+ \left(\frac{\tilde{\mu} - \xi_0 \nu}{1 - \xi_0} + b \right) \left(\log \left(\frac{\tilde{\mu} - \xi_0 \nu}{1 - \xi_0} + b \right) - 1 \right) + b \sum_{k=2}^{\infty} \left(\frac{b}{\nu} \right)^k \mathrm{Polyn}_{k-2}(\xi)$$

Here the variable μ has been shifted in such a way that the new origin is located at $\tilde{\mu} = \nu = 0$ and $\xi = \frac{\tilde{\mu}}{\nu}$.

2.4 Singular points of the Heun equation and eigenfunctions

We proceed to describe in more detail the equations that arise in higher orders of perturbation theory. Our goal is to explain the formula (33) and elaborate on it.

In the M-th order of perturbation theory we arrive at the following equation:

$$D_M P_M := \frac{d}{d\xi} \left(Q(\xi) \frac{dP_M}{d\xi} \right) - \left((M-2)^2 - 1 \right) \xi P_M = \text{r.h.s.}, \tag{34}$$

where

$$Q(\xi) = \xi^3 - \frac{3}{2} \xi^2 + d \tag{35}$$

and the right hand side depends on the previous orders of perturbation theory, i.e. on $P_{M-1}, \ldots, P_0$ and their derivatives. As discussed above, the del Pezzo cone corresponds to

$$d = \frac{16 + \sqrt{13}}{64}. \tag{36}$$

As we claimed in (31)-(32), the inhomogeneous equation (34) has a polynomial solution of degree $M-3$. The general solution, however, is produced by adding to this particular solution a general solution of the homogenized equation $D_M \Pi_M = 0$. The roots $\xi_i, i = 0, 1, 2$ of the polynomial $Q(\xi) = \prod_{i=0}^{2} (\xi - \xi_i)$, as well as ∞, are singular points of this equation. Hence $D_M \Pi_M = 0$ is a Fuchsian equation with 4 singular points – a particular case of the so-called Heun equation.

The question we wish to pose is whether the homogenized equation $D_M \Pi_M = 0$ has a nontrivial solution regular at *two* of the singular points, say ξ_1, ξ_2. This is necessary in order to comply with the regularity requirement of § 2.2. We claim that the answer is positive only for $M = 3, 4$:

$$\Pi_3 = \alpha \tag{37}$$
$$\Pi_4 = \beta(\xi - 1), \tag{38}$$
$$\text{where} \quad \alpha, \beta = \text{const.}$$

It will be convenient to parametrize the first two nonzero polynomials in (31) as follows:

$$\mathrm{Polyn}_0(\xi) = \alpha, \qquad \mathrm{Polyn}_1(\xi) = -\frac{2}{3} \alpha + \beta(\xi - 1) \tag{39}$$

Here α and β are the parameters of the metric.

Conjecture 1. The homogeneous Heun equation $D_M \Pi_M = 0$ has no <u>*polynomial*</u> solutions for $M \geq 5$ and d given by (36).

We have checked the validity of this conjecture numerically up to $M = 100$. Regarding non-polynomial solutions, in [?] we prove the following statement:

Proposition 1. The homogeneous Heun equation $D_M \Pi_M = 0$ has no *non-polynomial* solutions, which are analytic at the two singular points $\xi = \xi_1$, $\xi = \xi_2$ for $M \geq 5$ and d given by (36).

3 An example: the orthotoric metric

In the previous sections we have demonstrated that there exists a Ricci-flat metric with $U(2)\times U(1)$ isometry on the complex cone over $\mathbf{dP}_1$ with *at most* two parameters, which we termed α and β. There exists a closed expression for G, and hence for the metric, in a particular case when the parameters α and β are related in a certain way — this is the metric obtained in [?], as well as in [?] by means of the so-called 'orthotoric' ansatz developed in [?].

The dual potential for the orthotoric metric may be written as follows:

$$G_{\text{ortho}}=\sum_{i=1}^{3} \frac{(x-x_i)(y-x_i)}{1-x_i} \log|x-x_i|+\sum_{i=1}^{3} \frac{(x-y_i)(y-y_i)}{1-y_i} \log|y-y_i|-3\,(x+y), \tag{40}$$

where x_i, y_i are respectively the roots of the following two cubic polynomials:

$$T_c(x) = x^3 - \frac{3}{2}x^2 + c, \qquad T_d(y) = y^3 - \frac{3}{2}y^2 + d = Q(y) \tag{41}$$

In particular, $y_i = \xi_i$ are the roots of $Q(y)$ that we encountered before. The moment maps μ, ν are related to the auxiliary 'orthotoric' variables x, y by means of the following formulas:

$$\mu = x\,y, \quad \nu - \nu_0 = x + y - 1 \tag{42}$$

The potential (40), expressed in terms of μ, ν, satisfies the Ricci-flatness equation (6) with $a = -9$. One can now introduce new variables $\{\nu, \xi\}$, as before, and expand the function G at $\nu \to \infty$. The expansion of the orthotoric potential G in powers of $\frac{1}{\nu}$ has the following form:

$$\begin{aligned} G_{\text{ortho}} &= 3\nu\,(\log\nu - 1) - 3\xi_0 \log\left(\nu\,(\xi-\xi_0)\right) + \nu \sum_{i=0}^{2} \frac{1}{1-\xi_i}\,(\xi-\xi_i)\,\log\left(\xi-\xi_i\right) + \\ &+ \frac{9\xi_0^2(1-\xi_0)}{2(\xi-\xi_0)}\,\frac{1}{\nu} + \left(\frac{d-c}{2} + \frac{9\,\xi_0^3(1-\xi_0)^2}{2\,(\xi-\xi_0)^2}\right)\frac{1}{\nu^2} + \\ &+ \left(\frac{27\,(1-\xi_0)^3\xi_0^4}{4\,(\xi-\xi_0)^3} + (d-c)(\xi_0 + \frac{3}{4}(\xi-1))\right)\frac{1}{\nu^3} + \ldots \end{aligned} \tag{43}$$

We see that this expansion has the general structure of (31) with $b = -3\,\xi_0$. Moreover, we can identify the parameters α, β of (39):

$$\alpha = \frac{1}{2}\,\frac{d-c}{(-3\,\xi_0)^3}, \qquad \beta = \frac{3}{4}\,\frac{d-c}{(-3\,\xi_0)^4} \tag{44}$$

The fact that α and β are related in this way means that the orthotoric metric is a *special case* of a more general metric, in which the parameters α and β are independent.

It might seem from this discussion that the orthotoric potential G_{ortho} still possesses one nontrivial parameter c. However, it turns out that this parameter has to be fixed to a particular value by the requirement that the 3-rd line of the biangle in Fig. 1.2 passes at a correct angle with respect to the other two lines (meaning that the topology of the manifold is indeed the one of a cone over $\mathbf{dP}_1$). Even in the general case, when we do not impose the orthotoric relation (44) between α and β, we expect there to be an additional tolopogical relation between these parameters.

4 Conclusion

In the talk we presented a summary of the results reported in [**?**], related to the analysis of the parameter space of Ricci-flat metrics on the complex cone over a del Pezzo surface of rank one. Using an expansion at infinity, we have found two potential parameters, α and β, and proven, up to the validity of Conjecture 1, that there can be no further parameters in the metric. In general we conjecture that there is a particular relation between β and α that preserves the correct topology of the cone, i.e. $\beta = \beta(\alpha)$. In this case the remaining parameter is related to the size of the blown-up $\mathbb{CP}^1$ in the base of the cone, i.e. in the del Pezzo surface.

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