Ricci-flat metrics on complex cones

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Motivation

- Our wish is to understand the AdS/CFT correspondence away from the maximally supersymmetric case, but in situations where dual theories can still be under control ($\mathcal{N} = 1$)
- However, in this talk we will mainly concentrate on the geometric aspects of the problem

AdS/CFT with $\mathcal{N} = 1$ SUSY

- Classical example of AdS/CFT: $AdS_5 \times S^5$, dual to $\mathcal{N} = 4$ SYM in $d = 4$ Maldacena [1997]
- There exist extremal black hole solutions to IIB supergravity preserving $\mathcal{N} = 1$ SUSY, whose 'near-horizon' geometry is $AdS_5 \times X^5$ Morrison, Plesser [1998]
- X^5 is a Sasaki-Einstein manifold (this implies the existence of one Killing spinor)

Sasaki-Einstein manifolds

- X is Sasaki-Einstein iff the cone over it is Kähler and Ricci-flat:
  $$ds^2 = dr^2 + r^2\,(\widetilde{ds^2})_X$$
- ds^2 Kähler & Ricci-flat $\Leftrightarrow$ $(\widetilde{ds^2})_X$ Sasaki-Einstein, of positive curvature
- The metric can be written as
  $$(ds^2)_{X^5} = (d\phi - J)^2 + (ds^2)_{\mathcal{M}}$$
  where $(ds^2)_{\mathcal{M}}$ is Kähler-Einstein (but not necessarily smooth), J is the Kähler current
- $r = 0 \rightarrow$ singularity

Resolving the singularity of the cone

- It is possible to resolve the singularity of the conical metric by ‘blowing-up’ the vertex, i.e. by replacing it with a cycle of non-zero size
- The metric at infinity, i.e. at $r \to \infty$, will still be asymptotic to the cone:

$$ds^2 = dr^2 + r^2 \left(\widetilde{ds^2}\right)_X \quad \text{for} \quad r \to \infty$$

- Apart from simplest cases, resolved metrics on the cones are not known $\Rightarrow$ Our study

Some examples

- Eguchi, Hanson, 1978
  Complex dimension 2, singularity of the form
  $$\mathbb{C}^2/\mathbb{Z}_2 : \quad (z_1, z_2) \sim (-z_1, -z_2)$$
- Introducing invariant coordinates $X = z_1^2, Y = z_2^2, Z = z_1 z_2$, we get an equation $XY = Z^2 \quad \text{in} \quad \mathbb{C}^3$
- This corresponds to the cone in the embedding of $\mathbb{CP}^1$ by the linear system $|\mathcal{O}(2)|$, i.e. the *anticanonical* embedding

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The Eguchi-Hanson metric

- One can look for the Kähler potential of the form $K = K(|z_1|^2 + |z_2|^2)$.
  The metric is, as usual, $ds^2 = \partial_i \bar{\partial}_j K \, dz^i \, d\bar{z}^j$
- For a Kähler metric the Ricci tensor can be expressed as $R_{i\bar{j}} = -\partial_i \bar{\partial}_j \log \det g$
- Set $R_{i\bar{j}} = 0$, solve for the Kähler potential:

The Eguchi-Hanson metric

$$K = \sqrt{r^2 + 4x^2} + r \log\left(\frac{\sqrt{r^2+4x^2}-r}{2x}\right), \quad r > 0$$

More examples

- 2d case: Eguchi-Hanson=anticanonical cone over $\mathbf{CP}^1 \Rightarrow$ SE $X_3 = S^3/\mathbb{Z}_2$
- '3d Eguchi-Hanson'= anticanonical cone over $\mathbf{CP}^2 \Rightarrow$ SE $X_5 = S^5/\mathbb{Z}_3$
- 3d case: Candelas-de la Ossa [1990] = anticanonical cone over $\mathbf{CP}^1 \times \mathbf{CP}^1$ (resolved conifold) $\Rightarrow$ SE $X_5 := T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$

Other cones?

- One can only build Ricci-flat cones over complex manifolds of 'positive curvature' (i.e. with ample anticanonical class)
- For the cone to be of $\dim_{\mathbb{C}} = 3$, we take the underlying base to be of $\dim_{\mathbb{C}} = 2$
- Apart from $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$, there are only 8 other positively curved complex surfaces – the del Pezzo surfaces

$$dP_1, \ldots, dP_8$$

The del Pezzo surface $\mathbf{dP_1}$

- $\mathbf{dP}_n$ can be seen as $\mathbf{CP^2}$, blown-up in n sufficiently generic points
- We will consider the simplest non-homogeneous case, i.e. the cone over $\mathbf{dP_1}$
- Any metric on $\mathbf{dP_1}$ should have at least two parameters – the sizes of $\mathbf{CP^2}$ and of the blown-up $\mathbf{CP^1}$
- Do these parameters persist in the cone over $\mathbf{dP_1}$?

Isometries

- Whereas the automorphism group of $\mathbf{CP^2}$ is $\boldsymbol{PGL(3,\mathbf{C})}$, the automorphism group of the del Pezzo surface is reduced to

$$Aut(\mathrm{dP}_1) = P\begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{pmatrix} \tag{1}$$

- The isometry group of the metric on the *cone* is the maximal compact subgroup of the parabolic subgroup shown above, i.e. $\mathbf{Isom} = \boldsymbol{U(1)\times U(2)}$

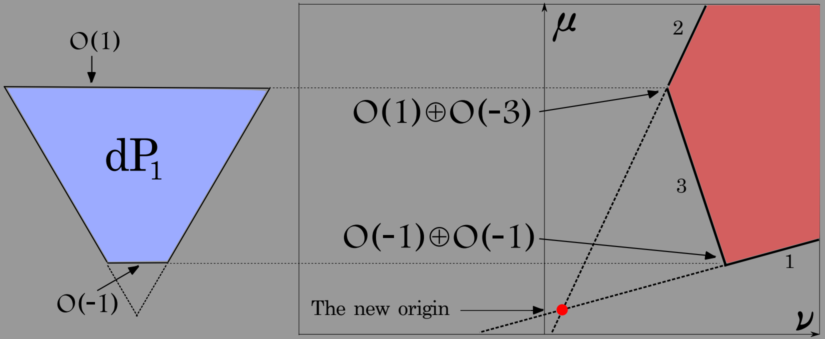
The main equation

- We will look for a Kähler potential of the form $K = K(|u|^2, |z_1|^2 + |z_2|^2) := K(e^t, e^s)$
- Just as in the case of the Eguchi-Hanson metric, we can write out a Ricci-flatness equation
- More convenient to perform a Legendre transform w.r.t. t, s, introducing the dual momentum maps $\mu = \frac{\partial K}{\partial t}$, $\nu = \frac{\partial K}{\partial s}$ and a dual potential $G = t\mu + s\nu - K$

The equation

$$e^{G_\mu+G_\nu}\ \left(G_{\mu\mu}G_{\nu\nu}-G_{\mu\nu}^2\right)=\mu$$

- The domain – the moment polygon



The expansion at ∞

- We can solve the equation exactly at large μ, ν with fixed 'angle' $\xi = \frac{\mu}{\nu}$, assuming the conical form of the metric
- This gives $G = 3\nu(\log \nu - 1) + \nu P_0(\xi)$
- $P_0(\xi)$ satisfies an ODE and can be found exactly. It provides a Sasaki-Einstein metric, which in the $\mathbf{dP_1}$ case is the $Y^{2,1}$ manifold

  Gauntlett, Martelli, Sparks, Waldram [2004]

M^{th} order and the Heun equation

- We can build a systematic perturbation theory

$$G = 3\nu(\log\nu - 1) + \nu\, P_0(\xi) + \log\nu + \sum_{k=0}^{\infty} \nu^{-k}\, P_{k+1}(\xi)$$

- In order ν^{-M} we obtain the equation

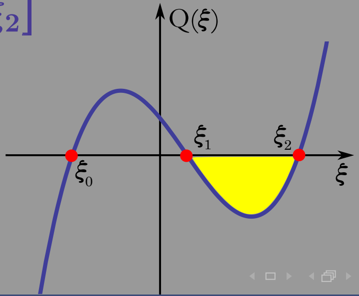
$$\frac{d}{d\xi}\left(Q(\xi)\frac{dP_M}{d\xi}\right) - \left((M-2)^2 - 1\right)\,\xi\, P_M = \text{r.h.s.},$$

$$\text{where} \quad Q(\xi) = \xi^3 - \frac{3}{2}\xi^2 + d$$

- This is a Heun equation – an analogue of hypergeometric equation with 4 Fuchsian singularities on $\mathbb{CP}^1$

Resolution parameters

- All resolution parameters should arise as (coefficients in front of) the solutions to the homogeneous equation in some order of perturbation theory
- The equation is solved in a 'physical' interval $\xi \in [\xi_1, \xi_2]$

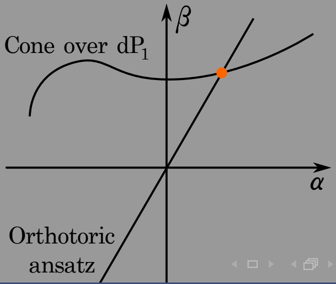


Resolution parameters. 2.

- Regularity of the metric at the boundaries of the moment polytope requires that the solutions should be regular at $\xi = \xi_1, \xi_2$
  $\Rightarrow$ Eigenvalue problem
- Solutions exist for $M = 3, 4$:
  $P_3 = \alpha, \ P_4 = \beta\,(\xi - 1)$
- Conjecture:
  For other M solutions do not exist

Resolution parameters. 3.

- When $\beta = -\frac{\alpha}{2\xi_0}$, the resolved metric is known Calderbank, Gauduchon [2006], Chen, Lu, Pope [2006]
- In general, topology imposes one more relation between β and α Martelli, Sparks [2007]
- Hence the general situation is as follows:



Questions / Answers

- Can one obtain an exact formula with both parameters α, β?
- As we discussed, there is an exact formula when $\beta = -\frac{\alpha}{2\xi_0}$. Is there a generalization?
- Dual field theories for $AdS_5 \times X^5$ have been conjectured Feng, Hanany, He, 2000
- Interpret the new parameter in terms of these $\mathcal{N} = 1$ gauge theories

Thank you!