A new version of Brakke's local regularity theorem

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Abstract

Consider an integral Brakke flow (μ_t), $t \in [0,T]$ inside some ball in Euclidean space. If μ_0 has small height, its measure does not deviate too much from that of a plane and if μ_T is non-empty, than Brakke's local regularity theorem yields that (μ_t) is actually smooth and graphical inside a smaller ball for times $t \in (C, T-C)$ for some constant C. Here we extend this result to times $t \in (C,T)$. The main idea is to prove that a Brakke flow that is initially locally graphical with small gradient will remain graphical for some time.

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1 Introduction

Overview Consider $g \in \mathcal{C}^\infty((t_1, t_2) \times \Omega, \mathbb{R}^{\mathbf{k}})$, $\Omega \subset \mathbb{R}^{\mathbf{n}}$ open. The family of graphs $M_t = \operatorname{graph}(g(t, \cdot))$ is called a smooth mean curvature flow, if

$$\partial_t g = \sum_{i,j=1}^{\mathbf{n}} \left(\delta_{ij} - \frac{D_i g D_j g}{1 + |Dg|^2} \right) D_i D_j g \tag{1}$$

at all points in $(t_1, t_2) \times \Omega$. This evolution equation can be generalised to $\mathbf{n}$-rectifiable Radon measures on $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$, see Definition 2.3. Such a weak solution will be called a Brakke flow. Here we want to show that under certain local assumptions a Brakke flow satisfies the smooth characterization from above.

The mean curvature flow was introduced in Brakke's poineering work [Bra78]. He described the evolution in the setting of geometric measure theory. This early work already contains an existence result as well as a regularity theory. However the arguments in [Bra78] often contain gaps or little errors. A new rigorous proof of the regularity results was given by Kasai and Tonegawa [KT14], [Ton15]. Also the author's thesis [Lah14] offers a completed version of Brakke's regularity theory following the original approach.

A major breakthrough in the studies of mean curvature flow was the monotonicity formula found by Huisken [Hui90] for smooth flows, which later was generalised to weak flows by Ilmanen [Ilm95] and localised by Ecker [Eck04]. Using the monotonicity, White proved a local regularity theorem [Whi05] stating that Gaussian density ratios close to one yields curvature estimates. White's theorem is formulated for smooth mean curvature flow and can be applied in a lot of singular situations as well, but not for arbitrary Brakke flows. Building up on White's curvature estimates, Ilmanen, Neves and Schulze showed in [INS14] which is locally initially graphical with small gradient remains graphical for some time. For related gradient and curvature estimates see [EH89], [EH91], [CM03], [Wan04], [CY07], [BH12].

Existence results for generalized solutions of mean curvature flow can be found in [Bra78], [CGG91], [ES91], [Ilm94], [KT15]. For an introduction to generalized solutions of mean curvature flow we recommend the work by Ilmanen [Ilm94] which also points out the similarities between Brakke flow and level set flow. We also want to mention the book by Ecker [Eck04] as a good reference for smooth mean curvature flow and regularity up to the first singular time.

Results of the present article We consider Brakke flows of $\mathbf{n}$-rectifiable Radon measures in $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$ see Definition 2.3 for the details. Note that all Brakke flows considered here are assumed to be integral. All constants below may depend on $\mathbf{n}$ and $\mathbf{k}$.

Our main result is a new version of Brakke' s local regularity theorem [Bra78, 6.10, 6.11], see also Kasai and Tonegawa [KT14, 8.7]. The statement says that a non-vanishing Brakke flow which initially locally lies in a small slab and consists of less then two sheets, becomes graphical in a small neighbourhood.

1.1 Theorem. *There exists a constant $\alpha_0 \in (0,1)$ and for every $\lambda \in (0,1)$ exists a $\gamma_0 \in (0,1)$ such that the following holds: Let $\gamma \in [0,\gamma_0]$, $\rho \in (0,\infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1 + \gamma^{\alpha_0}\rho^2, t_1 + \alpha_0\rho^2]$, $a = (\hat{a},\tilde{a}) \in \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{k}}$ and let $(\mu_t)_{t\in[t_1,t_2]}$ be a Brakke flow in $\mathbf{B}(a,2\rho)$ with $a \in \operatorname{spt}\mu_{t_2}$. Suppose*

$$\operatorname{spt}\mu_{t_1} \cap \mathbf{B}(a,2\rho) \subset \{(\hat{x},\tilde{x}) \in \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{k}} : |\tilde{x} - \tilde{a}| \le \gamma\rho\}, \tag{2}$$

$$\rho^{-\mathbf{n}}\mu_{t_1}(\mathbf{B}(a,\rho)) \le (2-\lambda)\omega_{\mathbf{n}}. \tag{3}$$

Set $I := (t_1 + \gamma^{\alpha_0}\rho^2, t_2)$.

Then there exists a $g \in \mathcal{C}^\infty\left(I \times \mathbf{B}^{\mathbf{n}}(\hat{a},\gamma_0\rho), \mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(a,\gamma_0\rho,\rho) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g(t,\cdot)) \quad \textit{for all}\, t \in I.$$

Moreover g satisfies (1) *and* $\sup|Dg(t,\cdot)| \le 2\sqrt[4]{\rho^{-2}(t-t_1)}$ *for all $t \in I$.*

The main difference to the existing versions is that here we obtain regularity up to the time t_2 at which we assumed the non-vanishing, were in Brakke's theorem measure bounds from below have to be assumed further in the future. Note that Brakke's theorem includes bounds on higher derivatives of g, which we don't get.

We also obtain a local regularity theorem similar to the one of White [Whi05], see also Ecker [Eck04, 5.6]. We show that a non-vanishing Brakke flow which locally has Gaussian density ratios close to one will become graphical in a small neighbourhood.

1.2 Theorem. *For every $\beta \in (0,1)$ there exists an $\eta \in (0,1)$ such that the following holds: Let $\rho \in (0,\infty)$, $\rho_0 \in [\rho,\infty)$, $t_0 \in \mathbb{R}$, $a \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t\in[t_0-\rho^2,t_0]}$ be a Brakke flow in $\mathbf{B}(a,(2+\sqrt{2\mathbf{n}})\rho+\rho_0)$. Suppose $a \in \operatorname{spt}\mu_{t_0}$ and for all $(s,y) \in (t_0-\rho^2,t_0] \times \mathbf{B}(a,\rho)$*

$$\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)}\varphi_{(s,y),\rho_0}\,\mathrm{d}\mu_{t_0-\rho^2} \le 1+\eta, \tag{4}$$

where Φ and φ are from Definition 2.7. Set $I := (t_0 - \eta^2\rho^2, t_0)$.

Then there exist $S \in \mathbf{O}(\mathbf{n}+\mathbf{k})$ and $g \in \mathcal{C}^\infty(I \times \mathbf{B^n}(0,\eta\rho), \mathbb{R}^{\mathbf{n}+\mathbf{k}})$, such that for $M_t = \operatorname{graph}(g(t,\cdot))$ we have

$$\mu_t \llcorner \mathbf{B}(a,\eta\rho) = \mathscr{H}^{\mathbf{n}} \llcorner (S[M_t] + a \cap \mathbf{B}(a,\eta\rho)) \quad \textit{for all}\, t \in I.$$

Moreover g satisfies (1) *and* $\sup|Dg| \le \beta$.

One key ingredient to obtain these regularity results is the observation that a non-vanishing Brakke flow, which is initially graphical with small gradient, will stay graphical for some time. This is basically the non-smooth version of a theorem by Ilmanen, Neves and Schulze [INS14, 1.5].

1.3 Theorem. *There exists a constant $l_0 \in (0,1)$ such that the following holds: Let $l \in [0,l_0]$, $\rho \in (0,\infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, t_1 + l_0\rho^2]$, $a \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t\in[t_1,t_2]}$ be a Brakke flow in $\mathbf{C}(a,2\rho,2\rho)$. Assume $a \in \operatorname{spt}\mu_{t_1}$ and*

$$\operatorname{spt}\mu_{t_2} \cap \mathbf{C}(a,\rho,\rho) \neq \emptyset. \tag{5}$$

Suppose there exists an $f \in \mathcal{C}^{0,1}\left(\mathbf{B^n}(\hat{a},2\rho),\mathbb{R}^{\mathbf{k}}\right)$ with $\operatorname{lip}(f) \le l$ *and*

$$\mu_{t_1} \llcorner \mathbf{C}(a,2\rho,2\rho) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(f). \tag{6}$$

Then there exists a $g \in \mathcal{C}^\infty\left((t_1,t_2) \times \mathbf{B^n}(\hat{a},\rho),\mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(a,\rho,\rho) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g(t,\cdot)) \quad \textit{for all}\, t \in (t_1,t_2).$$

Moreover g satisfies (1) *and* $\sup|Dg(t,\cdot)| \le \sqrt[4]{l + \rho^{-2}(t-t_1)}$ *for all $t \in (t_1,t_2)$.*

1.4 Remark. In all the above results g satisfies (1), thus the results for smooth graphical mean curvature flow can be applied to obtain bounds on $|D^2g|$. See for example estimates by Ecker and Huisken [EH91, 3.1] or Wang [Wan04, 4.1]. Note that in the above results we cannot expect to obtain a graphical representation at the final time see Example 2.5.

Having absolutely continuous first variation should imply that there are no boundary points. The following theorem formalizes this idea in the case of rectifiable Radon measures that are contained in a Lipschitz graph. This generalizes Simon's constancy theorem [Sim83, 8.4.1] to Lipschitz graphs, but additionally requires unit density.

1.5 Theorem. *Let $D \subset \mathbb{R}^{\mathbf{n}}$ be open and connected with ∂D is $(\mathbf{n}-1)$-rectifiable and set $U := D \times \mathbb{R}^{\mathbf{k}}$. Consider a unit density $\mathbf{n}$-rectifiable Radon measure μ and a Lipschitz function $f : D \to \mathbb{R}^{\mathbf{k}}$ such that*

$$\emptyset \neq \operatorname{spt}\mu \cap U \subset \operatorname{graph} f \tag{7}$$

$$\mu(A) = 0 \textit{ implies } \|\delta\mu\|(A) = 0 \textit{ for all } A \subset U \tag{8}$$

Then $\mu \llcorner U = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph} f$.

Organisation and sketch of proof We start by recalling some definitions and important results in the Preliminaries 2.

Then in section 3 we show Theorem 1.5. In the proof we employ the Gauss-Green theorem by Federer [Fed69, 4.5.6] to see that the projection of $\mathrm{spt}\mu \cap U$ onto $\mathbb{R}^{\mathbf{n}}$ is stationary, subsequently the result follows from Allard's constancy theorem [All72, 4.6.(3)].

The main part of this work is section 4, where Theorem 1.3 is established. Essentially we consider a Brakke flow in $\mathbf{C}(0,2,2)$ for times in $[0,\tau]$ such that $\mathrm{spt}\mu_\tau \cap \mathbf{C}(0,\delta,1) \neq \emptyset$. First assume as initial condition that $\mu_0 \llcorner \mathbf{C}(0,2,2)$ lies in a slab of height h and satisfies certain density ratio assumptions. Based on Brakke's local regularity theorem [Bra78, 6.11] and the estimates from the appendix we show that the flow is graphical inside $\mathbf{C}(0,h,1)$ for times in $[h,\tau-h]$, if h is small enough, $\delta \le h$ and $\tau \le \sqrt[4]{h}$. Under stronger density assumptions we actually obtain graphical representability inside $\mathbf{C}(0,1,1)$ for times in $[h,\tau - C\sqrt{h}]$, see Lemma 4.2.

Now exchange the initial condition to $\mu_0 \llcorner \mathbf{C}(0,2,2)$ is graphical with Lipschitz constant smaller than l. This allows to use Lemma 4.2 on arbitrary small scales, which yields that the flow is graphical inside $\mathbf{C}(0,1,1)$ for times in $[0,\tau - C\sqrt{l}]$, if l is small enough, $\delta \le l$ and $\tau \le \sqrt[4]{l}$. Iterating this result leads to Lemma 4.4, which says that the flow is graphical inside $\mathbf{B}(0,L\delta)$ for times in $[0,\tau-\delta^2]$, if we choose l small enough depending on L and suppose $\tau \le l$, $\delta \le l$. Using Lemma 4.4 with varying center points and arbitrary small δ we perceive that $\mathrm{spt}\mu_t \cap \mathbf{C}(0,1,1)$ is contained in a Lipschitz graph and has unit density for almost all $t \in [0,\tau]$. In view of Theorem 1.5 this lets us conclude Theorem 1.3.

Section 5 contains the proof of Theorem 1.1. First we see that Theorem 1.3 and Lemma 4.2 directly imply a version of Theorem 1.1, which assumes stronger density bounds at the beginning, see Lemma 5.1. Then we use Brakke's cylindrical growth Theorem [Bra78, 6.4] to simplify these assumptions, which establishes Theorem 1.1 in the desired form.

In section 6 Theorem 1.2 is proven. In order to do so we first employ Huiskin's monotonicty formula [Hui90, 3.1] to show that non-moving planes are the only Brakke flows in $\mathbf{R}^{\mathbf{n}+\mathbf{k}}$ that have Gaussian density ratios bounded by one everywhere. Then under the assumptions of Theorem 1.2 a blow up argument and Ilmanen's compactness theorem yield that in a small neighbourhood the conditions of Theorem 1.1 are satisfied, which yields the result.

Finally in the appendix A we show how a slab condition and bounds on area ratios at the initial time are maintained in the future.

Thanks I want to thank Ulrich Menne for his help and advice in particular for the proof of Theorem 1.5.

2 Preliminaries

Notation For an excellent introduction to geometric measure theory we recommend the lecture notes by Simon [Sim83]. Here we recall the most important definitions.

- We set $\mathbb{R}^+ := \{x \in \mathbb{R}, x \geq 0\}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $(a)_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$.
- We fix $\mathbf{n}, \mathbf{k} \in \mathbb{N}$. Quantities that only depend on $\mathbf{n}$ and/or $\mathbf{k}$ are considered constant. Such a constant may be denoted by C or c, in particular the value of C and c may change in each line.
- We denote the canonical basis of $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and $\mathbb{R}^{\mathbf{n}}$ by $(\mathbf{e}_i)_{1 \leq i \leq \mathbf{n}+\mathbf{k}}$ and $(\hat{\mathbf{e}}_i)_{1 \leq i \leq \mathbf{n}}$ respectively.
- For $a \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ the projections $\hat{a} \in \mathbb{R}^{\mathbf{n}}$ and $\tilde{a} \in \mathbb{R}^{\mathbf{k}}$ are given by $a = (\hat{a}, \tilde{a})$.

Let $n, k \in \mathbb{N}$.

- Let $\mathbf{O}(n)$ denote the space of rotations on $\mathbb{R}^n$. Let $\mathbf{G}(n+k, n)$ denote the space of n-dimensional subspaces of $\mathbb{R}^{n+k}$. For $T \in \mathbf{G}(n+k, n)$ set $T^\perp := \{x \in \mathbb{R}^{n+k} : \ x \cdot v = 0 \ \forall v \in T\}$. By $T_\natural : \mathbb{R}^{n+k} \to T$ we denote the projection onto T.
- For $R, r, h \in (0, \infty)$ and $a, b \in \mathbb{R}^n$ we set
$$\mathbf{B}^n(b, R) := \{x \in \mathbb{R}^n : |x - b| < R\}, \quad \mathbf{B}(b, r) := \mathbf{B}^{\mathbf{n}+\mathbf{k}}(b, r),$$
$$\mathbf{C}(a, r, h) := \mathbf{B}^{\mathbf{n}}(\hat{a}, r) \times \mathbf{B}^{\mathbf{k}}(\tilde{a}, h), \quad \mathbf{C}(a, r) := \mathbf{B}^{\mathbf{n}}(\hat{a}, r) \times \mathbb{R}^{\mathbf{k}}.$$
- Consider open sets $I \subset \mathbb{R}$ and $V \subset \mathbb{R}^n$ and $f \in \mathcal{C}^1(I \times V)$ then $\partial_t f$ denotes the derivative of f in I, while Df denotes the derivative of f in V. If $(\mu_t)_{t \in I}$ is a family of Radon measures on V we often abbreviate $\int_V f(t, x) \, \mathrm{d}\mu_t(x) = \int_V f \, \mathrm{d}\mu_t$.
- Let $\mathscr{L}^n$ denote the n-dimensional Lebesque measure and $\mathscr{H}^n$ denote the n-dimensional Hausdorf measure. Set $\omega_n := \mathscr{L}^n(B^n(0, 1))$.

Let $U \subset \mathbb{R}^{n+k}$ open and μ be a Radon measure on U

- Set $\mathrm{spt}\mu := \{x \in U : \ \mu(\mathbf{B}^{n+k}(x, r)) > 0, \text{ for all } r \in (0, \infty)\}$.

- Consider $x \in U$. We define the upper and lower density by

$$\Theta^{*n}(\mu,x) := \limsup_{r\searrow 0} \frac{\mu(\mathbf{B}^{n+k}(x,r))}{\omega_n r^n}, \quad \Theta^n_*(\mu,x) := \liminf_{r\searrow 0} \frac{\mu(\mathbf{B}^{n+k}(x,r))}{\omega_n r^n}$$

and if both coincide the value is denoted by $\Theta^n(\mu,x)$ and called the density of μ at x.

- Consider $y \in U$. If there exist $\theta(y) \in \mathbb{N}$ and $\mathbf{T}(\mu,y)\mu \in \mathbf{G}(n+k,n)$ such that

$$\lim_{\lambda\searrow 0} \lambda^{-n} \int_U \phi(\lambda^{-1}(x-y))\,\mathrm{d}\mu(x) = \theta(y) \int_{\mathbf{T}(\mu,y)} \phi(x)\, d\mathscr{H}^n(x)$$

for all $\phi \in \mathcal{C}^0_c\left(\mathbb{R}^{n+k}\right)$, then $\mathbf{T}(\mu,y)$ is called the (n-dimensional) approximate tangent space of μ at x with multiplicity $\theta(y)$.

- We say μ is n-rectifiable, if the approximate tangent space exists at μ-a.e. $x \in U$. Note that in this case $\theta(x) = \Theta^n(\mu,x)$ for μ-a.e. $x \in U$. We say μ is integer n-rectifiable, if μ is n-rectifiable and $\Theta^n(\mu,x) \in \mathbb{N}$ for μ-a.e. $x \in U$. We say μ has unit density, if μ is n-rectifiable and $\Theta^n(\mu,x) = 1$ for μ-a.e. $x \in U$.

Let μ be an n-rectifiable Radon measure on U

- Consider $\phi \in \mathcal{C}^1(U,\mathbb{R}^{n+k})$. For $x \in U$ such that $\mathbf{T}(\mu,x)$ exists set $\mathrm{div}_\mu\phi(x) := \sum_{i=1}^n D_{b_i}(\phi(x)\cdot b_i)$, where $(b_i)_{1\le i\le n}$ is an orthonormal basis of $\mathbf{T}(\mu,x)$.

- Denote the first variation of μ in U by $\delta\mu(\phi) := \int_U \mathrm{div}_\mu\phi\, d\mu$ for $\phi \in \mathcal{C}^1_c(U,\mathbb{R}^{n+k})$. Set $\|\delta\mu\|(A) := \sup\{\partial\mu(\phi), \phi \in \mathcal{C}^1_c(A,\mathbb{R}^{n+k}), |\phi| \le 1\}$ for $A \subset U$ open.

- If there exists $\mathbf{H}_\mu : \mathrm{spt}\mu \to \mathbb{R}^{n+k}$ such that $\mathbf{H}_\mu$ is locally μ-integrable and $\delta\mu(\phi) = \int_U \mathbf{H}_\mu \cdot \phi\, \mathrm{d}\mu$ for all $\phi \in \mathcal{C}^1_c(U,\mathbb{R}^{n+k})$, then $\mathbf{H}_\mu$ is called the generalised mean curvature vector of μ in U.

Brakke flow An introduction to the Brakke flow can be found in [Bra78], [Ilm94], [KT14], [Lah14].

2.1 Definition. For a Radon measure μ on $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and a $\phi \in \mathcal{C}^1_c(\mathbb{R}^{\mathbf{n}+\mathbf{k}})$ we define the Brakke variation $\mathscr{B}(\mu,\phi)$ as follows: If $\mu \llcorner \{\phi > 0\}$ is $\mathbf{n}$-rectifiable, has generalised mean curvature vector $\mathbf{H}_\mu$ in $\{\phi > 0\}$ and $\int_{\{\phi>0\}} |\mathbf{H}_\mu|^2 d\mu < \infty$ then set

$$\mathscr{B}(\mu, \phi) := \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \left((\mathbf{T}(\mu, x)^\perp)_\natural D\phi(x) \cdot \mathbf{H}_\mu(x) - \phi(x) |\mathbf{H}_\mu(x)|^2 \right) \, \mathrm{d}\mu(x).$$

Else we set $\mathscr{B}(\mu, \phi) := -\infty$. Note that in case μ is integer $\mathbf{n}$-rectifiable, by a deep theorem of Brakke [Bra78, 5.8], we have $\mathbf{H}_\mu(x) \perp \mathbf{T}(\mu, x)$ for μ-a.e. $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Hence in this case the projection can be left out.

2.2 Remark ([Bra78, 3.4],[Ilm94, 6.6])**.** If $\phi \in \mathcal{C}^2_c(\mathbb{R}^{\mathbf{n}+\mathbf{k}})$ and $\mathscr{B}(\mu, \phi) > -\infty$ we can estimate

$$\mathscr{B}(\mu, \phi) \leq \sup |D^2\phi| \; \mu(\{\phi > 0\}) - \frac{1}{2} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} |\mathbf{H}_\mu|^2 \phi \, \mathrm{d}\mu.$$

2.3 Definition. Let $U \subset \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ be open, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty)$ and $(\mu_t)_{t\in[t_1,t_2]}$ be a family of radon measures on $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$. We call $(\mu_t)_{t\in[t_1,t_2]}$ a Brakke flow in U if $\mu_t \llcorner U$ is integer $\mathbf{n}$-rectifiable for a.e. $t \in (t_1, t_2)$ and for all $t_1 \leq s_1 < s_2 \leq t_2$ we have

$$\mu_{s_2}(\phi(s_2, \cdot)) - \mu_{s_1}(\phi(s_1, \cdot)) \leq \int_{s_1}^{s_2} \left(\mathscr{B}(\mu_t, \phi(t, \cdot)) + \mu_t(\partial_t \phi(t, \cdot)) \right) \, \mathrm{d}t \tag{9}$$

for all $\phi \in \mathcal{C}^1((s_1, s_2) \times U) \cap \mathcal{C}^0([s_1, s_2] \times U)$ with $\cup_{t\in[s_1,s_2]} \mathrm{spt}\phi(t, \cdot) \subset\subset U$.

2.4 Remark. Suppose $(\mu_t)_{t\in[t_1,t_2]}$ is a Brakke flow in U:

- For a.e. $t \in (t_1, t_2)$ we have: $\mu_t \llcorner U$ is integer $\mathbf{n}$-rectifiable, has generalised mean curvature vector $\mathbf{H}_{\mu_t}$ in U and $\int_K |\mathbf{H}_{\mu_t}|^2 \, \mathrm{d}\mu_t < \infty$ for all $K \subset\subset U$.
- For $(s_0, y_0) \in \mathbb{R} \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and $r \in (0, \infty)$ set $\nu_t(A) := r^{-\mathbf{n}} \mu_{r^2 t + s_0}(rA + y_0)$, then $(\nu_t)_{t\in[r^{-2}(t_1-s_0), r^{-2}(t_2-s_0)]}$ is a Brakke flow in $r^{-1}(U - y_0)$.

The Brakke flow allows the sudden loss of mass. In particular we have

2.5 Example. For $0 < t_0 \leq T$ and $0 < \epsilon < \rho < \infty$ consider the Brakke flow $(\mu_t)_{t\in[0,T]}$ given by $\mu_t = \mathscr{H}^{\mathbf{n}} \llcorner (\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})$ for $t \in [0, t_0)$, $\mu_{t_0} = \mathscr{H}^{\mathbf{n}} \llcorner (\mathbf{B}^{\mathbf{n}}(0, \epsilon) \times \{0\}^{\mathbf{k}})$ and $\mu_t := \emptyset$ for $t \in (t_0, T]$. Note that μ_t is graphical with Lipschitz constant zero for $t \in [0, t_0)$ and $0 \in \mathrm{spt}\mu_{t_0}$ but $\mu_{t_0} \llcorner \mathbf{B}(0, \rho)$ is not graphical.

Important results Here we recall some important results that are crucial for the proofs in this article.

2.6 Lemma (Measure bound [Bra78, 3.7],[Eck04, 4.9])**.** *Let $R \in (0,\infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1,\infty)$, $z_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t\in[t_1,t_2]}$ be a Brakke flow in $\mathbf{B}(z_0, 2R)$.*

Then for all $t \in [t_1, t_1 + (2\mathbf{n})^{-1}R^2] \cap [t_1, t_2]$

$$\mu_t\left(\mathbf{B}(z_0,R)\right) \le 8\mu_{t_1}\left(\mathbf{B}(z_0,2R)\right).$$

2.7 Definition. Let $x_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$, $t_0 \in \mathbb{R}$, $\rho \in (0,\infty)$ be fixed. For $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and $t \in (-\infty, t_0)$ set

$$\Phi_{(t_0,x_0)}(t,x) := (4\pi(t_0-t))^{-\frac{\mathbf{n}}{2}} \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right).$$

$$\varphi_{(t_0,x_0),\rho}(t,x) := \left\{1-\rho^{-2}\left(|x-x_0|^2+2\mathbf{n}(t-t_0)\right)\right\}_+^3.$$

2.8 Theorem (Monotonicity formula [Hui90, 3.1],[Ilm95, 7][Eck04, 4.8])**.** *Consider $U \subset \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ open, $\rho, D \in (0,\infty)$, $(t_0,x_0) \in \mathbb{R}\times U$, $s_1 \in (-\infty,t_0)$ and $s_2 \in (s_1,t_0)$ and let $(\mu_t)_{t\in[s_1,s_2]}$ be a Brakke flow in U. Assume one of the following holds*

1. $\mathrm{spt}\varphi_{(t_0,x_0),\rho}(s_1,\cdot) \subset\subset U$.
2. $U = \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ *and* $\sup_{t\in[s_1,s_2]}\sup_{R\in(0,\infty)}\mu_t(\mathbf{B}(x_0,R)) \le DR^{\mathbf{n}}$.

Then

$$\int_U \Phi\varphi(s_2,x)\,\mathrm{d}\mu_{s_2}(x) - \int_U \Phi\varphi(s_1,x)\,\mathrm{d}\mu_{s_1}(x)$$
$$\le \int_{s_1}^{s_2}\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}}\left(\Phi\varphi(t,x)\left|\mathbf{H}_{\mu_t}(x)+\frac{(\mathbf{T}(\mu_t,x)^\perp)_\natural(x-x_0)}{2(t_0-t)}\right|^2\right)\mathrm{d}\mu_t(x)\,\mathrm{d}t,$$

where $\Phi = \Phi_{(t_0,x_0)}$, $\varphi = \varphi_{(t_0,x_0),\rho}$ if assumption 1 holds and $\varphi \equiv 1$ if assumption 2 holds. Here the term under the time integral is interpreted as $-\infty$ at times where one of the technical conditions fails, as in Definition 2.1.

2.9 Lemma (Clearing out [Bra78, 6.3])**.** *There exist constants $C \in (1,\infty)$ and $\alpha_1 := (\mathbf{n}+6)^{-1}$ such that the following holds: Let $\eta \in [0,\infty)$, $R \in (0,\infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1 + C\eta^{2\alpha_1}R^2, t_1 + (4\mathbf{n})^{-1}R^2)$, $x_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Let $U \subset \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ be open with $U \supset\supset \mathbf{B}(x_0,R)$ and let $(\mu_t)_{t\in[t_1,t_2]}$ be a Brakke flow in $\mathbf{B}(x_0,R)$. Suppose*

$$R^{-\mathbf{n}}\int_U (\{1-R^{-2}|x-x_0|^2\}_+)^3\,\mathrm{d}\mu_{t_1} \le \eta.$$

Set $R(t) := \sqrt{R^2 - 4\mathbf{n}(t - t_0)}$.

Then for all $t \in [t_1 + C\eta^{2\alpha_1} R^2, t_2]$

$$\mu_t(\mathbf{B}(x_0, R(t))) = 0.$$

2.10 Theorem (Local regularity [Bra78, 6.11],[KT14, 8.7],[Lah14, 9.2])**.** *For every* $\lambda \in (0,1]$ *there exist* $\Lambda \in (1,\infty)$ *and* $h_0 \in (0,1)$ *such that the following holds: Let* $K_0 \in [1,\infty)$*,* $h \in (0, K_0^{-1} h_0]$*,* $R \in (0,\infty)$*,* $t_1 \in \mathbb{R}$*,* $t_2 \in (t_1 + 2\Lambda R^2, \infty)$ $x_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$*, and let* $(\mu_t)_{t\in[t_1,t_2]}$ *be a Brakke flow in* $\mathbf{B}(x_0, 4R)$*. Suppose*

$$\operatorname{spt}\mu_t \cap \mathbf{B}(x_0, 4R) \subset \mathbf{C}(x_0, 4R, hR) \tag{10}$$

$$R^{-\mathbf{n}} \mu_t\left(\mathbf{B}(x_0, 4R)\right) \le K_0^2 \tag{11}$$

for all $t \in [t_1, t_2]$ *and*

$$R^{-\mathbf{n}} \mu_{t_1}\left(\mathbf{B}(x_0, (1+\lambda)R)\right) \le (2-\lambda)\omega_{\mathbf{n}} \tag{12}$$

$$R^{-\mathbf{n}} \mu_{t_2}\left(\mathbf{B}(x_0, R)\right) \ge \lambda\omega_{\mathbf{n}}. \tag{13}$$

Set $I := (t_1 + \Lambda R^2, t_2 - \Lambda R^2)$.

Then there exists a $g \in \mathcal{C}^\infty(I \times \mathbf{B}^{\mathbf{n}}(\hat{x}_0, h_0 R), \mathbb{R}^{\mathbf{k}})$ *such that*

$$\mu_t \llcorner \mathbf{C}(x_0, h_0 R, R) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g(t,\cdot))$$

for all $t \in I$*. Moreover* g *satisfies* (1) *and* $\sup|Dg| + R\sup|D^2 g| \le \Lambda K_0 h$*.*

To deduce this result from [Bra78, 6.11], [KT14, 8.7] or [Lah14, 9.2] you also need to use [Bra78, 6.6], [KT14, 5.7] or [Lah14, 7.6] to see that the density ratio bounds (12) and (13) actually hold at all times. Note that Brakke as well as Kasai and Tonegawa state this theorem for unit density Brakke flows, though their proofs only use integer density. For (1), smoothness and curvature estimate of g see [Ton15, 3.6].

2.11 Theorem (Compactness [Ilm94, 7.1])**.** *Let* $t_1 \in \mathbb{R}$ *and* $t_2 \in (t_1, \infty)$*. For* $i \in \mathbb{N}$ *consider an open set* $U_i \subset \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ *and a Brakke flow* $(\mu_t^i)_{t\in[t_1,t_2]}$ *in* U_i*. Assume* $U_i \subset U_{i+1}$ *for all* $i \in \mathbb{N}$ *and set* $U := \bigcup_{i=1}^\infty U_i$*. Suppose for every* $K \subset\subset U$ *there exists an* C_K *such that*

$$\sup_{i\in\mathbb{N}} \sup_{t\in[t_1,t_2]} \mu_t^i(K \cap U_i) \le C_K.$$

Then there exists a subsequence $\sigma : \mathbb{N} \to \mathbb{N}$ *and a Brakke flow* $(\mu_t)_{t\in[t_1,t_2]}$ *in* U *such that*

$$\mu_t(\phi) = \lim_{\substack{j\to\infty \\ j\ge j_0}} \mu_t^j(\phi) \quad \textit{for all } \phi \in \mathcal{C}_c^0(U_{\sigma(j_0)})$$

for all $t \in [t_1, t_2]$ *and all* $j_0 \in \mathbb{N}$*.*

Actually in [Ilm94] Ilmanen assumes $U_i \equiv M$, for a complete manifold M. To derive the above result from [Ilm94, 7.1] use a diagonal subsequence argument, see Remark A.5 for some more details.

2.12 Lemma (Tilt-bound [Bra78, 5.5])**.** *There exists a constant $C \in (0,\infty)$ such that the following holds: Let $U \subset \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ open and let μ be a integer $\mathbf{n}$-rectifiable Radon measure on U with $\mathcal{L}^2$-integrable mean curvature vector $\mathbf{H}_\mu$. Consider $g \in \mathcal{C}^1_c(U,\mathbb{R})$, $f,h \in \mathcal{C}^0_c(U,\mathbb{R})$ with $g^2 \le fh$.*

Then we have

$$\beta_g^2 \le C\left(\alpha_f\gamma_h + \xi_g^2\right),$$

where

$$\begin{aligned}
\alpha_f^2 &:= \int_U |\mathbf{H}(\mu,x)|^2 f(x)^2 \,\mathrm{d}\mu(x),\\
\beta_g^2 &:= \int_U \left\|(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_\natural - \mathbf{T}(\mu,x)_\natural\right\|^2 g(x)^2 \,\mathrm{d}\mu(x),\\
\gamma_h^2 &:= \int_U |\tilde{x}|^2 h(x)^2 \,\mathrm{d}\mu(x),\\
\xi_g^2 &:= \int_U |\tilde{x}|^2 |\nabla^\mu g(x)|^2 \,\mathrm{d}\mu(x).
\end{aligned}$$

2.13 Theorem (Cylindrical growth [Bra78, 6.4])**.** *Let $U \subset \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ open, $R_1 \in (0,\infty)$, $R_2 \in (R_1,\infty)$, $\alpha,\beta \in [0,\infty)$. Let μ be an integer $\mathbf{n}$-rectifiable Radon measure on U with $\mathcal{L}^2$-integrable mean curvature vector $\vec{H}_\mu$ and* $\operatorname{spt}\mu \cap \mathbf{C}(x_0,R_2) \subset\subset U$. *Consider $\psi \in \mathcal{C}^3_c([-1,1],\mathbb{R}^+)$. Suppose for all $r \in [R_1,R_2]$*

$$r^{-\mathbf{n}} \int_U |\mathbf{H}_{\mu_{s_1}}(x)|^2 \psi(r^{-1}|\hat{x}|) \,\mathrm{d}\mu(x) \le \alpha^2, \tag{14}$$

$$r^{-\mathbf{n}} \int_U \left\|(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_\natural - \mathbf{T}(\mu,x)_\natural\right\|^2 \psi(r^{-1}|\hat{x}|) \,\mathrm{d}\mu(x) \le \beta^2. \tag{15}$$

Then we have

$$\begin{aligned}
&\left| R_2^{-\mathbf{n}} \int_U \psi(R_2^{-1}|\hat{x}|) \,\mathrm{d}\mu(x) - R_1^{-\mathbf{n}} \int_U \psi(R_1^{-1}|\hat{x}|) \,\mathrm{d}\mu(x) \right| \\
&\qquad\qquad \le (\mathbf{n}\log(R_2/R_1) + \alpha(R_2-R_1) + \beta)\beta.
\end{aligned}$$

3 Graphs without holes

In this section we prove Theorem 1.5. Consider a unit density Radon measure μ such that the first variation $\delta\mu$ is absolutely continuous with respect to μ.

In some sense this should imply that μ has no 'boundary points'. Here we show that, if such a μ is contained in the graph of some Lipschitz function f, then μ actually coincides with the measure generated by the graph of f. For $f \in \mathcal{C}^2$ and stationary μ this is a direct consequence of the Allrd's constancy theorem [All72, 4.6.(3)] (see also Simon's notes [Sim83, 8.4.1]). Here we use the Gauss-Green theorem by Federer [Fed69, 4.5.6] to show that the projection of μ onto $\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}$ is stationary, which reduces our problem to the $\mathcal{C}^2$-setting.

3.1 Definition. Let μ be an n-rectifiable Radon measure on $\mathbb{R}^{n+k}$. We denote the associated general varifold by $\mathbf{V}(\mu)$, i.e. $\mathbf{V}(\mu)$ is the a Radon measure on $\mathbb{R}^{n+k} \times \mathbf{G}(n+k, n)$ given by

$$\mathbf{V}(\mu)(A) := \mu(\{x \in \mathbb{R}^{n+k} : (x, \mathbf{T}(\mu, x)) \in A\}).$$

For $y \in \mathbb{R}^{n+k}$ and $\lambda \in (0, \infty)$ we define the λ-blow-up around y by

$$\mu_{y,\lambda}(A) := \lambda^{-n}\mu(\lambda A + y).$$

for $A \subset \mathbb{R}^{n+k}$.

Proof of Theorem 1.5. This proof is based on ideas by Ulrich Menne. Set

$$\begin{aligned} &U_1 := \{x \in U : \ \Theta^{\mathbf{n}-1}(\|\delta\mu\|, x) = 0\}, \\ &Q_1 := \{x \in U : \ \Theta_*^{\mathbf{n}}(\mu, x) \geq 1\}, \quad Q_2 := Q_1 \cap U_1, \\ &R_1 := \{x \in U : \ \Theta^{\mathbf{n}}(\mu, x) = 0\}, \quad R_2 := R_1 \cap U_1. \end{aligned}$$

We claim

$$\mathscr{H}^{\mathbf{n}-1}(U \setminus (Q_2 \cup R_2)) = 0. \tag{16}$$

Note that by (7) we have

$$\Theta^{*\mathbf{n}}(\mu, x) \leq \operatorname{lip}(f) < \infty \text{ for all } x \in U. \tag{17}$$

Using a result by Menne [Men09, 2.11] we see $\mathscr{H}^{\mathbf{n}-1}(U \setminus (Q_1 \cup R_1)) = 0$. Hence, to establish the claim it remains to show

$$\mathscr{H}^{\mathbf{n}-1}(U \setminus U_1) = 0. \tag{18}$$

We proceed as Federer and Ziemer [FZ72, 8]. For $i \in \mathbb{N}$ set

$$B_i = \{x \in U \cap \mathbf{B}(0, i) : \Theta^{*\mathbf{n}-1}(\|\delta\mu\|, x) > i^{-1}\}.$$

Then by [Fed69, 2.10.19(3)] we have $i\|\partial\mu\| \geq \mathscr{H}^{\mathbf{n}-1}(B_i)$ for all $i \in \mathbb{N}$. This leads to the following chain of implications: B_i bounded, $\|\partial\mu\|(B_i) < \infty$, $\mathscr{H}^{\mathbf{n}-1}(B_i) < \infty$, $\mathscr{H}^{\mathbf{n}}(B_i) = 0$, $\|\partial\mu\|(B_i) = 0$, $\mathscr{H}^{\mathbf{n}-1}(B_i) = 0$. This shows (18) which completes the proof of (16).

Now set

$$\begin{aligned}
&A_0 := (\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_\natural(\mathrm{spt}\mu \cap U),\\
&Q_0 := \{\hat{x} \in \mathbb{R}^{\mathbf{n}} :\ \Theta^{\mathbf{n}}(\mathscr{L}^{\mathbf{n}} \llcorner (\mathbb{R}^{\mathbf{n}} \setminus A_0), \hat{x}) = 0\},\\
&R_0 := \{\hat{x} \in \mathbb{R}^{\mathbf{n}} :\ \Theta^{\mathbf{n}}(\mathscr{L}^{\mathbf{n}} \llcorner A_0, \hat{x}) = 0\}.
\end{aligned}$$

We want to use

$$(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_\natural Q_2 \subset Q_0 \text{ and } (\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_\natural R_2 \subset R_0. \tag{19}$$

We will prove this statement later. Suppose (19) holds, then (16) yields

$$\mathscr{H}^{\mathbf{n}-1}(D \setminus (Q_0 \cup R_0)) = 0. \tag{20}$$

We say $\hat{v} \in \partial\mathbf{B}^{\mathbf{n}}(0,1)$ is an external normal of A_0 at $\hat{y} \in \mathbb{R}^{\mathbf{n}}$, if

$$\begin{gathered}
\Theta^{\mathbf{n}}(\mathscr{L}^{\mathbf{n}} \llcorner \{\hat{x} \in \mathbb{R}^{\mathbf{n}} : (\hat{x} - \hat{y}) \cdot \hat{v} > 0\} \cap A_0, \hat{y}) = 0\\
\text{and } \Theta^{\mathbf{n}}(\mathscr{L}^{\mathbf{n}} \llcorner \{\hat{x} \in \mathbb{R}^{\mathbf{n}} : (\hat{x} - \hat{y}) \cdot \hat{v} < 0\} \setminus A_0, \hat{y}) = 0,
\end{gathered}$$

Let B_0 be the set consisting of all $\hat{y} \in \mathbb{R}^{\mathbf{n}}$ for which there exists an external normal of A_0 at $\hat{y}$. Then we have

$$B_0 \subset \mathbb{R}^{\mathbf{n}} \setminus (Q_0 \cup R_0). \tag{21}$$

To see this consider $\hat{y} \in Q_0$ and $\hat{v} \in \partial\mathbf{B}^{\mathbf{n}}(0,1)$. We can estimate

$$\begin{aligned}
&\mathscr{L}^{\mathbf{n}}(\{\hat{x} \in \mathbb{R}^{\mathbf{n}} : (\hat{x} - \hat{y}) \cdot \hat{v} > 0\} \cap A_0 \cap \mathbf{B}^{\mathbf{n}}(\hat{y}, r))\\
&\geq \mathscr{L}^{\mathbf{n}}(\{\hat{x} \in \mathbb{R}^{\mathbf{n}} : (\hat{x} - \hat{y}) \cdot \hat{v} > 0\} \cap \mathbf{B}^{\mathbf{n}}(\hat{y}, r)) - \mathscr{L}^{\mathbf{n}}((\mathbb{R}^{\mathbf{n}} \setminus A_0) \cap \mathbf{B}^{\mathbf{n}}(\hat{y}, r))\\
&\geq (2^{-1}\omega_{\mathbf{n}} - \epsilon) r^{\mathbf{n}}
\end{aligned}$$

for r small enough depending on ϵ. This yields $Q_0 \subset \mathbb{R}^{\mathbf{n}} \setminus B_0$. Similarly we can show $R_0 \subset \mathbb{R}^{\mathbf{n}} \setminus B_0$, which proves (21).

Let $K \subset \mathbb{R}^{\mathbf{n}}$ be compact. Using (20) we obtain

$$\mathscr{H}^{\mathbf{n}-1}(K \setminus (Q_0 \cup R_0)) \leq \mathscr{H}^{\mathbf{n}-1}((K \setminus D) \setminus R_0) \quad \leq \mathscr{H}^{\mathbf{n}-1}(\partial D \cap K) < \infty.$$

In view of [Fed69, 4.5.11] and [Fed69, 2.10.6] we can now use the general Gauss-Green theorem [Fed69, 4.5.6]. Combined with (20) and (21) this establishes

$$\int_{A_0} \mathrm{div}_{\mathbb{R}^{\mathbf{n}}}\phi \, \mathrm{d}\mathscr{L}^{\mathbf{n}} \leq \mathscr{H}^{\mathbf{n}-1}(D \cap B_0) \leq \mathscr{H}^{\mathbf{n}-1}(D \setminus (R_0 \cup Q_0)) = 0$$

for all $\phi \in \mathcal{C}^1_c(D, \mathbb{R}^{\mathbf{n}})$. Thus A_0 is stationary in D. Then the constancy theorem (see [Sim83, 8.4.1]) yields $A_0 = D$ which establishes the result. Hence it remains to prove (19).

We want to show $(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_\natural R_2 \subset R_0$. Consider $y \in R_2$. By (7) and as μ is integral we can estimate for $r \in (0, \infty)$

$$\begin{aligned} r^{-\mathbf{n}} \mathscr{L}^{\mathbf{n}}(A_0 \cap \mathbf{B}^{\mathbf{n}}(\hat{y}, r)) &= r^{-\mathbf{n}} \int_{\mathrm{graph}(f) \cap (A_0 \times \mathbb{R}^{\mathbf{k}}) \cap \mathbf{C}(y,r)} |Jf|^{-1} \, \mathrm{d}\mathscr{H}^{\mathbf{n}} \\ &\leq r^{-\mathbf{n}} \mu(\mathbf{B}(y, (1 + \mathrm{lip}(f))r)) \end{aligned}$$

and as $y \in R_2$ this goes to 0 for $r \searrow 0$. Thus $\hat{y} \in R_0$.

It remains to show $(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_\natural Q_2 \subset Q_0$. Suppose this is false, then there exists a $y_0 = (\hat{y}_0, \tilde{y}_0) \in Q_2$, an $\epsilon \in (0, 1)$ and a sequence $(r_m)_{m \in \mathbb{N}}$ with $r_m \searrow 0$ such that

$$r_m^{-\mathbf{n}} \mathscr{L}^{\mathbf{n}}(\mathbf{B}^{\mathbf{n}}(\hat{y}_0, r_m) \setminus A_0) > 2\epsilon \tag{22}$$

for all $m \in \mathbb{N}$. Consider the sequence $(\mu_m)_{m \in \mathbb{N}}$ given by $\mu_m = \mu_{y_0, r_m}$. By (7), unit density and as $y_0 \in U_1$ we have

$$\begin{aligned} \limsup_{m \to \infty} \mu_m(\mathbf{B}(0, R)) &= \limsup_{m \to \infty} r_m^{-\mathbf{n}} \mu(\mathbf{B}(y_0, R r_m)) \leq (1 + C \,\mathrm{lip} f) R^{\mathbf{n}}, \\ \limsup_{m \to \infty} \|\delta\mu_m\|(\mathbf{B}(0, R)) &= \lim_{m \to \infty} r_m^{-\mathbf{n}-1} \|\delta\mu\|(\mathbf{B}(y_0, R r_m)) = 0 \end{aligned}$$

for every $R \in (0, \infty)$. By varifold compactness (see [Sim83, 8.5.5] or [All72, 6.4]) there exists a stationary integer $\mathbf{n}$-rectifiable Radon measure ν such that for a subsequence we have

$$\mathbf{V}(\mu_m) \rightharpoonup \mathbf{V}(\nu) \ \text{ as radon measures on } \mathbb{R}^{\mathbf{n}+\mathbf{k}} \times \mathbf{G}(\mathbf{n} + \mathbf{k}, \mathbf{n}) \tag{23}$$

Moreover as $y_0 \in Q_1$ we have $y_0 \in \mathrm{spt}\mu$, then $0 \in \mathrm{spt}\nu$. Define $f_m \in \mathscr{C}^{0,1}(\mathbf{B}^{\mathbf{n}}(0, 1), \mathbb{R}^{\mathbf{k}})$ by $f_m(\hat{x}) := r_m^{-1} f(r_m(\hat{x} - \hat{y}_0))$. By the Arzela-Ascoli theorem exists a $g \in \mathscr{C}^{0,1}(\mathbf{B}^{\mathbf{n}}(0, 1), \mathbb{R}^{\mathbf{k}})$ such that for a subsequence $\|f_m - g\|_{C^0} \to 0$. We claim

$$\mathrm{spt}\nu \cap \mathbf{C}(0, 1) = \mathrm{graph}(g). \tag{24}$$

Suppose there exists a $z \in (\mathrm{spt}\nu \cap \mathbf{C}(0, 1)) \setminus \mathrm{graph}(g)$. Then we find $\rho \in (0, 1)$ with $\mathbf{B}(z, 4\rho) \cap \mathrm{graph}(g) = \emptyset$ and $\nu(\mathbf{B}(z, \rho)) > 0$. Thus for some large enough $m \in \mathbb{N}$ we have $\mathbf{B}(z, 3\rho) \cap \mathrm{graph}(f_m) = \emptyset$ and $\mu_m(\mathbf{B}(z, 2\rho)) > 0$. But by definition of f_m and μ_m combined with (7) we also have $\mathrm{spt}\mu_m \subset \mathrm{graph}(f_m)$, which yields a contradiction. Thus $\subset$ holds in (24)

Now suppose there exists an $z \in \operatorname{graph}(g) \setminus (\operatorname{spt}\nu \cap \mathbf{C}(0,1))$. As $\operatorname{spt}\nu$ is closed, we can find $\rho \in (0,\infty)$ and $z_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ such that

$$z_0 \in \partial\mathbf{C}(z,\rho) \cap \operatorname{spt}\nu \ \text{ and } \ \mathbf{C}(z,\rho) \cap \operatorname{spt}\nu = \emptyset. \tag{25}$$

Consider the sequence $(\nu_l)_{l\in\mathbb{N}}$ given by $\nu_l = \nu_{z_0,r_l}$. As above, by [Sim83, 8.5.5] there exists a stationary integer $\mathbf{n}$-rectifiable Radon measure ν_0 with $0 \in \operatorname{spt}\nu_0$ and such that for a subsequence we have

$$\mathbf{V}(\nu_l) \rightharpoonup \mathbf{V}(\nu_0) \ \text{ as radon measures on } \mathbb{R}^{\mathbf{n}+\mathbf{k}} \times \mathbf{G}(\mathbf{n}+\mathbf{k},\mathbf{n}) \tag{26}$$

Similar as above we also see

$$\operatorname{spt}\nu_0 \cap \mathbf{C}(0,1) \subset \operatorname{graph}(h) \tag{27}$$

for some $h \in \mathscr{C}^{0,1}(\mathbf{B}^{\mathbf{n}}(0,1),\mathbb{R}^{\mathbf{k}})$. Combining (25) and (26) we see $\operatorname{spt}\nu_0 \subset \{x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : \hat{x}\cdot(\hat{z}-\hat{z}_0) \leq 0\}$. As ν_0 is stationary this implies $\operatorname{spt}\nu_0 \subset \{x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : \hat{x}\cdot(\hat{z}-\hat{z}_0) = 0\}$. But in view of (27) this yields $\mathscr{H}^{\mathbf{n}}(\operatorname{spt}\nu_0) = 0$, hence $\operatorname{spt}\nu_0 = \emptyset$, which contradicts $0 \in \operatorname{spt}\nu_0$. This proves (24).

We continue to lead (22) to a contradiction. Using (7) and the unit density of μ we can estimate

$$\begin{aligned} L_m :=& \mathscr{L}^{\mathbf{n}}(A_0 \cap \mathbf{B}^{\mathbf{n}}(\hat{y}_0,r_m)) = \int_{\operatorname{graph}(f)\cap(A_0\times\mathbb{R}^{\mathbf{k}}\cap\mathbf{C}(y_0,r_m))} |Jf|^{-1}\,\mathrm{d}\mathscr{H}^{\mathbf{n}} \\ &\geq \int_{\mathbf{C}(y_0,r_m)} |\Lambda_{\mathbf{n}}\mathbf{T}(\mu,x)_\natural|^{-1}\,\mathrm{d}\mu(x) = r_m^{\mathbf{n}} \int_{\mathbf{C}(y_0,1)} |\Lambda_{\mathbf{n}}S_\natural|^{-1}\,\mathrm{d}\mathbf{V}(\mu_m)(x,S). \end{aligned}$$

Recall ϵ from (22). In view of (23) and (24) we obtain

$$r_m^{-\mathbf{n}}L_m + \epsilon \geq \int_{\mathbf{C}(0,1)} |\Lambda_{\mathbf{n}}S_\natural|^{-1}\,\mathrm{d}\mathbf{V}(\nu)(x,S) \geq \int_{\operatorname{graph}(g)\cap\mathbf{C}(0,1)} |Jg|^{-1}\,\mathrm{d}\mathscr{H}^{\mathbf{n}} = \omega_{\mathbf{n}}$$

for m large enough. Thus we see

$$r_m^{-\mathbf{n}}\mathscr{L}^{\mathbf{n}}(\mathbf{B}^{\mathbf{n}}(\hat{y}_0,r_m)\setminus A_0) = \omega_{\mathbf{n}} - r_m^{-\mathbf{n}}\mathscr{L}^{\mathbf{n}}(A_0\cap\mathbf{B}^{\mathbf{n}}(\hat{y}_0,r_m)) \leq \epsilon,$$

which contradicts (22).

This completes the proof of (19), which establishes the result. $\square$

4 Maintain graphical representability

In this section we prove Theorem 1.3. The main idea of the proof is to iterate Brakkes local regularity theorem (see Theorem 2.10) by choosing a time at which graphical representation is obtained as the new starting time. To do so we first show a version of Theorem 2.10 which only has assumptions at the initial and final time, see Proposition 4.1.

By Corollary A.2 initial height bounds yield weaker height bounds later on. Also by Huisken's monotonicity formula, Theorem 2.8 initial bounds on area ratio imply bounds on area ratio in the future (see Lemma A.3). Moreover by the clearing out lemma, Lemma 2.9 non-vanishing at some time yields a lower bound on measure a bit earlier. Thus with Brakke's local regularity theorem, Theorem 2.10 we obtain the Proposition below, which is an improved version of a result found in the author's thesis [Lah14, 11.7].

4.1 Proposition. *For every $\kappa \in (0,1)$ and $q_1, q_2 \in \mathbb{N}$ with $q_2 > \kappa q_1$ exist $\Sigma_1 \in (1,\infty)$ and $\sigma_1 \in (0, 2^{-2})$ such that the following holds: Let $\sigma \in (0,\sigma_1]$, $\rho \in (0,\infty)$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1 + 2\sigma^{2q_1}\rho^2, s_1 + 4\sigma\rho^2]$, $z_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t\in[s_1,s_2]}$ be a Brakke flow in $\mathbf{C}(z_0, 2\rho, 2\rho)$. Suppose*

$$\operatorname{spt}\mu_{s_2} \cap \mathbf{C}(z_0, \sigma_1\sigma^{q_1}\rho, \rho) \neq \emptyset, \tag{28}$$

$$\operatorname{spt}\mu_{s_1} \cap \mathbf{C}(z_0, 2\rho, 2\rho) \subset \mathbf{C}(z_0, 2\rho, \sigma^{q_1+q_2}\rho), \tag{29}$$

$$r^{-\mathbf{n}}\mu_{s_1}(\mathbf{B}(z_0, r)) \leq (2-\kappa)\omega_{\mathbf{n}} \quad \textit{for all} \quad r \in (\sigma_1\sigma^{q_1}\rho, \Sigma_1\sqrt{\sigma}\rho). \tag{30}$$

Set $I := (s_1 + \sigma^{2q_1}\rho^2, s_2 - \sigma^{2q_1}\rho^2)$.

Then there exists a $g \in \mathcal{C}^\infty\left(I \times \mathbf{B}^{\mathbf{n}}(\hat{z}_0, \sigma_1\sigma^{q_1}\rho), \mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(z_0, \sigma_1\sigma^{q_1}\rho, \rho) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g(t,\cdot)) \quad \textit{for all}\, t \in I.$$

Moreover g satisfies (1) *and* $\sup|Dg| + \sigma^{2q_1}\rho\sup|D^2g| \leq \Sigma_1\sigma^{q_2-\kappa q_1}$.

Proof. We may assume $s_1 = 0$, $z_0 = 0$ and $\rho = 1$. By Corollary A.2 with $p = q_1 + q_2$ and assumption (29) we have

$$\operatorname{spt}\mu_t \cap \mathbf{C}(0,1,1) \subset \mathbf{C}(0,1,2\sigma^{q_1+q_2}) \tag{31}$$

for all $t \in [0, s_2]$, where we estimated $s_2 \leq \sigma_1 \leq c_p$.

Choose $\lambda_1 \in (0, 2^{-3})$ depending on κ such that $(2-\kappa)(1+\lambda_1)^{\mathbf{n}} \leq 2 - \lambda_1$ and $C_2\lambda_1^{\frac{2}{\mathbf{n}+6}} \leq (16\mathbf{n})^{-1}$ where C_2 is the constant from Lemma 2.9. Let $\Lambda \in (1,\infty)$ be from Brakke's local regularity theorem, Theorem 2.10, chosen with respect to $\lambda = \lambda_1$. Consider the radius

$$\rho_1 := 4^{-1}\Lambda^{-\frac{1}{2}}\sigma^{q_1} \in (0, \sigma^{q_1}).$$

Set $t_2 := s_2 - (8\mathbf{n})^{-1}\rho_1^2$. We want to show

$$\mu_{t_2}(\mathbf{B}(0,\rho_1)) \geq \lambda_1\rho_1^{\mathbf{n}}. \tag{32}$$

Suppose this would be false, then we can use Lemma 2.9 with $\eta = \lambda_1$ to obtain $\mu_{s_2}(\mathbf{B}(0, 2^{-1}\rho_1)) = 0$. In view of (31) this contradicts (28), where we chose σ_1 small enough. Thus (32) has to be true.

Consider $\epsilon = \kappa$ and choose the corresponding δ according to Lemma A.3. We may assume $\delta^2 \leq (8\mathbf{n})^{-1}$. We want to use Lemma A.3 with $R = \delta^{-1}\sqrt{\sigma}$ and $r = 4\rho_1$. Note that $r \leq \sigma^{q_1} \leq \delta R$ and $s_2 \leq \sigma \leq (8\mathbf{n})^{-1}R^2$. For σ_1 small and Σ_1 large enough we have $R \leq 1$ and assumption (30) implies (90) with $K = 2\omega_{\mathbf{n}}$. Lemma A.3 then yields

$$(4\rho_1)^{-\mathbf{n}}\mu_t(\mathbf{B}(0, 4\rho_1)) \leq 2\omega_{\mathbf{n}}\rho_1^{-\kappa} \leq C_1\sqrt{\Lambda}\sigma^{-\kappa q_1} \tag{33}$$

for all $t \in [0, s_2]$ and some constant $C_1 \in (1, \infty)$.

For σ_1 small enough we have $\sigma_1\sigma^{q_1} < \rho_1 < \sqrt{\sigma}$. Then by assumption (30) and choice of λ_1 we can estimate

$$\rho_1^{-\mathbf{n}}\mu_0(\mathbf{B}(0, (1 + \lambda_1)\rho_1)) \leq (2 - \kappa)(1 + \lambda_1)^{\mathbf{n}}\omega_{\mathbf{n}} \leq (2 - \lambda_1)\omega_{\mathbf{n}} \tag{34}$$

Now choose h_0 according to Brakke's local regularity theorem, Theorem 2.10 with respect to $\lambda = \lambda_1$ as above. Set $h := 8\Lambda^{\frac{1}{2}}\sigma^{q_2}$. Note that $2\sigma^{q_1+q_2} \leq h\rho_1$ and for σ_1 small enough we have $(C_1\sqrt{\Lambda}\sigma^{-\kappa q_1})^{-\frac{1}{2}}h \leq C\Lambda\sigma^{q_2-\kappa q_1/2} \leq h_0$. Thus (31), (32), (33) and (34) let us apply Theorem 2.10 with $\rho = \rho_1$ which establishes the result. □

Looking at Proposition 4.1 we see that at time $t_2 - \sigma^4\rho_0^2$ we satisfy a non-vanishing condition in an increased cylinder. This allows to iterate above Proposition to obtain graphical representability inside a larger cylinder.

4.2 Lemma. *For every $\kappa \in (0, 1)$ exist $\Lambda_2 \in (1, \infty)$ and $\lambda_2 \in (0, (4\Lambda_2)^{-1})$ such that the following holds: Let $\lambda \in (0, \lambda_2]$, $\varrho_0 \in (0, \infty)$, $s_3 \in \mathbb{R}$, $s_4 \in (s_3 + 2\Lambda_2\lambda^2\varrho_0^2, s_3 + \lambda\varrho_0^2]$, $y_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t\in[s_3,s_4]}$ be a Brakke flow in $\mathbf{C}(y_0, 4\varrho_0, 2\varrho_0)$. Suppose*

$$\operatorname{spt}\mu_{s_4} \cap \mathbf{C}(y_0, \lambda_2\lambda^2\varrho_0, \varrho_0) \neq \emptyset, \tag{35}$$

$$\operatorname{spt}\mu_{s_3} \cap \mathbf{C}(y_0, 4\varrho_0, 2\varrho_0) \subset \mathbf{C}(y_0, 4\varrho_0, \lambda^4\varrho_0), \tag{36}$$

$$r^{-\mathbf{n}}\mu_{s_3}(\mathbf{B}(y, r)) \leq (2 - \kappa)\omega_{\mathbf{n}} \tag{37}$$

for all $y \in \mathbf{B}^{\mathbf{n}}(\hat{y}_0, 3\varrho_0) \times \{\tilde{y}_0\}$ and $r \in (\lambda_2\lambda^2\varrho_0, \Lambda_2\sqrt{\lambda}\varrho_0)$. Set $I := (s_3 + \lambda^4\varrho_0^2, s_4 - \Lambda_2\lambda^2\varrho_0^2)$.

Then there exists a $g \in \mathcal{C}^\infty\left(I \times \mathbf{B}^{\mathbf{n}}(\hat{y}_0, 2\varrho_0), \mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(y_0, 2\varrho_0, \varrho_0) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g(t, \cdot)) \quad \textit{for all}\, t \in I.$$

Moreover g satisfies (1) *and* $\sup|Dg| \leq \Lambda_2\lambda^{\frac{3}{2}}$.

Proof. We may assume $s_3 = 0$, $y_0 = 0$, $\varrho_0 = 2$ and $\kappa \leq 1/4$. Set $q_1 = q_2 := 2$. Let σ_1 be from Proposition 4.1 with respect to κ. For $m \in \mathbb{N}$ set $R_m := m\sigma_1\lambda^2$, $T_m := s_4 - 8m\lambda^4$ and $J_m := (2\lambda^4, T_m)$. Note that $T_m \geq s_4 - C\sigma_1^{-1}R_m\lambda^2$, in particular for $R_m \leq 5$ and Λ_2 large enough we have $T_m \geq (\Lambda_2 - C\sigma_1^{-1})\lambda^2 > 4\lambda^4$.

Consider the following statement:
$stat(m) :\Leftrightarrow$ There exists a $g_m \in \mathcal{C}^\infty\left(J_m \times \mathbf{B^n}(0, R_m), \mathbb{R}^{\mathbf{k}}\right)$ with

$$\mu_t \llcorner \mathbf{C}(0, R_m, 1) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g_m(t,\cdot)) \quad \text{for all } t \in J_m,$$
$$g_m \text{ satisfies (1) and } \sup|Dg_m| \leq \Lambda_2\sigma^{\frac{3}{2}}.$$

By Proposition 4.1 with $s_2 = s_4$, $\rho = 1$, $\sigma = \sqrt[4]{2}\lambda$ we see that $stat(1)$ is true. Now suppose $stat(m_0)$ holds for some $m_0 \in \mathbb{N}$ with $R_{m_0} \leq 5$. Using Proposition 4.1 with $s_2 = T_{m_0} - 2\lambda^4$, $\rho = 1$, $\sigma = \sqrt[4]{2}\lambda$ and arbitrary $z_0 \in \mathbf{B^n}(0, R_{m_0} + \sigma_1\lambda^2) \times \{0\}^{\mathbf{k}}$ yields that also $stat(m_0 + 1)$ is true. Thus $stat(m_1)$ holds for some $m_1 \in \mathbb{N}$ with $4 \leq R_{m_1} \leq 5$, which establishes the result. $\square$

Now consider a Brakke flow which is initially graphical with small Lipschitz constant. Then the conditions of Lemma 4.2 are satisfied for arbitrarily small scaling. Thus we can extend the interval of graphical representation up to the initial time.

4.3 Lemma. *There exist constants $C \in (1, \infty)$ and $\sigma_2 \in (0, 1)$ such that the following holds: Let $\sigma \in (0, \sigma_2]$, $\rho_0 \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1 + C\sigma^2\rho_0^2, t_1 + \sigma\rho_0^2]$, $z_0 \in \mathbb{R}^{\mathbf{n+k}}$ and let $(\mu_t)_{t\in[t_1,t_2]}$ be a Brakke flow in $\mathbf{C}(z_0, 4\rho_0, 2\varrho_0)$. Assume $z_0 \in \operatorname{spt}\mu_{t_1}$ and*

$$\operatorname{spt}\mu_{t_2} \cap \mathbf{C}(z_0, \sigma_2\sigma^2\rho_0, \rho_0) \neq \emptyset. \tag{38}$$

Suppose there exists an $f \in \mathcal{C}^{0,1}\left(\mathbf{B^n}(\hat{z}_0, 4\rho_0), \mathbb{R}^{\mathbf{k}}\right)$ with $\operatorname{lip}(f) \leq \sigma^4$ and

$$\mu_{t_1} \llcorner \mathbf{C}(z_0, 4\rho_0, 2\rho_0) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(f). \tag{39}$$

Set $I := (t_1, t_2 - C\sigma^2\rho_0^2)$.

Then there exists a $g \in \mathcal{C}^\infty\left(I \times \mathbf{B^n}(\hat{z}_0, 2\rho_0), \mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(z_0, 2\rho_0, \rho_0) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g(t,\cdot)) \quad \textit{for all}\, t \in I.$$

Moreover g satisfies (1) *and* $\sup|Dg| \leq \sigma$.

Proof. We may assume $t_1 = 0$, $z_0 = 0$ and $\rho_0 = 1$.

Let $C_2 \in (1,\infty)$ be a constant which we will choose later. For $s \in (0, 16\sigma^4]$ we consider the following statement:
$stat(s) :\Leftrightarrow$ There exists an $u_s \in \mathcal{C}^\infty\left((s, t_2 - C_2\sigma^2) \times \mathbf{B^n}(0,1), \mathbb{R}^\mathbf{k}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(0,2,1) = \mathscr{H}^\mathbf{n} \llcorner \operatorname{graph}(u_s(t,\cdot)) \tag{40}$$

$$u_s \text{ satisfies (1) and } \sup|Du_s| \leq \sigma \tag{41}$$

Suppose $stat(s_0)$ holds for some $s_0 \in (0, 16\sigma^2]$. We want to show that in this case also $stat(\frac{s_0}{2})$ holds. Let $\hat{y} \in \mathbf{B^n}(0,2)$ be arbitrary, set $y := (\hat{y}, f(\hat{y}))$ and $\varrho_0^2 := \sigma^{-1}s_0 \leq \sigma_2 \leq 1/4$. Using assumption (39) and $\operatorname{lip}(f) \leq \sigma^4$ yields

$$\operatorname{spt}\mu_0 \cap \mathbf{C}(y, 4\varrho_0, 3/2) \subset \mathbf{C}(y, 4\varrho_0, 4\sigma^4\varrho_0). \tag{42}$$

Then by Corollary A.2 with $R_1 = r_0 = 2\varrho_0$ and $R_2 = 5/4$ we obtain

$$\operatorname{spt}\mu_t \cap \mathbf{C}((\hat{y},0), 2\varrho_0, 1) \subset \mathbf{C}(y, 2\varrho_0, \varrho_0) \tag{43}$$

for all $t \in [0, 2s_0]$. Here we estimated $4\sigma^4\varrho_0 + 2\varrho_0^{-1}s_0 \leq C\sigma_1\varrho_0 \leq \varrho_0$ and $|f(\hat{y})| + 1 \leq 5/4$.

Set $J_2 := (s_0/2, (2-1/2)s_0)$. We want to use Lemma 4.2 with $\kappa = \frac{1}{2}$, $\lambda = 2\sigma$, $s_4 = 2s_0$ and $y_0 = y$. Choosing σ_2 small enough we obtain the following: $\Lambda_2\lambda^2\varrho_0^2 \leq s_0/2$; Statement (42) implies (36); Using assumption (39) and $\operatorname{lip}(f) \leq \sigma_2^4$, we see that (37) holds. Moreover by (43) and as $s_0 < 2s_0 < t_2 - C_2\sigma^2$ we can use assumption (40) to show (35). Then by Lemma 4.2 we obtain an $u_{s,\hat{y}} \in \mathcal{C}^\infty\left(J_2 \times \mathbf{B^n}(\hat{y}, \varrho_0), \mathbb{R}^\mathbf{k}\right)$ with $\sup|Du_{s,\hat{y}}| \leq C\sigma^{\frac{3}{2}} \leq \sigma$ and

$$\mu_t \llcorner \mathbf{C}((\hat{y},0), \varrho_0, 1) = \mu_t \llcorner \mathbf{C}(y, \varrho_0, \varrho_0) = \mathscr{H}^\mathbf{n} \llcorner \operatorname{graph}(u_{s,\hat{y}}(t,\cdot))$$

for all $t \in J_2$. Here we used (43) to obtain the first equality. As $\hat{y} \in \mathbf{B^n}(0,1)$ was arbitrary this shows $stat(\frac{s_0}{2})$ is true.

Similarly we can use Lemma 4.2 with $y_0 = 0$, $s_4 = t_2$, $\varrho_0 = 1$ and $\lambda = \sqrt{2}\sigma$ to obtain that $stat(4\sigma^4)$ is true for C_2 large enough. Hence we can start an iteration which yields that $stat(0)$ holds. This establishes the result. $\square$

Consider the situation of Lemma 4.3. If $z_0 \in \operatorname{spt}\mu_{t_2}$ and σ small enough we have that $(\mu_t)_{t\in[t_2 - C\sigma^2, t_2]}$ satisfies the conditions of Lemma 4.3 on the smaller scale $\rho_0/2$ with σ replaced by $\sqrt[4]{\sigma}$. Thus we can use $t_2 - C\sigma^2$ as the new starting time. This yields an iteration and by curvature bounds for graphical mean curvature flow (for example by Wang [Wan04, 4.1]), we can assure that the gradient does not blow up. This leads to the following:

4.4 Lemma. *There exists a constant $\beta_2 \in (0,1)$ such that the following holds: Let $\beta \in (0,\beta_2]$, $\epsilon \in (0,\beta^2)$, $\varrho_0 \in (0,\infty)$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1+\epsilon^2\varrho_0^2, s_1+\beta^4\varrho_0^2]$, $y_0 \in \mathbb{R}^{\mathbf{n+k}}$ and let $(\mu_t)_{t\in[s_1,s_2]}$ be a Brakke flow in $\mathbf{C}(y_0,2\varrho_0,2\varrho_0)$. Assume $y_0 \in \mathrm{spt}\mu_{s_1}$ and*

$$\mathrm{spt}\mu_{s_2} \cap \mathbf{C}(y_0,\beta^2\epsilon\varrho_0,\varrho_0) \neq \emptyset. \tag{44}$$

Suppose there exists an $f \in \mathcal{C}^{0,1}\left(\mathbf{B^n}(\hat{y}_0,2\varrho_0),\mathbb{R}^{\mathbf{k}}\right)$ with $\mathrm{lip}(f) \le \beta^4$ and

$$\mu_{s_1} \llcorner \mathbf{C}(y_0,2\varrho_0,2\varrho_0) = \mathscr{H}^{\mathbf{n}} \llcorner \mathrm{graph}(f). \tag{45}$$

Let $s \in (s_1, s_2-\epsilon^2\varrho_0^2)$ and $\varrho(s) := \beta^{-1/8}\sqrt{s_2-s}$.

Then there exists a $g_s \in \mathcal{C}^\infty\left((s_1,s)\times \mathbf{B^n}(\hat{y}_0,\varrho(s)),\mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(y_0,\varrho(s),\varrho_0) = \mathscr{H}^{\mathbf{n}} \llcorner \mathrm{graph}(g_s(t,\cdot)) \quad \text{for all}\, t \in (s_1,s).$$

Moreover g_s satisfies (1) and $\sup|Dg_s| \le \beta$.

Proof. We may assume $s_1 = 0$, $y_0 = 0$ and $\varrho_0 = 1$. First note that by (45), $\mathrm{lip}(f) \le \beta_2^4$, $s_2 \le \beta_2^4$ and Corollary A.2 with $R_1 = r_0 = R$ and $R_2 = 1$

$$\mathrm{spt}\mu_t \cap \mathbf{C}(0,R,1) \subset \mathbf{C}(0,R,2\beta_2^4 R + R^{-1}s_2) \subset \mathbf{C}(0,1,1/4) \tag{46}$$

for all $t \in [0,s_2]$, $R \in (0,1]$.

For $s \in (0,s_2)$, we consider the following statement:
stat(s) :⇔ There exists a $v_s \in \mathcal{C}^\infty\left((0,s)\times\mathbf{B^n}(0,\varrho(s)),\mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(0,\varrho(s),1) = \mathscr{H}^{\mathbf{n}} \llcorner \mathrm{graph}(v_s(t,\cdot)) \tag{47}$$

$$v_s \text{ satisfies (1) and } \sup|Dv_s| \le \beta. \tag{48}$$

First observe that *stat*(s) is true for all $s \in (0,(1-2^{-7})s_2]$. To see this use Lemma 4.3 with $t_1 = 0$, $t_2 = s_2$, $\sigma = \beta$ and $\rho_0 = \beta^{-1/8}\sqrt{s_2}$. Note that $s_2 - C\sigma^2\rho_0^2 \ge (1-C\beta_2)s_2 \ge (1-2^{-7})s_2$, $\sigma_2\sigma^2\rho_0 \ge \sigma_2\beta_2^{-1/8}\beta^2\epsilon \ge \beta^2\epsilon$ and $2\beta_2^4\rho_0 + \rho_0^{-1}s_2 \le \rho_0$ for β_2 small enough. In particular use (46) with $R = \rho_0$ to see that $\mathrm{spt}\mu_t \cap \mathbf{C}(0,\rho_0,\rho_0) = \mathrm{spt}\mu_t \cap \mathbf{C}(0,\rho_0,1)$.

Now assume *stat*(s_0) holds for some $s_0 \in [(1-2^{-7})s_2, s_2-\epsilon^2)$. We want to show that under this assumption *stat*(s) holds for all $s \in [s_0, s_0+\epsilon^5)$.

Set $\tau := s_2 - s_0 \in (\epsilon^2,\beta^4)$, $t_1 := s_0 - 63\tau$ and $a_0 := (0, v_{s_0}(t_1,0)) \in \mathbb{R}^{\mathbf{n+k}}$. Then $\varrho(t_1) = 8\varrho(s_0)$. Hence by (47) and (48)

$$\mathrm{spt}\mu_{t_1} \cap \mathbf{C}(0,8\varrho(s_0),1) \subset \mathbf{C}(a_0,8\varrho(s_0),8\beta\varrho(s_0)).$$

Thus by Corollary A.2 and (46)

$$\mathrm{spt}\mu_t \cap \mathbf{C}(0,4\varrho(s_0),1) \subset \mathbf{C}(a_0,4\varrho(s_0),\varrho(s_0)) \tag{49}$$

for all $t \in [t_1, s_2]$. Here we estimated $8\beta\varrho(s_0) + \varrho(s_0)^{-1}(s_2 - t_1) \le \varrho(s_0)$, for β_2 small enough.

Set $J := (t_1, s_2 - 2^{-1}\tau)$. Now use Lemma 4.3 with $\sigma = \sqrt[4]{\beta}$, $\rho_0 = 2\varrho(s_0)$, $t_2 = s_2$ and $z_0 = a_0$. Note that $s_2 - C\sigma^2\rho_0^2 \ge s_2 - C\sqrt[8]{\beta_2}\tau \ge s_2 - 2^{-1}\tau$ and $\sigma_2\sigma^2\rho_0 \ge \sigma_2\sqrt{\beta\tau} \ge \beta_2\beta\epsilon$ for β_2 small enough. Then we obtain an $u \in \mathcal{C}^\infty\left(J \times \mathbf{B}^{\mathbf{n}}(0, 4\varrho(s_0)), \mathbb{R}^{\mathbf{k}}\right)$ with $\sup|Du| \le \sqrt[4]{\beta}$ and

$$\mu_t \llcorner \mathbf{C}(0, 4\varrho(s_0), 1) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(u(t, \cdot)) \quad \text{for all } t \in J. \tag{50}$$

Here we also used (49). Moreover u satisfies (1). It remains to show the gradient bound for u.

For $t \in J$ consider $M_t := \operatorname{graph}(u(t, \cdot))$, which moves by smooth mean curvature flow. By Theorem [Lah15, 2.2.1] with $\rho = \varrho(s_0)$, $l = l_0$ and starting time t_1 we obtain a curvature bound

$$|\mathbf{A}(M_t, x)|^2 \le C|D^2u(t, \hat{x})| \le C(t - t_1)^{-1} \le C_1^2\tau^{-1} \tag{51}$$

for all $x \in M_t \cap \mathbf{C}(a_0, 2\varrho(s_0), 2\varrho(s_0))$ for all $t \in J \cap [s_0 - \tau, s_2]$ and some constant $C_1 \in (1, \infty)$. Here chose β_2 small enough such that $\beta_2 \le l_0$ and $s_2 - t_1 \le 64\sqrt[4]{\beta_2}\varrho(s_0)^2 \le l_0\varrho(s_0)^2$, where l_0 is from [Lah15, 2.2.1]. Moreover in view of (49) we see that (51) actually holds for all $x \in M_t \cap \mathbf{C}(0, 2\varrho(s_0), 1)$. Note that the curvature bounds in [Lah15, 2.2.1] are based on White's regularity theorem [Whi05]. Similarly we could use Wang's curvature estimate [Wan04, 4.1] to deduce (51).

Let $\hat{y} \in \mathbf{B}^{\mathbf{n}}(0, \varrho(s_0))$ be arbitrary, $y := (\hat{y}, u(s_0 - \beta^5\tau, \hat{y})) \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. In view of (47) and (48) we have

$$\operatorname{spt}\mu_{s_0 - \beta^5\tau} \cap \mathbf{C}((\hat{y}, 0), 2\beta^2\sqrt{\tau}, 1) \subset \mathbf{C}(y, 2\beta^2\sqrt{\tau}, 2\beta^3\sqrt{\tau}).$$

Then using Corollary A.2 with $R_1 = r_0 = \beta^2\sqrt{\tau}$, $R_2 = 1/2$ and (46) we obtain

$$\operatorname{spt}\mu_t \cap \mathbf{C}((\hat{y}, 0), \beta^2\sqrt{\tau}, 1) \subset \mathbf{C}(y, \beta^2\sqrt{\tau}, 4\beta^3\sqrt{\tau}) \tag{52}$$

for all $t \in [s_0 - \beta^5\tau, s_0 + \beta^5\tau]$. Note that $s_0 + \beta^5\tau \in J$ for $\beta_2^5 \le 1/2$. Combining (50), (51) and (52) we can use Lemma [Lah15, A.4] with $r = \beta^2\sqrt{\tau}$, $\xi^2 = 4\beta$, $K^2 = C_1\beta^2$ to obtain

$$|Du(t, \hat{y})|^2 \le C\beta^3 \le \beta^2$$

for all $t \in [s_0, s_0 + \beta^5\tau]$, where we chose β_2 small enough. By assumption $\tau = s_2 - s_0 \ge \epsilon^2$ and $\beta^2 \ge \epsilon$. As $\hat{y} \in \mathbf{B}^{\mathbf{n}}(0, \varrho(s_0))$ was arbitrary this shows that $stat(s)$ holds for all $s \in [s_0, s_0 + \epsilon^5)$, this establishes the result. □

Assume the setting of Theorem 1.3. Using Lemma 4.4 we can show that $\mathrm{spt}\mu_t \cap \mathbf{C}(0,\rho,\rho)$ is contained in a Lipschitz graph and has unit density for all $t \in (t_1,t_2)$. Then by Theorem 1.5 we find a sequence $\tau_m \nearrow t_2$ such that $\mathrm{spt}\mu_{\tau_m} \llcorner \mathbf{C}(0,\rho,\rho)$ is graphical. With Lemma 4.4 for arbitrary small ϵ we can conclude Theorem 1.3.

Proof of Theorem 1.3. We may assume $a=0$, $t_1=0$ and $\rho=3$. Using (6) and Corollary A.2 we obtain

$$\mathrm{spt}\mu_t \cap \mathbf{C}(0,4,4) \subset \mathbf{C}(0,4,h(t)) \tag{53}$$

for all $t \in [0,t_2]$ and $h(t) = l + t \leq 2l_0$.

Set $U := \mathbf{B}^{\mathbf{n}}(0,4) \times \mathbb{R}^{\mathbf{k}}$. By definition of a Brakke flow we find a sequence $(\tau_m)_{m\in\mathbb{N}}$ with $\tau_m \nearrow t_2$, $\tau_m \in (0,t_2]$ such that for all $m \in \mathbb{N}$ we have $\mu_m := \mu_{\tau_m} \llcorner \mathbf{C}(0,4,4)$ is integer $\mathbf{n}$-rectifiable and the generalised mean curvature vector $\mathbf{H}_{\mu_m}$ exists. In particular $\|\partial\mu_m\|$ is absolutely continuous with respect to μ_m inside U. Fix an arbitrary $m \in \mathbb{N}$. We want to show

$$\mathrm{spt}\mu_m \cap U \subset \mathrm{graph} f_m \tag{54}$$

for some Lipschitz function $f_m : \mathbf{B}^{\mathbf{n}}(0,4) \to \mathbb{R}^{\mathbf{k}}$

Let $x, y \in \mathrm{spt}\mu_m \cap U$ with $x \neq y$. Set $y_0 := (\hat{y}, f(\hat{y}))$. We want to show $|\tilde{x} - \tilde{y}| \leq L|\hat{x} - \hat{y}|$ for some constant $L \in (1,\infty)$ which will depend on l_0. By (53) we have $|\tilde{x} - \tilde{y}| \leq 2l_0$. Hence we may assume $|\hat{x} - \hat{y}| \leq l_0$.

First consider the case $\tau_m \leq 4|\hat{x} - \hat{y}|^2 \leq 4l_0^2$ and let $z \in \mathrm{spt}\mu_m \cap U$. Then $\mu_m(\mathbf{B}(z, 2\sqrt{\mathbf{n}\tau_m})) \neq \emptyset$, so by Lemma 2.6 we have $\mu_0(\mathbf{B}(z, 4\sqrt{\mathbf{n}\tau_m})) \neq \emptyset$. Thus by (6) and $\mathrm{lip}(f) \leq l_0$ we have $|\tilde{z} - \tilde{y}_0| \leq l_0|\hat{z} - \hat{y}| + 8\sqrt{\mathbf{n}\tau_m}$. For $z = x, y$ this yields the wanted estimate.

Now consider the case $0 < 4|\hat{x} - \hat{y}|^2 < \tau_m$. Set $\epsilon := |\hat{x} - \hat{y}|$. By (53) we have $y \in \mathrm{spt}\mu_m \cap \mathbf{C}(y_0, \beta^2\epsilon, 1)$ for all $\beta \in (0,\infty)$. Set $s_m := \tau_m - 2\epsilon^2$. Using Lemma 4.4 with $s_1 = 0$, $s_2 = \tau_m$, $\varrho_0 = 1$, $\beta = \sqrt[4]{l_0}$ we obtain a $g_m \in \mathcal{C}^\infty(\mathbf{B}^{\mathbf{n}}(\hat{y}, 8\mathbf{n}\epsilon))$ with $\sup|Dg_m| \leq \sqrt[4]{l_0}$ and

$$\mathrm{spt}\mu_{s_m} \cap \mathbf{C}(y_0, 8\mathbf{n}\epsilon, 1) = \mathscr{H}^{\mathbf{n}} \llcorner \mathrm{graph} g_m. \tag{55}$$

Let $z \in \mathrm{spt}\mu_m \cap U$ with $|\hat{z} - \hat{y}| \leq \epsilon$. Then $\mu_m(\mathbf{B}(z, 2\sqrt{\mathbf{n}}\epsilon)) \neq \emptyset$, so by Lemma 2.6 we have $\mu_{s_m}(\mathbf{B}(z, 4\sqrt{\mathbf{n}}\epsilon)) \neq \emptyset$. In view of (53) we have $|\tilde{z} - \tilde{y}_0| \leq 2l_0 < 1 - 4\sqrt{\mathbf{n}}$, hence we can use (55) to estimate $|\tilde{z} - g_m(\hat{y})| \leq l_0(1 + 4\mathbf{n})\epsilon$. For $z = x, y$ this proves (54).

Next we want to show that μ_m has unit density. Let $y \in \mathrm{spt}\mu_m \cap U$ and $r \in (0, \sqrt{\tau_m})$ be given. Set $\epsilon := \sqrt[32]{l_0} r$, $s_r := \tau_m - 16\sqrt[16]{l_0} r^2$ and $y_0 := (\hat{y}, f(\hat{y}))$. Note that by (53) we have $y \in \mathrm{spt}\mu_m \cap \mathbf{C}(y_0, \beta^2\epsilon, 1)$. Using Lemma 4.4 with

$s_1 = 0$, $s_2 = \tau_m$, $\varrho_0 = 1$, $\beta = \sqrt[4]{l_0}$ we obtain a $g_r \in \mathcal{C}^\infty(\mathbf{B^n}(\hat{y}, 4r))$ with $\sup |Dg_r| \leq \sqrt[4]{l_0}$ and

$$\operatorname{spt}\mu_{s_r} \cap \mathbf{C}(y_0, 4r, 1) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph} g_m, \tag{56}$$

Consider a radial cut-off function $\zeta_r \in \mathcal{C}^\infty\left(\mathbb{R}^{\mathbf{n+k}}, [0,1]\right)$ with $\sup |D^2\zeta_r| \leq Cr^{-2}$ and

$$\zeta_r(x) = \begin{cases} 1 & \text{for } \; 0 \leq |x-y| \leq r \\ 0 & \text{for } \; (1+2^{-\mathbf{n}-2})r \leq |x-y| \, . \end{cases}$$

By the Brakke flow equation (9), Remark 2.2 and Lemma 2.6 we can estimate

$$\begin{aligned} &\mu_m\left(\mathbf{B}(y,r)\right) - \mu_{s_r}\left(\mathbf{B}(y,(1+2^{-\mathbf{n}-2})r)\right) \\ &\leq \mu_m(\zeta_r) - \mu_{s_r}(\zeta_r) \leq C\int_{s_r}^{\tau_m} \left(\sup |D^2\zeta_r| \, \mu_t(\{\phi > 0\})\right) \mathrm{d}t \\ &\leq C\sqrt[16]{l_0} \sup\{\mu_t(\mathbf{B}(y,2r)), t \in [s_r, \tau_m]\} \leq C\sqrt[16]{l_0}\mu_{s_r}\left(\mathbf{B}(y,4r)\right). \end{aligned}$$

In view of (53) we have $|\tilde{y} - \tilde{y}_0| \leq 2l_0 < 1 - 8r$, hence we can use (56) and the above estimate to obtain

$$\mu_m\left(\mathbf{B}(y,r)\right) \leq (1 + C\sqrt[4]{l_0})\left(C\sqrt[16]{l_0}(4r)^{\mathbf{n}} + ((1+2^{-\mathbf{n}-2})r)^{\mathbf{n}}\right) \leq \frac{3}{2}r^{\mathbf{n}},$$

where we chose l_0 small enough. As we already know μ_m is integer rectifiable, this shows that μ_m even has unit density in U. Also, by (5) and Lemma 2.6 we have $\operatorname{spt}\mu_m \cap U \neq \emptyset$. Then Theorem 1.5 yields that in (54) actually holds equality. Hence

$$\mu_{\tau_m} \llcorner \mathbf{C}(0,4,4) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph} f_m$$

for all $m \in \mathbb{N}$, for some Lipschitz function $f_m : \mathbf{B^n}(0,4) \to \mathbb{R}^{\mathbf{k}}$. In view of this and (53) we can use Lemma 4.4 with $s_2 = \tau_m$, $\varrho_0 = 1$, $y_0 \in \mathbf{B^n}(0,4) \times \{0\}^{\mathbf{k}}$ and arbitrarily small ϵ to obtain graphical representability inside $\mathbf{C}(0,4,4)$ for times in $(0,\tau_m)$. As $\tau_m \nearrow t_2$ we can extend the time interval to $(0,t_2)$. Finally for the Lipschitz bound use Lemma 4.4 with $s_2 = t$ and $\beta = \sqrt[4]{l+t}$ for arbitrary $t \in (0,t_2)$. This completes the result. □

5 Brakke-type regularity theorem

Here we proof Theorem 1.1. Under slightly stronger assumptions on the starting density ratios the result directly follows from Lemma 4.2 and Theorem 1.3, see below:

5.1 Lemma. *There exists a constant $c \in (0,1)$ and for every $\kappa \in (0,1)$ exists an $h_2 \in (0, c^4)$ such that the following holds: Let $h \in (0, h_2]$, $\varrho \in (0,\infty)$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1 + \sqrt[4]{h}\varrho^2, s_1 + c\varrho^2]$, $x_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t\in[s_1,s_2]}$ be a Brakke flow in $\mathbf{C}(x_0, 9\varrho, 9\varrho)$. Suppose $x_0 \in \operatorname{spt}\mu_{s_2}$*

$$\operatorname{spt}\mu_{s_1} \cap \mathbf{C}(x_0, 9\varrho, 9\varrho) \subset \mathbf{C}(x_0, 9\varrho, h\varrho), \tag{57}$$

$$r^{-\mathbf{n}}\mu_{s_1}(\mathbf{B}(y,r)) \le (2-\kappa)\omega_{\mathbf{n}} \tag{58}$$

for all $y \in \mathbf{B}^{\mathbf{n}}(\hat{x}_0, 7\varrho) \times \{\tilde{x}_0\}$ and all $r \in (h\varrho, \varrho)$. Set $I := (s_1 + \sqrt[4]{h}\varrho^2, s_2)$.

Then there exists a $g \in \mathcal{C}^\infty\left(I \times \mathbf{B}^{\mathbf{n}}(\hat{x}_0, \varrho), \mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(x_0, \varrho, \varrho) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g(t,\cdot)) \quad \textit{for all } t \in I.$$

Moreover g satisfies (1) *and* $\sup|Dg(t,\cdot)| \le 2\sqrt[4]{\varrho^{-2}(t-s_1)}$ *for all $t \in I$.*

Proof. We may assume $x_0 = 0$, $s_1 = 0$ and $\varrho = 1$. Let Λ_2 and λ_2 be from Lemma 4.2 with respect to κ and set $s_4 := 8\Lambda_2\sqrt{h} < \sqrt[4]{h} < s_2$ for h_2 small enough. We see that $s_2 - s_4 \le (8\mathbf{n})^{-1}$, so Lemma 2.6 and $0 \in \operatorname{spt}\mu_{s_2}$ yield $\mu_{s_4}(\mathbf{B}(0,1)) > 0$. Thus there exists an $z_0 \in \operatorname{spt}\mu_{s_4} \cap \mathbf{B}(0,1)$.

Set $J := (4h, s_4 - 4\Lambda_2\sqrt{h})$. By Lemma 4.2 with $\varrho_0 = 2$, $y_0 = (\hat{z}_0, 0)$ and $\lambda = \sqrt[4]{h}$ there exists a Lipschitz function $g_1 \in \mathcal{C}^\infty(J \times \mathbf{B}^{\mathbf{n}}(\hat{z}_0, 4), \mathbb{R}^{\mathbf{k}})$ such that

$$\mu_t \llcorner \mathbf{C}((\hat{z}_0, 0), 4, 2) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g(t,\cdot)) \quad \text{for all } t \in J.$$

Moreover $\sup|Dg_1| \le \Lambda_2 h^{3/8} \le \sqrt[4]{h}$. Here we chose $h_2 \le \min\{\lambda_2^4, \Lambda_2^{-8}\}$. Set $a := (0, g_1(0))$ and $t_1 = 8h$. By Corollary A.2, assumption (57) and $|\hat{z}_0| \le 1$ we have $|a| < 1$ and $\operatorname{spt}\mu_{t_1} \cap \mathbf{C}((\hat{z}_0,0),4,2) \supset \operatorname{spt}\mu_{t_1} \cap \mathbf{C}(a,2,2)$ for h_2 small enough. Then Theorem 1.3 with $l = \sqrt[4]{h}$ and $\rho = 1$ yields the result. Here we chose h_2 small depending on Λ_2 such that $8h \in J$ and $\sqrt[4]{h} \le l_0$. □

Now under the assumptions of Theorem 1.1 we can find a time s_1 shortly after t_1 such that $\mu_{s_1} \llcorner \mathbf{C}(a,\rho,\rho)$ has bounded mean-curvature-excess and still has small height. By Lemma 2.12 then also the tilt-excess has to be small. Thus Brakke's cylindrical growth theorem, Theorem 2.13 can be used to show that the density assumptions of Lemma 5.1 hold, which then yields the conclusion of Theorem 1.1.

proof of Theorem 1.1. We may assume $a = 0$, $t_1 = 0$ and $\rho = 1$. First consider the case $\gamma > 0$. Set $U := \mathbf{B}(0,1)$ and $C(x,r) := \mathbf{C}(x,r) \cap U$ for $r \in (0,\infty)$, $x \in U$. In view of assumption (2) and as $\mathbf{C}(0,\sqrt{2},\sqrt{2}) \subset \mathbf{B}(0,2)$ we can use Corollary A.2 with $r_0 = \sqrt{2} - 1$ and $p = 4$ to obtain

$$\operatorname{spt}\mu_t \cap \mathbf{C}(0,1,1) \subset \mathbf{C}(0,1,2\gamma) \tag{59}$$

for all $t \in [0, \sqrt[4]{\gamma}]$ for γ_0 small enough. Fix a $\sigma \in (0, 2^{-5})$ such that $(1 - 8\sigma)^{-\mathbf{n}} \leq 1 + \lambda/8$ and $(1 + 4\sigma)^{\mathbf{n}} \leq 1 + \lambda/32$. In particular we can choose γ_0 small depending on σ. By Lemma 2.6 and assumption (3) we can estimate

$$\mu_t\left(\mathbf{B}(0, 1 - \sigma)\right) \leq C\sigma^{-\mathbf{n}-\mathbf{k}} \mu_0\left(\mathbf{B}(0, 1)\right) \leq C\sigma^{-\mathbf{n}-\mathbf{k}} \tag{60}$$

for all $t \in [0, (8\mathbf{n})^{-1}\sigma^2] \cap [0, t_2]$.

Fix a cut-off function $\psi \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ with $|\psi''| \leq C\sigma^{-2}$ and

$$\psi(t) = \begin{cases} 1 & \text{for } 0 \leq |t| \leq 1 - 2\sigma \\ 0 & \text{for } 1 - \sigma \leq |t|. \end{cases}$$

Consider $\zeta \in \mathcal{C}_c^\infty(\mathbf{B}(0, 1), [0, 1])$ given by $\zeta(x) = \psi(|x|)$. Consider $s \in (0, \sqrt[4]{\gamma}]$. Using the Brakke flow equation (9), and Remark 2.2 we can estimate

$$\begin{aligned} D &:= \mu_s(\zeta) + \frac{1}{2}\int_0^s \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} |\mathbf{H}_{\mu_t}|^2 \zeta \, \mathrm{d}\mu_t \, \mathrm{d}t \\ &\leq \mu_0(\zeta) + \sup |D^2\zeta| \int_0^s \mu_t\left(\{\zeta > 0\}\right) \, \mathrm{d}t. \end{aligned}$$

Hence by (3) and (60) we have

$$D \leq (2 - \lambda)\omega_{\mathbf{n}} + Cs\sigma^{-\mathbf{n}-\mathbf{k}-2} \leq (2 - \lambda/2)\omega_{\mathbf{n}}$$

where we used $s \leq \sqrt[4]{\gamma_0}$ and we chose γ_0 small enough. By (59) we have $\{\zeta = 1\} \supset \mathbf{B}(0, 1 - 2\sigma) \supset \mathrm{spt}\mu_s \cap C(0, 1 - 4\sigma)$, for $\gamma_0 \leq \sigma$. Thus

$$\mu_s\left(C(0, 1 - 4\sigma)\right) + \frac{1}{2}\int_0^s \int_{C(0,1-4\sigma)} |\mathbf{H}_{\mu_t}|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \leq (2 - \lambda/2)\omega_{\mathbf{n}} \tag{61}$$

for all $s \in (0, \sqrt[4]{\gamma}]$. In particular we find an $s_1 \in (0, \sqrt[4]{\gamma}]$ such that $\mu_{s_1} \llcorner U$ is integer $\mathbf{n}$-rectifiable, has $\mathcal{L}^2$-integrable mean curvature vector and

$$\int_{C(0,1-4\sigma)} |\mathbf{H}_{\mu_{s_1}}|^2 \, \mathrm{d}\mu_{s_1} \leq 2(2 - \lambda/2)\omega_{\mathbf{n}}\gamma^{-1/4} \leq C\gamma^{-1/4}. \tag{62}$$

Consider $y \in \mathbf{B}(0, \sigma)$. By (61) and choice of σ we can estimate

$$\mu_{s_1}(C(y, 1 - 8\sigma)) \leq (2 - \lambda/2)\omega_{\mathbf{n}} \leq (2 - \lambda/4)\omega_{\mathbf{n}}(1 - 8\sigma)^{\mathbf{n}}, \tag{63}$$

by definition of λ and for $\gamma_0 \leq \sigma$.

Let $f \in \mathcal{C}_c^\infty(C(0,1-4\sigma),[0,1])$ be such that $f(x) = \psi((1-4\sigma)^{-1}|\hat{x}|)$ for $x \in \mathrm{spt}\mu_{s_1} \cap U$. In view of (59),(61) and (62) we can use Lemma 2.12 with $f = g = h$ to obtain

$$\int_{C(0,1-6\sigma)} \left\|(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_\natural - \mathbf{T}(\mu_{s_1},x)_\natural\right\|^2 \,\mathrm{d}\mu_{s_1}(x) \le C\gamma^{7/8} \tag{64}$$

where we used $\sup|Df|^2 \le C\sigma^{-2} \le C\gamma_0^{-1} \le C\gamma^{-1}$.

Consider $y \in \mathbf{B}(0,\sigma)$ and $r_0 \in (\gamma^{4\alpha_0}\sigma/9,\sigma)$. Let $R_2 = 1-8\sigma$ and $R_1 = (1+4\sigma)r_0$. By (62) and (64) the assumptions of Theorem 2.13 are satisfied for $\alpha^2 = C\gamma^{-1/4}(\gamma^{4\alpha_0}\sigma)^{-2\mathbf{n}}$ and $\beta^2 = C\gamma^{7/8}(\gamma^{4\alpha_0}\sigma)^{-2\mathbf{n}}$. Hence we can estimate

$$\begin{aligned}&\left|R_2^{-\mathbf{n}}\int_U \psi(R_2^{-1}|\hat{x}-\hat{y}|)\,\mathrm{d}\mu(x) - R_1^{-\mathbf{n}}\int_U \psi(R_1^{-1}|\hat{x}-\hat{y}|)\,\mathrm{d}\mu(x)\right| \\ &\le C\gamma^{-1/8}(\gamma^{4\alpha_0}\sigma)^{-2\mathbf{n}}\gamma^{7/16} \le C\sigma^{-2\mathbf{n}}\gamma_0^{3/16} \le \lambda\omega_{\mathbf{n}}/4\end{aligned}$$

where we chose $\alpha_0 \le (64\mathbf{n})^{-1}$ and γ_0 small enough. By definition of ψ and (63) this yields

$$((1+4\sigma)r_0)^{-\mathbf{n}}\mu_0(\mathbf{B}(y,r_0)) \le (2-\lambda/8)\omega_{\mathbf{n}} \le (2-\lambda/16)\omega_{\mathbf{n}}(1+4\sigma)^{-\mathbf{n}}.$$

Now we can use Lemma 5.1 with $\varrho = \sigma/9$ and $h = \gamma^{2\alpha_0}$ to obtain the result. For the case $\gamma = 0$ use the above result with arbitrary small γ. □

5.2 Corollary. *Consider the situation of Theorem 1.1 without assumption* (3)*. Let $\delta \in (0,1)$ be such that $\gamma \le \delta\gamma_0$ and*

$$(\delta\rho)^{-\mathbf{n}}\mu_{t_1}(\mathbf{B}(y,\delta\rho)) \le (2-\lambda)\omega_{\mathbf{n}}. \tag{65}$$

for all $y \in \mathbf{C}(a,(2-\delta)\rho,\delta\rho)$. Moreover change the conditions on t_2 to $t_2 \in (t_1+\gamma^{\alpha_0}\delta^2\rho^2, t_1+\alpha_0\delta^2\rho^2)$ and set $J := (t_1+\gamma^{\alpha_0}\delta^2\rho^2, t_2)$.

Then there exists a $g \in \mathcal{C}^\infty\left(J \times \mathbf{B}^{\mathbf{n}}(\hat{a},(2-\delta)\rho),\mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}(a,(2-\delta)\rho,\rho) = \mathscr{H}^{\mathbf{n}} \llcorner \mathrm{graph}(g(t,\cdot)) \quad \textit{for all } t \in I.$$

Moreover g satisfies (1) *and* $\sup|Dg(t,\cdot)| \le 2\sqrt[4]{(\delta\rho)^{-2}(t-t_1)}$ *for all $t \in J$.*

Proof. We may assume $t_1 = 0$, $a = 0$ and $\rho = 1$. By Corollary A.2 with $r_0 = \delta$ we see that

$$\mu_t \llcorner \mathbf{C}(0,2-\delta,1) = \mu_t \llcorner \mathbf{C}(0,2-\delta,\delta/2) \tag{66}$$

for all $t \in [0,t_2]$. Here we chose α_0 and γ_0 small enough. By Brakke's continuity result [Bra78, 3.10] we have for almost every $s \in (0,t_2)$ that

$\mu_s(\phi) = \lim_{t\nearrow s}\mu_t(\phi)$ for all $\phi \in \mathcal{C}^0_c(\mathbf{C}(0,2,2))$. Consider such an s and $J_s = (h^{\alpha_0}\delta^2, s)$. Then for every $\hat{y} \in \mathbf{B^n}(0, 2-\delta)$ for which there exists an $z \in \operatorname{spt}\mu_s \cap \big(\{\hat{y}\} \times \mathbf{B^k}(0, \delta/2)\big)$ we can use Theorem 1.1 replacing ρ by δ, t_2 by s and a by z. This yields a function $g_{\hat{y}} \in \mathcal{C}^\infty\left(J_s \times \mathbf{B^n}(\hat{y}, \gamma_0\delta), \mathbb{R}^{\mathbf{k}}\right)$ such that

$$\mu_t \llcorner \mathbf{C}((\hat{y},0),\gamma_0\delta,1) = \mu_t \llcorner \mathbf{C}(z,\gamma_0\delta,\delta) = \mathscr{H}^{\mathbf{n}} \llcorner \operatorname{graph}(g_{\hat{y}}(t,\cdot)) \tag{67}$$

for all $t \in J_s$. Moreover $g_{\hat{y}}$ satisfies (1) and $\sup|Dg_{\hat{y}}(t,\cdot)| \le 2\sqrt[4]{\delta^{-2}t}$ for all $t \in J_s$. Here we used (66) for the first equality. Now by choice of s, connectedness of $\mathbf{B^n}(0,2-\delta)$ and (66), we see that either (67) holds for all $\hat{y} \in \mathbf{B^n}(0,2-\delta)$ or $\operatorname{spt}\mu_s \cap \mathbf{C}(0,2-\delta,\delta/2) = \emptyset$. As we can choose s arbitrary close to t_2 and as $0 \in \operatorname{spt}\mu_{t_2}$ only the first option remains, which proves the result. □

6 White-type regularity theorem

Here we want to prove Theorem 1.2. First we observe that a Brakke flow for which all Gaussian density ratios are one, has to be a plane. This mainly follows from Huisken's monotonicity formula, Theorem 2.8.

6.1 Lemma. *Let $M \in (1,\infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1,\infty)$ and $(\mu_t)_{t\in[t_1,t_2]}$ be a Brakke flow in $\mathbb{R}^{\mathbf{n+k}}$. Suppose $\mu_{t_2} \neq \emptyset$ and for all $(s,y) \in (t_1,t_2] \times \mathbb{R}^{\mathbf{n+k}}$*

$$\sup_{R\in(0,\infty)} R^{-\mathbf{n}}\mu_s(\mathbf{B}(y,R)) \le M, \tag{68}$$

$$\sup_{t\in[t_1,s)} \int_{\mathbb{R}^{\mathbf{n+k}}} \Phi_{(s,y)}\,\mathrm{d}\mu_t \le 1. \tag{69}$$

Then there exists a $T \in \mathbf{G}(\mathbf{n+k},\mathbf{n})$ and an $a \in \mathbb{R}^{\mathbf{n+k}}$ such that $\mu_t = \mathscr{H}^{\mathbf{n}} \llcorner (T+a)$ for all $t \in (t_1,t_2)$.

Proof. We may assume $t_1 = -1$ and $t_2 = 0$. For $t \in (-1,0)$ let $D(t)$ be the set of all $y \in \operatorname{spt}\mu_t$ such that $\Theta^{\mathbf{n}}(\mu_t,y) \ge 1$ and $\mathbf{T}(\mu_t,y)$ exists. Fix $s \in (-1,0)$ and $y \in D(s)$. For $\epsilon \in (0,1)$ there exist a cut-off function $\zeta \in \mathcal{C}^0_c\left(\mathbb{R}^{\mathbf{n+k}},[0,1]\right)$ and an $h_0 \in (0,s+1)$ such that

$$\int_{\mathbb{R}^{\mathbf{n}}\times\{0\}^{\mathbf{k}}} \Phi_{(0,0)}(-1,x)\zeta(x)\,\mathrm{d}\mathscr{H}^{\mathbf{n}}(x) \ge 1-\epsilon, \tag{70}$$

$$\int_{\mathbb{R}^{\mathbf{n+k}}} \Phi_{(s,y)}\,\mathrm{d}\mu_{s-h_0} \le \lim_{h\searrow 0}\int_{\mathbb{R}^{\mathbf{n+k}}} \Phi_{(s,y)}\,\mathrm{d}\mu_{s-h} + \epsilon. \tag{71}$$

By (70) and definition of the approximate tangent space we can estimate

$$\begin{aligned}(1-\epsilon)\Theta^{\mathbf{n}}(\mu_s,y) &\le \Theta^{\mathbf{n}}(\mu_s,y)\int_{\mathbf{T}(\mu_s,x)} \Phi_{(0,0)}(-1,x)\zeta(x)\,\mathrm{d}\mathscr{H}^{\mathbf{n}}(x)\\ &\le \lim_{\lambda\searrow 0}\lambda^{-\mathbf{n}}\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}}\Phi_{(0,0)}(-1,\lambda^{-1}(x-y))\,\mathrm{d}\mu_s(x)\\ &= \lim_{\lambda\searrow 0}\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}}\Phi_{(s+\lambda^2,y)}(s,x)\,\mathrm{d}\mu_s(x).\end{aligned}$$

Then with Huisken's monotonicity formula, Theorem 2.8, continuity of the integral and (71) we obtain

$$(1-\epsilon)\Theta^{\mathbf{n}}(\mu_s,y) \le \lim_{\lambda\searrow 0}\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}}\Phi_{(s+\lambda^2,y)}\,\mathrm{d}\mu_{s-h_0} \le \lim_{h\searrow 0}\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}}\Phi_{(s,y)}\,\mathrm{d}\mu_{s-h}+\epsilon.$$

Thus by (69) and as ϵ was arbitrary we have

$$1 \le \Theta^{\mathbf{n}}(\mu_s,y) \le \lim_{h\searrow 0}\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}}\Phi_{(s,y)}\,\mathrm{d}\mu_{s-h} \le 1 \tag{72}$$

for all $y \in D(s)$ for all $s \in (-1,0)$. Hence μ_t has unit density for a.e. $t \in (-1,0)$.

Fix an arbitrary $t_0 \in (-1,0)$ such that μ_{t_0} has unit density. Assumption $\mu_0 \neq \emptyset$ and Lemma 2.6 imply $\mathrm{spt}\mu_{t_0} \neq \emptyset$, so we can find $\mathbf{n}+1$ points $y_0,\dots,y_{\mathbf{n}}$ in $D(t_0)$ such that $v_i := y_i - y_0$, $i = 1,\dots,\mathbf{n}$ are linearly independent. Set $T := \mathrm{span}(v_i)_{1\le i\le \mathbf{n}}$. By estimates (69), (72) and Theorem 2.8 we obtain

$$\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}}\Phi_{(t_0,y_i)}\,\mathrm{d}\mu_t = 1$$

for all $t \in [-1,t_0)$ for all $i \in \{0,\dots,\mathbf{n}\}$.

Then Theorem 2.8 yields the existence of a $J \subset (-1,t_0)$ such that $\mathscr{L}^1((-1,t_0)\setminus J) = 0$ and for all $t \in J$ we have μ_t has unit density, the generalised mean curvature vector $\mathbf{H}_{\mu_t}$ exists with $\int |\mathbf{H}_{\mu_t}|^2\mu_t < \infty$ and

$$\mathbf{H}_{\mu_t}(x) + (2(t_0-t))^{-1}(\mathbf{T}(\mu_t,x)^{\perp})_\natural(x-y_i) = 0 \tag{73}$$

for μ_t-a.e. $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and all $i = 0,1,\dots,\mathbf{n}$.

Let $t \in J$ and let E_t be the set of points $x \in \mathrm{spt}\mu_t$ such that $\Theta^{\mathbf{n}}(\mu_t,x) \ge 1$, $\mathbf{T}(\mu_t,x)$ exists and (73) holds for all $i \in \{0,\dots,\mathbf{n}\}$. We see $\mu_t(\mathbb{R}^{\mathbf{n}+\mathbf{k}}\setminus E_t) = 0$. Consider $x \in E_t$ then by (73) we have

$$(\mathbf{T}(\mu_t,x)^{\perp})_\natural(y_0-y_i) = (\mathbf{T}(\mu_t,x)^{\perp})_\natural(x-y_i) - (\mathbf{T}(\mu_t,x)^{\perp})_\natural(x-y_0) = 0$$

for all $i \in \{1, \ldots, \mathbf{n}\}$. So $v_i \in \mathbf{T}(\mu_t, x)$ for all $i \in \{1, \ldots, \mathbf{n}\}$, hence $\mathbf{T}(\mu_t, x) = T$. As this holds for all $x \in E_t$ for all $t \in J$, we have $\mathbf{H}_{\mu_t} \equiv 0$ for a.e. $t \in (-1, t_0)$. This follows from Brakke's general regularity theorem [Bra78, 6.12] (see also [KT14, 3.2]). One could also deduce this from Menne's characterization of the mean curvature vector in [Men15, 15.6].

Now for a.e. $t \in (-1, t_0)$ equality (73) with $i = 0$ yields $E_t \subset T + y_0$. Thus $\mathrm{spt}\mu_t \subset T + y_0$. Then by Theorem 1.5 we have $\mu_t = \mathscr{H}^{\mathbf{n}} \llcorner (T + y_0)$. As this holds for a.e. t in $(-1, t_0)$ and by the continuity properties of the Brakke flow due to Brakke [Bra78, 3.10] we obtain $\mu_t = \mathscr{H}^{\mathbf{n}} \llcorner (T + y_0)$ for all $t \in (-1, t_0)$. As we can choose t_0 arbitrary close to 0 this establishes the result. □

Now suppose the Gaussian density ratios are locally bounded by $1 + \delta$. In view of the previous Lemma an indirect blow-up argument combined with Ilmanen's compactness theorem, Theorem 2.11, yields a small neighbourhood in which we have small height and density ratios close to one, see Lemma 6.2. In view of Theorem 1.1 this implies Theorem 1.2.

6.2 Lemma. *For all $\epsilon, \sigma \in (0, 1/2)$ there exists a $\delta \in (0, 1)$ such that the following holds: Let $\varrho_1 \in (0, \infty)$, $\varrho_2 \in [\varrho_1, \infty)$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t \in [t_0 - \varrho_1^2, t_0]}$ be a Brakke flow in $\mathbf{B}(x_0, (2+\sqrt{2\mathbf{n}})\varrho_1 + \varrho_2)$. Suppose $x_0 \in \mathrm{spt}\mu_{t_0}$ and for all $(s, y) \in (t_0 - \varrho_1^2, t_0] \times \mathbf{B}(x_0, \varrho_1)$*

$$\sup_{t \in [t_0 - \varrho_1^2, s)} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)} \varphi_{(s,y),\varrho_2} d\mu_t \leq 1 + \delta. \tag{74}$$

Then there exists a $T \in \mathbf{G}(\mathbf{n} + \mathbf{k}, \mathbf{n})$ such that both

$$\sup \Big\{ |(T^\perp)_\natural (x - x_0)|, x \in \mathrm{spt}\mu_{t_0 - \sigma\delta^2\varrho_1^2} \cap \mathbf{B}(x_0, 2\delta\varrho_1) \Big\} \leq \epsilon\delta\varrho_1,$$
$$\mu_{t_0 - \sigma\delta^2\varrho_1^2}(\mathbf{B}(x_0, \delta\varrho_1)) \leq \omega_{\mathbf{n}}(1 + \epsilon)(\delta\varrho_1)^{\mathbf{n}}.$$

Proof. We may assume $\varrho_1 = \delta^{-1}$, $t_0 = 0$ and $x_0 = 0$. Suppose the statement would be false. Then there exist $\epsilon, \sigma \in (0, 1/2)$ and for every $j \in \mathbb{N}$ we find a Brakke flow $(\nu_t^j)_{t \in [-j^2, 0]}$ in $B_j := \mathbf{B}(0, (2 + \sqrt{2\mathbf{n}})j + \varrho_j)$ and an $\varrho_j \in [j, \infty)$ such that $0 \in \mathrm{spt}\nu_0^j$,

$$\sup_{t \in [-j^2, s)} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)} \varphi_{(s,y),\varrho_j} \, \mathrm{d}\nu_t^j \leq 1 + \frac{1}{j} \tag{75}$$

for all $(s, y) \in (-j^2, 0] \times \mathbf{B}(0, j)$ and one of the following holds

$$\inf_{T \in \mathbf{G}(\mathbf{n}+\mathbf{k},\mathbf{n})} \sup \big\{ |(T^\perp)_\natural (x)|, x \in \mathrm{spt}\nu_{-\sigma} \cap \mathbf{B}(0, 2) \big\} > \epsilon \tag{76}$$

$$\nu_{-\sigma}^j(\mathbf{B}(0, 1)) > \omega_{\mathbf{n}}(1 + \epsilon). \tag{77}$$

We may assume the ϱ_j are monotonically increasing.

To obtain a converging subsequence of the (ν_t^j) we need uniform bounds on the measure of compact sets. We claim that for every $R \in (0,\infty)$ we can find a $D(R)$ such that

$$\sup_{j\in\mathbb{N}} \sup_{t\in[-1,0]} \nu_t^j(\mathbf{B}(0,R) \cap \mathbf{B}(0,j/2)) \le D(R). \tag{78}$$

First we show

$$\sup_{t\in[-1,0]} \sup_{y\in\mathbf{B}(0,j)} \sup_{R\in(0,j/4]} \nu_t^j(\mathbf{B}(y,R)) \le C_1 R^{\mathbf{n}} \tag{79}$$

for some constant $C_1 \in (1,\infty)$ and all $j \in \mathbb{N}, j \ge 2$. Set $c_2 := (2\mathbf{n})^{-1}$. To see (79) note that for $x \in \mathbf{B}(y,2R)$, $R \le j/4$ we have $\Psi_{(t,y)}(t - c_2R^2, x) \ge (4\pi c_2 R^2)^{-\frac{n}{2}} \exp(-1/c_2)$ and $\varphi_{(t,y),\varrho_j}(t - c_2R^2, x) \ge (1-1/4)^3$. Thus Lemma 2.6 and assumption (75) yield

$$\begin{aligned}\nu_t^j(\mathbf{B}(y,R)) &\le C\nu^j_{t-c_2R^2}(\mathbf{B}(y,2R)) \\ &\le CR^{\mathbf{n}} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(t,y)}\varphi_{(t,y),\varrho_j}\, \mathrm{d}\nu^j_{t-c_2R^2} \le CR^{\mathbf{n}}.\end{aligned}$$

To prove (78) note that by Lemma 2.6 we can estimate

$$\nu_t^j(\mathbf{B}(0,j/2)) \le \nu_t^j(\mathbf{B}(0,2\sqrt{\mathbf{n}}j)) \le \nu_{-1}^j(\mathbf{B}(0,4\sqrt{\mathbf{n}}j)) =: D_j(R)$$

for all $t \in [-1,0]$. Combined with (79) this proves (78).

Now we can use the compactness theorem by Ilmanen, Theorem 2.11 with $U_j = \mathbf{B}(0,j/2)$, to see that a subsequence of the (ν_t^j) converges to a Brakke flow $(\nu_t)_{t\in[-1,0]}$ in $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Note that we may assume that the whole sequence converges. In particular

$$\nu_t(\phi) = \lim_{\substack{j\to\infty \\ j\ge j_0}} \nu_t^j(\phi) \quad \text{for all } \phi \in \mathcal{C}_c^0(\mathbf{B}(0,j_0/2)) \tag{80}$$

for all $t \in [-1,0]$ and all $j_0 \in \mathbb{N}$. Combining this with (79) yields

$$\sup_{t\in[-1,0]} \sup_{y\in\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \sup_{R\in(0,\infty)} \nu_t(\mathbf{B}(y,R)) \le 2C_1 R^{\mathbf{n}}. \tag{81}$$

Next we want to show

$$\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)}\, \mathrm{d}\nu_t \le 1 \tag{82}$$

for all $(s,y) \in (-1,0] \times \mathbb{R}^{\mathbf{n+k}}$ and all $t \in [-1,s)$. To see this fix s, y, and t like that. First we see that by (81) we have

$$\int_{\mathbb{R}^{\mathbf{n+k}}} \Phi_{(s,y)} \,\mathrm{d}\nu_t < \infty. \tag{83}$$

In order to prove (83) consider $f_l : \mathbb{R}^{\mathbf{n+k}} \to \mathbb{R}^+$ given by $f_l(x) := \Phi_{(s,y)}(t,x)$ for $|x-y| < l$ and $f_l \equiv 0$ outside $\mathbf{B}(y,l)$. Obviously we have $f_{l+1} \geq f_l$. Now we can use (81) to estimate $\nu_t(\mathbf{B}(y,2l)) \leq 2C_1(2l)^{\mathbf{n}}$ for all $l \in \mathbb{N}$. Then for $l \geq l_0$ we can estimate

$$\begin{aligned}&\int_{\mathbb{R}^{\mathbf{n+k}}} f_{l+1} \,\mathrm{d}\nu_t - \int_{\mathbb{R}^{\mathbf{n+k}}} f_l \,\mathrm{d}\nu_t \leq \int_{\mathbf{B}(y,l+1)\setminus \mathbf{B}(y,l)} \Phi_{(s,y)} \,\mathrm{d}\nu_t \\ &\leq C(s-t)^{-\mathbf{n}/2} \exp(-l^2/(4(s-t)))\nu_t(\mathbf{B}(y,2l)) \leq l^{-3-\mathbf{n}}\nu_t(\mathbf{B}(y,2l)) \leq l^{-2},\end{aligned}$$

where we chose l_0 large enough depending on $s-t$. Thus $\lim_{l\to\infty} \int f_l d\nu_t < \infty$ and the monotone convergence theorem implies (83).

We continue to prove (82). Let $\gamma \in (0,1/2)$ be arbitrary. Note that y, s, t are still fixed. Using (83) and (80) we find $j_1, j_2, j_3 \in \mathbb{N}$, $j_3 > j_2 > j_1$ such that

$$\int_{\mathbb{R}^{\mathbf{n+k}}\setminus \mathbf{B}(y,\varrho_{j_1})} \Phi_{(s,y)} \,\mathrm{d}\nu_t \leq \gamma,$$

$$\int_{\mathbf{B}(y,\varrho_{j_1})} \Phi_{(s,y)} \,\mathrm{d}\nu_t - \int_{\mathbf{B}(y,\varrho_{j_2})} \Phi_{(s,y)} \,\mathrm{d}\nu_t^j \leq \gamma,$$

$$1 \leq \inf_{x\in \mathbf{B}(y,\varrho_{j_2})} \varphi_{(s,y),\varrho_j}(t,x) + \gamma\varrho_{j_2}^{-\mathbf{n}}$$

for all $j \geq j_3$. Combining these estimates with (79) we obtain

$$\int_{\mathbb{R}^{\mathbf{n+k}}} \Phi_{(s,y)} \,\mathrm{d}\nu_t \leq \int_{\mathbb{R}^{\mathbf{n+k}}} \Phi_{(s,y)}\varphi_{(s,y),\varrho_j} \,\mathrm{d}\nu_t^j + (2 + C(s-t)^{-\mathbf{n}/2})\gamma$$

for all $j \geq j_3$. By (75) and as s, t, y, γ were arbitrary this establishes (82).

In view of (81) and (82) we can use Lemma 6.1 to obtain $T \in \mathbf{G}(\mathbf{n}+\mathbf{k}, \mathbf{n})$ such that

$$\nu_t = \mathscr{H}^{\mathbf{n}} \llcorner T \tag{84}$$

for all $t \in (-1,0)$. Note that by (80) and as $0 \in \mathrm{spt}\nu_0^j$ for all $j \in \mathbb{N}$ we have that a in Lemma 6.1 has to be zero. Now we want to lead this to a contradiction.

First suppose that for infinitely many j inequality (76) holds, i.e. there exists a $z_j \in \mathrm{spt}\nu_{-\sigma}^j \cap \mathbf{B}(0,1)$ such that $(T^\perp)_\natural(z_j) > \epsilon$. Consider C_2 and

α_1 from the clearing out lemma, Lemma 2.9. Choose $\tau, \eta_1 \in (0, 1/2)$ such that $4\mathbf{n}\tau < (\epsilon/4)^2$ and $C_2\eta_1^{2\alpha_1}(\epsilon/4)^2 \leq \tau$. Then Lemma 2.9 with $R = \epsilon/4$ yields that $\nu^j_{-\sigma-\tau}(\mathbf{B}(z_j, \epsilon/4)) > \eta_1$ for infinitely many j. A subsequence of the z_j converges to some $z_0 \in \mathbf{B}(0, 2)$ with $(T^\perp)_\natural(z_0) \geq \epsilon$. Consider a cut-off function $\zeta_1 \in \mathcal{C}^\infty_c(\mathbf{B}(z_0, \epsilon/2), [0, 1])$ with $\{\zeta_1 = 1\} \supset \mathbf{B}(z_0, \epsilon/3)$. Then

$$\nu^j_{-\sigma-\tau}(\zeta_1) \geq \nu^j_{-\sigma-\tau}(\mathbf{B}(z_j, \epsilon/4)) > \eta_1 > 0 = \nu_{-\sigma-\tau}(\zeta_1)$$

for infinitely many j, where we used (84) for the last equality. In view of (80) this yields a contradiction.

Now suppose that for infinitely many j inequality (77) holds. Consider $\zeta_2 \in \mathcal{C}^\infty_c(\mathbf{B}(0, \sqrt[\mathbf{n}]{1 + \epsilon/2})), [0, 1])$ with $\{\zeta_2 = 1\} \supset \mathbf{B}(z_0, 1)$. In view of (84) we can estimate

$$\nu_{-\sigma}(\zeta_2) \leq \omega_{\mathbf{n}}(1 + \epsilon/2) < \omega_{\mathbf{n}}(1 + \epsilon) \leq \nu^j_{-\sigma}(\mathbf{B}(0, 1)) \leq \nu^j_{-\sigma}(\zeta_2)$$

for infinitely many j. Again, this contradicts (80), which establishes the result. □

Proof of Theorem 1.2. We may assume $t_0 = 0$, $a = 0$ and $\rho = 1$. Let α_0 and γ_0 be from Theorem 1.1 with respect to $\lambda = 1/2$. Choose $\epsilon \in (0, \gamma_0]$ such that $2\sqrt[4]{2\epsilon^{\alpha_0}} \leq \beta$ and set $\sigma := 2\epsilon^{\alpha_0}$. Let δ be chosen with respect to ϵ and σ according to Lemma 6.2 and choose $\eta \leq \delta$, $t_1 := -2\epsilon^{\alpha_0}\delta^2$. Then Lemma 6.2 yields the existence of a $T \in \mathbf{G}(\mathbf{n}+\mathbf{k}, \mathbf{n})$ such that

$$\begin{aligned} \sup\left\{|(T^\perp)_\natural(x)|, x \in \operatorname{spt}\mu_{t_1} \cap \mathbf{B}(0, 2\delta)\right\} &\leq \epsilon\delta. \\ \mu_{t_1}(\mathbf{B}(0, \delta)) &\leq \omega_{\mathbf{n}}(1 + \epsilon)\delta^{\mathbf{n}}. \end{aligned}$$

Then Theorem 1.1 with $\varrho = \delta$ and $\gamma = \epsilon$ yields the desired graphical representation for η small enough. □

A Appendix

For a Brakke flow initial local height bounds in a certain direction yield weaker height bounds later on in a decreased region. The result below follows directly from the Brakke flow equation, which here seems to resemble the maximum principle in some sense.

A.1 Proposition. *Fore every $p \in \mathbb{N}$ there exists a $C_p \in (1, \infty)$ such that the following holds: Let $R_0 \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty)$, $y_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$, $v \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{B}(y_0, 2R_0)$. Suppose*

$$\operatorname{spt}\mu_{t_1} \cap \mathbf{B}(y_0, 2R_0) \subset \{x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : (x - y_0) \cdot v \leq 0\}.$$

Then for all $t \in [t_1, t_2]$ we have

$$\mathrm{spt}\mu_t \cap \mathbf{B}(y_0, R_0) \subset \{x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : (x - y_0) \cdot v \leq C_p(t - t_1)^p R_0^{1-2p}\}.$$

Proof. We may assume $R_0 = 4$, $t_1 = 0$, $y_0 = 0$ and $v = \mathbf{e_{n+k}}$. Set $N := \mathbf{n}+\mathbf{k}-1$. We will prove this proposition by induction. Suppose the statement is true for some $p \in \mathbb{N}$. We want to show the statement holds for $p + 1$. We can assume $t_2 \leq 16(C_{p+1})^{-1/(p+1)}$, because the statement trivially holds for all later t. For $l \in \mathbb{R}$ set

$$H^-(l) := \{(\bar{x}, h) \in \mathbb{R}^N \times \mathbb{R} : h \leq l\}.$$

Fix an arbitrary $\bar{a}_0 \in \mathbf{B}^N(0, 4)$ and let $h_0 \in [0, 4]$. By the induction assumption we see

$$\mathrm{spt}\mu_t \cap \mathbf{B}((\bar{a}_0, h_0), 2) \subset H^-(h_0 + C_p t^p) \subset H^-(h_0 + 1/2) \tag{85}$$

for all $t \in [0, t_2]$. Here we estimated $C_p t_2^p \leq (16)^p C_p (C_{p+1})^{-p/(p+1)} \leq 1/2$, for C_{p+1} large enough. In particular as $h_0 \in [0, 4]$ was arbitrary this shows

$$\mathrm{spt}\mu_t \cap (\{\bar{a}\} \times [0, 4]) \subset H^-(1/2) \tag{86}$$

for all $t \in [0, t_2]$.

Consider the function $\eta \in \mathcal{C}^{0,1}(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}, \mathbb{R}^+) \cup \mathcal{C}^\infty(\{\eta > 0\})$ given by

$$\eta(t, \bar{x}, h) := \{h - C_p|\bar{x} - \bar{a}_0|^2 t^p - C_{p+1} t^{p+1}\}_+.$$

Treat η as a function on $\mathbb{R} \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Using $\mathrm{div}_{\mu_t}((\mathbb{R}^N \times \{0\})_\natural) \leq \mathbf{n}$ and choosing C_{p+1} large enough we can estimate

$$(\partial_t - \mathrm{div}_{\mu_t} D)\eta^3(t, x) \leq 6\eta^2(t, x)(2C_p \mathbf{n} t^p - (p + 1)C_{p+1} t^p) \leq 0 \tag{87}$$

for all $t \in [0, t_2]$ and all $x \in \mathrm{spt}\mu_t$ for which the approximate tangent space exists. Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^{\mathbf{n}+\mathbf{k}}, [0, 1])$ be a cut-off function such that

$$\mathbf{B}((\bar{a}_0, 0), 1) \subset \{\chi = 1\} \subset \mathrm{spt}\chi \subset \mathbf{B}((\bar{a}_0, 0), 2).$$

In view of (85) with $h_0 = 0$ and by definition of η we see

$$\mathrm{spt}\mu_t \cap \mathrm{spt}\eta(t, \cdot) \cap \mathrm{spt}D\chi = \emptyset \tag{88}$$

for all $t \in [t_1, t_2]$. Consider the test function $\phi \in \mathcal{C}^2(\mathbb{R} \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}, \mathbb{R}^+)$ given by $\phi := \eta^3 \chi$. Using (87), (88) and the Brakke flow equation (9) we obtain

$$\mu_s(\phi(s, \cdot)) - \mu_0(\phi(0, \cdot)) \leq \int_0^s \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \chi(\partial_t - \mathrm{div}_{\mu_t} D)\eta^3 \, \mathrm{d}\mu_t \, \mathrm{d}t \leq 0$$

for all $s \in (0, t_2]$. By assumption we have $\mu_0(\phi(0, \cdot)) = 0$, hence $\phi(t, x) = 0$ for all $t \in (0, t_2]$ and all $x \in \mathrm{spt}\mu_t$. By definition of η and χ this yields that $h \leq C_{p+1} t^p$ for all $t \in (0, t_2]$ and all $(\bar{x}, h) \in \mathrm{spt}\mu_t \cap (\{\bar{a}_0\} \times [0, 1))$. In view of (86) and as $\bar{a}_0 \in \mathbf{B}^N(0, 4)$ was arbitrary, this establishes the statement for $p + 1$.

It remains to prove the statement for $p = 1$. Let $\Upsilon \in \mathcal{C}^{0,1}(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}, \mathbb{R}^+) \cup \mathcal{C}^\infty(\{\Upsilon > 0\})$ be given by

$$\Upsilon(t, \bar{x}, h) := \{(25 - |(\bar{x}, h)|^2)\{h\}_+ - C_1 t\}_+.$$

Treat Υ as a function on $\mathbb{R} \times \mathbb{R}^{\mathbf{n+k}}$. Note that $\{\Upsilon(t, \cdot) > 0\} \subset \mathbf{B}(0, 5)$. Choosing C_1 large enough we can estimate

$$(\partial_t - \mathrm{div}_{\mu_t} D)\Upsilon^3(t, x) \leq 3\Upsilon^2(t, x)(\partial_t - \mathrm{div}_{\mu_t} D)\Upsilon(t, x) \leq 0$$

for all $t \in [0, t_2]$ and all $x \in \mathrm{spt}\mu_t$ for which the approximate tangent space exists. Thus by the Brakke flow equation (9) and our initial height assumption we obtain

$$\mu_t(\Upsilon^3(t, \cdot)) \leq \mu_0(\Upsilon^3(0, \cdot)) \leq 0$$

for all $t \in (0, t_2]$. For $(\bar{x}, h) \in \mathbf{B}(0, 4)$ we can estimate $(25 - |(\bar{x}, h)|^2) \geq 1$. Hence the definition of Υ establishes the result. $\square$

A.2 Corollary. *For every $p \in \mathbb{N}$ exists a $c_p \in (0, 1)$ such that the following holds: Let $R_1, r_0 \in (0, \infty)$, $h_1 \in (0, r_0/4]$, $R_2 \in [r_0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, t_1 + c_p r_0^2)$, $x_0 \in \mathbb{R}^{\mathbf{n+k}}$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{C}(x_0, R_1 + r_0, R_2 + r_0)$. Suppose*

$$\mathrm{spt}\mu_{t_1} \cap \mathbf{C}(x_0, R_1 + r_0, R_2 + r_0) \subset \mathbf{C}(x_0, R_1 + r_0, h_1).$$

Then for all $t \in [t_1, t_2]$ and $h(t) := h_1 + (t - t_1)^p r_0^{1-2p}$ we have

$$\mathrm{spt}\mu_t \cap \mathbf{C}(x_0, R_1, R_2) \subset \mathbf{C}(x_0, R_1, h(t)).$$

Proof. We may assume $t_1 = 0$ and $x_0 = 0$. Let $p \in \mathbb{N}$ be given and C_{p+1} be the value according to Proposition A.1. First we want to show

$$\mathrm{spt}\mu_t \cap \mathbf{C}(0, R_1, r_0/2) \subset \mathbf{C}(x_0, R_1, h(t)) \tag{89}$$

for all $t \in (0, t_2]$. To see this consider arbitrary $t \in (0, t_2]$ and $x = (\hat{x}, \tilde{x}) \in \mathrm{spt}\mu_t \cap \mathbf{C}(0, R_1, r_0/2)$. We want to show $|\tilde{x}| \leq h(t)$. Suppose $\tilde{x} \neq 0$. Set $\tilde{v} := |\tilde{x}|^{-1}\tilde{x}$, $v := (0, \tilde{v}) \in \mathbb{R}^{\mathbf{n+k}}$, and $y_0 := (\hat{x}, h_1\tilde{v})$. Note that $|x - y_0| \leq r_0/2$.

For arbitrary $y = (\hat{y}, \tilde{y}) \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ we have $(y - y_0) \cdot v \leq |\tilde{y}| - h_1$, hence by our initial height bound we see

$$\mathrm{spt}\mu_{t_1} \cap \mathbf{B}(y_0, r_0) \subset \{y \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : (y - y_0) \cdot v \leq 0\}.$$

Using Proposition A.1 with $p + 1$ and $R_0 = r_0/2$ yields

$$|\tilde{x}| - h_1 = (x - y_0) \cdot v \leq (r_0/2)^{-2p-1} C_{p+1} t^{p+1} \leq C_{p+1} c_p t^p r_0^{-2p+1} \leq t^p r_0^{-2p+1}$$

for c_p small enough. As t and x were arbitrary, this establishes (89).

Now consider $z = (\hat{z}, \tilde{z}) \in \mathbf{C}(0, R_1, R_2)$ with $\tilde{z} \geq r_0/2$. By our initial height bound and $h_1 \leq r_0/4$ we have $\mathrm{spt}\mu_0 \cap \mathbf{B}(z, r_0/4) = \emptyset$. Thus Lemma 2.6 with $R = r_0/8$ implies $z \notin \mathrm{spt}\mu_t$ for all $t \in [0, t_2]$, where we used $t_2 \leq c_p r_0^2 \leq (2\mathbf{n})^{-1}(r_0/8)^2$. This establishes the result. □

Based on Huisken's monotonicity formula [Hui90, 3.1] one can obtain bounds on area ratio at later times from initial area ratio bounds.

A.3 Lemma. *For every $\epsilon \in (0, 1]$ exists a $\delta \in (0, 1)$ such that the following holds: Let $K \in [0, \infty)$, $R \in (0, \infty)$, $r \in (0, \delta R]$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1, s_1 + (8\mathbf{n})^{-1} R^2]$, $y_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t \in [s_1, s_2]}$ be a Brakke flow in $\mathbf{B}(y_0, R)$. Suppose*

$$\varrho^{-\mathbf{n}} \mu_{s_1}(\mathbf{B}(y_0, \varrho)) \leq K \quad \text{for all } \varrho \in [r, R] \tag{90}$$

Then for all $t \in [s_1, s_2]$ we have

$$r^{-\mathbf{n}} \mu_t(\mathbf{B}(y_0, r)) \leq K(R/r)^{\epsilon}.$$

Proof. We may assume $R = 4$, $y_0 = 0$, $s_1 = 0$. Let $\epsilon \in (0, 1)$ be given. Fix an arbitrary $t \in [0, s_2]$ and set $s_0 := t + r^2$. Consider $\Phi = \Phi_{(s_0, 0)}$ and $\varphi = \varphi_{(s_0,0),3}$ from Definition 2.7. By $s_0 \leq s_2 + \delta \leq 3\mathbf{n}^{-1}$ we obtain

$$\mathrm{spt}\varphi(0, \cdot) \subset\subset \mathbf{B}(0, 4), \quad \sup_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \varphi(0, \cdot) \leq C, \quad \inf_{\mathbf{B}(0,1)} \varphi(t, \cdot) \geq 2^{-3}.$$

Thus by Huiskin's monotonicity formula, Theorem 2.8

$$\int_{\mathbf{B}(0,1)} \Phi \, \mathrm{d}\mu_t \leq 8 \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi\varphi \, \mathrm{d}\mu_t \leq 8 \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi\varphi \, \mathrm{d}\mu_0 \leq C \int_{\mathbf{B}(0,4)} \Phi \, \mathrm{d}\mu_0.$$

We have $s_0 - t = r^2$, hence $\inf_{\mathbf{B}(0,r)} \Phi(t, \cdot) \geq c r^{-\mathbf{n}}$. So by the above inequality

$$r^{-\mathbf{n}} \mu_t(\mathbf{B}(0, r)) \leq C \int_{\mathbf{B}(0,4)} \Phi(0, x) \, \mathrm{d}\mu_0(x) \tag{91}$$

Set $\alpha := \epsilon/(2\mathbf{n})$. Note that by definition of Φ and $s_0 = t + r^2 \geq r^2$ we have $\Phi(0,x) \leq Cr^{-\mathbf{n}}$ for all $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Then by assumption (90) we can estimate

$$\int_{\mathbf{B}(0,r^{1-\alpha})} \Phi(0,x)\,\mathrm{d}\mu_0(x) \leq Cr^{-\mathbf{n}}\mu_0(\mathbf{B}(0,r^{1-\alpha})) \leq CKr^{-\mathbf{n}\alpha} \leq \frac{K}{2r^{\epsilon}}, \tag{92}$$

where we used $\alpha = \epsilon/(2\mathbf{n})$, $r \leq \delta$ and chose δ small enough. By the properties of the exponential function we have $r^{-\mathbf{n}/2}\exp(-r^{-2\alpha}/4) \leq c$, were we again used $r \leq \delta$ and chose δ small enough depending on α and c. Hence for $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} \setminus \mathbf{B}(0,r^{1-\alpha})$ we have $\Phi(0,x) \leq 4^{-\mathbf{n}-1}$. Then by assumption (90) we can estimate

$$\int_{\mathbf{B}(0,4)\setminus\mathbf{B}(0,r^{1-\alpha})} \Phi(0,x)\,\mathrm{d}\mu_0(x) \leq 4^{-\mathbf{n}-1}\mu_0(\mathbf{B}(0,4)) \leq K/4$$

Inserting this and (92) into (91) establishes the result. □

A.4 Lemma. *There exists a constant $C \in (1,\infty)$ such that the following holds: Let $\varrho, M, \kappa \in (0,\infty)$, $\Lambda \in [1,\infty)$, $\delta \in (0,1/2]$, $s_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let μ be a Radon measure on $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Suppose $\delta \leq \min\{\Lambda^{-1}, C^{-1}\}$,*

$$\mu\left(\mathbf{B}(y_0, C\Lambda\varrho)\right) \leq M\varrho^{\mathbf{n}}, \tag{93}$$

$$\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s_0,y_0)}\varphi_{(s_0,y_0),\Lambda\varrho}(s_0-\varrho^2,x)\,\mathrm{d}\mu(x) \leq 1+\kappa. \tag{94}$$

Then for all $(s,y) \in (s_0-\delta^2\varrho^2, s_0] \times \mathbf{B}(y_0,\delta\varrho)$ we have

$$\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)}\varphi_{(s,y),\Lambda\varrho}(s_0-\varrho^2,x)\,\mathrm{d}\mu(x) \leq 1+\kappa+CM\Lambda\delta.$$

Proof. We may assume $s_0 = 0$, $y_0 = 0$ and $\varrho = 1$. Fix $(s,y) \in (-\delta^2, 0] \times \mathbf{B}(0,\delta)$. Note that $\mathrm{spt}(\varphi_{(s,y),\Lambda}(-1,\cdot)) \subset \mathbf{B}(0,(2\mathbf{n}+1)\Lambda)$. Let $x \in \mathbf{B}(0,(2\mathbf{n}+1)\Lambda)$. Direct calculations yield

$$\begin{aligned} 1 \leq (s+1)^{-\mathbf{n}/2} &\leq 1 + C\sqrt{-s} \leq 1 + C\delta \\ \exp\left(\left|\frac{|x|^2}{4} - \frac{|x-y|^2}{4(s+1)}\right|\right) &\leq \exp\left(C(\Lambda|y| + \Lambda^2|s|)\right) \leq 1 + C\Lambda\delta, \end{aligned}$$

where we used $\delta \leq \min\{\Lambda^{-1}, C^{-1}\}$. Thus we have

$$\begin{aligned} |\Phi_{(0,0)}(-1,x) - \Phi_{(s,y)}(-1,x)| &\leq C\Lambda\delta \\ |\varphi_{(0,0),\Lambda}(-1,x) - \varphi_{(s,y),\Lambda}(-1,x)| &\leq C\delta \end{aligned}$$

Combined with (93) and (94) this yields the result. □

A.5 Remark. Here we want to derive Theorem 2.11 from Ilmanen's work [Ilm94]. In case $U_i \equiv U$ the result directly follows from the proof of [Ilm94, 7.1]. Now consider the general case. We can find a subsequence $\lambda_1 : \mathbb{N} \to \mathbb{N}$ and a Brakke flow $(\nu_t^1)_{t\in[t_1,t_2]}$ in U_1 such that $\lim_{j\to\infty} \mu_t^{\lambda_1(j)}(\phi) = \nu_t^1(\phi)$ for all $\phi \in \mathcal{C}_c^0(U_1)$, for all $t \in [t_1, t_2]$. Inductively for all $l \in \mathbb{N}$, $l \geq 2$ we can find a subsequence $\lambda_l : \mathbb{N} \to \lambda_{l+1}[\mathbb{N}]$ and a Brakke flow $(\nu_t^l)_{t\in[t_1,t_2]}$ in U_l such that

$$\lim_{j\to\infty} \mu_t^{\lambda_l(j)}(\phi) = \nu_t^l(\phi) \quad \text{for all } \phi \in \mathcal{C}_c^0(U_l)$$

for all $t \in [t_1, t_2]$. In particular we have $\nu_t^{l_2} \llcorner U_{l_0} = \nu_t^{l_1} \llcorner U_{l_0}$ for all $l_0 \leq l_1 \leq l_2$ and all $t \in [t_1, t_2]$. Then $\mu_t(A) := \lim_{l\to\infty} \nu_t^l(A \cap U_l) \in [0, \infty]$ is well defined and gives the desired Brakke flow on U. With $\sigma(j) = \lambda_j(j)$ this establishes the result.

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