

# ON THE STRUCTURE OF LINEARIZED GRAVITY ON VACUUM SPACETIMES OF PETROV TYPE D

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ABSTRACT. In this paper we prove a new identity for linearized gravity on vacuum spacetimes of Petrov type D. The new identity yields a covariant version of the Teukolsky-Starobinsky identities for linearized gravity, which in addition to the two classical identities for the extreme linearized Weyl scalars includes three additional equations. By analogy with the spin-1 case, we expect the new identity to be relevant in deriving new conservation laws for linearized gravity.

## 1. INTRODUCTION

The black hole stability problem, i.e. the problem of proving dynamical stability of the Kerr vacuum black hole solution is one of the most important open problems in general relativity. Proving dispersive estimates for fields, including Maxwell and linearized gravity on the Kerr background is an essential step towards solving this problem. In the paper [4] a new conserved energy-momentum tensor for the Maxwell field on the Kerr spacetime was constructed. This construction relies on an analysis of the Teukolsky Master Equation (TME) and the Teukolsky-Starobinsky Identities (TSI) implied by the Maxwell field equations. In this paper we shall, motivated by the above considerations, study the TSI for linearized gravity on vacuum spacetimes of Petrov type D, which includes in particular the Kerr spacetime. Our main result is a fundamental new identity for linearized gravity which leads to a new, covariant form of the spin-2 TSI. This extends the two classically known TSI relations by three additional identities. In the following, we shall refer to the two classical TSI as the extreme TSI.

In its classical form, the Teukolsky-Press, or Teukolsky-Starobinsky relations [19, 18] relate the solutions of the radial Teukolsky equations for  $\pm s$ , and is thus valid only for the separated form of the equations. For the case of linearized gravity on the Kerr spacetime, a derivation of the extreme TSI using the Newman-Penrose formalism, which does not require a separation of variables, was given by Torres del Castillo [20], later corrected by Silva-Ortigoza [17], see the paper by Whiting and Price [22] for discussion and background. We shall refer to this pair of classical identities for the Maxwell and linearized gravity cases as the extreme TSI.

Consider a Petrov type D spacetime and let a principal null tetrad be given<sup>1</sup>. Recall that in this case, only one of the Newman-Penrose Weyl scalars,  $\Psi_2$  is non-zero. Let  $\kappa_1 \propto \Psi_2^{-1/3}$ , let  $\phi_i$ ,  $i = 0, 1, 2$  the Maxwell scalars and let  $\dot{\Psi}_0, \dot{\Psi}_4$  be the gauge invariant linearized Weyl scalars of spin weights  $\pm 2$ . In terms of the GHP operators  $\mathfrak{p}, \mathfrak{p}', \bar{\delta}, \bar{\delta}'$  the TME take the form

$$(\mathfrak{p} - \rho - \bar{\rho}) \mathfrak{p}'(\kappa_1 \phi_0) - (\bar{\delta} - \tau - \bar{\tau}') \bar{\delta}'(\kappa_1 \phi_0) = 0, \quad (1.1a)$$

$$(\mathfrak{p}' - \rho' - \bar{\rho}') \mathfrak{p}(\kappa_1 \phi_2) - (\bar{\delta}' - \bar{\tau} - \tau') \bar{\delta}(\kappa_1 \phi_2) = 0. \quad (1.1b)$$

for the spin-1 case, while the spin-2 TME for linearized gravity are given by

$$(\mathfrak{p} - 3\rho - \bar{\rho}) \mathfrak{p}'(\kappa_1 \dot{\Psi}_0) - (\bar{\delta} - 3\tau - \bar{\tau}') \bar{\delta}'(\kappa_1 \dot{\Psi}_0) - 3\Psi_2(\kappa_1 \dot{\Psi}_0) = 0, \quad (1.2a)$$

$$(\mathfrak{p}' - 3\rho' - \bar{\rho}') \mathfrak{p}(\kappa_1 \dot{\Psi}_4) - (\bar{\delta}' - 3\tau' - \bar{\tau}) \bar{\delta}(\kappa_1 \dot{\Psi}_4) - 3\Psi_2(\kappa_1 \dot{\Psi}_4) = 0, \quad (1.2b)$$

see [2]. The spin-1 extreme TSI are

$$\bar{\delta}' \bar{\delta}'(\kappa_1^2 \phi_0) = \mathfrak{p} \mathfrak{p}(\kappa_1^2 \phi_2) \quad (1.3a)$$

$$\mathfrak{p}' \mathfrak{p}'(\kappa_1^2 \phi_0) = \bar{\delta} \bar{\delta}(\kappa_1^2 \phi_2) \quad (1.3b)$$

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<sup>1</sup>In this paper, we shall use the GHP formalism and 2-spinor formalisms, see [3, 7, 15] for notation and background.

and the spin-2 extreme TSI are

$$\bar{\partial}'\bar{\partial}'\bar{\partial}'(\kappa_1^4\dot{\Psi}_0) = \flat\flat\flat(\kappa_1^4\dot{\Psi}_4) + \kappa_1^3\Psi_2\mathcal{L}_\xi\overline{\dot{\Psi}_0} \quad (1.4a)$$

$$\flat\flat\flat\flat(\kappa_1^4\dot{\Psi}_0) = \bar{\partial}\bar{\partial}\bar{\partial}\bar{\partial}(\kappa_1^4\dot{\Psi}_4) - \kappa_1^3\Psi_2\mathcal{L}_\xi\overline{\dot{\Psi}_4} \quad (1.4b)$$

where in the Kerr spacetime,  $\kappa_1^3\Psi_2 = M/27$ , with  $M$  the ADM mass, and the Killing field  $\xi^a \propto (\partial_t)^a$ , cf. Remark 2.2. A version of the spin-1 TSI in Newman-Penrose notation can be found in [10]. For the extreme spin-2 TSI, see [22]. However, this form does not make the  $\mathcal{L}_\xi$  term explicit. In general  $\kappa_1^3\Psi_2$  and  $\xi^a$  are a complex constant and Killing field, respectively. See (2.2) for the definition of  $\xi^a$ . Here we have stated the form of the equations for the case without sources. The general version of the TME, with sources, is given in Corollary 3.3, and the a version of TSI with sources can be deduced from Remark 4.3.

A naïve argument counting degrees of freedom indicates that it should be possible to represent the full dynamical degrees of freedom of the Maxwell and linearized gravity fields in terms of one complex scalar potential, solving the TME. However, although one may use a Debye potential construction to produce a new Maxwell or linearized gravity field from a solution of the second order TME [9, 12, 11, 13], both Maxwell and linearized gravity on a type D background have conserved charges, which correspond to non-radiative modes, and these cannot be represented in terms of Debye potentials. For the Maxwell field on the Kerr spacetime, the non-radiative mode is the well-known Coulomb solution, while for linearized gravity, the non-radiative modes correspond to variations of the moduli parameters  $M, a$ . Thus, it is only the radiative modes which can be represented by Debye potentials. In particular, it must be emphasized that the TME is not equivalent to the Maxwell or linearized gravity systems, and in particular, if we consider a Maxwell or linearized gravity field, it is not possible to reconstruct either of these fields given only one of the extreme scalars.

For the Maxwell field, the full set of TSI in fact contains a third relation, cf. [10], which can be written in the form

$$\bar{\tau}'\flat(\kappa_1^2\phi_2) + \flat\bar{\partial}(\kappa_1^2\phi_2) = \bar{\tau}'\flat'(\kappa_1^2\phi_0) + \flat'\bar{\partial}'(\kappa_1^2\phi_0). \quad (1.5)$$

As mentioned above, the full set of TME/TSI equations implied by the Maxwell field equation, has the important consequence that the symmetric tensor  $V_{ab}$  introduced in [4] is conserved. The tensor  $V_{ab}$  is, in contrast to the standard Maxwell stress-energy tensor independent of the non-radiative modes of the Maxwell field, and is therefore a suitable tool to construct dispersive estimates for the Maxwell field on the Kerr spacetime.

We now recall the Debye potential construction of solutions of linearized gravity. We first restrict to Minkowski space. Following Sachs and Bergmann [16], let  $\mathcal{H}_{abcd}$  be an anti-selfdual Weyl field<sup>2</sup>, i.e. a tensor with the symmetries of the Riemann tensor,  $\mathcal{H}_{abcd} = \mathcal{H}_{[ab]cd} = \mathcal{H}_{cdab}$ ,  $\mathcal{H}_{[abc]d} = 0$ , satisfying  $\mathcal{H}^a{}_{bac} = 0$  and  $\frac{1}{2}\epsilon_{ab}{}^{ef}\mathcal{H}_{efcd} = -i\mathcal{H}_{abcd}$ , and let

$$\mathfrak{g}_{ab} = \nabla^e\nabla^d\mathcal{H}_{acbd}. \quad (1.6)$$

Then, if  $\nabla^e\nabla_e\mathcal{H}_{abcd} = 0$ , it holds that  $\mathfrak{g}_{ab}$  solves the linearized vacuum Einstein equation.

The analogous construction for massless spin- $s$  fields on the 4-dimensional Minkowski space was discussed by Penrose [14]. In [6] this was used to prove decay estimates for such fields, based on decay estimates for the wave equation. We shall now describe the analog of the Sachs-Bergmann construction in the case of a vacuum type D metric.

Introduce the following complex anti-selfdual tensors with the symmetries of the Weyl tensor

$$\begin{aligned} Z_{abcd}^0 &= 4\bar{m}_{[a}n_{b]}\bar{m}_{[c}n_{d]}, \\ Z_{abcd}^4 &= 4l_{[a}m_{b]}l_{[c}m_{d]}, \end{aligned}$$

where  $(l^a, n^a, m^a, \bar{m}^a)$  constitutes a principal null tetrad. These are analogs of the anti-selfdual bivectors  $Z_{ab}^0, Z_{ab}^2$ , see [3, §2]. For a weighted scalar  $\chi_0$ , let  $\mathcal{H}_{abcd}$  be given by

$$\mathcal{H}_{abcd} = \kappa_1^4\chi_0 Z_{abcd}^0. \quad (1.7)$$

<sup>2</sup>Here we use a complex anti-selfdual Weyl field for consistency with the rest of the paper, although this is not used in [16].

Define the 1-form  $U_a$  by

$$U_a = -\nabla_a \log(\kappa_1),$$

cf. (2.7), and consider the following analog of (1.6), which sends  $\mathcal{H}_{abcd}$  to a 2-tensor  $\mathfrak{g}_{ab}$ ,

$$\mathfrak{g}_{ab} = \nabla^c(\nabla_d + 4U_d)\mathcal{H}_{(a}{}^d{}_{b)c}.$$

A calculation shows that  $\mathfrak{g}_{ab}$  is a complex solution to the linearized vacuum Einstein equation provided the scalar  $\chi_0$  solves the Teukolsky Master Equation (TME) for spin weight  $+2$  [12].

Applying the GHP prime operation  $l^a \leftrightarrow n^a$ ,  $m^a \leftrightarrow \bar{m}^a$  to the above construction, leads to the fact that the analogous construction with

$$\mathcal{H}_{abcd} = \kappa_1^4 \chi_4 Z_{abcd}^4 \quad (1.8)$$

yields a solution to the linearized Einstein equation in the same way, provided that now the scalar  $\kappa_1^4 \chi_4$  solves the TME for spin weight  $-2$ . Note that in general, the linearized metrics  $\mathfrak{g}_{ab}$  constructed from (1.7) and (1.8) are different. We are now able to state the tensor version of our main result, which describes this difference.

**Theorem 1.1** (Tensor version). *Let  $\delta g_{ab}$  be a solution to the linearized Einstein equation on a vacuum type D background, and let  $\dot{\Psi}_0, \dot{\Psi}_4$  be the corresponding linearized Weyl scalars of spin weight  $\pm 2$ . Let*

$$\widehat{\mathcal{H}}_{abcd} = \kappa_1^4 \dot{\Psi}_0 Z_{abcd}^0 - \kappa_1^4 \dot{\Psi}_4 Z_{abcd}^4, \quad (1.9)$$

and let

$$\mathcal{M}_{ab} = \nabla^c(\nabla_d + 4U_d)\widehat{\mathcal{H}}_{(a}{}^d{}_{b)c}. \quad (1.10)$$

Then there is a complex vector field  $\mathcal{A}_a$ , depending on up to three derivatives of the linearized metric  $\delta g_{ab}$ , and independent of the extreme linearized Weyl scalars  $\dot{\Psi}_0, \dot{\Psi}_4$ , such that

$$\mathcal{M}_{ab} = \nabla_{(a}\mathcal{A}_{b)} + \frac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\delta g_{ab}. \quad (1.11)$$

**Remark 1.2.** (1) From (1.11), it is clear that  $\mathcal{M}_{ab}$  is a complex solution of the linearized Einstein equation.

- (2) In the Maxwell case, the analogue of (1.11) is that the vector potential arising out of the Debye potential construction by taking the difference of the extreme Maxwell scalars from the same Maxwell field is pure gauge, see [1, Proposition 5.2.5]. In the spin-2 case above, the term involving  $\mathcal{L}_\xi\delta g_{ab}$  is a new feature, which indicates an important qualitative difference between the spin-1 and spin-2 cases.

As will be shown below, the identity (1.11) implies the the full TSI for linearized gravity, a set of five scalar equations of fourth order. To see that the extreme TSI follows from this identity, it is sufficient to recall that the extreme linearized Weyl scalars  $\dot{\Psi}_0, \dot{\Psi}_4$  are gauge invariant.

In order to discuss further details of the main result and its proof, we shall use the 2-spinor formalism, see [15] and [7] for background. We shall further make use of the variational formalism for spinors, which was introduced in [8].

The fundamental operators  $\mathcal{E}_{k,l}, \mathcal{E}_{k,l}^\dagger, \mathcal{F}_{k,l}, \mathcal{D}_{k,l}$  acting on symmetric spinors of valence  $k, l$  are defined as the irreducible parts of the covariant derivative  $\nabla_{AA'}\phi_{BC\dots DB'C'\dots D'}$  of the symmetric spinor  $\phi_{AB\dots DA'B'\dots D'}$  of valence  $k, l$ . They were introduced in the paper [5], which also contains a detailed discussion of their properties. The main features of the type D geometry is encoded in the Killing spinor  $\kappa_{AB} = \kappa_1 o_{(A}l_{B)}$ , satisfying

$$(\mathcal{F}_{2,0}\kappa)_{ABCA'} = 0.$$

We shall also make use of the extended operators  $\mathcal{E}_{k,l,m}, \mathcal{E}_{k,l,m}^\dagger, \mathcal{F}_{k,l,m}, \mathcal{D}_{k,l,m}$ , introduced here for the first time, defined in an analogous manner as the irreducible parts of

$$(\nabla_{AA'} + mU_{AA'})\phi_{BC\dots DB'C'\dots D'}, \quad (1.12)$$

see Section 2.2 below for details.

Given a 2-tensor  $\delta g_{ab}$  solving the linearized Einstein equations, define

$$G = \delta g^C{}_{C'}{}^{C'}{}_{C'}, \quad G_{ABA'B'} = \delta g_{(AB)(A'B')},$$

and let

$$\phi_{ABCD} = \frac{1}{2}(\mathcal{E}_{3,1}\mathcal{E}_{2,2}G)_{ABCD}. \quad (1.13)$$

Then  $\phi_{ABCD}$  is a modified linearized Weyl spinor, see (3.3). The extreme scalars  $\phi_0, \phi_4$  coincide with  $\dot{\Psi}_0, \dot{\Psi}_4$  introduced above.

**Theorem 1.1'** (Spinor version). *Let  $\delta g_{ABA'B'}$  be a solution to the linearized Einstein equation on a vacuum type D background, let  $\phi_{ABCD}$  be given by (1.13), and let  $\phi_0, \phi_4$  be the corresponding linearized Weyl scalars of spin weight  $\pm 2$ . Let  $o_A, \iota_A$  be a principal dyad. Define*

$$\widehat{\phi}_{ABCD} = \iota_A \iota_B \iota_C \iota_D \kappa_1^4 \phi_0 - o_A o_B o_C o_D \kappa_1^4 \phi_4, \quad (1.14)$$

and let

$$\mathcal{M}_{ABA'B'} = (\mathcal{C}_{3,1}^\dagger \mathcal{C}_{4,0,4}^\dagger \widehat{\phi})_{ABA'B'}. \quad (1.15)$$

Then, there is a complex vector field  $\mathcal{A}_{AA'}$  depending on up to three derivatives of the linearized metric  $\delta g_{ABA'B'}$ , and independent of the extreme linearized Weyl scalars  $\phi_0, \phi_4$ , such that

$$\mathcal{M}_{ABA'B'} = \frac{1}{2} \nabla_{AA'} \mathcal{A}_{BB'} + \frac{1}{2} \nabla_{BB'} \mathcal{A}_{AA'} + \frac{1}{2} \Psi_2 \kappa_1^3 (\mathcal{L}_\xi \delta g)_{ABA'B'}. \quad (1.16)$$

**Remark 1.3.** Equation (1.14) can be written purely in terms of the Killing spinor  $\kappa_{AB}$ , without reference to a specific dyad, see Example 2.8.

We now apply two more derivatives to both sides of equation (1.16) (or equivalently, to equation (1.11)). This gives the following result, which is the covariant form of the spin-2 TSI.

**Corollary 1.4.** *For a vacuum type D spacetime, we have the covariant generalization of the Teukolsky–Starobinski identities for linearized gravity*

$$(\mathcal{C}_{1,3}^\dagger \mathcal{C}_{2,2}^\dagger \mathcal{C}_{3,1}^\dagger \mathcal{C}_{4,0,4}^\dagger \widehat{\phi})_{A'B'C'D'} = \kappa_1^3 \Psi_2 (\mathcal{L}_\xi \bar{\phi})_{A'B'C'D'} + (\hat{\mathcal{L}}_A \bar{\Psi})_{A'B'C'D'}, \quad (1.17)$$

where  $\phi_{ABCD}$  is defined in (1.13), and

$$(\hat{\mathcal{L}}_A \bar{\Psi})_{A'B'C'D'} = \frac{1}{2} \bar{\Psi}_{A'B'C'D'} (\mathcal{D}_{1,1,0,6} \mathcal{A}) + 2 \bar{\Psi}_{(A'B'C'}{}^{F'} (\mathcal{C}_{1,1,0,2}^\dagger \mathcal{A})_{D')F'}. \quad (1.18)$$

is the Lie derivative on spinors.

**Remark 1.5.** *Given a pair  $\phi_0, \phi_4$  of spin weight  $\pm 2$  scalars, we say that they solve the spin-2 TME/TSI system, provided they solve the second order spin-2 TME system (1.2) and also that they solve the fourth order spin-2 TSI system (1.17), or alternatively the second order identity (1.16), in a suitable sense. In view of the result by Coll et al. mentioned above, it is natural to conjecture at this point that the combined spin-2 TME/TSI system (1.2) and (1.17) (or alternatively (1.2) and (1.16)) yields an equivalent system for linearized gravity, in the sense that given such a pair of scalars, it is possible, modulo gauge conditions and charges, to reconstruct the rest of the linearized gravitational field.*

Recall that in the case of the Kerr spacetime,  $\kappa_1^3 \Psi_2$  and  $\xi^a$  are real. This means that we can take the real and imaginary parts of  $\mathcal{M}_{ab}$ , yielding the following result.

**Corollary 1.6.** *On the Kerr spacetime, the imaginary part of (1.16) is gauge invariant and for the real part we get*

$$\text{Re} \mathcal{M}_{ABA'B'} = \frac{1}{2} \nabla_{AA'} \text{Re} \mathcal{A}_{BB'} + \frac{1}{2} \nabla_{BB'} \text{Re} \mathcal{A}_{AA'} + \frac{1}{54} M (\mathcal{L}_\xi \delta g)_{ABA'B'}. \quad (1.19)$$

**Remark 1.7.** *The spinor  $\text{Re} \mathcal{M}_{ABA'B'}$  can be interpreted as a linearized metric. It is independent of the coordinate gauge of the original metric and constructed only from the spin-2 part of the linearized Weyl curvature. Therefore it is gauge invariant. With a particular gauge choice one could hope to set  $\text{Re} \mathcal{A}_{AA'} = 0$ . One could then hope to reconstruct  $\delta g_{ABA'B'}$  from the curvature through integration in time.*

The paper is organized as follows. In Section 2 we give some background and preliminary results. Section 2.1 contains a review of the consequences of the existence of a Killing spinor on vacuum type D spacetimes. In particular we introduce extended fundamental operators and projection operators based on the Killing spinor. Example 2.8 introduces the spin-projected, sign-reversed linearized Weyl spinor  $\widehat{\phi}_{ABCD}$  which will play a central role in the paper. The section ends with two technical lemmata. In Section 3 a spinorial form of the field equations of linearized gravity is presented. Corollary 3.2 gives an equation for  $\widehat{\phi}_{ABCD}$  which is a consequence of the linearized Bianchi identity. This equation plays a central role in the proof of the main theorem,

given in section 4. In Appendix A we give the GHP form of the vector field  $A_{AA'}$  introduced in the main theorem, and of the spin-2 TSI.

## 2. PRELIMINARIES

**2.1. Geometric structure of Petrov type D spacetimes.** It is well known [21] that vacuum spacetimes of Petrov type D admit a non-trivial irreducible symmetric 2-spinor  $\kappa_{AB}$  solving the Killing spinor equation

$$(\mathcal{T}_{2,0}\kappa)_{ABCA'} = 0. \quad (2.1)$$

Defining the spinors

$$\xi_{AA'} = (\mathcal{C}_{2,0}^\dagger \kappa)_{AA'}, \quad (2.2)$$

$$\lambda_{A'B'} = (\mathcal{C}_{1,1}^\dagger \mathcal{C}_{2,0}^\dagger \kappa)_{A'B'}, \quad (2.3)$$

the complete table of derivatives reads

$$\nabla_{AA'} \kappa_{BC} = -\frac{1}{3} \xi_{CA'} \epsilon_{AB} - \frac{1}{3} \xi_{BA'} \epsilon_{AC}, \quad (2.4a)$$

$$\nabla_{AA'} \xi_{LL'} = -\frac{1}{2} \lambda_{A'L'} \epsilon_{AL} - \frac{3}{4} \kappa^{BC} \Psi_{ALBC} \bar{\epsilon}_{A'L'}, \quad (2.4b)$$

$$\nabla_{CC'} \lambda_{A'B'} = 2 \xi_C^{D'} \bar{\Psi}_{A'B'C'D'}. \quad (2.4c)$$

The fact that the system of equations (2.4) for  $(\kappa_{AB}, \xi_{AA'}, \lambda_{A'B'})$  is closed, implies in particular higher derivatives of  $\kappa_{AB}$  do not give any further information.

**Remark 2.1.** *If we furthermore assume the generalized Kerr-NUT condition that  $\xi_{AA'}$  is real, the middle equation simplifies to  $\lambda_{A'L'} = \frac{3}{2} \bar{\kappa}^{B'C'} \bar{\Psi}_{A'L'B'C'}$  so the  $\lambda_{A'B'}$  is not an independent field any more and the system reduces to*

$$\nabla_{AA'} \kappa_{BC} = -\frac{1}{3} \xi_{CA'} \epsilon_{AB} - \frac{1}{3} \xi_{BA'} \epsilon_{AC}, \quad (2.5a)$$

$$\nabla_{AA'} \xi_{LL'} = -\frac{3}{4} \bar{\kappa}^{B'C'} \bar{\Psi}_{A'L'B'C'} \epsilon_{AL} - \frac{3}{4} \kappa^{BC} \Psi_{ALBC} \bar{\epsilon}_{A'L'}. \quad (2.5b)$$

Using a principal dyad the Killing spinor takes the form

$$\kappa_{AB} = -2\kappa_1 o_{(A} l_{B)}, \quad (2.6)$$

with  $\kappa_1 \propto \Psi_2^{-1/3}$ . Beside the Killing vector field (2.2) another important vector field is defined by

$$U_{AA'} = -\frac{\kappa_{AB} \xi^B{}_{A'}}{3\kappa_1^2} = -\nabla_{AA'} \log(\kappa_1). \quad (2.7)$$

Because it is completely determined by the Killing spinor (2.6), we have the complete table of derivatives

$$(\mathcal{D}_{1,1}U) = -2\Psi_2 + \frac{\xi_{AA'} \xi^{AA'}}{9\kappa_1^2}, \quad (2.8a)$$

$$(\mathcal{C}_{1,1}U)_{AB} = 0, \quad (2.8b)$$

$$(\mathcal{C}_{1,1}^\dagger U)_{A'B'} = 0, \quad (2.8c)$$

$$(\mathcal{T}_{1,1}U)_{AB}{}^{A'B'} = \frac{\kappa_{AB} (\mathcal{C}_{1,1}^\dagger \xi)^{A'B'}}{6\kappa_1^2} + 2U_{(A} U_{B)}{}^{B')} - \frac{\xi_{(A} \xi_{B)}^{B'}}{9\kappa_1^2}, \quad (2.8d)$$

in particular  $U_{AA'}$  is closed.

From the integrability condition  $(\mathcal{L}_\xi \Psi)_{ABCD} = 0$  it follows that

$$(\mathcal{T}_{4,0}\Psi)_{ABCDFA'} = 5\Psi_{(ABCD} U_{F)A'}. \quad (2.9)$$

The curvature can be expressed in terms of the Killing spinor according to

$$\Psi_{ABCD} = \frac{3\Psi_2 \kappa_{(AB} \kappa_{CD)}}{2\kappa_1^2}. \quad (2.10)$$

**Remark 2.2.** *On Kerr with parameters  $(M, a)$  in Boyer-Lindquist coordinates  $\Psi_2 = -\frac{M}{(r-ia \cos \theta)^3}$  and we can set  $\kappa_1 = -\frac{1}{3}(r-ia \cos \theta)$ . Then  $\xi^a = \partial_t$ ,  $\Psi_2 \kappa_1^3 = \frac{1}{27}M$ .*

**2.2. Extended fundamental spinor operators.** Based on the irreducible decomposition of covariant derivatives on symmetric spinors, see [5], we define the extended fundamental operators with additional (extended) indices  $n, m$ ,

$$(\mathcal{D}_{k,l,n,m}\varphi)_{A_1\dots A_{k-1}}^{A'_1\dots A'_{l-1}} \equiv \left[ \nabla^{BB'} + nU^{BB'} + m\bar{U}^{BB'} \right] \varphi_{A_1\dots A_{k-1}B}^{A'_1\dots A'_{l-1}B'}, \quad (2.11a)$$

$$(\mathcal{C}_{k,l,n,m}\varphi)_{A_1\dots A_{k+1}}^{A'_1\dots A'_{l-1}} \equiv \left[ \nabla_{(A_1}{}^{B'} + nU_{(A_1}{}^{B'} + m\bar{U}_{(A_1}{}^{B')} \right] \varphi_{A_2\dots A_{k+1})}^{A'_1\dots A'_{l-1}B'}, \quad (2.11b)$$

$$(\mathcal{C}_{k,l,n,m}^\dagger\varphi)_{A_1\dots A_{k-1}}^{A'_1\dots A'_{l+1}} \equiv \left[ \nabla^{B(A'_1} + nU^{B(A'_1} + m\bar{U}^{B(A'_1} \right] \varphi_{A_1\dots A_{k-1}B}^{A'_2\dots A'_{l+1})}, \quad (2.11c)$$

$$(\mathcal{T}_{k,l,n,m}\varphi)_{A_1\dots A_{k+1}}^{A'_1\dots A'_{l+1}} \equiv \left[ \nabla_{(A_1}{}^{(A'_1} + nU_{(A_1}{}^{(A'_1} + m\bar{U}_{(A_1}{}^{(A'_1} \right] \varphi_{A_2\dots A_{k+1})}^{A'_2\dots A'_{l+1})}. \quad (2.11d)$$

For  $n = m = 0$  it coincides with the usual definition of the fundamental operators and the indices will be suppressed in that case. Because  $U_{AA'}$  is a logarithmic derivative we have

$$(\mathcal{D}_{k,l,n,m}\varphi)_{A_1\dots A_{k-1}}^{A'_1\dots A'_{l-1}} = \kappa_1^n \bar{\kappa}_1^m (\mathcal{D}_{k,l}\kappa_1^{-n} \bar{\kappa}_1^{-m} \varphi)_{A_1\dots A_{k-1}}^{A'_1\dots A'_{l-1}}, \quad (2.12a)$$

$$(\mathcal{C}_{k,l,n,m}\varphi)_{A_1\dots A_{k+1}}^{A'_1\dots A'_{l-1}} = \kappa_1^n \bar{\kappa}_1^m (\mathcal{C}_{k,l}\kappa_1^{-n} \bar{\kappa}_1^{-m} \varphi)_{A_1\dots A_{k+1}}^{A'_1\dots A'_{l-1}}, \quad (2.12b)$$

$$(\mathcal{C}_{k,l,n,m}^\dagger\varphi)_{A_1\dots A_{k-1}}^{A'_1\dots A'_{l+1}} = \kappa_1^n \bar{\kappa}_1^m (\mathcal{C}_{k,l}^\dagger\kappa_1^{-n} \bar{\kappa}_1^{-m} \varphi)_{A_1\dots A_{k-1}}^{A'_1\dots A'_{l+1}}, \quad (2.12c)$$

$$(\mathcal{T}_{k,l,n,m}\varphi)_{A_1\dots A_{k+1}}^{A'_1\dots A'_{l+1}} = \kappa_1^n \bar{\kappa}_1^m (\mathcal{T}_{k,l}\kappa_1^{-n} \bar{\kappa}_1^{-m} \varphi)_{A_1\dots A_{k+1}}^{A'_1\dots A'_{l+1}}. \quad (2.12d)$$

In particular it follows that the commutator of extended fundamental spinor operators with  $n_1 = n_2, m_1 = m_2$  reduces to the commutator of the usual fundamental spinor operators. For commutators of the extended operators with unequal weights  $n_1, n_2, m_1, m_2$  one simply splits into first derivatives and remainder with equal weights.

### 2.3. Projection operators and the spin decomposition.

**Definition 2.3.** Given the Killing spinor (2.6), define the operators  $\mathcal{K}_{k,l}^i : \mathcal{S}_{k,l} \rightarrow \mathcal{S}_{k-2i+2,l}, i = 0, 1, 2$  via

$$(\mathcal{K}_{k,l}^0\varphi)_{A_1\dots A_{k+2}A'_1\dots A'_l} = 2\kappa_1^{-1}\kappa_{(A_1A_2}\varphi_{A_3\dots A_{k+2})}A'_1\dots A'_l, \quad (2.13a)$$

$$(\mathcal{K}_{k,l}^1\varphi)_{A_1\dots A_kA'_1\dots A'_l} = \kappa_1^{-1}\kappa_{(A_1}{}^F\varphi_{A_2\dots A_k)F}A'_1\dots A'_l, \quad (2.13b)$$

$$(\mathcal{K}_{k,l}^2\varphi)_{A_1\dots A_{k-2}A'_1\dots A'_l} = -\frac{1}{2}\kappa_1^{-1}\kappa^{CD}\varphi_{A_1\dots A_{k-2}CD}A'_1\dots A'_l. \quad (2.13c)$$

**Example 2.4.** The “spin raising” operator  $\mathcal{K}_{k,l}^0$  on a symmetric  $(2, 0)$  spinor  $\varphi_{AB}$  has components

$$(\mathcal{K}_{2,0}^0\varphi)_0 = 0, \quad (\mathcal{K}_{2,0}^0\varphi)_1 = \varphi_0, \quad (\mathcal{K}_{2,0}^0\varphi)_2 = \frac{4}{3}\varphi_1, \quad (\mathcal{K}_{2,0}^0\varphi)_3 = \varphi_2, \quad (\mathcal{K}_{2,0}^0\varphi)_4 = 0.$$

The “sign flip” operator  $\mathcal{K}_{k,l}^1$  on a symmetric  $(4, 0)$  spinor  $\varphi_{ABCD}$  has components

$$(\mathcal{K}_{4,0}^1\varphi)_0 = \varphi_0, \quad (\mathcal{K}_{4,0}^1\varphi)_1 = \frac{1}{2}\varphi_1, \quad (\mathcal{K}_{4,0}^1\varphi)_2 = 0, \quad (\mathcal{K}_{4,0}^1\varphi)_3 = -\frac{1}{2}\varphi_3, \quad (\mathcal{K}_{4,0}^1\varphi)_4 = -\varphi_4.$$

The “spin lowering” operator  $\mathcal{K}_{k,l}^2$  on a symmetric  $(4, 0)$  spinor  $\varphi_{ABCD}$  has components

$$(\mathcal{K}_{4,0}^2\varphi)_0 = \varphi_1, \quad (\mathcal{K}_{4,0}^2\varphi)_1 = \varphi_2, \quad (\mathcal{K}_{4,0}^2\varphi)_2 = \varphi_3.$$

**Definition 2.5** (Spin decomposition). For any symmetric spinor  $\varphi_{A_1\dots A_{2s}}$ ,

- with integer  $s$ , define  $s+1$  symmetric valence  $2s$  spinors  $(\mathcal{P}_{2s,0}^i\varphi)_{A_1\dots A_{2s}}, i = 0 \dots s$  solving

$$\varphi_{A_1\dots A_{2s}} = \sum_{i=0}^s (\mathcal{P}_{2s,0}^i\varphi)_{A_1\dots A_{2s}}, \quad (2.14)$$

with  $(\mathcal{P}_{2s,0}^i\varphi)_{A_1\dots A_{2s}}$  depending only on the components  $\varphi_{s+i}$  and  $\varphi_{s-i}$ .

- with half-integer  $s$ , define  $s + \frac{1}{2}$  symmetric valence  $2s$  spinors  $(\mathcal{P}_{2s,0}^i\varphi)_{A_1\dots A_{2s}}, i = \frac{1}{2} \dots s$  solving

$$\varphi_{A_1\dots A_{2s}} = \sum_{i=1/2}^s (\mathcal{P}_{2s,0}^i\varphi)_{A_1\dots A_{2s}}, \quad (2.15)$$

with  $(\mathcal{P}_{2s,0}^i\varphi)_{A_1\dots A_{2s}}$  depending only on the components  $\varphi_{s+i}$  and  $\varphi_{s-i}$ .

**Remark 2.6.** *The spin decomposition can also be defined for spinors with primed indices and more generally for mixed valence. In that case the decompositions combine linearly.*

**Example 2.7.** (1) *For  $s = 2$  the decomposition is given by*

$$\varphi_{ABCD} = (\mathcal{P}_{4,0}^0 \varphi)_{ABCD} + (\mathcal{P}_{4,0}^1 \varphi)_{ABCD} + (\mathcal{P}_{4,0}^2 \varphi)_{ABCD} \quad (2.16)$$

*and the components, written as vectors, are*

$$\begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \varphi_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \varphi_1 \\ 0 \\ \varphi_3 \\ 0 \end{pmatrix} + \begin{pmatrix} \varphi_0 \\ 0 \\ 0 \\ 0 \\ \varphi_4 \end{pmatrix}. \quad (2.17)$$

*In terms of the operators (2.13) they read*

$$(\mathcal{P}_{4,0}^0 \varphi)_{ABCD} = \frac{3}{8} (\mathcal{K}_{2,0}^0 \mathcal{K}_{0,0}^0 \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \varphi)_{ABCD}, \quad (2.18a)$$

$$(\mathcal{P}_{4,0}^1 \varphi)_{ABCD} = (\mathcal{K}_{2,0}^0 \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \varphi)_{ABCD}, \quad (2.18b)$$

$$(\mathcal{P}_{4,0}^2 \varphi)_{ABCD} = (\mathcal{K}_{4,0}^1 \mathcal{K}_{4,0}^1 \mathcal{K}_{4,0}^1 \mathcal{K}_{4,0}^1 \varphi)_{ABCD} - \frac{1}{16} (\mathcal{K}_{2,0}^0 \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \varphi)_{ABCD}. \quad (2.18c)$$

(2) *For  $s = 3/2$  on a  $(3, 1)$  spinor the decomposition is given by*

$$\varphi_{ABCA'} = (\mathcal{P}_{3,1}^{1/2} \varphi)_{ABCA'} + (\mathcal{P}_{3,1}^{3/2} \varphi)_{ABCA'} \quad (2.19)$$

*and the components, written as vectors, are*

$$\begin{pmatrix} \varphi_{0A'} \\ \varphi_{1A'} \\ \varphi_{2A'} \\ \varphi_{3A'} \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_{1A'} \\ \varphi_{2A'} \\ 0 \end{pmatrix} + \begin{pmatrix} \varphi_{0A'} \\ 0 \\ 0 \\ \varphi_{3A'} \end{pmatrix}. \quad (2.20)$$

*In terms of the operators (2.13) they read*

$$(\mathcal{P}_{3,1}^{1/2} \varphi)_{ABCA'} = \frac{3}{4} (\mathcal{K}_{1,1}^0 \mathcal{K}_{3,1}^2 \varphi)_{ABCA'}, \quad (2.21a)$$

$$(\mathcal{P}_{3,1}^{3/2} \varphi)_{ABCA'} = -\frac{1}{12} (\mathcal{K}_{1,1}^0 \mathcal{K}_{3,1}^2 \varphi)_{ABCA'} + (\mathcal{K}_{3,1}^1 \mathcal{K}_{3,1}^1 \varphi)_{ABCA'}. \quad (2.21b)$$

**Example 2.8.** *Given a symmetric spinor  $\varphi_{ABCD}$ , let*

$$\widehat{\varphi}_{ABCD} = \kappa_1^4 (\mathcal{K}_{4,0}^1 \mathcal{P}_{4,0}^2 \varphi)_{ABCD}.$$

*Then the components of  $\widehat{\varphi}_{ABCD}$  are*

$$\begin{pmatrix} \widehat{\varphi}_0 \\ \widehat{\varphi}_1 \\ \widehat{\varphi}_2 \\ \widehat{\varphi}_3 \\ \widehat{\varphi}_4 \end{pmatrix} = \begin{pmatrix} \kappa_1^4 \varphi_0 \\ 0 \\ 0 \\ 0 \\ -\kappa_1^4 \varphi_4 \end{pmatrix}.$$

**Lemma 2.9.** *For any symmetric spinors  $\varphi_{ABCD}$ ,  $\varphi_{AB}$ ,  $\varphi$ ,  $\varphi_{AA'}$ ,  $\varphi_{ABCA'}$  and an integer  $w$ , we have the algebraic identities*

$$(\mathcal{K}_{4,0}^1 \mathcal{P}_{4,0}^1 \varphi)_{ABCD} = \frac{1}{2} (\mathcal{K}_{2,0}^0 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \varphi)_{ABCD}, \quad (2.22a)$$

$$(\mathcal{K}_{1,1}^1 \mathcal{K}_{1,1}^1 \varphi)_{AA'} = \varphi_{AA'}, \quad (2.22b)$$

$$(\mathcal{K}_{2,0}^2 \mathcal{K}_{2,0}^1 \varphi) = 0, \quad (2.22c)$$

$$(\mathcal{P}_{3,1}^{3/2} \mathcal{K}_{1,1}^0 \phi)_{ABCA'} = 0, \quad (2.22d)$$

$$(\mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \phi)_{AB} = (\mathcal{K}_{2,0}^1 \phi)_{AB}, \quad (2.22e)$$

$$(\mathcal{K}_{3,1}^1 \mathcal{K}_{3,1}^1 \mathcal{K}_{3,1}^1 \phi)_{ABCA'} = -\frac{2}{9} (\mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \phi)_{ABCA'} + (\mathcal{K}_{3,1}^1 \phi)_{ABCA'}, \quad (2.22f)$$

and the first order differential identities

$$(\mathcal{C}_{4,0,w}^\dagger \mathcal{K}_{4,0}^\dagger \varphi)_{ABCA'} = (\mathcal{K}_{3,1}^1 \mathcal{C}_{4,0,w}^\dagger \varphi)_{ABCA'} + \frac{1}{2}(\mathcal{T}_{2,0,-4+w} \mathcal{K}_{4,0}^2 \varphi)_{ABCA'}, \quad (2.23a)$$

$$(\mathcal{C}_{4,0,w}^\dagger \mathcal{K}_{2,0}^0 \varphi)_{ABCA'} = (\mathcal{K}_{1,1}^0 \mathcal{C}_{2,0,-1+w}^\dagger \varphi)_{ABCA'} - (\mathcal{T}_{2,0,-4+w} \mathcal{K}_{2,0}^1 \varphi)_{ABCA'}, \quad (2.23b)$$

$$(\mathcal{C}_{3,1,w}^\dagger \mathcal{K}_{1,1}^0 \varphi)_{ABA'B'} = (\mathcal{K}_{0,2}^0 \mathcal{C}_{1,1,-1+w}^\dagger \varphi)_{ABA'B'} - \frac{4}{3}(\mathcal{T}_{1,1,-3+w} \mathcal{K}_{1,1}^1 \varphi)_{ABA'B'}, \quad (2.23c)$$

$$(\mathcal{C}_{2,0,w}^\dagger \mathcal{K}_{2,0}^1 \varphi)_{AA'} = (\mathcal{K}_{1,1}^1 \mathcal{C}_{2,0,w}^\dagger \varphi)_{AA'} + (\mathcal{T}_{0,0,-2+w} \mathcal{K}_{2,0}^2 \varphi)_{AA'}, \quad (2.23d)$$

$$(\mathcal{C}_{2,0,w}^\dagger \mathcal{K}_{4,0}^2 \varphi)_{AA'} = (\mathcal{K}_{3,1}^2 \mathcal{C}_{4,0,1+w}^\dagger \varphi)_{AA'}, \quad (2.23e)$$

$$(\mathcal{C}_{2,0,w}^\dagger \mathcal{K}_{0,0}^0 \varphi)_{AA'} = -2(\mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,-2+w} \varphi)_{AA'}, \quad (2.23f)$$

$$(\mathcal{D}_{1,1,w} \mathcal{K}_{1,1}^1 \phi) = 2(\mathcal{K}_{2,0}^2 \mathcal{C}_{1,1,-2+w} \phi), \quad (2.23g)$$

$$(\mathcal{D}_{1,1,w} \mathcal{K}_{3,1}^2 \phi) = (\mathcal{K}_{2,0}^2 \mathcal{D}_{3,1,1+w} \phi), \quad (2.23h)$$

$$(\mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathcal{T}_{2,0,w} \phi)_{ABCA'} = -\frac{1}{4}(\mathcal{C}_{4,0,4+w}^\dagger \mathcal{K}_{2,0}^0 \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \phi)_{ABCA'} + \frac{3}{4}(\mathcal{T}_{2,0,w} \mathcal{K}_{2,0}^1 \phi)_{ABCA'}. \quad (2.23i)$$

*Proof.* For (2.22a) we calculate

$$\begin{aligned} (\mathcal{K}_{4,0}^1 \mathcal{P}_{4,0}^1 \varphi)_{ABCD} &= (\mathcal{K}_{4,0}^1 \mathcal{K}_{2,0}^0 \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \varphi)_{ABCD} \\ &= \frac{1}{2}(\mathcal{K}_{2,0}^0 \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \varphi)_{ABCD} \\ &= \frac{1}{2}(\mathcal{K}_{2,0}^0 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \varphi)_{ABCD}. \end{aligned}$$

In the first step uses (2.18b), the second one is a commutator of  $\mathcal{K}^1$  and  $\mathcal{K}^0$  and the third step makes use of the fact that three sign-flips are equal to one sign-flip. For (2.22b) we note that  $\mathcal{K}^1$  on a (1, 1) spinor changes sign in two of the four components,

$$(\mathcal{K}_{1,1}^1 \varphi)_{00'} = \varphi_{00'}, \quad (\mathcal{K}_{1,1}^1 \varphi)_{01'} = \varphi_{01'}, \quad (\mathcal{K}_{1,1}^1 \varphi)_{10'} = -\varphi_{10'}, \quad (\mathcal{K}_{1,1}^1 \varphi)_{11'} = -\varphi_{11'},$$

so  $\mathcal{K}_{1,1}^1 \mathcal{K}_{1,1}^1 = \text{Id}$ . Equation (2.22c) is true because  $\mathcal{K}_{2,0}^1$  cancels the middle component of  $\varphi_{AB}$  and  $\mathcal{K}_{2,0}^2$  singles out that middle component. The rest of the algebraic identities are proved analogously. The proof of the differential identities relies on a straightforward but tedious expansion of projectors (2.13) and fundamental operators (2.11). We only calculate (2.23e).

$$\begin{aligned} (\mathcal{C}_{2,0,-1}^\dagger \mathcal{K}_{4,0}^2 \phi)_{AA'} - (\mathcal{K}_{3,1}^2 \mathcal{C}_{4,0}^\dagger \phi)_{AA'} &= \frac{U^B{}_{A'} \kappa^{CD} \phi_{ABCD}}{2\kappa_1} + \frac{\kappa^{BC} (\mathcal{C}_{4,0}^\dagger \phi)_{ABCA'}}{2\kappa_1} + (\mathcal{C}_{2,0}^\dagger \mathcal{K}_{4,0}^2 \phi)_{AA'} \\ &= \frac{U^B{}_{A'} \kappa^{CD} \phi_{ABCD}}{2\kappa_1} - \frac{\phi_{ABCD} (\mathcal{T}_{2,0} \kappa)^{BCD}{}_{A'}}{2\kappa_1} \\ &\quad + \frac{\kappa^{CD} \phi_{ABCD} (\mathcal{T}_{0,0} \kappa_1)^B{}_{A'}}{2\kappa_1^2} \\ &= 0. \end{aligned}$$

The rest are proved along similar lines.  $\square$

**Lemma 2.10.** *The following identities hold*

$$\begin{aligned} 0 &= (\mathcal{K}_{0,2}^0 \mathcal{C}_{1,1,-5}^\dagger \mathcal{T}_{0,0,-3} \varphi)_{ABA'B'} + \frac{4}{3}(\mathcal{T}_{1,1,-4} \mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,-6} \varphi)_{ABA'B'} \\ &\quad - \frac{4}{3}(\mathcal{T}_{1,1,-7} \mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,-3} \varphi)_{ABA'B'}, \end{aligned} \quad (2.24a)$$

$$\begin{aligned} 0 &= (\mathcal{C}_{3,1,-1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{T}_{0,0} \varphi)_{ABA'B'} + 2(\mathcal{K}_{0,2}^0 \mathcal{C}_{1,1}^\dagger \mathcal{T}_{0,0} \varphi)_{ABA'B'} + 12(\mathcal{K}_{2,2}^1 \mathcal{T}_{1,1} \mathcal{T}_{0,0} \varphi)_{ABA'B'} \\ &\quad - \frac{32}{3}(\mathcal{T}_{1,1,-1} \mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,3/2} \varphi)_{ABA'B'}, \end{aligned} \quad (2.24b)$$

$$\begin{aligned} 0 &= -\frac{1}{12}(\mathcal{C}_{3,1,-1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{D}_{2,2,4} \varphi)_{ABA'B'} + (\mathcal{C}_{3,1,-1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{C}_{2,2,1} \varphi)_{ABA'B'} - \Psi_2(\mathcal{K}_{2,2}^1 \varphi)_{ABA'B'} \\ &\quad - (\mathcal{K}_{2,2}^1 \mathcal{C}_{3,1}^\dagger \mathcal{C}_{2,2} \varphi)_{ABA'B'} - \frac{1}{3}(\mathcal{K}_{2,2}^1 \mathcal{T}_{1,1} \mathcal{D}_{2,2} \varphi)_{ABA'B'} + \frac{(\mathcal{L}_\xi \varphi)_{ABA'B'}}{3\kappa_1} \\ &\quad + \frac{2}{9}(\mathcal{T}_{1,1,-1} \mathcal{K}_{1,1}^1 \mathcal{D}_{2,2,1} \varphi)_{ABA'B'} - \frac{2}{3}(\mathcal{T}_{1,1,-1} \mathcal{K}_{3,1}^2 \mathcal{C}_{2,2,-2} \varphi)_{ABA'B'}. \end{aligned} \quad (2.24c)$$

*Proof.* This can be verified by expanding in components.  $\square$



## 3. SPINORIAL FORMULATION OF LINEARIZED GRAVITY

In this section we review the field equations of linearized gravity for a general vacuum background and allow for sources of the linearized field. The spinor variational operator  $\vartheta$  developed in [8] will be used. Let  $\delta g_{ABA'B'}$  be a solution of the linearized Einstein equations

$$12\epsilon_{AB}\bar{\epsilon}_{A'B'}\vartheta\Lambda - 4\vartheta\Phi_{ABA'B'} = \square\delta g_{ABA'B'} - \nabla_{AA'}F_{BB'} - \nabla_{BB'}F_{AA'} - 2\bar{\Psi}_{A'B'C'D'}\delta g_{AB}{}^{C'D'} - 2\Psi_{ABCD}\delta g^{CD}{}_{A'B'}, \quad (3.1)$$

with  $\Lambda$  the Ricci scalar,  $\Phi_{ABA'B'}$  the trace-free Ricci spinor,  $\Psi_{ABCD}$  the Weyl spinor and  $F^{AA'} = -\frac{1}{2}\nabla^{AA'}\delta g^B{}_{B'}{}^{B'}{}_{B'} + \nabla_{BB'}\delta g^{ABA'B'}$  the gauge source function. Observe that we make the variation with the indices down, and raise them and take traces afterwards. We define the irreducible parts of the linearized metric as

$$G_{ABA'B'} = \delta g_{(AB)(A'B')}, \quad G = \delta g^C{}_{C'}{}^{C'}{}_{C'}, \quad (3.2)$$

and introduce

$$\phi_{ABCD} = \frac{1}{4}G\Psi_{ABCD} + \vartheta\Psi_{ABCD}. \quad (3.3)$$

as a modification of the varied Weyl spinor  $\vartheta\Psi_{ABCD}$ <sup>3</sup>. Then we get, see [8], for a general vacuum background

$$\phi_{ABCD} = \frac{1}{2}(\mathcal{C}_{3,1}\mathcal{C}_{2,2}G)_{ABCD}, \quad (3.4a)$$

$$\begin{aligned} \vartheta\Phi_{ABA'B'} &= \frac{1}{2}G^{CD}{}_{A'B'}\Psi_{ABCD} + \frac{1}{2}(\mathcal{C}_{3,1}^\dagger\mathcal{C}_{2,2}G)_{ABA'B'} + \frac{1}{6}(\mathcal{T}_{1,1}\mathcal{D}_{2,2}G)_{ABA'B'} \\ &\quad - \frac{1}{8}(\mathcal{T}_{1,1}\mathcal{T}_{0,0}G)_{ABA'B'}, \end{aligned} \quad (3.4b)$$

$$\vartheta\Lambda = -\frac{1}{24}(\mathcal{D}_{1,1}\mathcal{D}_{2,2}G) + \frac{1}{32}(\mathcal{D}_{1,1}\mathcal{T}_{0,0}G). \quad (3.4c)$$

In particular (3.4b) and (3.4c) together are equivalent to (3.1). Note also that  $\vartheta\Lambda = 0 = \vartheta\Phi_{ABA'B'}$  in the source-free case. As a consequence of (3.4) we derive the linearized Binachi identity.

**Lemma 3.1.** *For a general vacuum background the modified Weyl spinor (3.4a) satisfies*

$$\begin{aligned} (\mathcal{C}_{4,0}^\dagger\phi)_{ABCA'} &= (\mathcal{C}_{2,2}\vartheta\Phi)_{ABCA'} + \frac{1}{2}\Psi_{ABCD}(\mathcal{D}_{2,2}G)^D{}_{A'} - \frac{3}{2}\Psi_{(AB}{}^{DF}(\mathcal{C}_{2,2}G)_C)DFA' \\ &\quad - \frac{1}{8}\Psi_{ABCD}(\mathcal{T}_{0,0}G)^D{}_{A'} + \frac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathcal{T}_{4,0}\Psi)_{ABCDFB'}. \end{aligned} \quad (3.5)$$

Restricting to a type D background this simplifies to

$$\begin{aligned} (\mathcal{C}_{4,0}^\dagger\phi)_{ABCA'} &= (\mathcal{C}_{2,2}\vartheta\Phi)_{ABCA'} - \frac{3}{2}\Psi_2(\mathcal{C}_{2,2,1}G)_{ABCA'} - \frac{3}{8}\Psi_2(\mathcal{K}_{1,1}^0\mathcal{K}_{1,1}^1\mathcal{D}_{2,2,4}G)_{ABCA'} \\ &\quad + \frac{3}{32}\Psi_2(\mathcal{K}_{1,1}^0\mathcal{K}_{1,1}^1\mathcal{T}_{0,0}G)_{ABCA'} + \frac{9}{4}\Psi_2(\mathcal{K}_{1,1}^0\mathcal{K}_{3,1}^2\mathcal{C}_{2,2,1}G)_{ABCA'}. \end{aligned} \quad (3.6)$$

*Proof.* We apply  $\mathcal{C}_{2,2}$  on (3.4b), commute  $\mathcal{C}_{2,2}\mathcal{C}_{3,1}^\dagger$ , use (3.4a) to get

$$\begin{aligned} (\mathcal{C}_{2,2}\vartheta\Phi)_{ABCA'} &= \frac{1}{2}(\mathcal{C}_{2,2}\mathcal{C}_{3,1}^\dagger\mathcal{C}_{2,2}G)_{ABCA'} + \frac{1}{6}(\mathcal{C}_{2,2}\mathcal{T}_{1,1}\mathcal{D}_{2,2}G)_{ABCA'} - \frac{1}{8}(\mathcal{C}_{2,2}\mathcal{T}_{1,1}\mathcal{T}_{0,0}G)_{ABCA'} \\ &\quad - \frac{1}{3}\Psi_{ABCD}(\mathcal{D}_{2,2}G)^D{}_{A'} + \frac{1}{2}\Psi_{(AB}{}^{DF}(\mathcal{C}_{2,2}G)_C)DFA' \\ &\quad - \frac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathcal{T}_{4,0}\Psi)_{ABCDFB'} \end{aligned} \quad (3.7)$$

$$\begin{aligned} &= \frac{1}{6}(\mathcal{C}_{2,2}\mathcal{T}_{1,1}\mathcal{D}_{2,2}G)_{ABCA'} - \frac{1}{8}(\mathcal{C}_{2,2}\mathcal{T}_{1,1}\mathcal{T}_{0,0}G)_{ABCA'} + (\mathcal{C}_{4,0}^\dagger\phi)_{ABCA'} \\ &\quad - \frac{1}{3}\Psi_{ABCD}(\mathcal{D}_{2,2}G)^D{}_{A'} + \frac{3}{2}\Psi_{(AB}{}^{DF}(\mathcal{C}_{2,2}G)_C)DFA' \\ &\quad - \frac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathcal{T}_{4,0}\Psi)_{ABCDFB'} - \frac{1}{8}(\mathcal{T}_{2,0}\mathcal{D}_{3,1}\mathcal{C}_{2,2}G)_{ABCA'}. \end{aligned} \quad (3.8)$$

<sup>3</sup>In a type D principal frame this modification only affects the middle component.

We then commute the  $\mathcal{C}_{2,2}\mathcal{T}_{1,1}$  operators, and in the last step we commute  $\mathcal{D}_{3,1}\mathcal{C}_{2,2}$  and use  $\mathcal{C}_{1,1}\mathcal{T}_{0,0} = 0$  to get

$$\begin{aligned} (\mathcal{C}_{2,2}\vartheta\Phi)_{ABCA'} &= (\mathcal{C}_{4,0}^\dagger\phi)_{ABCA'} - \frac{1}{2}\Psi_{ABCD}(\mathcal{D}_{2,2}G)^D{}_{A'} + \frac{3}{2}\Psi_{(AB}{}^{DF}(\mathcal{C}_{2,2}G)_{C)DFA'} \\ &\quad + \frac{1}{8}\Psi_{ABCD}(\mathcal{T}_{0,0}G)^D{}_{A'} - \frac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathcal{T}_{4,0}\Psi)_{ABCD}{}^{FB'} \\ &\quad + \frac{1}{12}(\mathcal{T}_{2,0}\mathcal{C}_{1,1}\mathcal{D}_{2,2}G)_{ABCA'} - \frac{1}{16}(\mathcal{T}_{2,0}\mathcal{C}_{1,1}\mathcal{T}_{0,0}G)_{ABCA'} \\ &\quad - \frac{1}{8}(\mathcal{T}_{2,0}\mathcal{D}_{3,1}\mathcal{C}_{2,2}G)_{ABCA'} \end{aligned} \quad (3.9)$$

$$\begin{aligned} &= (\mathcal{C}_{4,0}^\dagger\phi)_{ABCA'} - \frac{1}{2}\Psi_{ABCD}(\mathcal{D}_{2,2}G)^D{}_{A'} + \frac{3}{2}\Psi_{(AB}{}^{DF}(\mathcal{C}_{2,2}G)_{C)DFA'} \\ &\quad + \frac{1}{8}\Psi_{ABCD}(\mathcal{T}_{0,0}G)^D{}_{A'} - \frac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathcal{T}_{4,0}\Psi)_{ABCD}{}^{FB'}. \end{aligned} \quad (3.10)$$

This gives (3.5). On a type D spacetime, we can use (2.9) and (2.10). The resulting  $U_{AA'}$  spinors can be incorporated as extended indices, and the  $\kappa_{AB}$  spinors can then be rewritten in terms the  $\mathcal{K}^i$  operators to get (3.6).  $\square$

Note that on a Minkowski background and without sources the right hand side of (3.5) vanishes and the linearized Bianchi identity reduces to the spin-2 equation. The linearized Bianchi identity (3.6) is of fundamental importance and next we derive some differential identities for it which are needed for the main theorem.

**Corollary 3.2.** *The rescaled spin-projected Weyl spinor  $\widehat{\phi}_{ABCD} = \kappa_1^4(\mathcal{K}_{4,0}^1\mathcal{P}_{4,0}^2\phi)_{ABCD}$  satisfies*

$$\begin{aligned} (\mathcal{C}_{4,0,4}^\dagger\widehat{\phi})_{ABCA'} &= -(\kappa_1^4\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathcal{C}_{4,0}^\dagger\mathcal{P}_{4,0}^1\phi)_{ABCA'} - \frac{3}{2}(\kappa_1^4\Psi_2\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathcal{C}_{2,2,1}G)_{ABCA'} \\ &\quad + (\kappa_1^4\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathcal{C}_{2,2}\vartheta\Phi)_{ABCA'}. \end{aligned} \quad (3.11)$$

*Proof.* Applying the operator  $\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\kappa_1^4\mathcal{C}_{4,0}^\dagger$  to (2.16) and using (2.23a) gives the identity

$$(\mathcal{C}_{4,0,4}^\dagger\widehat{\phi})_{ABCA'} = -(\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathcal{C}_{4,0,4}^\dagger(\kappa_1^4\mathcal{P}_{4,0}^1\phi))_{ABCA'} + (\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}(\kappa_1^4\mathcal{C}_{4,0}^\dagger\phi))_{ABCA'}. \quad (3.12)$$

The result follows from  $(\mathcal{P}_{3,1}^{3/2}\mathcal{K}_{1,1}^0\phi)_{ABCA'} = 0$  together with (3.6).  $\square$

**Corollary 3.3** (Covariant TME). *The covariant form of the spin-2 Teukolsky Master equation with source is given by*

$$(\mathcal{C}_{3,1}\mathcal{C}_{4,0,4}^\dagger\widehat{\phi})_{ABCD} = -3\Psi_2\widehat{\phi}_{ABCD} + \kappa_1^4(\mathcal{P}_{4,0}^2\mathcal{K}_{4,0}^1\mathcal{C}_{3,1,-4}\mathcal{C}_{2,2}\vartheta\Phi)_{ABCD}. \quad (3.13)$$

*Proof.* Apply the operator  $\mathcal{P}_{4,0}^2\mathcal{C}_{3,1}$  to (3.11) and use

$$(\mathcal{P}_{4,0}^2\mathcal{C}_{3,1}\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathcal{C}_{4,0,4}^\dagger(\kappa_1^4\mathcal{P}_{4,0}^1\phi))_{ABCD} = 0, \quad (3.14a)$$

$$(\mathcal{P}_{4,0}^2\mathcal{C}_{3,1}\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}(\kappa_1^4\mathcal{C}_{2,2}\vartheta\Phi))_{ABCD} = \kappa_1^4(\mathcal{P}_{4,0}^2\mathcal{K}_{4,0}^1\mathcal{C}_{3,1,-4}\mathcal{C}_{2,2}\vartheta\Phi)_{ABCD}, \quad (3.14b)$$

$$(\mathcal{P}_{4,0}^2\mathcal{C}_{3,1}\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}(\kappa_1^4\Psi_2\mathcal{C}_{2,2,1}G))_{ABCD} = \Psi_2\kappa_1^4(\mathcal{P}_{4,0}^2\mathcal{K}_{4,0}^1\mathcal{C}_{3,1}\mathcal{C}_{2,2}G)_{ABCD}. \quad (3.14c)$$

$\square$

#### 4. MAIN THEOREM

We shall now prove our main theorem. The following is the detailed statement of Theorem 1.1'.

**Theorem 4.1.** *Let  $\phi_{ABCD}$  be the modified linearized Weyl spinor given in (3.3) and let*

$$\widehat{\phi}_{ABCD} = \kappa_1^4(\mathcal{K}_{4,0}^1\mathcal{P}_{4,0}^2\phi)_{ABCD} \quad (4.1)$$

*be the rescaled spin-projected field as in example 2.8, and let*

$$\mathcal{M}_{ABA'B'} = (\mathcal{C}_{3,1}^\dagger\mathcal{C}_{4,0,4}^\dagger\widehat{\phi})_{ABA'B'}. \quad (4.2)$$

*Then we have*

$$\mathcal{M}_{ABA'B'} = \frac{1}{2}\nabla_{AA'}\mathcal{A}_{BB'} + \frac{1}{2}\nabla_{BB'}\mathcal{A}_{AA'} + \frac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi\delta g)_{ABA'B'}, \quad (4.3)$$

*where the complex vector field  $A^{AA'}$  is given by*

$$\begin{aligned} \mathcal{A}_{AA'} &= -\frac{1}{2}\Psi_2\kappa_1^3\xi^{BB'}(\mathcal{K}_{0,2}^0\mathcal{K}_{2,2}^2G)_{ABA'B'} + \frac{2}{3}\kappa_1^3\xi^B{}_{A'}(\mathcal{K}_{2,0}^1\mathcal{K}_{4,0}^2\phi)_{AB} \\ &\quad + (\mathcal{K}_{1,1}^1\mathcal{T}_{0,0}\mathcal{K}_{2,0}^2\mathcal{K}_{4,0}^2(\kappa_1^4\phi))_{AA'} - \frac{1}{4}\Psi_2\kappa_1^4(\mathcal{K}_{1,1}^1\mathcal{T}_{0,0,2}G)_{AA'}. \end{aligned} \quad (4.4)$$

*Proof.* Apply the operator  $\mathcal{C}^\dagger$  to (3.11) and moving out the scalars  $\Psi_2$  and  $\kappa_1$  we get

$$\begin{aligned} (\mathcal{C}_{3,1}^\dagger \mathcal{E}_{4,0,4}^\dagger \widehat{\phi})_{ABA'B'} &= \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathcal{E}_{2,2} \vartheta \Phi)_{ABA'B'} - \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathcal{E}_{4,0}^\dagger \mathcal{P}_{4,0}^1 \phi)_{ABA'B'} \\ &\quad - \frac{3}{2} \Psi_2 \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathcal{E}_{2,2,1} G)_{ABA'B'}. \end{aligned} \quad (4.5)$$

The second term can be rewritten by expanding the spin-1 projector according to (2.18b) and commuting out the  $\mathcal{K}_{2,0}^0$  using (2.23b) together with (2.22d),(2.22e),

$$\begin{aligned} \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathcal{E}_{4,0}^\dagger \mathcal{P}_{4,0}^1 \phi)_{ABA'B'} &= -\kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathcal{T}_{2,0,-4} \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'} \\ &= \frac{1}{4} \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^0 \mathcal{E}_{2,0,-1}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'} \\ &\quad - \frac{2}{3} \kappa_1^4 (\mathcal{T}_{1,1,-4} \mathcal{E}_{2,0,-4}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}. \end{aligned} \quad (4.6)$$

In the second step (2.23i) and (2.23b) with (2.22e) and a commutator is used. To commute out the  $\mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2$  in the first term, we first use (2.23d) and (2.23e) to get

$$\begin{aligned} (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{E}_{2,0,-1}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'} &= (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathcal{E}_{4,0}^\dagger \phi)_{ABA'B'} \\ &\quad + (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{T}_{0,0,-3} \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}, \\ &= (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathcal{E}_{4,0}^\dagger \phi)_{ABA'B'} \\ &\quad - \frac{4}{3} (\mathcal{T}_{1,1,-4} \mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,-6} \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}. \end{aligned} \quad (4.7)$$

In the second step (2.23c) is used together with (2.24a). Using (4.7) in (4.6) and the linearized Bianchi identity (3.6) in the first term of (4.7) yields

$$\begin{aligned} \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathcal{E}_{4,0}^\dagger \mathcal{P}_{4,0}^1 \phi)_{ABA'B'} &= \frac{1}{4} \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathcal{E}_{2,2} \vartheta \Phi)_{ABA'B'} \\ &\quad - \frac{1}{8} \kappa_1^4 \left( \mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 (\Psi_2 (\mathcal{K}_{1,1}^1 \mathcal{D}_{2,2,4} G)) \right)_{ABA'B'} \\ &\quad + \frac{1}{32} \kappa_1^4 \left( \mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 (\Psi_2 (\mathcal{K}_{1,1}^1 \mathcal{T}_{0,0} G)) \right)_{ABA'B'} \\ &\quad + \frac{3}{8} \kappa_1^4 \left( \mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 (\Psi_2 (\mathcal{K}_{3,1}^2 \mathcal{E}_{2,2,1} G)) \right)_{ABA'B'} \\ &\quad - \frac{2}{3} \kappa_1^4 (\mathcal{T}_{1,1,-4} \mathcal{E}_{2,0,-4}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'} \\ &\quad - \frac{1}{3} \kappa_1^4 (\mathcal{T}_{1,1,-4} \mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,-6} \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}. \end{aligned} \quad (4.8)$$

The second and third term on the right hand side can be simplified further using (2.22b). Using (4.8) in (4.5) and expanding the spin decomposition in the last term of (4.5) leads to

$$\begin{aligned} (\mathcal{C}_{3,1}^\dagger \mathcal{E}_{4,0,4}^\dagger \widehat{\phi})_{ABA'B'} &= \left( \mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} (\kappa_1^4 (\mathcal{E}_{2,2} \vartheta \Phi)) \right)_{ABA'B'} \\ &\quad + \frac{1}{8} \Psi_2 \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{D}_{2,2,4} G)_{ABA'B'} \\ &\quad - \frac{1}{4} \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathcal{E}_{2,2} \vartheta \Phi)_{ABA'B'} \\ &\quad - \frac{1}{3} \Psi_2 \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathcal{E}_{2,2,1} G)_{ABA'B'} \\ &\quad - \frac{1}{32} \Psi_2 \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{T}_{0,0} G)_{ABA'B'} \\ &\quad - \frac{3}{2} \Psi_2 \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{K}_{3,1}^1 \mathcal{K}_{3,1}^1 \mathcal{E}_{2,2,1} G)_{ABA'B'} \\ &\quad + \frac{2}{3} \kappa_1^4 (\mathcal{T}_{1,1,-4} \mathcal{E}_{2,0,-4}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'} \\ &\quad + \frac{1}{3} \kappa_1^4 (\mathcal{T}_{1,1,-4} \mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,-6} \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}. \end{aligned} \quad (4.9)$$

The fourth and sixth term on the right hand side can be combined via (2.22f). Defining the complex vector field

$$\begin{aligned} A_{AA'} &= -\frac{1}{2} \kappa_1^4 (\mathcal{E}_{2,0,-4}^\dagger \mathcal{K}_{0,0}^0 \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{AA'} + \frac{2}{3} \kappa_1^4 (\mathcal{E}_{2,0,-4}^\dagger \mathcal{K}_{4,0}^2 \phi)_{AA'} + \frac{1}{3} \Psi_2 \kappa_1^4 (\mathcal{K}_{1,1}^1 \mathcal{D}_{2,2,1} G)_{AA'} \\ &\quad - \frac{1}{3} \Psi_2 \kappa_1^4 (\mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,3/2} G)_{AA'} - \Psi_2 \kappa_1^4 (\mathcal{K}_{3,1}^2 \mathcal{E}_{2,2,-2} G)_{AA'} \end{aligned} \quad (4.10a)$$

$$\begin{aligned} &= -\frac{1}{2} \Psi_2 \kappa_1^3 \zeta^{BB'} (\mathcal{K}_{0,2}^0 \mathcal{K}_{2,2}^2 G)_{ABA'B'} + \frac{2}{3} \kappa_1^3 \zeta^B{}_{A'} (\mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{AB} \\ &\quad + (\mathcal{K}_{1,1}^1 \mathcal{T}_{0,0} \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 (\kappa_1^4 \phi))_{AA'} - \frac{1}{4} \Psi_2 \kappa_1^4 (\mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,2} G)_{AA'} + \frac{2}{3} \kappa_1^4 (\mathcal{K}_{3,1}^2 \mathcal{E}_{2,2} \vartheta \Phi)_{AA'} \end{aligned} \quad (4.10b)$$

(for the second version we used the linearized Bianchi identity (3.6)) we find

$$\begin{aligned} (\mathcal{C}_{3,1}^\dagger \mathcal{C}_{4,0,4}^\dagger \widehat{\phi})_{ABA'B'} &= (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} (\kappa_1^4 \mathcal{C}_{2,2} \vartheta \Phi))_{ABA'B'} - 3\Psi_2 \kappa_1^4 (\mathcal{K}_{2,2}^1 \vartheta \Phi)_{ABA'B'} \\ &\quad - \frac{1}{4} \kappa_1^4 (\mathcal{C}_{3,1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathcal{C}_{2,2} \vartheta \Phi)_{ABA'B'} + \frac{1}{2} \Psi_2 \kappa_1^3 (\mathcal{L}_\xi G)_{ABA'B'} \\ &\quad + (\mathcal{T}_{1,1} \mathcal{A})_{ABA'B'} \end{aligned} \quad (4.11)$$

by using (2.24). Setting sources to zero,  $\vartheta \Phi_{ABA'B'} = 0$ , and using (4.13) proves the theorem. In general the third term on the right hand side can be rewritten using (2.21a), see Remark 4.3.  $\square$

The following lemma collects some properties of the vector field  $\mathcal{A}$ .

**Lemma 4.2.**

$$\mathcal{A}^{AA'} U_{AA'} = -\frac{1}{3} \kappa_1^3 (\mathcal{L}_\xi \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \vartheta \Psi), \quad (4.12)$$

$$(\mathcal{D}_{1,1} \mathcal{A}) = -\frac{1}{54} M \mathcal{L}_\xi G, \quad (4.13)$$

*Proof.* Equation (4.12) can be verified by a direct calculation. To prove (4.13), we start with

$$\begin{aligned} \mathcal{A}_{AA'} &= -\frac{1}{2} (\mathcal{C}_{2,0}^\dagger \mathcal{K}_{0,0}^0 \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 (\kappa_1^4 \phi))_{AA'} + \frac{2}{3} (\mathcal{C}_{2,0}^\dagger \mathcal{K}_{4,0}^2 (\kappa_1^4 \phi))_{AA'} + \frac{1}{3} (\mathcal{K}_{1,1}^1 \mathcal{D}_{2,2,2} (\kappa_1^4 \Psi_2 G))_{AA'} \\ &\quad - \frac{1}{3} (\mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,5/2} (\kappa_1^4 \Psi_2 G))_{AA'} - (\mathcal{K}_{3,1}^2 \mathcal{C}_{2,2,-1} (\kappa_1^4 \Psi_2 G))_{AA'}. \end{aligned} \quad (4.14)$$

Applying  $\mathcal{D}_{1,1}$  to this and using the commutator relation  $\mathcal{D}_{1,1} \mathcal{C}_{2,0}^\dagger = 0$  gives

$$\begin{aligned} (\mathcal{D}_{1,1} \mathcal{A}) &= \frac{1}{3} (\mathcal{D}_{1,1} \mathcal{K}_{1,1}^1 \mathcal{D}_{2,2,2} (\kappa_1^4 \Psi_2 G)) - \frac{1}{3} (\mathcal{D}_{1,1} \mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,5/2} (\kappa_1^4 \Psi_2 G)) \\ &\quad - (\mathcal{D}_{1,1} \mathcal{K}_{3,1}^2 \mathcal{C}_{2,2,-1} (\kappa_1^4 \Psi_2 G)). \end{aligned} \quad (4.15)$$

Using (2.23g) on the first two terms, and (2.23h) on the last term gives

$$\begin{aligned} (\mathcal{D}_{1,1} \mathcal{A}) &= \frac{2}{3} (\mathcal{K}_{2,0}^2 \mathcal{C}_{1,1,-2} \mathcal{D}_{2,2,2} (\kappa_1^4 \Psi_2 G)) - \frac{2}{3} (\mathcal{K}_{2,0}^2 \mathcal{C}_{1,1,-2} \mathcal{T}_{0,0,5/2} (\kappa_1^4 \Psi_2 G)) \\ &\quad - (\mathcal{K}_{2,0}^2 \mathcal{D}_{3,1,1} \mathcal{C}_{2,2,-1} (\kappa_1^4 \Psi_2 G)). \end{aligned} \quad (4.16)$$

The first and the last term cancel due to the general identity

$$(\mathcal{K}_{2,0}^2 \mathcal{D}_{3,1,1} \mathcal{C}_{2,2,-1} \phi) = \frac{2}{3} (\mathcal{K}_{2,0}^2 \mathcal{C}_{1,1,-2} \mathcal{D}_{2,2,2} \phi). \quad (4.17)$$

This identity can be proven by expanding the extended indices and commuting the derivatives. In the same way we can also prove the identity

$$(\mathcal{K}_{2,0}^2 \mathcal{C}_{1,1,v} \mathcal{T}_{0,0,w} \phi) = \frac{(w-v)(\mathcal{L}_\xi \phi)}{6\kappa_1}, \quad (4.18)$$

for arbitrary weights  $v$  and  $w$ . This finally gives

$$(\mathcal{D}_{1,1} \mathcal{A}) = -\frac{1}{2} \Psi_2 \kappa_1^3 (\mathcal{L}_\xi G), \quad (4.19)$$

where we in the last step commuted  $\Psi_2 \kappa_1^3$  through the Lie derivative.  $\square$

**Remark 4.3.** If we allow for sources the general form of (4.3) is given by

$$\begin{aligned} (\mathcal{C}_{3,1}^\dagger \mathcal{C}_{4,0,4}^\dagger \widehat{\phi})_{ABA'B'} &= -(\mathcal{C}_{3,1}^\dagger (\kappa_1^4 \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{1/2} \mathcal{C}_{2,2} \vartheta \Phi))_{ABA'B'} + (\mathcal{C}_{3,1}^\dagger (\kappa_1^4 \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathcal{C}_{2,2} \vartheta \Phi))_{ABA'B'} \\ &\quad - 3\Psi_2 \kappa_1^4 (\mathcal{K}_{2,2}^1 \vartheta \Phi)_{ABA'B'} + \frac{1}{2} \Psi_2 \kappa_1^3 (\mathcal{L}_\xi G)_{ABA'B'} + (\mathcal{T}_{1,1} \mathcal{A})_{ABA'B'}. \end{aligned} \quad (4.20)$$

**Lemma 4.4.** For linearized diffeomorphism of the background metric, generated by a real vector  $\zeta_{AA'}$  of the original metric we get

$$G_{ABA'B'} = 2(\mathcal{T}_{1,1} \zeta)_{ABA'B'}, \quad G = 2(\mathcal{D}_{1,1} \zeta). \quad (4.21)$$

For the curvature and  $\mathcal{A}_{AA'}$  we get

$$\vartheta \Phi_{ABA'B'} = 0, \quad (4.22a)$$

$$\phi_{ABCD} = \frac{3}{16} \Psi_2 (\mathcal{K}_{2,0}^0 \mathcal{K}_{0,0}^0 \mathcal{D}_{1,1,6} \zeta)_{ABCD} + \frac{3}{2} \Psi_2 (\mathcal{K}_{2,0}^0 \mathcal{K}_{2,0}^1 \mathcal{C}_{1,1,2} \zeta)_{ABCD}, \quad (4.22b)$$

$$\mathcal{A}_{AA'} = -\Psi_2 \kappa_1^3 (\mathcal{L}_\xi \zeta)_{AA'}. \quad (4.22c)$$

*Proof.* For the curvature, one can use the results of [8] and transform it to the operators of this paper using the type D structure of the curvature. Applying  $\mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2$  or  $\mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2$  onto  $\phi_{ABCD}$  gives

$$(\mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi) = \frac{1}{2} \Psi_2 (\mathcal{D}_{1,1,6} \zeta), \quad (4.23)$$

$$(\mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{AB} = \frac{3}{2} \Psi_2 (\mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{C}_{1,1,2} \zeta)_{AB}. \quad (4.24)$$

These relations can then be used in the definition of  $\mathcal{A}_{AA'}$  to yield

$$\begin{aligned} \mathcal{A}_{AA'} = & -\Psi_2 \kappa_1^3 \xi^{BB'} (\mathcal{K}_{0,2}^0 \mathcal{K}_{2,2}^2 \mathcal{T}_{1,1} \zeta)_{ABA'B'} + \Psi_2 \kappa_1^3 \xi^B{}_{A'} (\mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{C}_{1,1,2} \zeta)_{AB} \\ & - \frac{1}{2} \Psi_2 \kappa_1^4 (\mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,2} \mathcal{D}_{1,1} \zeta)_{AA'} + \frac{1}{2} \Psi_2 \kappa_1^4 (\mathcal{K}_{1,1}^1 \mathcal{T}_{0,0,-1} \mathcal{D}_{1,1,6} \zeta)_{AA'}. \end{aligned} \quad (4.25)$$

An expansion of the extended indices and a reformulation of the Lie derivative in terms of fundamental spinor operators gives the gauge dependence of  $\mathcal{A}_{AA'}$ .  $\square$

**Remark 4.5.** (1) *In view of (4.22c),  $\text{Im} \mathcal{A}_{AA'}$  is gauge invariant, and further,  $\text{Re} \mathcal{A}_{AA'} = 0$  is essentially pure gauge.*

(2) *Moving the last term in (1.19) to the left hand side, we can interpret*

$$\text{Re} \mathcal{M}_{ABA'B'} - \frac{1}{54} M (\mathcal{L}_\xi \delta g)_{ABA'B'}$$

*as a pure linearized diffeomorphism since it is a symmetrized covariant derivative of a real vector field.*

#### APPENDIX A. GHP FORM OF SOME EXPRESSIONS

The components of the complex vector field  $\mathcal{A}_{AA'}$  defined in (4.4) are given by

$$\mathcal{A}_{00'} = -3G_{11'} \Psi_2 \kappa_1^4 \rho + 3G_{10'} \Psi_2 \kappa_1^4 \tau - 2\kappa_1^4 \tau' \phi_1 - \frac{1}{4} \Psi_2 \kappa_1^4 (\mathfrak{p} + 2\rho) G + \kappa_1^4 (\mathfrak{p} - 4\rho) \phi_2, \quad (A.1a)$$

$$\mathcal{A}_{01'} = -3G_{12'} \Psi_2 \kappa_1^4 \rho + 3G_{11'} \Psi_2 \kappa_1^4 \tau - 2\kappa_1^4 \rho' \phi_1 - \frac{1}{4} \Psi_2 \kappa_1^4 (\bar{\delta} + 2\tau) G + \kappa_1^4 (\bar{\delta} - 4\tau) \phi_2, \quad (A.1b)$$

$$\mathcal{A}_{10'} = 3G_{10'} \Psi_2 \kappa_1^4 \rho' - 3G_{11'} \Psi_2 \kappa_1^4 \tau' + 2\kappa_1^4 \rho \phi_3 + \frac{1}{4} \Psi_2 \kappa_1^4 (\bar{\delta}' + 2\tau') G - \kappa_1^4 (\bar{\delta}' - 4\tau') \phi_2, \quad (A.1c)$$

$$\mathcal{A}_{11'} = 3G_{11'} \Psi_2 \kappa_1^4 \rho' - 3G_{12'} \Psi_2 \kappa_1^4 \tau' + 2\kappa_1^4 \tau \phi_3 + \frac{1}{4} \Psi_2 \kappa_1^4 (\mathfrak{p}' + 2\rho') G - \kappa_1^4 (\mathfrak{p}' - 4\rho') \phi_2. \quad (A.1d)$$

The dyad components of (1.17) on a Kerr background are given by

$$\frac{1}{27} M \mathcal{L}_\xi \bar{\phi}_4 = \mathfrak{p}' \mathfrak{p}' \mathfrak{p}' \mathfrak{p}' \hat{\phi}_0 + \bar{\delta} \bar{\delta} \bar{\delta} \bar{\delta} \hat{\phi}_4, \quad (A.2a)$$

$$\begin{aligned} \frac{1}{27} M \mathcal{L}_\xi \bar{\phi}_3 = & (\bar{\delta}' (\mathfrak{p}' - \bar{\rho}') - 6\bar{\rho}' \bar{\tau}) (\mathfrak{p}' + 2\bar{\rho}') (\mathfrak{p}' + 2\bar{\rho}') \hat{\phi}_0 \\ & + (\mathfrak{p} (\bar{\delta} - \bar{\tau}') - 6\bar{\rho} \bar{\tau}') (\bar{\delta} + 2\bar{\tau}') (\bar{\delta} + 2\bar{\tau}') \hat{\phi}_4 \\ & + \frac{3}{2} \bar{\Psi}_2 (\mathfrak{p}' - \rho' + 2\bar{\rho}') \mathcal{A}_{01'} - \frac{3}{2} \bar{\Psi}_2 (\bar{\delta} - \tau + 2\bar{\tau}') \mathcal{A}_{11'}, \end{aligned} \quad (A.2b)$$

$$\begin{aligned} \frac{1}{27} M \mathcal{L}_\xi \bar{\phi}_2 = & ((\bar{\delta}' - \bar{\tau}) \mathfrak{p}' - 12\bar{\rho}' \bar{\tau}) ((\bar{\delta}' + 2\bar{\tau}') (\mathfrak{p}' + 3\bar{\rho}') - 2\bar{\rho}' \bar{\tau}) \hat{\phi}_0 \\ & + ((\bar{\delta} - \bar{\tau}') \mathfrak{p} - 12\bar{\rho} \bar{\tau}') ((\bar{\delta} + 2\bar{\tau}') (\mathfrak{p} + 3\bar{\rho}) - 2\bar{\rho} \bar{\tau}') \hat{\phi}_4 \\ & - \frac{1}{2} \bar{\Psi}_2 (\mathfrak{p} - \rho + 5\bar{\rho}) \mathcal{A}_{11'} - \frac{1}{2} \bar{\Psi}_2 (\mathfrak{p}' - \rho' + 5\bar{\rho}') \mathcal{A}_{00'} \\ & + \frac{1}{2} \bar{\Psi}_2 (\bar{\delta} - \tau + 5\bar{\tau}') \mathcal{A}_{10'} + \frac{1}{2} \bar{\Psi}_2 (\bar{\delta}' - \tau' + 5\bar{\tau}') \mathcal{A}_{01'}, \end{aligned} \quad (A.2c)$$

$$\begin{aligned} \frac{1}{27} M \mathcal{L}_\xi \bar{\phi}_1 = & (\mathfrak{p}' (\bar{\delta}' - \bar{\tau}) - 6\bar{\rho}' \bar{\tau}) (\bar{\delta}' + 2\bar{\tau}') (\bar{\delta}' + 2\bar{\tau}') \hat{\phi}_0 \\ & + (\bar{\delta} (\mathfrak{p} - \bar{\rho}) - 6\bar{\rho} \bar{\tau}') (\mathfrak{p} + 2\bar{\rho}) (\mathfrak{p} + 2\bar{\rho}) \hat{\phi}_4 \\ & + \frac{3}{2} \bar{\Psi}_2 (\mathfrak{p} - \rho + 2\bar{\rho}) \mathcal{A}_{10'} - \frac{3}{2} \bar{\Psi}_2 (\bar{\delta}' - \tau' + 2\bar{\tau}') \mathcal{A}_{00'}, \end{aligned} \quad (A.2d)$$

$$\frac{1}{27} M \mathcal{L}_\xi \bar{\phi}_0 = \mathfrak{p} \mathfrak{p} \mathfrak{p} \mathfrak{p} \hat{\phi}_4 + \bar{\delta}' \bar{\delta}' \bar{\delta}' \bar{\delta}' \hat{\phi}_0. \quad (A.2e)$$

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