NEW IDENTITIES FOR LINEARIZED GRAVITY ON THE KERR SPACETIME

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ABSTRACT. In this paper we prove a new identity for linearized gravity on the Kerr spacetime and more generally on vacuum spacetimes of Petrov type D. The new identity yields a covariant version of the Teukolsky-Starobinsky identities for linearized gravity which, in addition to the two classical identities for linearized Weyl scalars with extreme spin weights, includes three additional equations. By analogy with the spin-1 case, we expect the new identity to be relevant in deriving new conservation laws for linearized gravity and in particular for proving integrated local decay estimates, as well as pointwise decay estimates for the linearized gravitational field on the vacuum spacetimes of Petrov type D, including the Kerr spacetime.

1. INTRODUCTION

The black hole stability problem, i.e. the problem of proving dynamical stability of the Kerr vacuum black hole solution is one of the most important open problems in general relativity. Proving dispersive estimates, in particular integrated local energy decay or Morawetz estimates for test fields with spin such as Maxwell and linearized gravity on the Kerr background is an essential step towards proving the pointwise decay estimates needed for solving the black hole stability problem. For the spin-0 case of scalar fields on the Kerr spacetime, Morawetz and pointwise decay estimates are known [7, 21]. For spin-1 (Maxwell) and spin-2 (linearized gravity) test fields on the spherically symmetric Schwarzschild spacetime, Morawetz and pointwise decay estimates are known [17, 20], see also [12] for a different approach in the Maxwell case. For spinning fields on the rotating Kerr spacetime, the problem is more difficult. Morawetz and energy estimates were proven for Maxwell fields on very slowly rotating Kerr spacetimes in [8]. However, for the case of linearized gravity on Kerr, results of this type are not available. The only stability results for linearized gravity on Kerr in this direction known to date are the mode stability result of Whiting [39] and its generalization to the case of real frequencies [13].

The vector fields method of Klainerman, and its generalization incorporating the hidden symmetries present in the Kerr geometry [7] is an important tool in constructing Morawetz estimates. In this approach, currents constructed via the energy-momentum tensor play a central role. As discussed in e.g. [12, 8], in order to prove a Morawetz estimate for a spinning field on a black hole background, it is necessary to construct currents where the non-radiating modes are eliminated. In the Maxwell case, the non-radiating modes correspond to the conserved charges, while in the case of linearized gravity on Kerr, they correspond to linearized mass and angular momentum.

In the paper [14] a new symmetric conserved tensor V_{ab} for the Maxwell field on the Kerr spacetime was constructed, which can be viewed as a higher order energy-momentum tensor. It has properties which are desirable from the point of view of Morawetz estimates. In particular, V_{ab} is quadratic in the Maxwell field strength and its first derivative, and independent of the non-radiating modes of the Maxwell field. Further, in contrast to the classical symmetric Maxwell energy-momentum tensor it has non-vanishing trace, which for technical reasons is important in the construction of Morawetz currents. In addition, to leading order V_{ab} satisfies the dominant

energy condition. Work on applying Morawetz currents constructed in terms of V_{ab} using the approach developed in [7] based on generalized vector fields defined in terms of second order symmetry operators for the Maxwell field, is ongoing. Here the classification of symmetry operators for the Maxwell equation [9], generalizing the classical results of Carter [18] for the scalar field case plays a crucial role.

At this point it is important to recall that the Kerr spacetime is algebraically special. In particular it belongs to the family of Petrov type D vacuum spacetimes, and shares many properties with other members of that family. See [11, 29] for background. We emphasize that the facts mentioned in the previous paragraph, as well as the work in the present paper are valid not just in Kerr but in Petrov type D vacuum spacetimes. Since our results require only the presence of a non-null conformal Killing-Yano tensor[1] they are valid in Minkowski space[2] as well.

For the analysis of Maxwell and linearized gravitational fields on Kerr, the Teukolsky Master Equation (TME) [34] and the Teukolsky-Starobinsky Identities (TSI) [35, 33] play a crucial role. For a spin-$\mathfrak{s}$ field, $\mathfrak{s} = 1, 2$, the TME are wave equations governing the components (here we refer to the Maxwell or linearized Weyl Newman-Penrose scalars [27, 23] defined with respect a principal null tetrad) of the field with extreme spin weights $\pm\mathfrak{s}$, while the classical TSI are differential relations of order $2\mathfrak{s}$ between these components.

In its classical form, the TSI [35, 33] relate the solutions of the radial Teukolsky equations for fields of spin-weights $\pm\mathfrak{s}$, and are thus valid only for the separated form of the equations. In that context the TSI are sometimes referred to as the Teukolsky-Press identities. For the case of linearized gravity on a Petrov type D vacuum spacetime, a derivation of the TSI using the Newman-Penrose formalism, which does not require a separation of variables, was given by Torres del Castillo [36], later corrected by Silva-Ortigoza [32]. See the paper by Whiting and Price [40] for discussion and background.

The TME and TSI are consequences of the spin-$\mathfrak{s}$ field equations and may thus be viewed as integrability conditions. As pointed out by Coll *et al.* [19], in the spin-1 case the classical TSI system must be completed by adding one equation in order for the system of integrability conditions given by the TME and TSI systems to be *equivalent* to the Maxwell system, modulo charge. Examining the full TSI system, one finds that those equations which correspond to the classical TSI have extreme spin weights. For this reason we shall use the term *extreme TSI* when referring to the classical form of the TSI.

Due to the fact that the TME and TSI involve only the Maxwell scalars of extreme spin weight the non-radiating mode carrying the charge, also known as the Coulomb solution, cancels out of the TME and TSI systems. Thus, in order to reconstruct a Maxwell field from a solution of the TME and full TSI systems, it is necessary to specify the charge as an additional parameter.

The fact that the tensor V_{ab} defined in [14] is conserved can be seen by direct computation to be a consequence of the full TSI system for the Maxwell field. Hence, this tensor may be viewed as an energy-momentum tensor for the TSI system. Due to the fact that V_{ab} satisfies the dominant energy condition to leading order makes it plausible that the full TSI system is hyperbolic. In fact, as shall be demonstrated in a separate paper [5], the TME as well as the full TSI system independently yield hyperbolic systems both for Maxwell and linearized gravity. Further, as will be shown in future work, there are actions which yield the TME and TSI systems as Euler-Lagrange equations. Remarkably, the tensor V_{ab} appears as the symmetric energy-momentum tensor for the spin-1 TSI action [6].

The above discussion makes it interesting to develop the corresponding ideas for the spin-2 or linearized gravity case. In particular, we would like to find an analogue of the tensor V_{ab} for the spin-2 case. An important step, which we carry out in this paper, is to derive the full TSI system for linearized gravity, thus generalizing the result of Coll *et al.* to the spin-2 case. As we shall see, the full TSI system for linearized gravity contains, in addition to the extreme TSI, three additional equations.

In order to understand how to derive the full TSI for the spin-2 case, the following remarks are helpful. In the Maxwell case, the Debye potential construction [37] on the Kerr background

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[1] A conformal Killing-Yano tensor is non-null if it is of algebraic type $\{1, 1\}$.

[2] In this case it is necessary to specify a conformal Killing-Yano tensor on Minkowski space.

can be used to construct from the Maxwell field a complex pure gauge vector potential

$$\boldsymbol{\alpha}_a = (df)_a$$

given by a first order differential operator acting on the Maxwell field strength. The field strength of $\boldsymbol{\alpha}_a$ has vanishing complex anti-self dual and self dual parts, which correspond to the TME and full TSI systems respectively. In the case of linearized gravity, the role of the vector potential is played by the linearized metric. Here, the analogous situation holds. The Debye potential construction can be used to construct from a solution $\dot{g}_{ab}$ of the linearized vacuum Einstein equations[3], a complex, traceless, symmetric 2-tensor $\mathcal{M}_{ab}$ which is essentially a pure gauge metric satisfying the linearized vacuum Einstein equations. This result is the main theorem of the present paper, cf. Theorem 1.1.

The self dual Weyl curvature of $\mathcal{M}_{ab}$ yields the full TSI for linearized gravity on vacuum spacetimes of Petrov type D, and in particular on the Kerr spacetime. The full TSI system is a differential relation of order four in the linearized Weyl curvature. We also note that the anti-self dual Weyl curvature of $\mathcal{M}_{ab}$ yields a fourth order identity related to the TME. A new feature is encountered compared to the spin-1 case, since the resulting identities contain terms involving the Lie derivative of the background curvature.

As just mentioned, the intermediate metric $\mathcal{M}_{ab}$ is defined in terms of Debye potentials for the linearized vacuum Einstein equations. We now recall this construction, first restricting to Minkowski space. Following Sachs and Bergmann [30], let $\mathcal{H}_{abcd}$ be an anti-self dual Weyl field[4], i.e. a tensor with the symmetries of the Riemann tensor, $\mathcal{H}_{abcd} = \mathcal{H}_{[ab]cd} = \mathcal{H}_{cdab}$, $\mathcal{H}_{[abc]d} = 0$, satisfying $\mathcal{H}^a{}_{bac} = 0$ and $\frac{1}{2}\epsilon_{ab}{}^{ef}\mathcal{H}_{efcd} = -i\mathcal{H}_{abcd}$, and let

$$\mathfrak{g}_{ab} = \nabla^c\nabla^d\mathcal{H}_{acbd}. \tag{1.1}$$

Then, if $\nabla^e\nabla_e\mathcal{H}_{abcd} = 0$, it follows that $\mathfrak{g}_{ab}$ solves the linearized vacuum Einstein equation.

The analogous construction for massless spin-$\mathfrak{s}$ fields on the 4-dimensional Minkowski space was discussed by Penrose [28]. In [10] this was used to prove decay estimates for such fields, based on decay estimates for the wave equation. We shall now describe the analogue of the Sachs-Bergmann construction in the case of a vacuum Petrov type D metric.

Introduce the following complex anti-self dual tensors with the symmetries of the Weyl tensor

$$Z^0_{abcd} = 4\bar{m}_{[a}n_{b]}\bar{m}_{[c}n_{d]},$$
$$Z^4_{abcd} = 4l_{[a}m_{b]}l_{[c}m_{d]},$$

where $(l^a, n^a, m^a, \bar{m}^a)$ constitutes a principal null tetrad. These are analogues of the anti-self dual bivectors Z^0_{ab}, Z^2_{ab}, see [2, §2]. For a complex scalar χ_0, let $\mathcal{H}_{abcd}$ be given by

$$\mathcal{H}_{abcd} = \kappa_1^4\chi_0 Z^0_{abcd}. \tag{1.2}$$

with the complex function κ_1 being the Killing spinor coefficient, see (2.11) for details. Define the 1-form U_a by

$$U_a = -\nabla_a\log(\kappa_1),$$

cf. (2.12), and consider the following analogue[5] of (1.1), which sends $\mathcal{H}_{abcd}$ to a 2-tensor $\mathfrak{g}_{ab}$,

$$\mathfrak{g}_{ab} = \nabla^c(\nabla_d + 4U_d)\mathcal{H}_{(a}{}^d{}_{b)c}.$$

A calculation shows that $\mathfrak{g}_{ab}$ is a complex solution to the linearized vacuum Einstein equation provided the scalar $\kappa_1^4\chi_0$ solves the TME for spin weight $+2$ [24]. See corollary 3.4 below for the covariant form of the TME system, see also equation (A.2) for the component form. The analogous construction with

$$\mathcal{H}_{abcd} = \kappa_1^4\chi_4 Z^4_{abcd} \tag{1.3}$$

yields a solution to the linearized Einstein equation in the same way, provided that now the scalar $\kappa_1^4\chi_4$ solves the TME for spin weight -2. Note that in general, the linearized metrics $\mathfrak{g}_{ab}$

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[3]We shall sometimes refer to the linearized vacuum Einstein equations as the source-free linearized Einstein equations.

[4]Here we use a complex anti-self dual Weyl field for consistency with the rest of the paper, although this is not used in [30].

[5]More precisely it differs by a gauge transformation of *third kind*, cf. [24], so that the scalar potential solves the TME.

constructed from (1.2) and (1.3) are different. We are now able to state the tensor version of our main result, which describes this difference. Given scalars $\dot{\Psi}_0, \dot{\Psi}_4$ of spin weights 2 and -2 respectively, define

$$\mathcal{H}^{\pm}_{abcd} = \kappa_1^4 \dot{\Psi}_0 Z^0_{abcd} \pm \kappa_1^4 \dot{\Psi}_4 Z^4_{abcd}. \tag{1.4}$$

Theorem 1.1 (Tensor version)**.** *Let $\dot{g}_{ab}$ be a solution to the source-free linearized Einstein equation on a vacuum background of Petrov type D, and let $\dot{\Psi}_0, \dot{\Psi}_4$ be the linearized Weyl scalars of spin weights ± 2 defined with respect to a principal null tetrad. Let*

$$\mathcal{M}_{ab} = \nabla^c(\nabla_d + 4U_d)\mathcal{H}^-_{(a}{}^d{}_{b)c}. \tag{1.5}$$

Then, there is a complex vector field $\mathcal{A}_a$ depending on up to three derivatives of the linearized metric $\dot{g}_{ab}$, such that

$$\mathcal{M}_{ab} = \nabla_{(a}\mathcal{A}_{b)} + \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\dot{g}_{ab}. \tag{1.6}$$

Here, ξ^a is a Killing vector defined in (2.7) and Ψ_2 is the only non-vanishing component of the background curvature.

Remark 1.2. *In the Maxwell case, the analogue of (1.6) is that the vector potential arising out of the Debye potential construction by taking the difference of the extreme Maxwell scalars from the same Maxwell field, is pure gauge, see [1, eq. (5.40)]. In the spin-2 case above, the term involving $\mathcal{L}_\xi\dot{g}_{ab}$ is a new feature, which indicates an important qualitative difference between the spin-1 and spin-2 cases.*

In a Petrov type D vacuum spacetime we have that ξ^a is Killing and $\Psi_2\kappa_1^3$ is constant, see (2.11) below. Hence, the intermediate metric $\mathcal{M}_{ab}$ given in (1.6) is a complex solution of the linearized Einstein equation[6]. It is natural to ask for the remaining (Weyl) curvature and we find

Corollary 1.3. *The TSI is the self dual Weyl curvature of the metric $\mathcal{M}_{ab}$ given in (1.6). With* $\mathrm{TF}^{\mathrm{sd}}$ *the projection operator on the self dual trace free part, it is given by*

$$(\mathrm{TF}^{\mathrm{sd}}\dot{R}[\mathcal{M}])_{abcd} = i(\mathrm{TF}^{\mathrm{sd}}\mathcal{L}_{\mathrm{Im}\mathcal{A}}R)_{abcd} + \frac{\Psi_2\kappa_1^3}{\overline{\kappa_1}^4}(\mathcal{L}_\xi\overline{\mathcal{H}^+})_{abcd}, \tag{1.7}$$

where $\dot{R}[\mathcal{M}]_{abcd}$ is defined using

$$\dot{R}[\mathcal{M}]_{abcd} = 2g_{e[d}\nabla_{c]}\nabla_{[a}\mathcal{M}_{b]}{}^e - \tfrac{2}{3}R_{[cd]}{}^f{}_{[a}\mathcal{M}_{b]f} + \tfrac{2}{3}R^f{}_{[cd][a}\mathcal{M}_{b]f}. \tag{1.8}$$

Overview of this paper. In section 2 we give some background and preliminary results, in particular we introduce the 2-spinor formalism which shall be used throughout the paper. Section 2.2 contains a review of the consequences of the existence of a Killing spinor on vacuum type D spacetimes. In sections 2.3 and 2.4 we introduce a set of geometrically defined operators together with commutation rules, which allow us to exploit the special geometry in Petrov type D spacetimes. In section 3 a spinorial form of the field equations of linearized gravity is presented. We derive a convenient form of the linearized Bianchi identity in corollary 3.3. This equation plays a central role in the proof of the main theorem given in section 4. In lemma 4.4 we analyze the curvature of the intermediate metric $\mathcal{M}_{ab}$. In particular, its self dual linearized Weyl curvature gives a covariant form of the full spin-2 TSI. Finally, in corollary 4.7 we give a simplified form of the TSI for the Kerr case, containing only gauge invariant quantities. Appendix A contains the GHP component form of various equations discussed in this paper.

2. PRELIMINARIES

2.1. **2-spinors and irreducible decompositions.** Let $(\mathcal{N}, g_{ab})$ be a Lorentzian 3+1 dimensional spin manifold with metric of signature $+ - - -$. The spacetimes we are interested in here are spin, in particular any orientable, globally hyperbolic 3+1 dimensional spacetime is spin, cf. [22, page 346]. We shall throughout the paper make use of 2-spinor formalism, which simplifies calculations and makes many geometrical structures more transparent. See [29] and [11] for background.

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[6]The first term on the right-hand side of equation (1.6) is pure gauge since it is the action of a linearized diffeomorphism generated by $\mathcal{A}_a$.

If $\mathcal{N}$ is spin, then the orthonormal frame bundle $\mathrm{SO}(\mathcal{N})$ admits a lift to $\mathrm{Spin}(\mathcal{N})$, a principal $\mathrm{SL}(2,\mathbb{C})$-bundle. The group $\mathrm{SL}(2,\mathbb{C})$ has two fundamental inequivalent representations $\mathbb{C}^2$ and $\bar{\mathbb{C}}^2$. We denote sections of the corresponding spinor bundles with unprimed primed uppercase indices, respectively. The action of $\mathrm{SL}(2,\mathbb{C})$ leaves invariant an anti-symmetric 2-spinor ϵ_{AB}, the spin-metric.

The associated bundle construction now gives vector bundles over $\mathcal{N}$ corresponding to the representations of $\mathrm{SL}(2,\mathbb{C})$, in particular we have bundles of valence (k,l) spinors with sections $\varphi_{A \cdots D A' \cdots D'}$. Here k,l are the number of unprimed and primed indices. An important aspect of the 2-spinor formalism is the correspondence between tensors and spinors. An example is provided by the correspondence between metric and spin-metric $g_{ab} = \epsilon_{AB}\bar{\epsilon}_{A'B'}$. The Levi-Civita connection lifts to act on sections of the spinor bundles,

$$\nabla_{AA'} : \varphi_{B \cdots D B' \cdots D'} \to \nabla_{AA'}\varphi_{B \cdots D B' \cdots D'} \tag{2.1}$$

where we have used the tensor-spinor correspondence to replace the index a by AA'.

Irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ and hence also of $\mathrm{SO}(1,3)$ correspond exactly to symmetric spinors, which are automatically traceless. The space of symmetric spinors of valence (k,l) is denoted by $\mathcal{S}_{k,l}$. The correspondence between symmetric spinors and irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ yields efficient methods for decomposition of geometric expressions into irreducible pieces, which can be used for canonicalization. The SymManipulator package [15], which has been developed by one of the authors (T.B.) for the *Mathematica* based symbolic differential geometry suite *xAct* [26], exploits in a systematic way the above mentioned decompositions and allows one to carry out investigations which are not feasible to do by hand. The related *SpinFrames* package [3] developed by two of the authors (S.A. and T.B.) implements computations in tetrad components using the Newman-Penrose (NP) and Geroch-Held-Penrose (GHP) [23] formalisms.

The above mentioned correspondence between spinors and tensors, and the decomposition into irreducible pieces, can be applied to the Riemann curvature tensor. In this case, they correspond to the scalar curvature R, traceless Ricci tensor S_{ab}, and the Weyl tensor C_{abcd}. The Riemann tensor then takes the form

$$R_{abcd} = -\tfrac{1}{12}g_{ad}g_{bc}R + \tfrac{1}{12}g_{ac}g_{bd}R + \tfrac{1}{2}g_{bd}S_{ac} - \tfrac{1}{2}g_{bc}S_{ad} - \tfrac{1}{2}g_{ad}S_{bc} + \tfrac{1}{2}g_{ac}S_{bd} + C_{abcd}, \tag{2.2}$$

and the spinor equivalents of these tensors are

$$C_{abcd} = \Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}, \tag{2.3a}$$

$$S_{ab} = -2\Phi_{ABA'B'}, \tag{2.3b}$$

$$R = 24\Lambda. \tag{2.3c}$$

The irreducible decomposition into symmetric spinors in particular applies to covariant derivatives of symmetric spinors $\varphi_{A \cdots D A' \cdots D'} \in \mathcal{S}_{k,l}$. Decomposing (2.1) into its irreducible parts leads to

$$\begin{aligned}
\nabla_A{}^{A'}\varphi_{A_1 \cdots A_k}{}^{A'_1 \cdots A'_l} = {} & (\mathscr{T}_{k,l}\varphi)_{AA_1 \cdots A_k}{}^{A'A'_1 \cdots A'_l} \\
& - \tfrac{l}{l+1}\bar{\epsilon}^{A'(A'_1}(\mathscr{C}_{k,l}\varphi)_{AA_1 \cdots A_k}{}^{A'_2 \cdots A'_l)} \\
& - \tfrac{k}{k+1}\epsilon_{A(A_1}(\mathscr{C}^{\dagger}_{k,l}\varphi)_{A_2 \cdots A_k)}{}^{A'A'_1 \cdots A'_l} \\
& + \tfrac{kl}{(k+1)(l+1)}\epsilon_{A(A_1}\bar{\epsilon}^{A'(A'_1}(\mathscr{D}_{k,l}\varphi)_{A_2 \cdots A_k)}{}^{A'_2 \cdots A'_l)}
\end{aligned} \tag{2.4}$$

with coefficients given by the following four *fundamental spinor operators* [9, §2.1], also implemented in the *SymManipulator* package [15].

Definition 2.1. *The differential operators*

$$\mathscr{D}_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k-1,l-1}, \quad \mathscr{C}_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k+1,l-1}, \quad \mathscr{C}^{\dagger}_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k-1,l+1}, \quad \mathscr{T}_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k+1,l+1}$$

are defined as

$$(\mathscr{D}_{k,l}\varphi)_{A_1 \cdots A_{k-1}}{}^{A'_1 \cdots A'_{l-1}} \equiv \nabla^{BB'}\varphi_{A_1 \cdots A_{k-1}B}{}^{A'_1 \cdots A'_{l-1}}{}_{B'}, \tag{2.5a}$$

$$(\mathscr{C}_{k,l}\varphi)_{A_1 \cdots A_{k+1}}{}^{A'_1 \cdots A'_{l-1}} \equiv \nabla_{(A_1}{}^{B'}\varphi_{A_2 \cdots A_{k+1})}{}^{A'_1 \cdots A'_{l-1}}{}_{B'}, \tag{2.5b}$$

$$(\mathscr{C}^{\dagger}_{k,l}\varphi)_{A_1 \cdots A_{k-1}}{}^{A'_1 \cdots A'_{l+1}} \equiv \nabla^{B(A'_1}\varphi_{A_1 \cdots A_{k-1}B}{}^{A'_2 \cdots A'_{l+1})}, \tag{2.5c}$$

$$(\mathscr{T}_{k,l}\varphi)_{A_1 \cdots A_{k+1}}{}^{A'_1 \cdots A'_{l+1}} \equiv \nabla_{(A_1}{}^{(A'_1}\varphi_{A_2 \cdots A_{k+1})}{}^{A'_2 \cdots A'_{l+1})}. \tag{2.5d}$$

The operators are called respectively the divergence, curl, curl-dagger, and twistor operators.

With respect to complex conjugation, the operators $\mathscr{D}, \mathscr{T}$ satisfy $\overline{\mathscr{D}_{k,l}} = \mathscr{D}_{l,k}$, $\overline{\mathscr{T}_{k,l}} = \mathscr{T}_{l,k}$, while $\overline{\mathscr{C}_{k,l}} = \mathscr{C}^{\dagger}_{l,k}$, $\overline{\mathscr{C}^{\dagger}_{k,l}} = \mathscr{C}_{l,k}$. A complete set of commutation formulas for the fundamental operators has been given in [9, §2.2].

2.2. Geometric structure of Petrov type D spacetimes.

It is well known [38] that vacuum spacetimes of Petrov type D admit a non-trivial irreducible symmetric 2-spinor κ_{AB} solving the Killing spinor equation

$$(\mathscr{T}_{2,0}\kappa)_{ABCA'} = 0. \tag{2.6}$$

Defining the spinors

$$\xi_{AA'} = (\mathscr{C}^{\dagger}_{2,0}\kappa)_{AA'}, \tag{2.7}$$

$$\lambda_{A'B'} = (\mathscr{C}^{\dagger}_{1,1}\mathscr{C}^{\dagger}_{2,0}\kappa)_{A'B'}, \tag{2.8}$$

the complete table of derivatives reads

$$\nabla_{AA'}\kappa_{BC} = -\tfrac{1}{3}\xi_{CA'}\epsilon_{AB} - \tfrac{1}{3}\xi_{BA'}\epsilon_{AC}, \tag{2.9a}$$

$$\nabla_{AA'}\xi_{LL'} = -\tfrac{1}{2}\lambda_{A'L'}\epsilon_{AL} - \tfrac{3}{4}\kappa^{BC}\Psi_{ALBC}\bar{\epsilon}_{A'L'}, \tag{2.9b}$$

$$\nabla_{CC'}\lambda_{A'B'} = 2\xi_C{}^{D'}\bar{\Psi}_{A'B'C'D'}. \tag{2.9c}$$

The fact that the system of equations (2.9) for $(\kappa_{AB}, \xi_{AA'}, \lambda_{A'B'})$ is closed, implies in particular that higher derivatives of κ_{AB} do not contain any further information. Tensor symmetrizing (2.9b) leads to zero on the right-hand side and shows that $\xi^{AA'}$ is a Killing vector.

Remark 2.2. *If we furthermore assume the generalized Kerr-NUT condition that $\xi_{AA'}$ is real, the middle equation simplifies to $\lambda_{A'L'} = \tfrac{3}{2}\bar{\kappa}^{B'C'}\bar{\Psi}_{A'L'B'C'}$ so the $\lambda_{A'B'}$ is not an independent field anymore and the complete table reduces to*

$$\nabla_{AA'}\kappa_{BC} = -\tfrac{1}{3}\xi_{CA'}\epsilon_{AB} - \tfrac{1}{3}\xi_{BA'}\epsilon_{AC}, \tag{2.10a}$$

$$\nabla_{AA'}\xi_{LL'} = -\tfrac{3}{4}\bar{\kappa}^{B'C'}\bar{\Psi}_{A'L'B'C'}\epsilon_{AL} - \tfrac{3}{4}\kappa^{BC}\Psi_{ALBC}\bar{\epsilon}_{A'L'}. \tag{2.10b}$$

Using a principal dyad [7] the Killing spinor takes the form

$$\kappa_{AB} = -2\kappa_1 o_{(A}\iota_{B)}, \tag{2.11}$$

with $\kappa_1 \propto \Psi_2^{-1/3}$. Beside the Killing vector field (2.7) another important vector field is defined by

$$U_{AA'} = -\frac{\kappa_{AB}\xi^B{}_{A'}}{3\kappa_1^2} = -\nabla_{AA'}\log(\kappa_1). \tag{2.12}$$

Because it is completely determined by the Killing spinor (2.11), we have the complete table of derivatives

$$(\mathscr{D}_{1,1}U) = -2\Psi_2 + \frac{\xi_{AA'}\xi^{AA'}}{9\kappa_1^2}, \tag{2.13a}$$

$$(\mathscr{C}_{1,1}U)_{AB} = 0, \tag{2.13b}$$

$$(\mathscr{C}^{\dagger}_{1,1}U)_{A'B'} = 0, \tag{2.13c}$$

$$(\mathscr{T}_{1,1}U)_{AB}{}^{A'B'} = \frac{\kappa_{AB}(\mathscr{C}^{\dagger}_{1,1}\xi)^{A'B'}}{6\kappa_1^2} + 2U_{(A}{}^{(A'}U_{B)}{}^{B')} - \frac{\xi_{(A}{}^{(A'}\xi_{B)}{}^{B')}}{9\kappa_1^2}, \tag{2.13d}$$

in particular $U_{AA'}$ is closed, $(dU)_{ab} = 0$.

From the integrability condition $(\mathcal{L}_\xi \Psi)_{ABCD} = 0$ it follows that

$$(\mathscr{T}_{4,0}\Psi)_{ABCDFA'} = 5\Psi_{(ABCD}U_{F)A'}. \tag{2.14}$$

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[7] Note that κ_1 and Ψ_2 can be expressed covariantly via the relations $\kappa_{AB}\kappa^{AB} = -2\kappa_1{}^2$ and $\Psi_{ABCD}\Psi^{ABCD} = 6\Psi_2^2$. Hence, we can allow κ_1 and Ψ_2 in covariant expressions.

The curvature can be expressed in terms of the Killing spinor according to

$$\Psi_{ABCD} = \frac{3\Psi_2 \kappa_{(AB}\kappa_{CD)}}{2\kappa_1^2}. \tag{2.15}$$

Remark 2.3. *On Kerr spacetime with parameters (M, a) in a principal tetrad in Boyer-Lindquist coordinates (t, r, θ, ϕ), the curvature scalar is given by $\Psi_2 = -M(r - ia\cos\theta)^{-3}$ and we can set $\kappa_1 = -\frac{1}{3}(r - ia\cos\theta)$. Then one finds $\Psi_2\kappa_1^3 = \frac{1}{27}M$ and $\xi^a = (\partial_t)^a$.*

2.3. **Extended fundamental spinor operators.** As we often need to rescale with powers of κ_1 and $\bar\kappa_1$ we introduce extended fundamental operators with additional (extended) indices n, m:

$$(\mathscr{D}_{k,l,n,m}\varphi)_{A_1...A_{k-1}}{}^{A'_1...A'_{l-1}} \equiv \kappa_1^n\bar\kappa_1^m(\mathscr{D}_{k,l}\kappa_1^{-n}\bar\kappa_1^{-m}\varphi)_{A_1...A_{k-1}}{}^{A'_1...A'_{l-1}}, \tag{2.16a}$$

$$(\mathscr{C}_{k,l,n,m}\varphi)_{A_1...A_{k+1}}{}^{A'_1...A'_{l-1}} \equiv \kappa_1^n\bar\kappa_1^m(\mathscr{C}_{k,l}\kappa_1^{-n}\bar\kappa_1^{-m}\varphi)_{A_1...A_{k+1}}{}^{A'_1...A'_{l-1}}, \tag{2.16b}$$

$$(\mathscr{C}^\dagger_{k,l,n,m}\varphi)_{A_1...A_{k-1}}{}^{A'_1...A'_{l+1}} \equiv \kappa_1^n\bar\kappa_1^m(\mathscr{C}^\dagger_{k,l}\kappa_1^{-n}\bar\kappa_1^{-m}\varphi)_{A_1...A_{k-1}}{}^{A'_1...A'_{l+1}}, \tag{2.16c}$$

$$(\mathscr{T}_{k,l,n,m}\varphi)_{A_1...A_{k+1}}{}^{A'_1...A'_{l+1}} \equiv \kappa_1^n\bar\kappa_1^m(\mathscr{T}_{k,l}\kappa_1^{-n}\bar\kappa_1^{-m}\varphi)_{A_1...A_{k+1}}{}^{A'_1...A'_{l+1}}. \tag{2.16d}$$

For $n = m = 0$ it coincides with the definition (2.5) of the fundamental operators and the indices will be suppressed in that case. Because $U_{AA'}$ is a logarithmic derivative, (2.12), we can equivalently express them as

$$(\mathscr{D}_{k,l,n,m}\varphi)_{A_1...A_{k-1}}{}^{A'_1...A'_{l-1}} = \left[\nabla^{BB'} + nU^{BB'} + m\bar U^{BB'}\right]\varphi_{A_1...A_{k-1}B}{}^{A'_1...A'_{l-1}}{}_{B'}, \tag{2.17a}$$

$$(\mathscr{C}_{k,l,n,m}\varphi)_{A_1...A_{k+1}}{}^{A'_1...A'_{l-1}} = \left[\nabla_{(A_1}{}^{B'} + nU_{(A_1}{}^{B'} + m\bar U_{(A_1}{}^{B'}\right]\varphi_{A_2...A_{k+1})}{}^{A'_1...A'_{l-1}}{}_{B'}, \tag{2.17b}$$

$$(\mathscr{C}^\dagger_{k,l,n,m}\varphi)_{A_1...A_{k-1}}{}^{A'_1...A'_{l+1}} = \left[\nabla^{B(A'_1} + nU^{B(A'_1} + m\bar U^{B(A'_1}\right]\varphi_{A_1...A_{k-1}B}{}^{A'_2...A'_{l+1})}, \tag{2.17c}$$

$$(\mathscr{T}_{k,l,n,m}\varphi)_{A_1...A_{k+1}}{}^{A'_1...A'_{l+1}} = \left[\nabla_{(A_1}{}^{(A'_1} + nU_{(A_1}{}^{(A'_1} + m\bar U_{(A_1}{}^{(A'_1}\right]\varphi_{A_2...A_{k+1})}{}^{A'_2...A'_{l+1})}. \tag{2.17d}$$

It follows that the commutator of extended fundamental spinor operators with $n_1 = n_2, m_1 = m_2$ reduces to the commutator of the usual fundamental spinor operators [9, §2.2]. For commutators of the extended operators with unequal weights n_1, n_2, m_1, m_2 one simply splits into first derivatives and remainder with equal weights.

2.4. **Projection operators and the spin decomposition.** The Killing spinor κ_{AB} plays a central role in the geometry of Petrov type D spaces. The tensor product of κ_{AB} with a symmetric spinor has at most three different irreducible components. These involve either zero, one or two contractions and symmetrization. For these operations we introduce the $\mathcal{K}$-operators in

Definition 2.4. *Given the Killing spinor (2.11), define the operators $\mathcal{K}^i_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k-2i+2,l}, i = 0, 1, 2$ via*

$$(\mathcal{K}^0_{k,l}\varphi)_{A_1...A_{k+2}A'_1...A'_l} = 2\kappa_1^{-1}\kappa_{(A_1 A_2}\varphi_{A_3...A_{k+2})A'_1...A'_l}, \tag{2.18a}$$

$$(\mathcal{K}^1_{k,l}\varphi)_{A_1...A_k A'_1...A'_l} = \kappa_1^{-1}\kappa_{(A_1}{}^F\varphi_{A_2...A_k)FA'_1...A'_l}, \tag{2.18b}$$

$$(\mathcal{K}^2_{k,l}\varphi)_{A_1...A_{k-2}A'_1...A'_l} = -\frac{1}{2}\kappa_1^{-1}\kappa^{CD}\varphi_{A_1...A_{k-2}CDA'_1...A'_l}. \tag{2.18c}$$

Note that the complex conjugated operators act on the primed indices in the analogous way. The action of the $\mathcal{K}$-operators does have an interpretation in terms of the resulting components with respect to a principal dyad.

Example 2.5. *The "spin raising" operator[8] $\mathcal{K}^0_{k,l}$ on a symmetric $(2, 0)$ spinor φ_{AB} has components*

$$(\mathcal{K}^0_{2,0}\varphi)_0 = 0, \quad (\mathcal{K}^0_{2,0}\varphi)_1 = \varphi_0, \quad (\mathcal{K}^0_{2,0}\varphi)_2 = \frac{4}{3}\varphi_1, \quad (\mathcal{K}^0_{2,0}\varphi)_3 = \varphi_2, \quad (\mathcal{K}^0_{2,0}\varphi)_4 = 0.$$

---

[8]The name spin raising and lowering is due to the fact that multiplication and symmetrization or contraction of a spin-$\mathfrak{s}$ field with a valence-2 Killing spinor leads to a spin-$\mathfrak{s} + 1$ or spin-$\mathfrak{s} - 1$ field respectively, see [29, Sec. 6.4].

The "sign flip" operator $\mathcal{K}^1_{k,l}$ on a symmetric $(4,0)$ spinor φ_{ABCD} has components

$$(\mathcal{K}^1_{4,0}\varphi)_0 = \varphi_0, \quad (\mathcal{K}^1_{4,0}\varphi)_1 = \tfrac{1}{2}\varphi_1, \quad (\mathcal{K}^1_{4,0}\varphi)_2 = 0, \quad (\mathcal{K}^1_{4,0}\varphi)_3 = -\tfrac{1}{2}\varphi_3, \quad (\mathcal{K}^1_{4,0}\varphi)_4 = -\varphi_4.$$

The "spin lowering" operator $\mathcal{K}^2_{k,l}$ on a symmetric $(4,0)$ spinor φ_{ABCD} has components

$$(\mathcal{K}^2_{4,0}\varphi)_0 = \varphi_1, \qquad\qquad (\mathcal{K}^2_{4,0}\varphi)_1 = \varphi_2, \qquad\qquad (\mathcal{K}^2_{4,0}\varphi)_2 = \varphi_3.$$

Definition 2.6 (Spin decomposition). *For any symmetric spinor $\varphi_{A_1\ldots A_{2s}}$,*
- *with integer s, define $s+1$ symmetric valence $2s$ spinors $(\mathcal{P}^i_{2s,0}\varphi)_{A_1\ldots A_{2s}}, i = 0\ldots s$ solving*

$$\varphi_{A_1\ldots A_{2s}} = \sum_{i=0}^{s}(\mathcal{P}^i_{2s,0}\varphi)_{A_1\ldots A_{2s}}, \tag{2.19}$$

  with $(\mathcal{P}^i_{2s,0}\varphi)_{A_1\ldots A_{2s}}$ depending only on the components φ_{s+i} and φ_{s-i}.
- *with half-integer s, define $s + \tfrac{1}{2}$ symmetric valence $2s$ spinors $(\mathcal{P}^i_{2s,0}\varphi)_{A_1\ldots A_{2s}}, i = \tfrac{1}{2}\ldots s$ solving*

$$\varphi_{A_1\ldots A_{2s}} = \sum_{i=1/2}^{s}(\mathcal{P}^i_{2s,0}\varphi)_{A_1\ldots A_{2s}}, \tag{2.20}$$

  with $(\mathcal{P}^i_{2s,0}\varphi)_{A_1\ldots A_{2s}}$ depending only on the components φ_{s+i} and φ_{s-i}.

Remark 2.7. *The spin decomposition can also be defined for spinors with primed indices and more generally for mixed valence. In that case the decompositions combine linearly.*

Example 2.8.   (1) *For $s = 2$ the decomposition is given by*

$$\varphi_{ABCD} = (\mathcal{P}^0_{4,0}\varphi)_{ABCD} + (\mathcal{P}^1_{4,0}\varphi)_{ABCD} + (\mathcal{P}^2_{4,0}\varphi)_{ABCD} \tag{2.21}$$

and the components, written as vectors, are

$$\begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \varphi_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \varphi_1 \\ 0 \\ \varphi_3 \\ 0 \end{pmatrix} + \begin{pmatrix} \varphi_0 \\ 0 \\ 0 \\ 0 \\ \varphi_4 \end{pmatrix}. \tag{2.22}$$

In terms of the operators (2.18) they read

$$(\mathcal{P}^0_{4,0}\varphi)_{ABCD} = \tfrac{3}{8}(\mathcal{K}^0_{2,0}\mathcal{K}^0_{0,0}\mathcal{K}^2_{2,0}\mathcal{K}^2_{4,0}\varphi)_{ABCD}, \tag{2.23a}$$

$$(\mathcal{P}^1_{4,0}\varphi)_{ABCD} = (\mathcal{K}^0_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\varphi)_{ABCD}, \tag{2.23b}$$

$$(\mathcal{P}^2_{4,0}\varphi)_{ABCD} = (\mathcal{K}^1_{4,0}\mathcal{K}^1_{4,0}\mathcal{K}^1_{4,0}\mathcal{K}^1_{4,0}\varphi)_{ABCD} - \tfrac{1}{16}(\mathcal{K}^0_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\varphi)_{ABCD}. \tag{2.23c}$$

(2) *For $s = 3/2$ on a $(3,1)$ spinor the decomposition is given by*

$$\varphi_{ABCA'} = (\mathcal{P}^{1/2}_{3,1}\varphi)_{ABCA'} + (\mathcal{P}^{3/2}_{3,1}\varphi)_{ABCA'} \tag{2.24}$$

and the components, written as vectors, are

$$\begin{pmatrix} \varphi_{0A'} \\ \varphi_{1A'} \\ \varphi_{2A'} \\ \varphi_{3A'} \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_{1A'} \\ \varphi_{2A'} \\ 0 \end{pmatrix} + \begin{pmatrix} \varphi_{0A'} \\ 0 \\ 0 \\ \varphi_{3A'} \end{pmatrix}. \tag{2.25}$$

In terms of the operators (2.18) they read

$$(\mathcal{P}^{1/2}_{3,1}\varphi)_{ABCA'} = \tfrac{3}{4}(\mathcal{K}^0_{1,1}\mathcal{K}^2_{3,1}\varphi)_{ABCA'}, \tag{2.26a}$$

$$(\mathcal{P}^{3/2}_{3,1}\varphi)_{ABCA'} = -\tfrac{1}{12}(\mathcal{K}^0_{1,1}\mathcal{K}^2_{3,1}\varphi)_{ABCA'} + (\mathcal{K}^1_{3,1}\mathcal{K}^1_{3,1}\varphi)_{ABCA'}. \tag{2.26b}$$

For the proof of the main theorem we need various commutator relations of the above introduced operators. The complete set of commutators of $\mathcal{K}$-operators with extended fundamental operators can be found in [4, Appendix B] and here we restrict to the special cases needed for the proof.

Lemma 2.9. *For any symmetric spinors* φ_{ABCD}, φ_{AB}, φ, $\varphi_{AA'}$, $\varphi_{ABCA'}$ *and an integer* w, *we have the algebraic identities*

$$(\mathcal{K}^1_{4,0}\mathcal{P}^1_{4,0}\varphi)_{ABCD} = \tfrac{1}{2}(\mathcal{K}^0_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\varphi)_{ABCD}, \tag{2.27a}$$

$$(\mathcal{K}^1_{1,1}\mathcal{K}^1_{1,1}\varphi)_{AA'} = \varphi_{AA'}, \tag{2.27b}$$

$$(\mathcal{K}^2_{2,0}\mathcal{K}^1_{2,0}\varphi) = 0, \tag{2.27c}$$

$$(\mathcal{P}^{3/2}_{3,1}\mathcal{K}^0_{1,1}\varphi)_{ABCA'} = 0, \tag{2.27d}$$

$$(\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\varphi)_{AB} = (\mathcal{K}^1_{2,0}\varphi)_{AB}, \tag{2.27e}$$

$$(\mathcal{K}^1_{3,1}\mathcal{K}^1_{3,1}\mathcal{K}^1_{3,1}\varphi)_{ABCA'} = -\tfrac{2}{9}(\mathcal{K}^0_{1,1}\mathcal{K}^1_{1,1}\mathcal{K}^2_{3,1}\varphi)_{ABCA'} + (\mathcal{K}^1_{3,1}\varphi)_{ABCA'}, \tag{2.27f}$$

and the first order differential identities

$$(\mathscr{C}^\dagger_{4,0,w}\mathcal{K}^1_{4,0}\varphi)_{ABCA'} = (\mathcal{K}^1_{3,1}\mathscr{C}^\dagger_{4,0,w}\varphi)_{ABCA'} + \tfrac{1}{2}(\mathscr{T}_{2,0,-4+w}\mathcal{K}^2_{4,0}\varphi)_{ABCA'}, \tag{2.28a}$$

$$(\mathscr{C}^\dagger_{4,0,w}\mathcal{K}^0_{2,0}\varphi)_{ABCA'} = (\mathcal{K}^0_{1,1}\mathscr{C}^\dagger_{2,0,-1+w}\varphi)_{ABCA'} - (\mathscr{T}_{2,0,-4+w}\mathcal{K}^1_{2,0}\varphi)_{ABCA'}, \tag{2.28b}$$

$$(\mathscr{C}^\dagger_{3,1,w}\mathcal{K}^0_{1,1}\varphi)_{ABA'B'} = (\mathcal{K}^0_{0,2}\mathscr{C}^\dagger_{1,1,-1+w}\varphi)_{ABA'B'} - \tfrac{4}{3}(\mathscr{T}_{1,1,-3+w}\mathcal{K}^1_{1,1}\varphi)_{ABA'B'}, \tag{2.28c}$$

$$(\mathscr{C}^\dagger_{2,0,w}\mathcal{K}^1_{2,0}\varphi)_{AA'} = (\mathcal{K}^1_{1,1}\mathscr{C}^\dagger_{2,0,w}\varphi)_{AA'} + (\mathscr{T}_{0,0,-2+w}\mathcal{K}^2_{2,0}\varphi)_{AA'}, \tag{2.28d}$$

$$(\mathscr{C}^\dagger_{2,0,w}\mathcal{K}^2_{4,0}\varphi)_{AA'} = (\mathcal{K}^2_{3,1}\mathscr{C}^\dagger_{4,0,1+w}\varphi)_{AA'}, \tag{2.28e}$$

$$(\mathscr{C}^\dagger_{2,0,w}\mathcal{K}^0_{0,0}\varphi)_{AA'} = -2(\mathcal{K}^1_{1,1}\mathscr{T}_{0,0,-2+w}\varphi)_{AA'}, \tag{2.28f}$$

$$(\mathscr{D}_{1,1,w}\mathcal{K}^1_{1,1}\varphi) = 2(\mathcal{K}^2_{2,0}\mathscr{C}_{1,1,-2+w}\varphi), \tag{2.28g}$$

$$(\mathscr{D}_{1,1,w}\mathcal{K}^2_{3,1}\varphi) = (\mathcal{K}^2_{2,0}\mathscr{D}_{3,1,1+w}\varphi), \tag{2.28h}$$

$$(\mathcal{K}^1_{3,1}\mathcal{P}^{3/2}_{3,1}\mathscr{T}_{2,0,w}\varphi)_{ABCA'} = -\tfrac{1}{4}(\mathscr{C}^\dagger_{4,0,4+w}\mathcal{K}^0_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\varphi)_{ABCA'} + \tfrac{3}{4}(\mathscr{T}_{2,0,w}\mathcal{K}^1_{2,0}\varphi)_{ABCA'}. \tag{2.28i}$$

Proof. For (2.27a) we calculate

$$\begin{aligned}(\mathcal{K}^1_{4,0}\mathcal{P}^1_{4,0}\varphi)_{ABCD} &= (\mathcal{K}^1_{4,0}\mathcal{K}^0_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\varphi)_{ABCD}\\ &= \tfrac{1}{2}(\mathcal{K}^0_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\varphi)_{ABCD}\\ &= \tfrac{1}{2}(\mathcal{K}^0_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\varphi)_{ABCD}.\end{aligned}$$

In the first step uses (2.23b), the second one is a commutator of $\mathcal{K}^1$ and $\mathcal{K}^0$ and the third step makes use of the fact that three sign-flips are equal to one sign-flip. For (2.27b) we note that $\mathcal{K}^1$ on a $(1,1)$ spinor changes sign in two of the four components,

$$(\mathcal{K}^1_{1,1}\varphi)_{00'} = \varphi_{00'}, \quad (\mathcal{K}^1_{1,1}\varphi)_{01'} = \varphi_{01'}, \quad (\mathcal{K}^1_{1,1}\varphi)_{10'} = -\varphi_{10'}, \quad (\mathcal{K}^1_{1,1}\varphi)_{11'} = -\varphi_{11'},$$

so $\mathcal{K}^1_{1,1}\mathcal{K}^1_{1,1} = \mathrm{Id}$. Equation (2.27c) is true because $\mathcal{K}^1_{2,0}$ cancels the middle component of φ_{AB} and $\mathcal{K}^2_{2,0}$ singles out that middle component. The rest of the algebraic identities are proved analogously. The proof of the differential identities relies on a straightforward but tedious expansion of projectors (2.18) and extended fundamental operators (2.17). We only calculate (2.28e).

$$\begin{aligned}(\mathscr{C}^\dagger_{2,0,-1}\mathcal{K}^2_{4,0}\varphi)_{AA'} - (\mathcal{K}^2_{3,1}\mathscr{C}^\dagger_{4,0}\varphi)_{AA'} &= \frac{U^B{}_{A'}\kappa^{CD}\varphi_{ABCD}}{2\kappa_1} + \frac{\kappa^{BC}(\mathscr{C}^\dagger_{4,0}\varphi)_{ABCA'}}{2\kappa_1} + (\mathscr{C}^\dagger_{2,0}\mathcal{K}^2_{4,0}\varphi)_{AA'}\\ &= \frac{U^B{}_{A'}\kappa^{CD}\varphi_{ABCD}}{2\kappa_1} - \frac{\varphi_{ABCD}(\mathscr{T}_{2,0}\kappa)^{BCD}{}_{A'}}{2\kappa_1}\\ &\quad + \frac{\kappa^{CD}\varphi_{ABCD}(\mathscr{T}_{0,0}\kappa_1)^B{}_{A'}}{2\kappa_1^2}\\ &= 0.\end{aligned}$$

The other identities are proved along the same lines. $\qquad\square$

Lemma 2.10. *For any symmetric spinors* $\varphi, \varphi_{ABA'B'}$ *the following identities hold*

$$0 = (\mathcal{K}_{0,2}^0 \mathscr{C}_{1,1,-5}^\dagger \mathscr{T}_{0,0,-3}\varphi)_{ABA'B'} + \tfrac{4}{3}(\mathscr{T}_{1,1,-4}\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,-6}\varphi)_{ABA'B'}$$
$$- \tfrac{4}{3}(\mathscr{T}_{1,1,-7}\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,-3}\varphi)_{ABA'B'}, \tag{2.29a}$$

$$0 = (\mathscr{C}_{3,1,-1}^\dagger \mathcal{K}_{1,1}^0 \mathscr{T}_{0,0}\varphi)_{ABA'B'} + 2(\mathcal{K}_{0,2}^0 \mathscr{C}_{1,1}^\dagger \mathscr{T}_{0,0}\varphi)_{ABA'B'} + 12(\mathcal{K}_{2,2}^1 \mathscr{T}_{1,1} \mathscr{T}_{0,0}\varphi)_{ABA'B'}$$
$$- \tfrac{32}{3}(\mathscr{T}_{1,1,-1}\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,3/2}\varphi)_{ABA'B'}, \tag{2.29b}$$

$$0 = -\tfrac{1}{12}(\mathscr{C}_{3,1,-1}^\dagger \mathcal{K}_{1,1}^0 \mathscr{D}_{2,2,4}\varphi)_{ABA'B'} + (\mathscr{C}_{3,1,-1}^\dagger \mathcal{K}_{3,1}^1 \mathscr{C}_{2,2,1}\varphi)_{ABA'B'} - \Psi_2 (\mathcal{K}_{2,2}^1\varphi)_{ABA'B'}$$
$$- (\mathcal{K}_{2,2}^1 \mathscr{C}_{3,1}^\dagger \mathscr{C}_{2,2}\varphi)_{ABA'B'} - \tfrac{1}{3}(\mathcal{K}_{2,2}^1 \mathscr{T}_{1,1} \mathscr{D}_{2,2}\varphi)_{ABA'B'} + \frac{(\mathcal{L}_\xi \varphi)_{ABA'B'}}{3\kappa_1}$$
$$+ \tfrac{2}{9}(\mathscr{T}_{1,1,-1}\mathcal{K}_{1,1}^1 \mathscr{D}_{2,2,1}\varphi)_{ABA'B'} - \tfrac{2}{3}(\mathscr{T}_{1,1,-1}\mathcal{K}_{3,1}^2 \mathscr{C}_{2,2,-2}\varphi)_{ABA'B'}. \tag{2.29c}$$

Proof. This can be verified by expanding all operators in terms of the non-extended fundamental spinor operators. Alternatively the verification can be done by expanding in components using the *SpinFrames* package [3]. □

3. Spinorial formulation of linearized gravity

In this section we review the field equations of linearized gravity for a general vacuum background and allow for sources of the linearized field. The spinor variational operator ϑ developed in [16] will be used. Let $\dot{g}_{ab}$ be a linearized metric and $\dot{g}_{ABA'B'}$ the spinorial version. Observe that we make the variation with the indices down, and raise them and take traces afterwards. We define the irreducible parts of the linearized metric as

$$G_{ABA'B'} = \dot{g}_{(AB)(A'B')}, \qquad\qquad \slashed{G} = \dot{g}^C{}_C{}^{C'}{}_{C'}, \tag{3.1}$$

so the decomposition into traceless and trace parts is given by

$$\dot{g}_{ABA'B'} = G_{ABA'B'} + \tfrac{1}{4}\slashed{G}\epsilon_{AB}\bar{\epsilon}_{A'B'}. \tag{3.2}$$

For the spinor variation of the irreducible parts of the curvature we get, see [16], for a general vacuum background

$$\vartheta\Psi_{ABCD} = \tfrac{1}{2}(\mathscr{C}_{3,1}\mathscr{C}_{2,2}G)_{ABCD} - \tfrac{1}{4}\slashed{G}\Psi_{ABCD} \tag{3.3a}$$

$$\vartheta\Phi_{ABA'B'} = \tfrac{1}{2}G^{CD}{}_{A'B'}\Psi_{ABCD} + \tfrac{1}{2}(\mathscr{C}_{3,1}^\dagger \mathscr{C}_{2,2}G)_{ABA'B'} + \tfrac{1}{6}(\mathscr{T}_{1,1}\mathscr{D}_{2,2}G)_{ABA'B'}$$
$$- \tfrac{1}{8}(\mathscr{T}_{1,1}\mathscr{T}_{0,0}\slashed{G})_{ABA'B'}, \tag{3.3b}$$

$$\vartheta\Lambda = -\tfrac{1}{24}(\mathscr{D}_{1,1}\mathscr{D}_{2,2}G) + \tfrac{1}{32}(\mathscr{D}_{1,1}\mathscr{T}_{0,0}\slashed{G}). \tag{3.3c}$$

Note also that $\vartheta\Lambda = 0 = \vartheta\Phi_{ABA'B'}$ in the source-free case, i.e. when $\dot{g}_{ABA'B'}$ is a solution to the linearized vacuum Einstein equations.

For later use, see lemma 4.4 below, we introduce the notation $\vartheta\Psi[\varphi, \slashed{\phi}]_{ABCD}$ for the linearized Weyl curvature operator acting on a symmetric tensor field with irreducible parts $\varphi_{ABA'B'}, \slashed{\phi}$ (and analogously the other curvature operators). In case the field is given by $G_{ABA'B'}, \slashed{G}$ or 0, we suppress the additional argument. It will also be convenient to introduce

$$\phi_{ABCD} = \tfrac{1}{2}(\mathscr{C}_{3,1}\mathscr{C}_{2,2}G)_{ABCD} = \vartheta\Psi_{ABCD} + \tfrac{1}{4}\slashed{G}\Psi_{ABCD}. \tag{3.4}$$

as a modification of the varied Weyl spinor $\vartheta\Psi_{ABCD}$[9]. In some equations it will be more convenient to use ϕ_{ABCD} and in others to use $\vartheta\Psi_{ABCD}$.

As a consequence of (3.3) we derive the linearized Bianchi identity.

Lemma 3.1. *For a general vacuum background the modified Weyl spinor* (3.4) *satisfies*

$$(\mathscr{C}_{4,0}^\dagger \phi)_{ABCA'} = (\mathscr{C}_{2,2}\vartheta\Phi)_{ABCA'} + \tfrac{1}{2}\Psi_{ABCD}(\mathscr{D}_{2,2}G)^D{}_{A'} - \tfrac{3}{2}\Psi_{(AB}{}^{DF}(\mathscr{C}_{2,2}G)_{C)DFA'}$$
$$- \tfrac{1}{8}\Psi_{ABCD}(\mathscr{T}_{0,0}\slashed{G})^D{}_{A'} + \tfrac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathscr{T}_{4,0}\Psi)_{ABCDFB'}. \tag{3.5}$$

Restricting to a type D background this simplifies to

$$(\mathscr{C}_{4,0}^\dagger \phi)_{ABCA'} = (\mathscr{C}_{2,2}\vartheta\Phi)_{ABCA'} - \tfrac{3}{2}\Psi_2(\mathscr{C}_{2,2,1}G)_{ABCA'} - \tfrac{3}{8}\Psi_2(\mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathscr{D}_{2,2,4}G)_{ABCA'}$$
$$+ \tfrac{3}{32}\Psi_2(\mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathscr{T}_{0,0}\slashed{G})_{ABCA'} + \tfrac{9}{4}\Psi_2(\mathcal{K}_{1,1}^0 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2,1}G)_{ABCA'}. \tag{3.6}$$

---

[9]In a type D principal frame this modification only affects the middle component.

Proof. We apply $\mathscr{C}_{2,2}$ on (3.3b), commute $\mathscr{C}_{2,2}\mathscr{C}_{3,1}^\dagger$, use (3.4) to get

$$
\begin{aligned}
(\mathscr{C}_{2,2}\vartheta\Phi)_{ABCA'} =\ & \tfrac{1}{2}(\mathscr{C}_{2,2}\mathscr{C}_{3,1}^\dagger\mathscr{C}_{2,2}G)_{ABCA'} + \tfrac{1}{6}(\mathscr{C}_{2,2}\mathscr{T}_{1,1}\mathscr{D}_{2,2}G)_{ABCA'} - \tfrac{1}{8}(\mathscr{C}_{2,2}\mathscr{T}_{1,1}\mathscr{T}_{0,0}\mathscr{G})_{ABCA'} \\
& - \tfrac{1}{3}\Psi_{ABCD}(\mathscr{D}_{2,2}G)^D{}_{A'} + \tfrac{1}{2}\Psi_{(AB}{}^{DF}(\mathscr{C}_{2,2}G)_{C)DFA'} \\
& - \tfrac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathscr{T}_{4,0}\Psi)_{ABCDFB'}
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
=\ & \tfrac{1}{6}(\mathscr{C}_{2,2}\mathscr{T}_{1,1}\mathscr{D}_{2,2}G)_{ABCA'} - \tfrac{1}{8}(\mathscr{C}_{2,2}\mathscr{T}_{1,1}\mathscr{T}_{0,0}\mathscr{G})_{ABCA'} + (\mathscr{C}_{4,0}^\dagger\phi)_{ABCA'} \\
& - \tfrac{1}{3}\Psi_{ABCD}(\mathscr{D}_{2,2}G)^D{}_{A'} + \tfrac{3}{2}\Psi_{(AB}{}^{DF}(\mathscr{C}_{2,2}G)_{C)DFA'} \\
& - \tfrac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathscr{T}_{4,0}\Psi)_{ABCDFB'} - \tfrac{1}{8}(\mathscr{T}_{2,0}\mathscr{D}_{3,1}\mathscr{C}_{2,2}G)_{ABCA'}.
\end{aligned}
\tag{3.8}
$$

We then commute the $\mathscr{C}_{2,2}\mathscr{T}_{1,1}$ operators, and in the last step we commute $\mathscr{D}_{3,1}\mathscr{C}_{2,2}$ and use $\mathscr{C}_{1,1}\mathscr{T}_{0,0} = 0$ to get

$$
\begin{aligned}
(\mathscr{C}_{2,2}\vartheta\Phi)_{ABCA'} =\ & (\mathscr{C}_{4,0}^\dagger\phi)_{ABCA'} - \tfrac{1}{2}\Psi_{ABCD}(\mathscr{D}_{2,2}G)^D{}_{A'} + \tfrac{3}{2}\Psi_{(AB}{}^{DF}(\mathscr{C}_{2,2}G)_{C)DFA'} \\
& + \tfrac{1}{8}\Psi_{ABCD}(\mathscr{T}_{0,0}\mathscr{G})^D{}_{A'} - \tfrac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathscr{T}_{4,0}\Psi)_{ABCDFB'} \\
& + \tfrac{1}{12}(\mathscr{T}_{2,0}\mathscr{C}_{1,1}\mathscr{D}_{2,2}G)_{ABCA'} - \tfrac{1}{16}(\mathscr{T}_{2,0}\mathscr{C}_{1,1}\mathscr{T}_{0,0}\mathscr{G})_{ABCA'} \\
& - \tfrac{1}{8}(\mathscr{T}_{2,0}\mathscr{D}_{3,1}\mathscr{C}_{2,2}G)_{ABCA'}
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
=\ & (\mathscr{C}_{4,0}^\dagger\phi)_{ABCA'} - \tfrac{1}{2}\Psi_{ABCD}(\mathscr{D}_{2,2}G)^D{}_{A'} + \tfrac{3}{2}\Psi_{(AB}{}^{DF}(\mathscr{C}_{2,2}G)_{C)DFA'} \\
& + \tfrac{1}{8}\Psi_{ABCD}(\mathscr{T}_{0,0}\mathscr{G})^D{}_{A'} - \tfrac{1}{2}G^{DF}{}_{A'}{}^{B'}(\mathscr{T}_{4,0}\Psi)_{ABCDFB'}.
\end{aligned}
\tag{3.10}
$$

This gives (3.5). On a type D spacetime, we can use (2.14) and (2.15). The resulting $U_{AA'}$ spinors can be incorporated as extended indices, and the κ_{AB} spinors can then be rewritten in terms the $\mathcal{K}^i$ operators to get (3.6). $\qquad\square$

Note that on a Minkowski background and without sources the right-hand side of (3.5) vanishes and the linearized Bianchi identity reduces to the spin-2 equation. The linearized Bianchi identity (3.6) is of fundamental importance and next we derive some differential identities for it which are needed for the main theorem. The following variable appears naturally.

Definition 3.2. *Define the symmetric spinor $\widehat{\phi}_{ABCD}$ as the rescaled, sign-flipped and spin-2 projected linearized Weyl spinor,*

$$
\widehat{\phi}_{ABCD} = \kappa_1^4(\mathcal{K}_{4,0}^1\mathcal{P}_{4,0}^2\vartheta\Psi)_{ABCD}.
\tag{3.11}
$$

The components of $\widehat{\phi}_{ABCD}$ in a principal dyad are

$$
\begin{pmatrix} \widehat{\phi}_0 \\ \widehat{\phi}_1 \\ \widehat{\phi}_2 \\ \widehat{\phi}_3 \\ \widehat{\phi}_4 \end{pmatrix} = \begin{pmatrix} \kappa_1^4\vartheta\Psi_0 \\ 0 \\ 0 \\ 0 \\ -\kappa_1^4\vartheta\Psi_4 \end{pmatrix}.
$$

Corollary 3.3. *An alternative form of the linearized Bianchi identity (3.6), involving the variable (3.11), is given by*

$$
\begin{aligned}
(\mathscr{C}_{4,0,4}^\dagger\widehat{\phi})_{ABCA'} =\ & -\left(\kappa_1^4\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathscr{C}_{4,0}^\dagger\mathcal{P}_{4,0}^1\phi\right)_{ABCA'} - \tfrac{3}{2}\left(\kappa_1^4\Psi_2\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathscr{C}_{2,2,1}G\right)_{ABCA'} \\
& + \left(\kappa_1^4\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathscr{C}_{2,2}\vartheta\Phi\right)_{ABCA'}.
\end{aligned}
\tag{3.12}
$$

Proof. Applying the operator $\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\kappa_1^4\mathscr{C}_{4,0}^\dagger$ to (2.21) and using (2.28a) gives the identity

$$
(\mathscr{C}_{4,0,4}^\dagger\widehat{\phi})_{ABCA'} = -\left(\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}\mathscr{C}_{4,0,4}^\dagger(\kappa_1^4\mathcal{P}_{4,0}^1\phi)\right)_{ABCA'} + \left(\mathcal{K}_{3,1}^1\mathcal{P}_{3,1}^{3/2}(\kappa_1^4\mathscr{C}_{4,0}^\dagger\phi)\right)_{ABCA'}.
\tag{3.13}
$$

The result follows from $(\mathcal{P}_{3,1}^{3/2}\mathcal{K}_{1,1}^0\phi)_{ABCA'} = 0$ together with (3.6). $\qquad\square$

Corollary 3.4 (Covariant TME). *The covariant form of the spin-2 Teukolsky Master equation with source on a vacuum type D background is given by*

$$
(\mathscr{C}_{3,1}\mathscr{C}_{4,0,4}^\dagger\widehat{\phi})_{ABCD} = -3\Psi_2\widehat{\phi}_{ABCD} + \kappa_1^4(\mathcal{P}_{4,0}^2\mathcal{K}_{4,0}^1\mathscr{C}_{3,1,-4}\mathscr{C}_{2,2}\vartheta\Phi)_{ABCD}.
\tag{3.14}
$$

Proof. Apply the operator $\mathcal{P}^2_{4,0}\mathscr{C}_{3,1}$ to (3.12) and use

$$\left(\mathcal{P}^2_{4,0}\mathscr{C}_{3,1}\mathcal{K}^1_{3,1}\mathcal{P}^{3/2}_{3,1}\mathscr{C}^\dagger_{4,0,4}(\kappa^4_1\mathcal{P}^1_{4,0}\phi)\right)_{ABCD} = 0, \tag{3.15a}$$

$$\left(\mathcal{P}^2_{4,0}\mathscr{C}_{3,1}\mathcal{K}^1_{3,1}\mathcal{P}^{3/2}_{3,1}(\kappa^4_1\mathscr{C}_{2,2}\vartheta\Phi)\right)_{ABCD} = \kappa^4_1(\mathcal{P}^2_{4,0}\mathcal{K}^1_{4,0}\mathscr{C}_{3,1,-4}\mathscr{C}_{2,2}\vartheta\Phi)_{ABCD}, \tag{3.15b}$$

$$\left(\mathcal{P}^2_{4,0}\mathscr{C}_{3,1}\mathcal{K}^1_{3,1}\mathcal{P}^{3/2}_{3,1}(\kappa^4_1\Psi_2\mathscr{C}_{2,2,1}G)\right)_{ABCD} = \Psi_2\kappa^4_1(\mathcal{P}^2_{4,0}\mathcal{K}^1_{4,0}\mathscr{C}_{3,1}\mathscr{C}_{2,2}G)_{ABCD}. \tag{3.15c}$$

$\square$

4. MAIN THEOREM

We shall now prove our main theorem. The following is the detailed statement of Theorem 1.1 including source terms.

Theorem 4.1. *Let $\dot{g}_{ABA'B'}$ be a solution to the linearized Einstein equations with linearized Weyl curvature $\vartheta\Psi_{ABCD}$ and linearized source $\vartheta\Phi_{ABA'B'}$ on a vacuum background of Petrov type D. Furthermore, let $\widehat{\phi}_{ABCD} = \kappa^4_1(\mathcal{K}^1_{4,0}\mathcal{P}^2_{4,0}\vartheta\Psi)_{ABCD}$ be the modified linearized Weyl spinor and let*

$$\mathcal{M}_{ABA'B'} = (\mathscr{C}^\dagger_{3,1}\mathscr{C}^\dagger_{4,0,4}\widehat{\phi})_{ABA'B'}. \tag{4.1}$$

Then we have

$$\mathcal{M}_{ABA'B'} = \tfrac{1}{2}\nabla_{AA'}\mathcal{A}_{BB'} + \tfrac{1}{2}\nabla_{BB'}\mathcal{A}_{AA'} + \tfrac{1}{2}\Psi_2\kappa^3_1(\mathcal{L}_\xi\dot{g})_{ABA'B'} + (\mathcal{N}_{2,2}\vartheta\Phi)_{ABA'B'}, \tag{4.2}$$

where the complex one form $\mathcal{A}_{AA'}$ and the source term $(\mathcal{N}_{2,2}\vartheta\Phi)_{ABA'B'}$ are given by

$$\mathcal{A}_{AA'} = -\tfrac{1}{4}\not{\mathscr{C}}\Psi_2\kappa^3_1\xi_{AA'} - \tfrac{1}{2}\Psi_2\kappa^3_1\xi^{BB'}(\mathcal{K}^0_{0,2}\mathcal{K}^2_{2,2}G)_{ABA'B'} + \tfrac{2}{3}\kappa^3_1\xi^B{}_{A'}(\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\vartheta\Psi)_{AB}$$
$$+ (\mathcal{K}^1_{1,1}\mathcal{T}_{0,0}\mathcal{K}^2_{2,0}\mathcal{K}^2_{4,0}\kappa^4_1\vartheta\Psi)_{AA'} + \tfrac{2}{3}\kappa^4_1(\mathcal{K}^2_{3,1}\mathscr{C}_{2,2}\vartheta\Phi)_{AA'}, \tag{4.3a}$$

$$(\mathcal{N}_{2,2}\vartheta\Phi)_{ABA'B'} = -\left(\mathscr{C}^\dagger_{3,1}(\kappa^4_1\mathcal{K}^1_{3,1}\mathcal{P}^{1/2}_{3,1}\mathscr{C}_{2,2}\vartheta\Phi)\right)_{ABA'B'} + \left(\mathscr{C}^\dagger_{3,1}(\kappa^4_1\mathcal{K}^1_{3,1}\mathcal{P}^{3/2}_{3,1}\mathscr{C}_{2,2}\vartheta\Phi)\right)_{ABA'B'}$$
$$- 3\Psi_2\kappa^4_1(\mathcal{K}^1_{2,2}\vartheta\Phi)_{ABA'B'}. \tag{4.3b}$$

Before proving the theorem we collect some algebraic and differential identities for $\mathcal{A}_{AA'}$.

Lemma 4.2. *The one-form $\mathcal{A}_{AA'}$ defined in (4.3a) has the following properties:*

$$\mathcal{A}^{AA'}U_{AA'} = -\tfrac{1}{3}\kappa^3_1(\mathcal{L}_\xi\mathcal{K}^2_{2,0}\mathcal{K}^2_{4,0}\vartheta\Psi) + \tfrac{2}{3}\kappa^4_1 U^{AA'}(\mathcal{K}^2_{3,1}\mathscr{C}_{2,2}\vartheta\Phi)_{AA'} \tag{4.4a}$$

$$(\mathscr{D}_{1,1}\mathcal{A}) = -\tfrac{1}{2}\Psi_2\kappa^3_1(\mathcal{L}_\xi\not{\mathscr{C}}) \tag{4.4b}$$

$$(\mathscr{C}_{1,1,2}\mathcal{K}^1_{1,1}\mathcal{A})_{AB} = -\tfrac{1}{8}\Psi_2\kappa_1{}^3(\mathcal{L}_\xi\mathcal{K}^0_{0,0}\not{\mathscr{C}})_{AB} - \tfrac{2}{3}\kappa_1{}^3(\mathcal{L}_\xi\mathcal{K}^2_{4,0}\vartheta\Psi)_{AB}$$
$$+ \tfrac{2}{3}\kappa_1{}^4(\mathscr{C}_{1,1,-2}\mathcal{K}^1_{1,1}\mathcal{K}^2_{3,1}\mathscr{C}_{2,2}\vartheta\Phi)_{AB} - \tfrac{2}{3}\kappa_1{}^3\xi_{(A}{}^{A'}(\mathcal{K}^2_{3,1}\mathscr{C}_{2,2}\vartheta\Phi)_{B)A'}. \tag{4.4c}$$

Proof. Equation (4.4a) can be verified by a direct calculation using (2.12). To prove (4.4b), we make use of the form of $\mathcal{A}_{AA'}$ given in equation (4.19a) below,

$$\mathcal{A}_{AA'} = -\tfrac{1}{2}\left(\mathscr{C}^\dagger_{2,0}\mathcal{K}^0_{0,0}\mathcal{K}^2_{2,0}\mathcal{K}^2_{4,0}(\kappa^4_1\phi)\right)_{AA'} + \tfrac{2}{3}\left(\mathscr{C}^\dagger_{2,0}\mathcal{K}^2_{4,0}(\kappa^4_1\phi)\right)_{AA'} + \tfrac{1}{3}\left(\mathcal{K}^1_{1,1}\mathscr{D}_{2,2,2}(\kappa^4_1\Psi_2 G)\right)_{AA'}$$
$$- \tfrac{1}{3}\left(\mathcal{K}^1_{1,1}\mathcal{T}_{0,0,5/2}(\kappa^4_1\Psi_2\not{\mathscr{C}})\right)_{AA'} - \left(\mathcal{K}^2_{3,1}\mathscr{C}_{2,2,-1}(\kappa^4_1\Psi_2 G)\right)_{AA'}. \tag{4.5}$$

Applying $\mathscr{D}_{1,1}$ to this and using the commutator relation $\mathscr{D}_{1,1}\mathscr{C}^\dagger_{2,0} = 0$ gives

$$(\mathscr{D}_{1,1}\mathcal{A}) = \tfrac{1}{3}\left(\mathscr{D}_{1,1}\mathcal{K}^1_{1,1}\mathscr{D}_{2,2,2}(\kappa^4_1\Psi_2 G)\right) - \tfrac{1}{3}\left(\mathscr{D}_{1,1}\mathcal{K}^1_{1,1}\mathcal{T}_{0,0,5/2}(\kappa^4_1\Psi_2\not{\mathscr{C}})\right)$$
$$- \left(\mathscr{D}_{1,1}\mathcal{K}^2_{3,1}\mathscr{C}_{2,2,-1}(\kappa^4_1\Psi_2 G)\right). \tag{4.6}$$

Using (2.28g) on the first two terms, and (2.28h) on the last term gives

$$(\mathscr{D}_{1,1}\mathcal{A}) = \tfrac{2}{3}\left(\mathcal{K}^2_{2,0}\mathscr{C}_{1,1,-2}\mathscr{D}_{2,2,2}(\kappa^4_1\Psi_2 G)\right) - \tfrac{2}{3}\left(\mathcal{K}^2_{2,0}\mathscr{C}_{1,1,-2}\mathcal{T}_{0,0,5/2}(\kappa^4_1\Psi_2\not{\mathscr{C}})\right)$$
$$- \left(\mathcal{K}^2_{2,0}\mathscr{D}_{3,1,1}\mathscr{C}_{2,2,-1}(\kappa^4_1\Psi_2 G)\right). \tag{4.7}$$

The first and the last term cancel due to the general identity

$$(\mathcal{K}^2_{2,0}\mathscr{D}_{3,1,1}\mathscr{C}_{2,2,-1}\phi) = \tfrac{2}{3}(\mathcal{K}^2_{2,0}\mathscr{C}_{1,1,-2}\mathscr{D}_{2,2,2}\phi). \tag{4.8}$$

This identity can be proven by expanding the extended indices and commuting the derivatives. In the same way we can also prove the identity

$$(\mathcal{K}_{2,0}^2 \mathscr{C}_{1,1,v} \mathscr{T}_{0,0,w} \varphi) = \frac{(w-v)(\mathcal{L}_\xi \varphi)}{6\kappa_1}, \tag{4.9}$$

for arbitrary weights v and w. This finally gives (4.4b), where we in the last step commuted $\Psi_2 \kappa_1^3$ through the Lie derivative.

To prove (4.4c), we first note the following relation that is a consequence of linearized Bianchi

$$(\mathscr{C}_{2,0}^\dagger \mathcal{K}_{4,0}^2 \phi)_{AA'} = \tfrac{3}{2}\Psi_2(\mathscr{C}_{0,2,2}\mathcal{K}_{2,2}^2 G)_{AA'} + \tfrac{1}{8}\Psi_2(\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0}\mathcal{\not{G}})_{AA'} + U^B{}_{A'}(\mathcal{K}_{4,0}^2 \phi)_{AB}$$
$$+ (\mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{AA'}. \tag{4.10}$$

Applying a $\mathcal{K}_{1,1}^1$ to (4.3a) gives after minor algebraic manipulations

$$(\mathcal{K}_{1,1}^1 \mathcal{A})_{AA'} = -\tfrac{1}{4}\mathcal{\not{G}}\Psi_2 U_{AA'}\kappa_1{}^4 + \tfrac{2}{3}\kappa_1{}^4(\mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{AA'} + \Psi_2\kappa_1{}^3\xi_A{}^{B'}(\mathcal{K}_{2,2}^2 G)_{A'B'}$$
$$+ \tfrac{2}{3}\kappa_1{}^3\xi^B{}_{A'}(\mathcal{K}_{4,0}^2 \phi)_{AB} + (\mathscr{T}_{0,0,2}\mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \kappa_1{}^4\phi)_{AA'} - \tfrac{1}{4}(\mathscr{T}_{0,0,2}\Psi_2\kappa_1{}^4\mathcal{\not{G}})_{AA'}. \tag{4.11}$$

Applying $\mathscr{C}_{1,1,2}$ on this, using the commutator relation $\mathscr{C}_{1,1,2}\mathscr{T}_{0,0,2} = 0$ and translating $\xi^{AA'}\mathscr{T}$ terms to Lie derivatives, we end up with

$$(\mathscr{C}_{1,1,2}\mathcal{K}_{1,1}^1 \mathcal{A})_{AB} = \tfrac{2}{3}(\mathscr{C}_{1,1,2}\kappa_1{}^4\mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{AB} - \tfrac{1}{24}\Psi_2\kappa_1{}^3(\mathcal{K}_{0,0}^0 \mathcal{L}_\xi\mathcal{\not{G}})_{AB}$$
$$+ \tfrac{1}{9}\kappa_1{}^2\xi_{CA'}\xi^{CA'}(\mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{AB} - \tfrac{2}{3}\kappa_1{}^3(\mathcal{L}_\xi\mathcal{K}_{4,0}^2 \phi)_{AB}$$
$$- \tfrac{2}{3}\kappa_1{}^3\xi_{(A}{}^{A'}(\mathscr{C}_{2,0}^\dagger \mathcal{K}_{4,0}^2 \phi)_{B)A'} + \Psi_2\kappa_1{}^3\xi_{(A}{}^{A'}(\mathscr{C}_{0,2,2}\mathcal{K}_{2,2}^2 G)_{B)A'}$$
$$+ \tfrac{1}{12}\Psi_2\kappa_1{}^3\xi_{(A}{}^{A'}(\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0}\mathcal{\not{G}})_{B)A'}. \tag{4.12}$$

Using (4.10) to eliminate the $\mathscr{C}_{2,0}^\dagger \mathcal{K}_{4,0}^2 \phi$ terms yields

$$(\mathscr{C}_{1,1,2}\mathcal{K}_{1,1}^1 \mathcal{A})_{AB} = -\tfrac{1}{24}\Psi_2\kappa_1{}^3(\mathcal{K}_{0,0}^0 \mathcal{L}_\xi\mathcal{\not{G}})_{AB} - \tfrac{2}{3}\kappa_1{}^3(\mathcal{L}_\xi\mathcal{K}_{4,0}^2 \phi)_{AB}$$
$$+ \tfrac{2}{3}\kappa_1{}^4(\mathscr{C}_{1,1,-2}\mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{AB} - \tfrac{2}{3}\kappa_1{}^3\xi_{(A}{}^{A'}(\mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{B)A'}. \tag{4.13}$$

Inserting (3.4) leads to the result. $\qquad\qquad\square$

Proof of theorem 4.1. Apply the operator $\mathscr{C}^\dagger$ to (3.12) and moving out the scalars Ψ_2 and κ_1 we get

$$(\mathscr{C}_{3,1}^\dagger \mathscr{C}_{4,0,4}^\dagger \widehat{\phi})_{ABA'B'} = \kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2}\mathscr{C}_{2,2}\vartheta\Phi)_{ABA'B'} - \kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2}\mathscr{C}_{4,0}^\dagger \mathcal{P}_{4,0}^1 \phi)_{ABA'B'}$$
$$- \tfrac{3}{2}\Psi_2\kappa_1^4(\mathscr{C}_{3,1,-1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2}\mathscr{C}_{2,2,1}G)_{ABA'B'}. \tag{4.14}$$

The second term can be rewritten by expanding the spin-1 projector according to (2.23b) and commuting out the $\mathcal{K}_{2,0}^0$ using (2.28b) together with (2.27d),(2.27e),

$$\kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2}\mathscr{C}_{4,0}^\dagger \mathcal{P}_{4,0}^1 \phi)_{ABA'B'} = -\kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2}\mathscr{T}_{2,0,-4}\mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}$$
$$= \tfrac{1}{4}\kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathscr{C}_{2,0,-1}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}$$
$$- \tfrac{2}{3}\kappa_1^4(\mathscr{T}_{1,1,-4}\mathscr{C}_{2,0,-4}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}. \tag{4.15}$$

In the second step (2.28i) and (2.28b) with (2.27e) and a commutator is used. To commute out the $\mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2$ in the first term, we first use (2.28d) and (2.28e) to get

$$(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathscr{C}_{2,0,-1}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'} = (\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{4,0}^\dagger \phi)_{ABA'B'}$$
$$+ (\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathscr{T}_{0,0,-3}\mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'},$$
$$= (\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{4,0}^\dagger \phi)_{ABA'B'}$$
$$- \tfrac{4}{3}(\mathscr{T}_{1,1,-4}\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,-6}\mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}. \tag{4.16}$$

In the second step (2.28c) is used together with (2.29a). Using (4.16) in (4.15) and the linearized Bianchi identity (3.6) in the first term of (4.16) yields

$$
\begin{aligned}
\kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2} \mathscr{C}_{4,0}^\dagger \mathcal{P}_{4,0}^1 \phi)_{ABA'B'} = {} & \tfrac{1}{4}\kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{ABA'B'} \\
& - \tfrac{1}{8}\kappa_1^4\big(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 (\Psi_2 \mathcal{K}_{1,1}^1 \mathscr{D}_{2,2,4}G)\big)_{ABA'B'} \\
& + \tfrac{1}{32}\kappa_1^4\big(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 (\Psi_2 \mathcal{K}_{1,1}^1 \mathscr{T}_{0,0}\not{G})\big)_{ABA'B'} \\
& + \tfrac{3}{8}\kappa_1^4\big(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 (\Psi_2 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2,1}G)\big)_{ABA'B'} \\
& - \tfrac{2}{3}\kappa_1^4(\mathscr{T}_{1,1,-4}\mathscr{C}_{2,0,-4}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'} \\
& - \tfrac{1}{3}\kappa_1^4(\mathscr{T}_{1,1,-4}\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,-6}\mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}. \quad (4.17)
\end{aligned}
$$

The second and third term on the right-hand side can be simplified further using (2.27b). Using (4.17) in (4.14) and expanding the spin decomposition in the last term of (4.14) leads to

$$
\begin{aligned}
(\mathscr{C}_{3,1}^\dagger \mathscr{C}_{4,0,4}^\dagger \widehat{\phi})_{ABA'B'} = {} & \big(\mathscr{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2}(\kappa_1^4 \mathscr{C}_{2,2}\vartheta\Phi)\big)_{ABA'B'} \\
& + \tfrac{1}{8}\Psi_2 \kappa_1^4(\mathscr{C}_{3,1,-1}^\dagger \mathcal{K}_{1,1}^0 \mathscr{D}_{2,2,4}G)_{ABA'B'} \\
& - \tfrac{1}{4}\kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{ABA'B'} \\
& - \tfrac{1}{3}\Psi_2 \kappa_1^4(\mathscr{C}_{3,1,-1}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2,1}G)_{ABA'B'} \\
& - \tfrac{1}{32}\Psi_2 \kappa_1^4(\mathscr{C}_{3,1,-1}^\dagger \mathcal{K}_{1,1}^0 \mathscr{T}_{0,0}\not{G})_{ABA'B'} \\
& - \tfrac{3}{2}\Psi_2 \kappa_1^4(\mathscr{C}_{3,1,-1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{K}_{3,1}^1 \mathcal{K}_{3,1}^1 \mathscr{C}_{2,2,1}G)_{ABA'B'} \\
& + \tfrac{2}{3}\kappa_1^4(\mathscr{T}_{1,1,-4}\mathscr{C}_{2,0,-4}^\dagger \mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'} \\
& + \tfrac{1}{3}\kappa_1^4(\mathscr{T}_{1,1,-4}\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,-6}\mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{ABA'B'}. \quad (4.18)
\end{aligned}
$$

The fourth and sixth term on the right-hand side can be combined via (2.27f). Defining the complex vector field

$$
\begin{aligned}
\mathcal{A}_{AA'} = {} & -\tfrac{1}{2}\kappa_1^4(\mathscr{C}_{2,0,-4}^\dagger \mathcal{K}_{0,0}^0 \mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 \phi)_{AA'} + \tfrac{2}{3}\kappa_1^4(\mathscr{C}_{2,0,-4}^\dagger \mathcal{K}_{4,0}^2 \phi)_{AA'} + \tfrac{1}{3}\Psi_2 \kappa_1^4(\mathcal{K}_{1,1}^1 \mathscr{D}_{2,2,1}G)_{AA'} \\
& - \tfrac{1}{3}\Psi_2 \kappa_1^4(\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,3/2}\not{G})_{AA'} - \Psi_2 \kappa_1^4(\mathcal{K}_{3,1}^2 \mathscr{C}_{2,2,-2}G)_{AA'} \quad (4.19\mathrm{a}) \\
= {} & \big(\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0}\mathcal{K}_{2,0}^2 \mathcal{K}_{4,0}^2 (\kappa_1^4 \phi)\big)_{AA'} - \tfrac{1}{4}\Psi_2 \kappa_1^4(\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,2}\not{G})_{AA'} + \tfrac{2}{3}\kappa_1^4(\mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{AA'} \\
& - \tfrac{1}{2}\Psi_2 \kappa_1^3 \xi^{BB'}(\mathcal{K}_{0,2}^0 \mathcal{K}_{2,2}^2 G)_{ABA'B'} + \tfrac{2}{3}\kappa_1^3 \xi^B{}_{A'}(\mathcal{K}_{2,0}^1 \mathcal{K}_{4,0}^2 \phi)_{AB} \quad (4.19\mathrm{b})
\end{aligned}
$$

(for the second version we used the linearized Bianchi identity (3.6)) we find

$$
\begin{aligned}
(\mathscr{C}_{3,1}^\dagger \mathscr{C}_{4,0,4}^\dagger \widehat{\phi})_{ABA'B'} = {} & \big(\mathscr{C}_{3,1}^\dagger \mathcal{K}_{3,1}^1 \mathcal{P}_{3,1}^{3/2}(\kappa_1^4 \mathscr{C}_{2,2}\vartheta\Phi)\big)_{ABA'B'} - 3\Psi_2 \kappa_1^4(\mathcal{K}_{2,2}^1 \vartheta\Phi)_{ABA'B'} \\
& - \tfrac{1}{4}\kappa_1^4(\mathscr{C}_{3,1,-4}^\dagger \mathcal{K}_{1,1}^0 \mathcal{K}_{1,1}^1 \mathcal{K}_{3,1}^2 \mathscr{C}_{2,2}\vartheta\Phi)_{ABA'B'} + \tfrac{1}{2}\Psi_2 \kappa_1^3(\mathcal{L}_\xi G)_{ABA'B'} \\
& + (\mathscr{T}_{1,1}\mathcal{A})_{ABA'B'} \quad (4.20)
\end{aligned}
$$

by using (2.29) (the third term on the right-hand side can be rewritten using (2.26a)). Since

$$
\tfrac{1}{2}\nabla_{AA'}\mathcal{A}_{BB'} + \tfrac{1}{2}\nabla_{BB'}\mathcal{A}_{AA'} = \tfrac{1}{4}\epsilon_{AB}\bar{\epsilon}_{A'B'}(\mathscr{D}_{1,1}\mathcal{A}) + (\mathscr{T}_{1,1}\mathcal{A})_{ABA'B'}, \quad (4.21)
$$

(4.4b) for the trace terms and (3.2) finally proves the theorem.   □

We will restrict to the source-free case $\vartheta\Phi_{ABA'B'} = 0, \vartheta\Lambda = 0$ for the rest of this section. In this case the last term in (4.2) is zero and $\mathcal{M}_{ABA'B'}$ has the following property.

Corollary 4.3. *The complex field $\mathcal{M}_{ABA'B'}$ is a traceless (by definition (4.1)) solution to the source-free linearized vacuum Einstein equations because the first two terms on the right-hand side of (4.2) form a linearized diffeomorphism and the third term is a symmetry operator on $\dot{g}_{ABA'B'}$ (remember that $\Psi_2 \kappa_1^3$ is a constant) which is itself a solution.*

We can more generally derive all curvature components of the complex metric $\mathcal{M}_{ABA'B'}$.

Lemma 4.4. *In the source-free case, the curvature, see (3.3), of the complex metric (4.1) is given by*

$$\vartheta\Psi[\mathcal{M}]_{ABCD} = \tfrac{1}{2}(\hat{\mathcal{L}}_{\mathcal{A}}\Psi)_{ABCD} + \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi\phi)_{ABCD} \tag{4.22a}$$

$$= \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi\mathcal{P}_{4,0}^2\phi)_{ABCD} \tag{4.22b}$$

$$\overline{\vartheta\Psi}[\mathcal{M}]_{A'B'C'D'} = \tfrac{1}{2}(\hat{\mathcal{L}}_{\mathcal{A}}\bar\Psi)_{A'B'C'D'} + \tfrac{1}{2}\kappa_1^3\Psi_2(\mathcal{L}_\xi\bar\phi)_{A'B'C'D'} \tag{4.22c}$$

$$\vartheta\Phi[\mathcal{M}]_{ABA'B'} = 0 \tag{4.22d}$$

$$\vartheta\Lambda[\mathcal{M}] = 0 \tag{4.22e}$$

where

$$(\hat{\mathcal{L}}_{\mathcal{A}}\bar\Psi)_{A'B'C'D'} = \tfrac{1}{2}\bar\Psi_{A'B'C'D'}(\mathscr{D}_{1,1,0,6}\mathcal{A}) + 2\bar\Psi_{(A'B'C'}{}^{F'}(\mathscr{C}_{1,1,0,2}^\dagger\mathcal{A})_{D')F'}, \tag{4.23a}$$

$$(\hat{\mathcal{L}}_{\mathcal{A}}\Psi)_{ABCD} = \tfrac{1}{2}\Psi_{ABCD}(\mathscr{D}_{1,1,6}\mathcal{A}) + 2\Psi_{(ABC}{}^F(\mathscr{C}_{1,1,2}\mathcal{A})_{D)F}. \tag{4.23b}$$

Remark 4.5. *Expanding the left-hand side of equation (4.22c) using the complex conjugate of equation (3.3a) gives*

$$(\mathscr{C}_{1,3}^\dagger\mathscr{C}_{2,2}^\dagger\mathscr{C}_{3,1}^\dagger\mathscr{C}_{4,0,4}^\dagger(\kappa_1^4\mathcal{K}_{4,0}^1\mathcal{P}_{4,0}^2\vartheta\Psi))_{A'B'C'D'} = (\hat{\mathcal{L}}_{\mathcal{A}}\bar\Psi)_{A'B'C'D'} + \kappa_1^3\Psi_2(\mathcal{L}_\xi\bar\phi)_{A'B'C'D'} \tag{4.24}$$

which is the covariant form of the full TSI for source-free linearized gravity on a general Petrov type D vacuum background, see also corollary 4.7 for a manifestly gauge-invariant form of the covariant TSI on a Kerr background.

Proof. Commuting two derivatives in vacuum type D leads to the operator identity

$$(\mathscr{C}_{3,1}\mathscr{C}_{2,2}\mathscr{T}_{1,1}\varphi)_{ABCD} = \tfrac{1}{2}\Psi_{ABCD}(\mathscr{D}_{1,1,6}\varphi) + 2\Psi_{(ABC}{}^F(\mathscr{C}_{1,1,2}\varphi)_{D)F}. \tag{4.25}$$

Using this identity and its complex conjugate together with the source-free version of (4.20) and the fact that $\mathcal{M}$ is traceless, the curvature relations (4.22a) and (4.22c) follow.

In the source-free case, we find using (4.4c), (4.4b) and (4.4a) that

$$\begin{aligned}(\hat{\mathcal{L}}_{\mathcal{A}}\Psi)_{ABCD} &= \tfrac{3}{16}\Psi_2(\mathcal{K}_{2,0}^0\mathcal{K}_{0,0}^0\mathscr{D}_{1,1,6}\mathcal{A})_{ABCD} + \tfrac{3}{2}\Psi_2(\mathcal{K}_{2,0}^0\mathcal{K}_{2,0}^0\mathscr{C}_{1,1,2}\mathcal{A})_{ABCD}\\ &= \tfrac{3}{2}\Psi_2(\mathcal{K}_{2,0}^0\mathscr{C}_{1,1,2}\mathcal{K}_{1,1}^0\mathcal{A})_{ABCD} - \tfrac{3}{8}\Psi_2(\mathcal{K}_{2,0}^0\mathcal{K}_{0,0}^0\mathscr{D}_{1,1,4}\mathcal{A})_{ABCD}\\ &\quad + \tfrac{3}{16}\Psi_2(\mathcal{K}_{2,0}^0\mathcal{K}_{0,0}^0\mathscr{D}_{1,1,6}\mathcal{A})_{ABCD}\\ &= \tfrac{1}{8}\Psi_2\kappa_1^3(\mathcal{K}_{2,0}^0\mathcal{K}_{0,0}^0\mathcal{L}_\xi\mathcal{K}_{2,0}^2\mathcal{K}_{4,0}^2\phi)_{ABCD} - \Psi_2\kappa_1^3(\mathcal{K}_{2,0}^0\mathcal{L}_\xi\mathcal{K}_{4,0}^2\phi)_{ABCD}\\ &= -\Psi_2\kappa_1^3(\mathcal{L}_\xi\phi)_{ABCD} + \Psi_2\kappa_1^3(\mathcal{L}_\xi\mathcal{P}_{4,0}^2\phi)_{ABCD}. \tag{4.26}\end{aligned}$$

from which (4.22b) follows. Equations (4.22d) and (4.22e) follow from corollary 4.3. $\qquad\square$

An important property of the linearized curvature scalars $\vartheta\Psi_0, \vartheta\Psi_4$ entering $\mathcal{M}_{ABA'B'}$ is that they are invariant under infinitesimal diffeomorphisms and so is $\mathcal{M}_{ABA'B'}$ itself. We find the following behavior under gauge transformations.

Lemma 4.6. *For linearized diffeomorphism of the background metric, generated by a real vector $\zeta^{AA'}$ of the original metric we get*

$$G_{ABA'B'} = 2(\mathscr{T}_{1,1}\zeta)_{ABA'B'}, \qquad\qquad \not{G} = 2(\mathscr{D}_{1,1}\zeta). \tag{4.27}$$

For the curvature and $\mathcal{A}_{AA'}$ we get

$$\vartheta\Phi_{ABA'B'} = 0, \tag{4.28a}$$

$$\vartheta\Lambda = 0, \tag{4.28b}$$

$$\phi_{ABCD} = \tfrac{3}{16}\Psi_2(\mathcal{K}_{2,0}^0\mathcal{K}_{0,0}^0\mathscr{D}_{1,1,6}\zeta)_{ABCD} + \tfrac{3}{2}\Psi_2(\mathcal{K}_{2,0}^0\mathcal{K}_{2,0}^1\mathscr{C}_{1,1,2}\zeta)_{ABCD}, \tag{4.28c}$$

$$\mathcal{A}_{AA'} = -\Psi_2\kappa_1^3(\mathcal{L}_\xi\zeta)_{AA'}. \tag{4.28d}$$

Proof. For the curvature, one can use the results of [16] and transform it to the operators of this paper using the type D structure of the curvature. Applying $\mathcal{K}_{2,0}^2\mathcal{K}_{4,0}^2$ or $\mathcal{K}_{2,0}^1\mathcal{K}_{4,0}^2$ onto ϕ_{ABCD} gives

$$(\mathcal{K}_{2,0}^2\mathcal{K}_{4,0}^2\phi) = \tfrac{1}{2}\Psi_2(\mathscr{D}_{1,1,6}\zeta), \tag{4.29}$$

$$(\mathcal{K}_{2,0}^1\mathcal{K}_{4,0}^2\phi)_{AB} = \tfrac{3}{2}\Psi_2(\mathcal{K}_{2,0}^1\mathcal{K}_{2,0}^2\mathscr{C}_{1,1,2}\zeta)_{AB}. \tag{4.30}$$

These relations can then be used in (4.19b) to yield

$$\mathcal{A}_{AA'} = -\Psi_2 \kappa_1^3 \xi^{BB'}(\mathcal{K}_{0,2}^0 \mathcal{K}_{2,2}^2 \mathscr{T}_{1,1}\zeta)_{ABA'B'} + \Psi_2 \kappa_1^3 \xi^{B}{}_{A'}(\mathcal{K}_{2,0}^1 \mathcal{K}_{2,0}^1 \mathscr{C}_{1,1,2}\zeta)_{AB}$$
$$-\tfrac{1}{2}\Psi_2 \kappa_1^4 (\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,2}\mathscr{D}_{1,1}\zeta)_{AA'} + \tfrac{1}{2}\Psi_2 \kappa_1^4 (\mathcal{K}_{1,1}^1 \mathscr{T}_{0,0,-1}\mathscr{D}_{1,1,6}\zeta)_{AA'}. \qquad (4.31)$$

An expansion of the extended indices and a reformulation of the Lie derivative in terms of fundamental spinor operators gives the gauge dependence of $\mathcal{A}_{AA'}$. $\qquad \square$

Corollary 4.7. *For linearized diffeomorphism generated by a real vector $\zeta^{AA'}$ on a Kerr background, we have*

$$\mathrm{Im}\mathcal{A}_{AA'} = 0. \qquad (4.32)$$

Then we have the covariant form of the Teukolsky-Starobinsky identities for linearized gravity on Kerr in terms of manifestly gauge invariant quantities,

$$(\mathscr{C}_{1,3}^\dagger \mathscr{C}_{2,2}^\dagger \mathscr{C}_{3,1}^\dagger \mathscr{C}_{4,0,4}^\dagger (\kappa_1^4 \mathcal{K}_{4,0}^1 \mathcal{P}_{4,0}^2 \vartheta\Psi))_{A'B'C'D'} = \Psi_2 \kappa_1^3 (\mathcal{L}_\xi \mathcal{P}_{0,4}^2 \overline{\vartheta\Psi})_{A'B'C'D'} + 2i(\hat{\mathcal{L}}_{\mathrm{Im}\mathcal{A}}\overline{\Psi})_{A'B'C'D'}. \qquad (4.33)$$

Proof. Equation (4.32) follows from reality of $\xi^{AA'}, \zeta^{AA'}$ and $\Psi_2\kappa_1^3$ on Kerr , see remark 2.3. Expand the curvature operator in (4.22c) and subtract the complex conjugate of the vacuum identity between (4.22a) and (4.22b). To end up with the identity in terms of $\vartheta\Psi_{ABCD}$, use (3.11) and (3.4). $\qquad \square$

5. Conclusions and outlook

In theorem 4.1 we have shown how the Debye potential construction for linearized gravity on a vacuum spacetime of Petrov type D (generalizing the Sachs-Bergmann super-potential for linearized gravity on Minkowski space) can be used to define a complex solution $\mathcal{M}_{ab}$ to the field equations of linearized gravity, which in view of the identity (4.2) is essentially pure gauge. In particular,

$$\mathcal{M}_{ABA'B'} - \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi \dot{g})_{ABA'B'}$$

is a pure gauge metric.

Calculating the self-dual linearized Weyl curvature *intermediate metric* $\mathcal{M}_{ab}$ on the Kerr background leads to a covariant form of the Teukolsky-Starobinsky Identities for linearized gravity (4.33), which when viewed as a system of scalar equations in terms of Newman-Penrose scalars includes three additional equations compared to the classical form of the TSI. As will be shown in a future paper [5], the full TSI system can be seen as a hyperbolic evolution equation.

In view of the fundamental role played by the full TSI for the spin-1 case in analyzing the conservation laws implied by the Maxwell field equations [14], we expect that this new covariant identity providing the full TSI for the spin-2 case will lead to a more complete understanding of the structure of linearized gravity on the Kerr spacetime, and in particular the conservation laws implied by this system. It is worth emphasizing that when restricted to the Kerr background, the identity (4.2) yields a manifestly gauge invariant form of the full TSI, given in (4.33). In particular, $\mathrm{Im}\mathcal{A}_{AA'}$ is gauge invariant in that case.

The fact that $\mathrm{Im}\mathcal{A}_{AA'}$ is gauge invariant is relevant for the problem of classifying gauge invariant quantities for linearized gravity on the Kerr spacetime. Work on this problem, extending and completing analysis of gauge invariant quantities on the Schwarzschild spacetime given in [31], is ongoing.

Recall that the Bianchi identity in differential geometry is a geometric identity which is valid independently of any field equation. The same is true for the linearized Bianchi identity with sources (3.5), and its specialization to Petrov type D backgrounds (3.6). It is important to note that several of the fundamental identities including (3.14) and (4.2) presented here are *operator identities* derived from the Bianchi identity by applying suitable fundamental operators and making use of commutation rules, and are therefore valid for any linearized metric, not necessarily a solution of the linearized Einstein equations.

Operator identities as the ones just mentioned have many interesting applications. For example, they lead to symmetry operators for linearized gravity via the method of adjoint operators, see [4]. Further, expect that they will play a major role in the derivation of differential complexes

for fields on algebraically special backgrounds, for example the complex extending the Killing operator, cf. [25].

The black hole stability problem has provided a major motivation for the present investigation, and we believe that the results proved here, and their consequences mentioned above, such as the hyperbolic formulation and conservation laws for TSI, as well as construction of symmetry operators for linearized gravity are relevant for proving stability of the Kerr black hole.

Appendix A. GHP form of some expressions

In this section we present the components of certain spinor equations with respect to a principal null dyad in a Petrov type D spacetime. Recall that in this case, only one of the Newman-Penrose Weyl scalars, Ψ_2, is non-zero. We are using the compact GHP formalism [23] in which the operators $\text{þ}, \text{þ}', \eth, \eth'$ are weighted directional derivatives along the tetrad. The computations have been performed using the *SpinFrames* package [3].

We note the relation between components of the varied Weyl spinor $\vartheta\Psi_{ABCD}$ used in this paper and the linearized Newman-Penrose Weyl scalars (the difference is due to the variation of the tetrad/dyad).

$$\vartheta\Psi_0 = \dot{\Psi}_0, \quad \vartheta\Psi_1 = \dot{\Psi}_1 + 3\Psi_2(\vartheta o)_0, \quad \vartheta\Psi_2 = \dot{\Psi}_2, \quad \vartheta\Psi_3 = \dot{\Psi}_3 - 3\Psi_2(\vartheta\iota)_1, \quad \vartheta\Psi_4 = \dot{\Psi}_4. \quad \text{(A.1)}$$

The components of the spin-2 TME (3.14) in the source-free case take the compact form

$$\left((\text{þ}-3\rho-\bar{\rho})\,\text{þ}' - (\eth-3\tau-\bar{\tau}')\,\eth' - 3\Psi_2\right)(\kappa_1\vartheta\Psi_0) = 0, \quad \text{(A.2a)}$$

$$\left((\text{þ}'-3\rho'-\bar{\rho}')\,\text{þ} - (\eth'-3\tau'-\bar{\tau})\,\eth - 3\Psi_2\right)(\kappa_1\vartheta\Psi_4) = 0. \quad \text{(A.2b)}$$

The components of the complex vector field $\mathcal{A}_{AA'}$ defined in (4.3a) in the source-free case are given by

$$\mathcal{A}_{00'} = -\tfrac{3}{4}\mathcal{C}\!\!\!/\,\Psi_2\kappa_1{}^4\rho - 3G_{11'}\Psi_2\kappa_1{}^4\rho + 3G_{10'}\Psi_2\kappa_1{}^4\tau - 2\kappa_1^4\vartheta\Psi_1\tau' + \text{þ}(\kappa_1^4\vartheta\Psi_2), \quad \text{(A.3a)}$$

$$\mathcal{A}_{01'} = -\tfrac{3}{4}\mathcal{C}\!\!\!/\,\Psi_2\kappa_1{}^4\tau - 3G_{12'}\Psi_2\kappa_1{}^4\rho + 3G_{11'}\Psi_2\kappa_1{}^4\tau - 2\kappa_1^4\vartheta\Psi_1\rho' + \eth(\kappa_1^4\vartheta\Psi_2), \quad \text{(A.3b)}$$

$$\mathcal{A}_{10'} = \tfrac{3}{4}\mathcal{C}\!\!\!/\,\Psi_2\kappa_1{}^4\tau' + 3G_{10'}\Psi_2\kappa_1{}^4\rho' - 3G_{11'}\Psi_2\kappa_1{}^4\tau' + 2\kappa_1^4\vartheta\Psi_3\rho - \eth'(\kappa_1^4\vartheta\Psi_2), \quad \text{(A.3c)}$$

$$\mathcal{A}_{11'} = \tfrac{3}{4}\mathcal{C}\!\!\!/\,\Psi_2\kappa_1{}^4\rho' + 3G_{11'}\Psi_2\kappa_1{}^4\rho' - 3G_{12'}\Psi_2\kappa_1{}^4\tau' + 2\kappa_1^4\vartheta\Psi_3\tau - \text{þ}'(\kappa_1^4\vartheta\Psi_2). \quad \text{(A.3d)}$$

We also state an explicit form of the imaginary part on Kerr, since it is gauge invariant,

$$\text{Im}\mathcal{A}_{00'} = -\tfrac{3}{2}iG_{10'}\Psi_2\kappa_1{}^4\tau - \tfrac{3}{2}iG_{01'}\Psi_2\kappa_1{}^4\tau' + i\kappa_1^4\vartheta\Psi_1\tau' - i\bar{\kappa}_{1'}^4\overline{\vartheta\Psi}_{1'}\bar{\tau}' + \text{þ}\,\text{Im}(\kappa_1^4\vartheta\Psi_2), \quad \text{(A.4a)}$$

$$\text{Im}\mathcal{A}_{01'} = \tfrac{3}{2}iG_{12'}\Psi_2\kappa_1{}^4\rho + \tfrac{3}{2}iG_{01'}\Psi_2\kappa_1{}^4\rho' + i\kappa_1^4\vartheta\Psi_1\rho' + i\bar{\kappa}_{1'}^4\overline{\vartheta\Psi}_{3'}\bar{\rho} - i\,\eth\,\text{Re}(\kappa_1^4\vartheta\Psi_2), \quad \text{(A.4b)}$$

$$\text{Im}\mathcal{A}_{10'} = -\tfrac{3}{2}iG_{21'}\Psi_2\kappa_1{}^4\rho - \tfrac{3}{2}iG_{10'}\Psi_2\kappa_1{}^4\rho' - i\kappa_1^4\vartheta\Psi_3\rho - i\bar{\kappa}_{1'}^4\overline{\vartheta\Psi}_{1'}\bar{\rho}' + i\,\eth'\,\text{Re}(\kappa_1^4\vartheta\Psi_2), \quad \text{(A.4c)}$$

$$\text{Im}\mathcal{A}_{11'} = \tfrac{3}{2}iG_{21'}\Psi_2\kappa_1{}^4\tau + \tfrac{3}{2}iG_{12'}\Psi_2\kappa_1{}^4\tau' - i\kappa_1^4\vartheta\Psi_3\tau + i\bar{\kappa}_{1'}^4\overline{\vartheta\Psi}_{3'}\bar{\tau} - \text{þ}'\,\text{Im}(\kappa_1^4\vartheta\Psi_2). \quad \text{(A.4d)}$$

The dyad components of the full covariant TSI (4.33) on a Kerr background are given by

$$0 = \text{þ}'\,\text{þ}'\,\text{þ}'\,\text{þ}'(\kappa_1^4\vartheta\Psi_0) - \eth\,\eth\,\eth\,\eth(\kappa_1^4\vartheta\Psi_4) - \tfrac{M}{27}\mathcal{L}_\xi\overline{\vartheta\Psi}_4, \quad \text{(A.5a)}$$

$$\begin{aligned}
0 = {}&\left(\eth'(\text{þ}' - \bar{\rho}') - 6\bar{\rho}'\bar{\tau}\right)(\text{þ}' + 2\bar{\rho}')(\text{þ}' + 2\bar{\rho}')(\kappa_1^4\vartheta\Psi_0) \\
&- \left(\text{þ}(\eth - \bar{\tau}') - 6\bar{\rho}\bar{\tau}'\right)(\eth + 2\bar{\tau}')(\eth + 2\bar{\tau}')(\kappa_1^4\vartheta\Psi_4) \\
&- 3i\bar{\Psi}_2(\eth - \tau + 2\bar{\tau}')\text{Im}\mathcal{A}_{11'} + 3i\bar{\Psi}_2(\text{þ}' - \rho' + 2\bar{\rho}')\text{Im}\mathcal{A}_{01'},
\end{aligned} \quad \text{(A.5b)}$$

$$\begin{aligned}
0 = {}&\left((\eth' - \bar{\tau})\,\text{þ}' - 12\bar{\rho}'\bar{\tau}\right)\!\left((\eth' + 2\bar{\tau})(\text{þ}' + 3\bar{\rho}') - 2\bar{\rho}'\bar{\tau}\right)(\kappa_1^4\vartheta\Psi_0) \\
&- \left((\eth - \bar{\tau}')\,\text{þ} - 12\bar{\rho}\bar{\tau}'\right)\!\left((\eth + 2\bar{\tau}')(\text{þ} + 3\bar{\rho}) - 2\bar{\rho}\bar{\tau}'\right)(\kappa_1^4\vartheta\Psi_4) \\
&+ i\bar{\Psi}_2(\eth' + 5\bar{\tau} - \tau')\text{Im}\mathcal{A}_{01'} + i\bar{\Psi}_2(\eth - \tau + 5\bar{\tau}')\text{Im}\mathcal{A}_{10'} \\
&- i\bar{\Psi}_2(\text{þ} - \rho + 5\bar{\rho})\text{Im}\mathcal{A}_{11'} - i\bar{\Psi}_2(\text{þ}' - \rho' + 5\bar{\rho}')\text{Im}\mathcal{A}_{00'},
\end{aligned} \quad \text{(A.5c)}$$

$$\begin{aligned}
0 = {}&\left(\text{þ}'(\eth' - \bar{\tau}) - 6\bar{\rho}'\bar{\tau}\right)(\eth' + 2\bar{\tau})(\eth' + 2\bar{\tau})(\kappa_1^4\vartheta\Psi_0) \\
&- \left(\eth(\text{þ} - \bar{\rho}) - 6\bar{\rho}\bar{\tau}'\right)(\text{þ} + 2\bar{\rho})(\text{þ} + 2\bar{\rho})(\kappa_1^4\vartheta\Psi_4) \\
&- 3i\bar{\Psi}_2(\eth' + 2\bar{\tau} - \tau')\text{Im}\mathcal{A}_{00'} + 3i\bar{\Psi}_2(\text{þ} - \rho + 2\bar{\rho})\text{Im}\mathcal{A}_{10'},
\end{aligned} \quad \text{(A.5d)}$$

$$0 = \eth'\,\eth'\,\eth'\,\eth'(\kappa_1^4\vartheta\Psi_0) - \text{þ}\,\text{þ}\,\text{þ}\,\text{þ}(\kappa_1^4\vartheta\Psi_4) - \tfrac{M}{27}\mathcal{L}_\xi\overline{\vartheta\Psi}_0. \quad \text{(A.5e)}$$

Remark A.1. *Let ϕ_{AB} be a solution to the source-free Maxwell equations and let ϕ_i, $i = 0, 1, 2$ be the Maxwell scalars. The spin-1 TME is given by*

$$\left((\text{þ} - \rho - \bar{\rho})\,\text{þ}' - (\eth - \tau - \bar{\tau}')\,\eth'\,\right)(\kappa_1 \phi_0) = 0, \tag{A.6a}$$

$$\left((\text{þ}' - \rho' - \bar{\rho}')\,\text{þ} - (\eth' - \tau' - \bar{\tau})\,\eth\,\right)(\kappa_1 \phi_2) = 0, \tag{A.6b}$$

and the spin-1 extreme TSI are

$$\eth'\eth'(\kappa_1^2 \phi_0) = \text{þ}\,\text{þ}(\kappa_1^2 \phi_2) \tag{A.7a}$$

$$\text{þ}'\text{þ}'(\kappa_1^2 \phi_0) = \eth\,\eth(\kappa_1^2 \phi_2). \tag{A.7b}$$

For the Maxwell field, the full set of TSI in fact contains a third relation, cf. [19], which can be written in the form

$$(\text{þ}\,\eth + \bar{\tau}'\,\text{þ})(\kappa_1^2 \phi_2) = (\text{þ}'\,\eth' + \bar{\tau}\,\text{þ}')(\kappa_1^2 \phi_0). \tag{A.8}$$

As mentioned above, the full set of TME/TSI equations implied by the Maxwell field equation, has the important consequence that the symmetric tensor V_{ab} introduced in [14] is conserved. The tensor V_{ab} is, in contrast to the standard Maxwell stress-energy tensor, independent of the non-radiative modes of the Maxwell field, and is therefore a suitable tool to construct dispersive estimates for the Maxwell field on the Kerr spacetime.

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