The Harper-Hofstadter Hamiltonian and conical diffraction in photonic lattices with grating assisted tunneling

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We introduce a grating assisted tunneling scheme for tunable synthetic magnetic fields in photonic lattices, which can be implemented at optical frequencies in optically induced one- and two-dimensional dielectric photonic lattices. We demonstrate a conical diffraction pattern in particular realization of these lattices which possess Dirac points in k-space, as a signature of the synthetic magnetic fields. The two-dimensional photonic lattice with grating assisted tunneling constitutes the realization of the Harper-Hofstadter Hamiltonian.

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Synthetic magnetism for photons is a unique tool for the manipulation and control of light, and for the design of novel topological phases and states in photonics [1–15]. Topological photonics is a rapidly growing field [15], advancing in parallel to analogous efforts in ultracold atomic gases [16, 17], inspired by the development of topological insulators in condensed matter physics [18]. One motivating aspect of topological photonic systems is the existence of unidirectional backscattering immune states [1–4], which are robust to imperfections, and thus may serve as novel waveguides and for building integrated photonic devices. The first experimental observations of such edge states were in the microwave domain, in magneto-optical photonic crystals [4], theoretically proposed in Refs. [1–3]. In the optical domain, imaging of topological edge states was reported in Floquet topological insulators, implemented in modulated honeycomb photonic lattices [5], and in the two-dimensional array of coupled optical-ring resonators [6]. The strategies for obtaining synthetic magnetic/gauge fields and topological phases for optical photons are closely related to the system at hand. In systems of coupled optical resonators, the strategy is to tune the phase of the tunneling between coupled cavities [6–9]; for example, by using link resonators of different length [6, 7] or time-modulation of the coupling [9]. Photonic topological insulators were also proposed in superlattices of metamaterials with strong magneto-electric coupling [13]. In 2D photonic lattices (waveguide arrays), pseudomagnetic fields have been demonstrated by inducing 'strain' in optical graphene [10]. By modulating 1D photonic lattices along the propagation axis, one can choose the sign of the hopping parameter between neighboring sites [11], whereas topological states were achieved using 1D photonic quasicrystals [12]. However, even though photonic lattices possess a great potential for exploring synthetic magnetism and topological effects, a viable scheme for arbitrary design of the phases of the complex tunneling matrix elements still needs to be developed. Here we introduce one such scheme termed grating assisted tunneling, and propose its implementation in optically induced photonic lattices [19–23]. We demonstrate the conical diffraction pattern [24, 25] in particular realizations of one- (1D) and two-dimensional (2D) square photonic lattices with grating assisted tunneling. The latter constitutes demonstration of the the ('single particle') Harper-Hofstadter Hamiltonian (HHH) [26, 27] in these systems.

The proposed method is inspired by the so called laser assisted tunneling scheme, which was implemented in optical lattices with ultracold atoms [28–30]. The scheme is based on theoretical proposals in Refs. [31, 32], subsequently developed to experimentally realize the HHH [29, 30], and staggered magnetic fields in optical superlattices [28]. Many of the results obtained with ultracold atoms are viable in photonic lattices, and perhaps even more of them are feasible, because heating by spontaneous emission that is present in ultracold atomic systems, is absent in photonic systems.

We consider the paraxial propagation of light in a photonic lattice defined by the index of refraction $n = n_0 + \delta n(x, y, z)$ ($\delta n \ll n_0$), where n_0 is the constant background index of refraction, and $\delta n(x, y, z)$ describes small spatial variations, which are slow along the propagation z-axis. The slowly varying amplitude of the electric field $\psi(x, y, z)$ follows the (continuous) Schrödinger equation:

$$i\frac{\partial \psi}{\partial z} = -\frac{1}{2k}\nabla^2\psi - \frac{k\delta n}{n_0}\psi; \tag{1}$$

here, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and $k = 2\pi n_0/\lambda$, where λ is the wavelength in vacuum. From now on we will refer to $\delta n(x, y, z)$ as the potential or index of refraction. In the simulations we use $n_0 = 2.3$, corresponding to the systems that were used to implement the optical induction technique [20, 21], and $\lambda = 500$ nm.

For clarity, let us first present the grating assisted tunneling method in a 1D photonic lattice. If the potential is a periodic lattice, $\delta n_L(x) = \delta n_L(x+a)$, where a is the lattice constant, and if the lattice is sufficiently 'deep', the propagation of light can be approximated by using a discrete Schrödinger equation [23, 33]:

$$i\frac{d\psi_m}{dz} = -(J_x\psi_{m-1} + J_x\psi_{m+1}), \tag{2}$$

where J_x quantifies the tunneling between adjacent waveguides and $\psi_m(z)$ is the amplitude at the mth lattice site, i.e., waveguide. To create a synthetic magnetic field, we adopt the strategy to tune the phase of the tunneling between lattice sites. For this 1D lattice, we seek a scheme which effectively renormalizes J_x to get $K_x \exp(i\phi_m)$, where ϕ_m denotes the phase for tunneling from site m to site $m+1$. The scheme is illustrated in Fig. 1. In Fig. 1(a) we show the potential $\delta n(x,z)$, which can be modeled as a discrete lattice with complex tunneling parameters $K_x \exp(i\phi_m)$ shown in Fig. 1(b).

Let us gradually explain the idea behind the variation of the index of refraction as in Fig. 1(a). In Fig. 1(e), we show the propagation of intensity in a continuous 1D model in the lattice potential $\delta n_L(x) = \delta n_{L0}\cos^2(\pi x/a)$, with $\psi(x,0) = \sqrt{I}e^{-x^2/(3a)^2}$; $\delta n_{L0} = 4\times 10^{-4}$, $a = 10\,\mu$m. We see the usual diffraction pattern for a spatially broad excitation covering several lattice sites [23]. Next, suppose that we introduce a linear gradient of index of refraction along the x direction $\delta n_T(x) = -\eta x$ in addition to the lattice potential, such that $\delta n = \delta n_L(x) + \delta n_T(x)$. For a sufficiently large tilt, the tunneling is *suppressed*. This can be seen from Fig. 1(f) which shows the propagation of intensity in a tilted potential with $\eta = 0.1\delta n_{L0}/a$; the tilt should be smaller than the gap between the first two bands. Finally, let us introduce an additional small grating potential at a small angle θ with respect to the z axis, $\delta n_G(x,z) = \delta n_{G0}\cos^2((q_x x - \kappa z)/2)$, such that $\delta n(x,z) = \delta n_L(x) + \delta n_T(x) + \delta n_G(x,z)$. This total potential $\delta n(x,z)$ is illustrated in Fig. 1(a). The 'frequency' κ is determined by the angle θ of the grating with respect to the z-axis, which is chosen such that $\kappa = \eta a k/n_0$ and the grating forms a z-dependent perturbation resonant with the index offset between neighboring lattice sites ηa (Fig. 1(a)). The grating *restores* the tunneling along the x-axis, hence the term *grating assisted tunneling*. Restored tunneling is seen in Fig. 1(g) which shows diffraction for identical initial conditions as in Figs. 1(e) and (f); the grating parameters are $q_x = \pi/a$ and $\delta n_{G0} = 0.1\delta n_{L0}$.

However, the diffraction pattern is drastically changed. In order to interpret it, we point out that, for resonant tunneling where $\kappa = \eta a k/n_0$ and a sufficiently large tilt ($J \ll \eta$), z-averaging over the rapidly oscillating terms shows that the system can be modeled by an effective discrete Schrödinger equation (e.g., see [29] for ultracold atoms):

$$i\frac{d\psi_m}{dz} = -(K_x e^{i\phi_{m-1}}\psi_{m-1} + K_x e^{-i\phi_m}\psi_{m+1}), \tag{3}$$

where $\phi_m = \mathbf{q}\cdot\mathbf{R}_m = q_x m a$. In Fig. 1(g) we used $q_x = \pi/a$, i.e., $\phi_m = m\pi$. Such a discrete lattice is illustrated in Fig. 1(c); its dispersion having a 1D Dirac cone at $k_x = 0$. For a wavepacket that initially excites modes close to $k_x = 0$, the diffraction in the discrete model (3) yields the so-called (1D) conical diffraction pattern [11, 24], as illustrated in Fig. 1(h). The initial conditions for propagation in the discrete model corresponds to the initial conditions in the continuous system, $\psi_m(0) = \sqrt{I}e^{-(m/3)^2}$, and $K_x = 0.053$ mm^{-1}. Thus, we interpret the diffraction pattern in Fig. 1(g) as 1D conical diffraction, a signature of the discrete model depicted in Fig. 1(c). A comparison of the discrete [Fig. 1(h)] and the realistic continuous model [Fig. 1(g)], clearly shows that we can use grating assisted tunneling to tune the phases of the tunneling parameters in the discrete Schrödinger equation, thereby realizing synthetic magnetic fields.

Before proceeding to 2D systems, we discuss the amplitude of the tunneling matrix elements K_x as a function of the strength of the grating δn_{G0}. Figure 1(d) shows K_x (green squares) and J_x (blue circles) versus δn_{G0}, where J_x corresponds to the potential which includes the lattice and the grating, but no tilt. The amplitudes J_x and K_x are obtained by comparing the diffraction pattern of the discrete with the continuous model, and adjusting J_x and K_x until the two patterns coincide, as in Figs. 1(g) and (h). Our results in Fig. 1(d) are in agreement with those in ultracold atoms [e.g., see Fig. 3(a) in Ref. [29]].

The extension of the scheme to 2D lattices is straightforward. We consider a square lattice, $\delta n_L = \delta n_{L0}(\cos^2(\pi x/a) + \cos^2(\pi y/a))$, the tilt in the x direction, $\delta n_T(x) = -\eta x$, and the grating which has the form $\delta n_G(x,y,z) = \delta n_{G0}\cos^2((q_x x + q_y y - \kappa z)/2)$. Propagation of light in the total potential $\delta n(x,y,z) = \delta n_L(x,y) + \delta n_T(x) + \delta n_G(x,y,z)$ can be modeled by the discrete Schrödinger equation (the derivation is equivalent to that in Ref. [29] for ultracold atoms):

$$\begin{aligned} i\frac{d\psi_{m,n}}{dz} = -(&K_x e^{i\phi_{m-1,n}}\psi_{m-1,n} + K_x e^{-i\phi_{m,n}}\psi_{m+1,n} \\ &+ J_y\psi_{m,n-1} + J_y\psi_{m,n+1}), \end{aligned} \tag{4}$$

where $\phi_{m,n} = \mathbf{q}\cdot\mathbf{R}_{m,n} = q_x m a + q_y n a$. Note that the tunneling along y does not yield a phase because there is no tilt in the y direction; the tunneling amplitude along y depends on the depth of the grating, as illustrated in Fig. 1(d) with blue circles.

In order to demonstrate that the propagation of light in the continuous 2D potential $\delta n(x,y,z)$ is indeed equivalent to the dynamics of Eq. (4), we compare propagation in the discrete model (4), with that of the

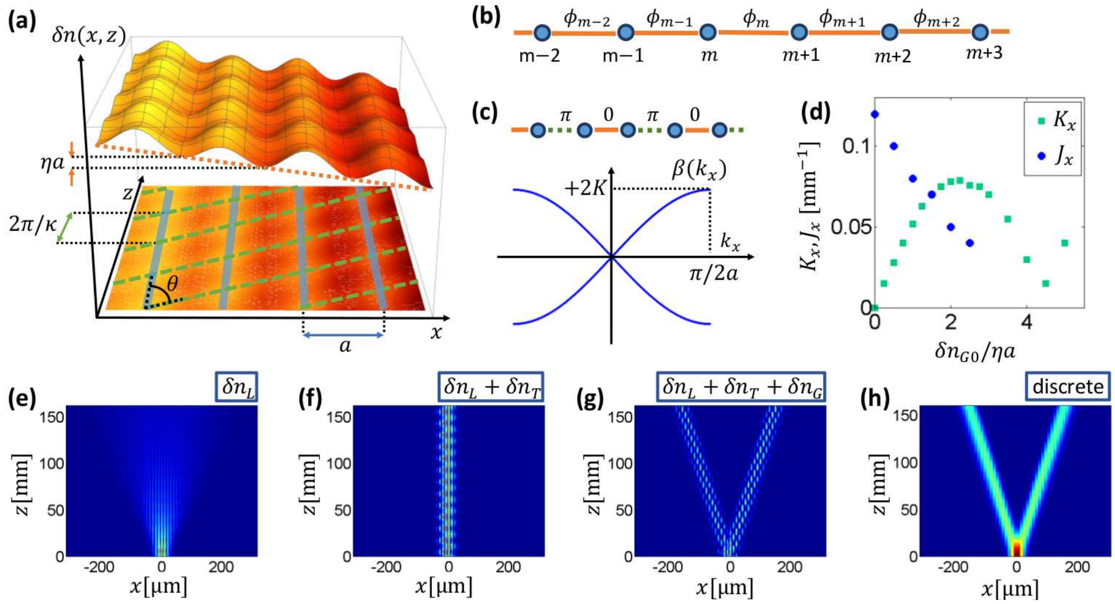

FIG. 1. (color online) Illustration of the grating assisted tunneling scheme. (a) Sketch of the spatial dependence of the index of refraction $\delta n_L(x,z) = \delta n_L + \delta n_T + \delta n_G$, which creates a synthetic magnetic field in a photonic lattice. A 1D photonic lattice $[\delta n_L = \delta n_{L0}\cos^2(\pi x/a)]$ is superimposed with a linear gradient index in the x direction ($\delta n_T = -\eta x$, orange dotted line), and an additional small grating potential $[\delta n_G = \delta n_{G0}\cos^2((q_x x - \kappa z)/2)]$, at a small angle ($\theta$ is on the order of 1°) with respect to the z axis. In the xz plane we plot the projection of δn_L (blue bold lines), and the grating (green dashed lines). (b) A discrete lattice with complex tunneling matrix elements between sites, $K_x e^{i\phi_m}$, and spatially dependent phases ϕ_m, can model the system in (a). (c) A particular choice of θ (see text), yields a lattice with $\phi_m = \pi m$, and a dispersion with a 1D Dirac cone at $k_x = 0$. (d) The amplitude of the tunneling K_x as a function of the strength of the grating δn_{G0} (green squares). The tunneling amplitude J_x versus δn_{G0} for a system without the tilt. (e,f,g) Numerical simulation of the evolution of a wavepacket that initially excites modes close to $k_x = 0$ in the continuous 1D photonic lattice. (e) Diffraction in a periodic photonic lattice δn_L. (f) Propagation in the tilted system, $\delta n_L + \delta n_T$, shows that the tunneling (diffraction) is suppressed. (g) The tunneling is restored by an additional grating potential with $q_x = \pi/a$. (h) Propagation of the wavepacket in the corresponding discrete model illustrated in (c).

continuous equation (1). The lattice parameters are $\delta n_{L0} = 4 \times 10^{-4}$, $a = 13\ \mu$m; the tilt is given by $\eta = 0.1\delta n_{L0}/a$; the grating is defined by $\delta n_{G0} = 0.17\delta n_{L0}$ and $q_x = -q_y = \pi/a$, which yields $\phi_{m,n} = (m-n)\pi$, and κ is chosen to yield resonant tunneling. The discrete lattice which corresponds to this choice of phases is illustrated in Figure 2(a). It has two bands, $\beta = \pm 2\sqrt{K_x^2 \sin^2(k_x a) + J_y^2 \cos^2(k_y a)}$ (β is the propagation constant), touching at two 2D Dirac points at $(k_x, k_y) = (0, \pm\pi/2a)$ in the Brillouin zone [34], as depicted in Fig. 2(b). Suppose that the incoming beam at $z = 0$, excites the modes which are in the vicinity of these two Dirac points. Around these points, for a given $\hat{\mathbf{k}} = \mathbf{k}/k$, the group velocity, $\nabla_{\mathbf{k}}\beta(k_x, k_y)$, is constant. Thus, the beam will undergo conical diffraction, which has been thoroughly addressed with Dirac points in honeycomb optical lattices [24]. To demonstrate this effect, we consider the propagation of a beam with the initial profile given by $\psi(x,y,0) = \sqrt{I}\cos(y\pi/2a)\exp(-(x/3a)^2 - (y/3a)^2)$ in the 2D potential $\delta n(x,y,z)$ (the cosine term ensures that we excite modes close to the two Dirac points). In Fig. 2(c) we show this beam after propagation for $z = 162$ mm. The two concentric rings are a clear evidence of conical diffraction [24, 25]. We also compare this with the propagation in the discrete model (4), with tunneling phases plotted in Fig. 2(a); $K_x = 0.11$ mm^{-1}, $J_y = 0.14$ mm^{-1}. The results of the propagation in the discrete model are shown in Fig. 2(d). It is evident that for our choice of the tilt and the grating, we have effectively realized the lattice plotted in 2(a). This is in fact a realization of the HHH for $\alpha = 1/2$, where α is the flux per plaquette in units of the flux quantum [29, 30].

For the experimental implementation of the scheme, we propose the so-called optical induction technique in photosensitive materials, which can be implemented in photorefractives [19–23]. In these systems, both the lattice $\delta n_L(x,y)$ and the grating potential $\delta n_G(x,y,z)$ can be obtained in a straightforward fashion by using interference of plane waves in the medium [23]. The lattice constant a, the grating parameter $\mathbf{q}$, and hence the hopping phase $\phi_{m,n} = \mathbf{q}\cdot\mathbf{R}_{m,n}$, are tunable by changing the angle between the interfering beams. This could enable manipulation of the phases of the tunneling matrix elements in real time [20].

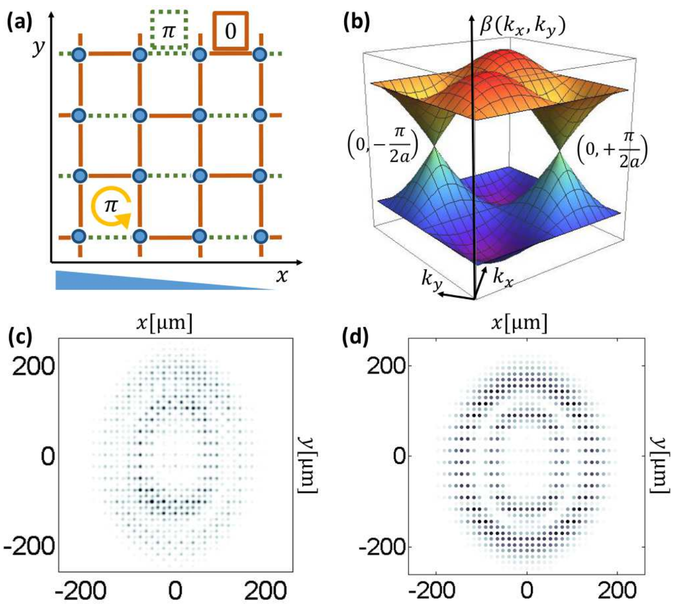

FIG. 2. (color online) Grating assisted tunneling and conical diffraction in a square photonic lattice. (a) Sketch of the 2D lattice with grating assisted tunneling along the x direction. The resulting nontrivial hopping phases π and 0 are denoted with dashed and solid lines, respectively. A wavepacket that makes one loop around the plaquette accumulates the phase π. (b) The lattice possesses two 2D Dirac cones at $(k_x, k_y) = (0, \pm\pi/2a)$ in the dispersion $\beta(k_x, k_y)$, where β is the propagation constant. (c) Intensity of a beam, which initially excites modes in the vicinity of the Dirac points, after propagation for $z = 162$ mm. The intensity has two concentric rings corresponding to the conical diffraction pattern. (d) Simulation of the diffraction pattern in the discrete model corresponding to the lattice in (a), which also exhibits conical diffraction (see text for details).

The most challenging part of the implementation appears to be creation of the linear tilt potential. To observe the conical diffraction effects, one needs a linear gradient over the ~ 20 lattice sites, which implies a total index tilt $\Delta n = 20\eta a \sim 20 \times 0.1\delta n_{L0} \sim 8 \times 10^{-4}$. In principle, the linear tilt could be achieved by using a spatial light modulator. Another possibility which could create a linear tilt is to use crystals with linear dependence of the index of refraction on temperature, and a temperature gradient across the crystal.

However, it should be emphasized that if one uses a superlattice for δn_L, rather than the linear tilt potential, as in Ref. [28] for ultracold atomic gases, the grating assisted tunneling scheme proposed here would straightforwardly yield *staggered* synthetic magnetic fields for photons. The achievement of such staggered fields is straightforward with optically induced lattices proposed here.

Before closing, we emphasize that the spatial phase between the periodic lattice potential δn_L and the grating δn_G is a relevant parameter. This point has recently been examined in detail in the context of ultracold atoms [35]. To translate it to our system, suppose that in our 1D system, we shift the grating along the x-direction, such that $\delta n_G = \delta n_{G0} \cos^2((q_x x - \kappa z + \xi)/2)$. This simply shifts the phases in the discrete lattice [shown in Fig. 1(c)] such that ϕ_m is replaced by $\phi_m + \xi$, which moves the 1D Dirac point in k_x-space by $\Delta k_x = \xi/a$. We have numerically verified that this indeed happens.

In conclusion, we have introduced a grating assisted tunneling scheme for tunable synthetic magnetic fields in photonic lattices, and proposed its implementation at optical frequencies in optically induced one- and two-dimensional dielectric photonic lattices. As a signature of the synthetic magnetic fields, we have demonstrated the conical diffraction pattern in particular realization of these lattices [shown in Fig. 1(c) and Fig. 2(a)]. The two-dimensional photonic lattice with grating assisted tunneling constitutes the realization of the HHH. The scheme is well suited for realization of staggered synthetic magnetic fields in photorefracives. We envision that this proposal will open the way to other studies of light propagation in photonic lattices with complex tunneling matrix elements, including nonlinear propagation, solitons and instabilities in synthetic magnetic fields, and creation of synthetic dimensions in photonic lattices [36] by employing synthetic fields.

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Note added. A few days before this paper was submitted to arXiv, a paper by a different group [37] appeared on the arXiv. It is also inspired by laser assisted tunneling in cold gases, and experimentally demonstrated modulation-assisted tunneling in laser-fabricated photonic Wannier-Stark ladders.

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