Dual Pairs of Generalized Lyapunov Inequalities and Balanced Truncation of Stochastic Linear Systems

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***Abstract*—We consider two approaches to balanced truncation of stochastic linear systems, which follow from different generalizations of the reachability Gramian of deterministic systems. Both preserve mean-square asymptotic stability, but only the second leads to a stochastic H^∞-type bound for the approximation error of the truncated system.**

***Index Terms*—Asymptotic mean square stability, balanced truncation, generalized Lyapunov equation, model order reduction, stochastic linear system.**

I. INTRODUCTION

OPTIMIZATION and (feedback) control of dynamical systems is often computationally infeasible for high dimensional plant models. Therefore, one tries to reduce the order of the system, so that the input-output mapping is still computable with sufficient accuracy, but at considerably smaller cost than for the original system [1]–[5]. To guarantee the desired accuracy, computable error bounds are required. Moreover, system properties which are relevant in the context of control system design like asymptotic stability need to be preserved. It has long been known that for linear time-invariant (LTI) systems the method of balanced truncation preserves asymptotic stability and provides an error bound for the L^2-induced input-output norm, i.e., the H^∞-norm of the associated transfer function; see [6], [7]. When considering model order reduction of more general system classes, it is natural to try to extend this approach. This has been worked out for descriptor systems in [8], for time-varying systems in [9]–[11], for bilinear systems in [12]–[14] and general nonlinear systems, e.g., in [15]. Yet another generalization of LTI systems is obtained considering dynamics driven by noise processes. This leads to the class of stochastic systems, which have been considered in a system theoretic context, e.g., in [16]–[18]. Quite recently, balanced truncation has also been described for linear stochastic systems of Itô type in [14], [19], and [20]. Already the formulation of the method leads to two different variants that are equivalent in the deterministic case, but not so for stochastic systems. It is natural to ask which of the above-mentioned properties of balanced truncation also hold for these variants. The aim of this paper is to answer this question.

Let us recapitulate balanced truncation for linear control systems of the form

$$\dot{x} = Ax + Bu \qquad y = Cx \qquad \sigma(A) \subset \mathbb{C}_-. \tag{1}$$

Here $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$ are the state, output, and input of the system, respectively. Moreover $\sigma(A)$ denotes the spectrum of A and $\mathbb{C}_-$ the open left half complex plane. Let

$$\mathcal{L}_A : X \mapsto A^T X + XA$$

denote the Lyapunov operator and

$$\mathcal{L}_A^* : X \mapsto AX + XA^T$$

its adjoint with respect to the Frobenius inner product $\langle Z, Y\rangle = \text{trace}(Y^T Z)$. Then $\sigma(A) \subset \mathbb{C}_-$ if and only if there exists a positive definite solution X of the Lyapunov inequality $\mathcal{L}_A(X) < 0$, by Lyapunov's classical stability theorem, see, e.g., [21].

Balanced truncation means truncating a balanced realization. This realization is obtained by a state space transformation computed from the Gramians P and Q, which solve the dual pair of *Lyapunov equations*

$$\mathcal{L}_A(Q) = A^T Q + QA = -C^T C \tag{2a}$$

$$\mathcal{L}_A^*(P) = AP + PA^T = -BB^T \tag{2b}$$

or more generally the *inequalities*

$$\mathcal{L}_A(Q) \le -C^T C \qquad \mathcal{L}_A^*(P) \le -BB^T. \tag{3}$$

These (in)equalities are essential in the characterization of stability, controllability and observability of system (1). If $\det P \neq 0$, the inequalities (3) can be written as

$$\mathcal{L}_A(Q) \le -C^T C \tag{4a}$$

$$\mathcal{L}_A(P^{-1}) = P^{-1}A + A^T P^{-1} \le -P^{-1}BB^T P^{-1}. \tag{4b}$$

In the present paper we discuss extensions of (3) and (4) for stochastic linear systems.

As indicated above, the equivalent formulations (3) and (4) lead to different generalizations, if we consider Itô-type stochastic systems of the form

$$dx = Ax\,dt + Nx\,dw + Bu\,dt, \quad y = Cx \tag{5}$$

where A, B, C are as in (1) and $N \in \mathbb{R}^{n\times n}$. System (5) is asymptotically mean-square stable (e.g., [18], [22], [23]), if and

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only if there exists a positive definite solution X of the generalized Lyapunov inequality

$$(\mathcal{L}_A + \Pi_N)(X) = A^T X + XA + N^T XN < 0.$$

Here $\Pi_N : X \mapsto N^T XN$ and $\Pi_N^* : X \mapsto NXN^T$. This stability criterion indicates that in the stochastic context, the generalized Lyapunov operator $\mathcal{L}_A + \Pi_N$ takes over the role of $\mathcal{L}_A$. Substituting $\mathcal{L}_A$ by $\mathcal{L}_A + \Pi_N$ in (3) and (4), we obtain two different dual pairs of generalized Lyapunov inequalities. We call them *type I*

$$(\mathcal{L}_A + \Pi_N)(Q) = A^T Q + QA + N^T QN \le -C^T C \quad \text{(6a)}$$
$$(\mathcal{L}_A + \Pi_N)^*(P) = AP + PA^T + NPN^T \le -BB^T \quad \text{(6b)}$$

and *type II*

$$\begin{aligned}(\mathcal{L}_A + \Pi_N)(Q) &= A^T Q + QA + N^T QN \\ &\le -C^T C \quad \text{(7a)}\\ (\mathcal{L}_A + \Pi_N)(P^{-1}) &= A^T P^{-1} + P^{-1}A + N^T P^{-1} N \\ &\le -P^{-1}BB^T P^{-1}. \quad \text{(7b)}\end{aligned}$$

Note that (6) corresponds to (3) in the sense that $\mathcal{L}_A^*(P)$ has been replaced by $(\mathcal{L}_A + \Pi_N)^*(P)$, while (7) corresponds to (4), where $\mathcal{L}_A(P^{-1})$ has been replaced by $(\mathcal{L}_A + \Pi_N)(P^{-1})$. In general (if N and P do not commute), the inequalities (6b) and (7b) are not equivalent. At first glance it is not clear which generalization is more appropriate.

If the system is asymptotically mean-square stable, then for both types there are solutions $Q, P > 0$. By a suitable state space-transformation, it is possible to balance the system such that $Q = P = \Sigma > 0$ is diagonal. Consequently, the usual procedure of balanced truncation can be applied to reduce the order of (5). For simplicity, let us refer to this as *type I* or *type II balanced truncation*.

Under natural assumptions, this reduction preserves mean-square asymptotic stability. For type I, this nontrivial fact has been proven in [24]. Moreover, in [20], an H^2-error bound has been provided. However, different from the deterministic case, there is no H^∞-type error bound in terms of the truncated entries in Σ. This will be shown in Example I.3.

In contrast, for type II, an H^∞-type error bound has been obtained in [19]. In the present paper, as one of our main contributions, we show in Theorem II.2 that type II balanced truncation also preserves mean-square asymptotic stability. The proof differs significantly from the one given for type I. Using this result, we are able to give a more compact proof of the error bound, Theorem II.4, which exploits the stochastic bounded real lemma [17].

We illustrate our results by analytical and numerical examples in Section IV.

II. Type I Balanced Truncation

Consider a stochastic linear control system of Itô-type

$$dx = Ax\,dt + \sum_{j=1}^{k} N_j x\,dw_j + Bu\,dt, \quad y = Cx \quad \text{(8)}$$

where $w_j = (w_j(t))_{t\in\mathbb{R}_+}$ are uncorrelated zero-mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ (e.g., [25], [26]).

To simplify the notation, we only consider the case $k = 1$ and set $w = w_1$, $N = N_1$. But all results can immediately be generalized for $k > 1$.

Let $L_w^2(\mathbb{R}_+, \mathbb{R}^q)$ denote the corresponding space of nonanticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|_{L_w^2}^2 := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2\,dt\right) < \infty$$

where $\mathcal{E}$ denotes expectation.

Let the homogeneous equation $dx = Axdt + Nxdw$ be asymptotically mean-square-stable, i.e., $\mathcal{E}(\|x(t)\|^2) \overset{t\to\infty}{\longrightarrow} 0$, for all solutions x.

Then, by Theorem A.1, the equations

$$A^T Q + QA + N^T QN = -C^T C$$
$$AP + PA^T + NPN^T = -BB^T$$

have unique solutions $Q \ge 0$ and $P \ge 0$. If the system is observable and reachable (see Theorem A.8), then Q and P are nonsingular, and thus positive definite.

A similarity transformation

$$(A, N, B, C) \mapsto (S^{-1}AS, S^{-1}NS, S^{-1}B, CS)$$

of the system implies the contragradient transformation as

$$(Q, P) \mapsto (S^T QS, S^{-1}PS^{-T}).$$

Choosing, e.g., $S = LV\Sigma^{-1/2}$, with Cholesky factorizations $LL^T = P$, $R^T R = Q$ and a singular value decomposition $RL = U\Sigma V^T$, we obtain $S^{-1} = \Sigma^{-1/2}U^T R$ and

$$S^T QS = S^{-1}PS^{-T} = \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n).$$

After suitable partitioning

$$\Sigma = \begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix} \qquad S = [S_1 \quad S_2] \qquad S^{-1} = \begin{bmatrix}T_1\\ T_2\end{bmatrix}$$

a truncated system is given in the form

$$(A_{11}, N_{11}, B_1, C_1) = (T_1 AS_1, T_1 NS_1, T_1 B, CS_1).$$

The following result has been proven in [24].

Theorem I.1: Let $A, N \in \mathbb{R}^{n\times n}$ satisfy

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-.$$

For a block-diagonal matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, assume that

$$A^T\Sigma + \Sigma A + N^T\Sigma N \le 0 \text{ and } A\Sigma + \Sigma A^T + N\Sigma N^T \le 0.$$

Then, with the usual partitioning of A and N, we have

$$\sigma(I \otimes A_{11} + A_{11} \otimes I + N_{11} \otimes N_{11}) \subset \mathbb{C}_-.$$

Its implication for mean-square stability of the truncated system is immediate.

Corollary I.2: Consider an asymptotically mean square stable stochastic linear system

$$dx = Ax\,dt + Nx\,dw.$$

Assume that a matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$ is given as in Theorem I.1 and A and N are partitioned accordingly. Then the truncated system

$$dx_r = A_{11}x_r\,dt + N_{11}x_r\,dw$$

is also asymptotically mean square stable.

If the diagonal entries of Σ_2 are small, it is expected that the truncation error is small. In fact this is supported by an H^2-error bound obtained in [20]. Additionally, however, from the deterministic situation (see [2], [6]), one would also hope for an H^∞-type error bound of the form

$$\|y - y_r\|_{L^2_w(\mathbb{R}_+,\mathbb{R}^p)} \overset{?}{\leq} \alpha(\operatorname{trace}\Sigma_2)\|u\|_{L^2_w(\mathbb{R}_+,\mathbb{R}^m)} \tag{9}$$

with some real number $\alpha > 0$. The following example shows that no such general α exists.

Example I.3: Let $A = -\begin{bmatrix} 1 & 0 \\ 0 & a^2 \end{bmatrix}$ with $a > 1$, $N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = [0 \quad 1]$.

Solving (6) with equality, we get $P = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4a^2} \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{4a^2} & 0 \\ 0 & \frac{1}{2a^2} \end{bmatrix}$ with $\sigma(PQ) = \{1/8a^2, 1/8a^4\}$ so that $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2)$, where $\sigma_1 = 1/\sqrt{8}a$ and $\sigma_2 = 1/\sqrt{8}a^2$. The system is balanced by the transformation $S = \begin{bmatrix} 2a^2 & 0 \\ 0 & 1/2 \end{bmatrix}^{1/4}$.

Then $CS = (1/2^{1/4})[0 \quad 1]$ so that $C_r = 0$ for the truncated system of order 1. Thus, the output of the reduced system is $y_r \equiv 0$, and the truncation error $\|\mathbb{L} - \mathbb{L}_r\|$ is equal to the stochastic H^∞-norm (see [17]) of the original system

$$\|\mathbb{L}\| = \sup_{x(0)=0, \|u\|_{L^2_w}=1} \|y\|_{L^2_w}.$$

We show now that this norm is equal to $1/\sqrt{2}a = 2a\sigma_2$. Thus, depending on a, the ratio of the truncation error and $\operatorname{trace}\Sigma_2 = \sigma_2$ can be arbitrarily large.

According to the stochastic bounded real lemma, Theorem A.5, $\|\mathbb{L}\|$ is the infimum over all γ so that the Riccati inequality

$$\begin{aligned} 0 &< A^TX + XA + N^TXN - C^TC - \frac{1}{\gamma^2}XBB^TX \\ &= \begin{bmatrix} -2x_1 + x_3 - \frac{1}{\gamma^2}x_1^2 & -(a^2+1)x_2 - \frac{1}{\gamma^2}x_1x_2 \\ -(a^2+1)x_2 - \frac{1}{\gamma^2}x_1x_2 & -2a^2x_3 - \frac{1}{\gamma^2}x_2^2 - 1 \end{bmatrix} \end{aligned} \tag{10}$$

possesses a solution $X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} < 0$.

If a given matrix X satisfies this condition, then so does the same matrix with x_2 replaced by 0. Hence we can assume that $x_2 = 0$, and end up with the two conditions $x_3 < -(1/2a^2)$ and (after multiplying the upper left entry with $-\gamma^2$)

$$\begin{aligned} 0 &> x_1^2 + 2\gamma^2x_1 - \gamma^2x_3 = (x_1+\gamma^2)^2 - \gamma^2(\gamma^2 + x_3) \\ &> (x_1+\gamma^2)^2 - \gamma^2\left(\gamma^2 - \frac{1}{2a^2}\right). \end{aligned}$$

Thus necessarily $\gamma^2 > 1/2a^2$, i.e., $\gamma > 1/\sqrt{2}a$. This already proves that $\|\mathbb{L}\| \geq 1/\sqrt{2}a = 2a\sigma_2$, which suffices to disprove the existence of a general bound α in (9). Taking infima, it is easy to show that indeed $\|\mathbb{L}\| = 1/\sqrt{2}a$. AQ1

III. Type II Balanced Truncation

We now consider the inequalities (7).

Lemma II.1: Assume that $dx = Axdt + Nxdw$ is asymptotically mean-square-stable. Then inequality (7b) is solvable with $P > 0$.

Proof: By Theorem A.1, for a given $Y < 0$, there exists a $\tilde{P} > 0$, so that $A^T\tilde{P}^{-1} + \tilde{P}^{-1}A + N^T\tilde{P}^{-1}N = Y$. Then $P = \varepsilon^{-1}\tilde{P}$, for sufficiently small $\varepsilon > 0$, satisfies

$$A^TP^{-1} + P^{-1}A + N^TP^{-1}N = \varepsilon Y < -\varepsilon^2\tilde{P}^{-1}BB^T\tilde{P}^{-1}$$

so that (7b) holds even in the strict form. □

It is easy to see that like in the previous section a state space transformation

$$(A, N, B, C) \mapsto (S^{-1}AS, S^{-1}NS, S^{-1}B, CS)$$

leads to a contragradient transformation $Q \mapsto S^TQS$, $P \mapsto S^{-1}PS^{-T}$ of the solutions. That is, Q and P satisfy (7a) and (7b), if and only if S^TQS and $S^{-1}PS^{-T}$ do so for the transformed data. As before, we can assume the system to be balanced with

$$Q = P = \Sigma = \operatorname{diag}(\sigma_1 I, \ldots, \sigma_\nu I) = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \tag{11}$$

where $\sigma_1 > \cdots > \sigma_\nu > 0$ and $\sigma(\Sigma_1) = \{\sigma_1, \ldots, \sigma_r\}$, $\sigma(\Sigma_2) = \{\sigma_{r+1}, \ldots, \sigma_\nu\}$. Hence, we will now assume (after balancing) that a diagonal matrix Σ as in (11) is given which satisfies

$$A^T\Sigma + \Sigma A + N^T\Sigma N \leq -C^TC \tag{12a}$$

$$A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N \leq -\Sigma^{-1}BB^T\Sigma^{-1}. \tag{12b}$$

Partitioning A, N, B, C like Σ, we write the system as

$$\begin{aligned} dx_1 &= (A_{11}x_1 + A_{12}x_2 + B_1u)\,dt + (N_{11}x_1 + N_{12}x_2)\,dw \\ dx_2 &= (A_{21}x_1 + A_{22}x_2 + B_2u)\,dt + (N_{21}x_1 + N_{22}x_2)\,dw \\ y &= C_1x_1 + C_2x_2. \end{aligned}$$

The reduced system obtained by truncation is

$$dx_r = (A_{11}x_r + B_1u)\,dt + N_{11}x_r\,dw \qquad y_r = C_1x_r.$$

The index r is the number of different singular values σ_j that have been kept in the reduced system. In the following subsections, we consider matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

$\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$ as in (11), and equations of the form

$$A^T\Sigma + \Sigma A + N^T \Sigma N = -\tilde{C}^T\tilde{C} \tag{13a}$$

$$A^T\Sigma^{-1} + \Sigma^{-1} A + N^T \Sigma^{-1} N = -\tilde{B}\tilde{B}^T \tag{13b}$$

with arbitrary right-hand sides $-\tilde{C}^T\tilde{C} \le 0$ and $-\tilde{B}\tilde{B}^T \le 0$.

A. *Preservation of Asymptotic Stability*

The following theorem is the main new result of this paper.

Theorem II.2: Let A and N be given such that

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-. \tag{14}$$

Assume further that for a block-diagonal matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, we have

$$A^T\Sigma + \Sigma A + N^T \Sigma N \le 0 \tag{15a}$$

$$A^T\Sigma^{-1} + \Sigma^{-1} A + N^T \Sigma^{-1} N \le 0. \tag{15b}$$

Then, with the usual partitioning of A and N, we have

$$\sigma(I \otimes A_{11} + A_{11} \otimes I + N_{11} \otimes N_{11}) \subset \mathbb{C}_-. \tag{16}$$

Again we have an immediate interpretation in terms of mean-square stability of the truncated system.

Corollary II.3: Consider an asymptotically mean square stable stochastic linear system

$$dx = Ax\, dt + Nx\, dw.$$

Assume that a matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$ is given as in Theorem II.2 and A and N are partitioned accordingly. Then the truncated system

$$dx_r = A_{11}x_r\, dt + N_{11}x_r\, dw$$

is also asymptotically mean square stable.

Proof of Theorem II.2: Note that the inequalities (15) are equivalent to the equations (13) with appropriate right-hand sides $-\tilde{C}^T\tilde{C}$ and $-\tilde{B}\tilde{B}^T$. In accordance with the partitioning of A, N, and Σ, each matrix equation (13a) and (13b) consists of three blocks.

By way of contradiction, we assume that (16) does not hold. Then by Theorem A.3, there exist $V \ge 0$, $V \ne 0$, $\alpha \ge 0$ such that

$$A_{11}V + VA_{11}^T + N_{11}VN_{11}^T = \alpha V. \tag{17}$$

Taking the scalar product of the left upper block of (13a) with V, we obtain $0 \ge \alpha \operatorname{trace}(\Sigma_1 V)$ whence $\alpha = 0$ and $\tilde{C}_1 V = 0$, $N_{21}V = 0$ by Corollary A.4. Hence

$$\left(A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + N_{11}^T\Sigma_1 N_{11}\right) V = 0. \tag{18}$$

Analogously, we have $\tilde{B}_1^T V = 0$.

In particular, from $N_{21}V = 0$, we get

$$(\mathcal{L}_A^* + \Pi_N^*)\left(\begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & VA_{21}^T \\ A_{21}V & 0 \end{bmatrix}.$$

We will show that $A_{21}V = 0$, which implies

$$0 \in \sigma(I \otimes A + A \otimes I + N \otimes N) \tag{19}$$

in contradiction to (14), and thus finishes the proof.

We first show that $\operatorname{Im}V$ is invariant under A_{11} and N_{11}. To this end, let $Vz = 0$. Then by (17)

$$0 = z^T\left(A_{11}V + VA_{11}^T + N_{11}VN_{11}^T\right)z = z^T N_{11}VN_{11}^T z$$

whence also $VN_{11}^T z = 0$, i.e., $N_{11}^T z \in \operatorname{Ker}V$. From this, we have

$$0 = \left(A_{11}V + VA_{11}^T + N_{11}VN_{11}^T\right)z = VA_{11}^T z$$

implying $A_{11}^T z \in \operatorname{Ker}V$. Thus, $A_{11}^T \operatorname{Ker}V \subset \operatorname{Ker}V$ and $N_{11}^T \operatorname{Ker}V \subset \operatorname{Ker}V$.

Since $\operatorname{Ker}V = (\operatorname{Im}V)^\top$, it follows further that $\operatorname{Im}V$ is invariant under A_{11} and N_{11}.

Let $V = V_1V_1^T$, where V_1 has full column rank, i.e., $\det V_1^TV_1 \ne 0$. Then by the invariance, there exist square matrices X and Y, such that

$$A_{11}V_1 = V_1X \qquad N_{11}V_1 = V_1Y.$$

It follows that

$$\begin{aligned} 0 &= A_{11}V_1V_1^T + V_1V_1^TA_{11}^T + N_{11}V_1V_1^TN_{11}^T \\ &= V_1(X + X^T + YY^T)V_1^T \end{aligned}$$

whence $X + X^T + YY^T = 0$. Moreover, from (18), we get

$$\begin{aligned} A_{11}^T\Sigma_1V_1 &= -\Sigma_1A_{11}V_1 - N_{11}^T\Sigma_1N_{11}V_1 \\ &= -\Sigma_1V_1X - N_{11}^T\Sigma_1V_1Y. \end{aligned} \tag{20}$$

Using this substitution in the following computation, we obtain

$$\begin{aligned} 0 \ge{} & V_1^T\Sigma_1^2\left(A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11}\right)\Sigma_1^2V_1 \\ ={} & -V_1^T\Sigma_1^3V_1X - X^TV_1^T\Sigma_1^3V_1 \\ & - V_1^T\Sigma_1^2N_{11}^T\Sigma_1V_1Y - Y^TV_1^T\Sigma_1N_{11}\Sigma_1^2V_1 \\ & + V_1^T\Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2V_1. \end{aligned} \tag{21}$$

Taking the trace in (21), we have

$$0 = \operatorname{trace}\begin{bmatrix} V_1Y \\ V_1 \end{bmatrix}^T M \begin{bmatrix} V_1Y \\ V_1 \end{bmatrix}$$

where

$$M = \begin{bmatrix} \Sigma_1^3 & -\Sigma_1N_{11}\Sigma_1^2 \\ -\Sigma_1^2N_{11}^T\Sigma_1 & \Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2 \end{bmatrix}$$

is positive semidefinite

$$\begin{bmatrix} \Sigma_1^3 & -\Sigma_1N_{11}\Sigma_1^2 \\ -\Sigma_1^2N_{11}^T\Sigma_1 & \Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2 \end{bmatrix}\begin{bmatrix} V_1Y \\ V_1 \end{bmatrix} = 0.$$

The first block row then implies $N_{11}\Sigma_1^2 V_1 = \Sigma_1^2 V_1 Y$. From (21), using also (20) again, we thus have

$$\begin{aligned}0 &= \left(A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11}\right)\Sigma_1^2 V_1\\ &= -\Sigma_1 V_1 X - N_{11}^T\Sigma_1 V_1 Y + \Sigma_1^{-1}A_{11}\Sigma_1^2 V_1 + N_{11}^T\Sigma_1 V_1 Y\\ &= -\Sigma_1 V_1 X + \Sigma_1^{-1}A_{11}\Sigma_1^2 V_1\end{aligned}$$

i.e., $A_{11}\Sigma_1^2 V_1 = \Sigma_1^2 V_1 X$. It follows that for arbitrary $k \in \mathbb{N}$, the eigenvector V in (17) can be replaced by

$$\Sigma_1^{2k} V \Sigma_1^{2k} = \Sigma_1^{2k} V_1 V_1^T \Sigma_1^{2k}$$

because

$$\begin{aligned}0 &= \Sigma_1^2 V_1 (X + X^T + YY^T) V_1^T \Sigma_1^2\\ &= A_{11}\left(\Sigma_1^2 V_1 V_1^T \Sigma_1^2\right) + \left(\Sigma_1^2 V_1 V_1^T \Sigma_1^2\right) A_{11}^T\\ &\quad + N_{11}\left(\Sigma_1^2 V_1 V_1^T \Sigma_1^2\right) N_{11}^T.\end{aligned}$$

Induction leads to

$$\begin{aligned}0 = A_{11}\left(\Sigma_1^{2k} V_1 V_1^T \Sigma_1^{2k}\right) &+ \left(\Sigma_1^{2k} V_1 V_1^T \Sigma_1^{2k}\right) A_{11}^T\\ &+ N_{11}\left(\Sigma_1^{2k} V_1 V_1^T \Sigma_1^{2k}\right) N_{11}^T.\end{aligned}$$

As above, we conclude that $N_{21}\Sigma_1^{2k}V_1 = 0$, $\tilde{C}_1\Sigma_1^{2k}V_1 = 0$, and $\tilde{B}_1^T\Sigma_1^{2k}V_1 = 0$. Multiplying the lower left blocks of (13a) and (13b) with $\Sigma_1^{2(k-1)}V_1$ and $\Sigma_1^{2k}V_1$, respectively, we get

$$\begin{aligned}A_{12}^T\Sigma_1^{2k-1}V_1 + \Sigma_2 A_{21}\Sigma_1^{2(k-1)}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1 Y &= 0\\ A_{12}^T\Sigma_1^{2k-1}V_1 + \Sigma_2^{-1} A_{21}\Sigma_1^{2k}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1 Y &= 0.\end{aligned}$$

Hence (after multiplication with Σ_2), for all $k \geq 1$, we have

$$\begin{aligned}\Sigma_2^2 A_{21}\Sigma_1^{2(k-1)}V_1 &= -\Sigma_2\left(A_{12}^T\Sigma_1^{2k-1}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1 Y\right)\\ &= A_{21}\Sigma_1^{2k}V_1.\end{aligned}$$

Applying this identity repeatedly, we get

$$A_{21}\Sigma_1^{2k}V_1 = \Sigma_2^{2k}A_{21}V_1 \quad \text{for all} \quad k \in \mathbb{N}.$$

If μ is the minimal polynomial of Σ_1^2, then $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$ implies $\det \mu(\Sigma_2^2) \neq 0$ and

$$0 = A_{21}\mu\left(\Sigma_1^2\right)V_1 = \mu\left(\Sigma_2^2\right)A_{21}V_1$$

whence $A_{21}V_1 = 0$ and also $A_{21}V = 0$. Hence we obtain the contradiction (19). □

B. Error Estimate

The following theorem has been proven in [19] using LMI-techniques. Exploiting the stability result in the previous subsection, we can give a slightly more compact proof based on the stochastic bounded real lemma, Theorem A.6.

Theorem II.4: Let A and N satisfy

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-.$$

Assume furthermore that for $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}I, \ldots, \sigma_\nu I)$ and $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, the following Lyapunov inequalities hold:

$$A^T\Sigma + \Sigma A + N^T\Sigma N \leq -C^T C$$

$$A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N \leq -\Sigma^{-1}BB^T\Sigma^{-1}.$$

If $x(0) = 0$ and $x_r(0) = 0$, then for all $T > 0$, it holds that

$$\|y - y_r\|_{L^2_w([0,T])} \leq 2(\sigma_{r+1} + \cdots + \sigma_\nu)\|u\|_{L^2_w([0,T])}.$$

Proof: We adapt a proof for deterministic systems, e.g., [2, Th. 7.9]. In the central argument we treat the case where $\Sigma_2 = \sigma_\nu I$ and show that

$$\|y - y_{\nu-1}\|_{L^2_w[0,T]} \leq 2\sigma_\nu\|u\|_{L^2_w[0,T]}. \qquad (22)$$

From the left upper blocks of (13a) and (13b), we can see that also

$$A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + N_{11}^T\Sigma_1 N_{11} \leq -C_1^T C_1$$

$$A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11} \leq -\Sigma_1^{-1}B_1B_1^T\Sigma_1^{-1}.$$

Hence we can repeat the above argument to remove $\sigma_{\nu-1}, \ldots, \sigma_{r+1}$ successively. By the triangle inequality we find that

$$\begin{aligned}\|y - y_r\|_{L^2_w[0,T]} &\leq \sum_{j=r}^{\nu-1}\|y_{j+1} - y_j\|_{L^2_w[0,T]}\\ &\leq 2(\sigma_\nu + \cdots + \sigma_{r+1})\|u\|_{L^2_w[0,T]}.\end{aligned}$$

which then concludes the proof.

To prove (22), we make use of the stochastic bounded real lemma. In the following let $r = \nu - 1$ and consider the error system defined by:

$$\begin{aligned}dx_e &= A_e x_e\, dt + N_e x_e\, dw + B_e u\, dt\\ y_e &= C_e x_e = y - y_r\end{aligned}$$

where

$$x_e = \begin{bmatrix} x_1\\ x_2\\ x_r\end{bmatrix} \qquad A_e = \begin{bmatrix} A_{11} & A_{12} & 0\\ A_{21} & A_{22} & 0\\ 0 & 0 & A_{11}\end{bmatrix}$$

$$N_e = \begin{bmatrix} N_{11} & N_{12} & 0\\ N_{21} & N_{22} & 0\\ 0 & 0 & N_{11}\end{bmatrix} \qquad B_e = \begin{bmatrix} B_1\\ B_2\\ B_1\end{bmatrix}$$

$$C_e = [C_1 \quad C_2 \quad -C_1].$$

Applying the state space transformation

$$\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_r\end{bmatrix} = \begin{bmatrix} x_1 - x_r\\ x_2\\ x_1 + x_r\end{bmatrix} = \underbrace{\begin{bmatrix} I_r & 0 & -I_r\\ 0 & I_{n-r} & 0\\ I_r & 0 & I_r\end{bmatrix}}_{=S^{-1}}\begin{bmatrix} x_1\\ x_2\\ x_r\end{bmatrix}$$

we obtain the transformed system

$$\tilde{A}_e = S^{-1} A_e S = \begin{bmatrix} A_{11} & A_{12} & 0 \\ \frac{1}{2}A_{21} & A_{22} & \frac{1}{2}A_{21} \\ 0 & A_{12} & A_{11} \end{bmatrix}$$

$$\tilde{N}_e = S^{-1} N_e S = \begin{bmatrix} N_{11} & N_{12} & 0 \\ \frac{1}{2}N_{21} & N_{22} & \frac{1}{2}N_{21} \\ 0 & N_{12} & N_{11} \end{bmatrix}$$

$$\tilde{B}_e = S^{-1} B \begin{bmatrix} 0 \\ B_2 \\ 2B_1 \end{bmatrix}$$

$$\tilde{C}_e = C_e S = [C_1 \quad C_2 \quad 0].$$

By Theorem A.6, we have $\|\mathbb{L}_e\| \le 2\sigma_\nu$, if the Riccati inequality

$$\mathcal{R}_{\sigma_\nu}(X) = \tilde{A}_e^T X + X\tilde{A}_e + \tilde{N}_e^T X \tilde{N}_e + \tilde{C}_e^T \tilde{C}_e + \frac{1}{4\sigma_\nu^2} X \tilde{B}_e \tilde{B}_e^T X \le 0 \quad (23)$$

possesses a solution $X \ge 0$. In fact, such a solution is given by the block-diagonal matrix

$$X = \operatorname{diag}\left(\Sigma_1, 2\Sigma_2, \sigma_\nu^2 \Sigma_1^{-1}\right) = \operatorname{diag}\left(\Sigma_1, 2\sigma_\nu I, \sigma_\nu^2 \Sigma_1^{-1}\right) > 0.$$

To verify this, we set $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and

$$M = J(A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N + \Sigma^{-1}BB^T\Sigma^{-1})J$$

where $M \le 0$ by (13b). Considering all blocks of (13a) and (13b), a straight-forward computation yields

$$\mathcal{R}_{\sigma_\nu}(X) = \begin{bmatrix} A^T\Sigma + \Sigma A + N^T\Sigma N + C^T C & 0 \\ 0 & 0 \end{bmatrix} - \frac{\sigma_\nu}{2} \begin{bmatrix} N_{21}^T \\ 0 \\ -N_{21}^T \end{bmatrix} \begin{bmatrix} N_{21}^T \\ 0 \\ -N_{21}^T \end{bmatrix}^T + \sigma_\nu^2 \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \le 0$$

which is inequality (23). □

Example II.5: Let the system (A, N, B, C) and Q be as in Example I.3. The matrix

$$P = \begin{bmatrix} 1 + \sqrt{1-p} & 0 \\ 0 & p \end{bmatrix}^{-1} > 0, \text{ where } 0 < p \le 1$$

satisfies inequality (7b). As in Example I.3, we have $\mathbb{L}_r = 0$ for the corresponding reduced system of order 1, so that the truncation error again is $1/\sqrt{2}a$, independently of $p \in]0, 1]$.

On the other hand we have

$$\sigma_2^2 = \min \sigma(PQ) = \frac{1}{4a^2(1 + \sqrt{1-p})} \ge \frac{1}{8a^2}$$

with equality for $p \to 0$. Theorem II.4 thus gives the sharp error bound $2\sigma_2 = 1/\sqrt{2}a$. Note, that there is no $P > 0$ satisfying (7b).

The previous example illustrates the problem of optimizing over all solutions of inequality (7b).

IV. Numerical Examples

To compare the reduction methods, we need to compute Q, P from (6) or (7). Instead of the inequalities (6a), (6b), (7a) we can consider the corresponding equations, for which quite efficient algorithms have been developed recently, e.g., [27]–[30]. These also allow for a low-rank approximation of the solutions. In contrast we cannot replace (7b) by the corresponding equation, because this may not be solvable (see Example II.5). Even worse, we neither have any solvability or uniqueness criteria nor reliable algorithms.

Therefore, in general, we have to work with the inequality (7b), which is solvable according to Lemma II.1, but of course not uniquely solvable.

In view of our application, we aim at a solution P of (7b), so that (some of) the eigenvalues of PQ are particularly small, since they provide the error bound. Choosing a matrix $Y < 0$ and a very small ε along the lines of the proof of Lemma II.1 can be contrary to this aim. Hence some optimization over all solutions of (7b) is required.

Note also that a matrix $P > 0$ satisfies (7b), if and only if it satisfies the linear matrix inequality (LMI)

$$\begin{bmatrix} PA^T + AP + BB^T & PN^T \\ NP & -P \end{bmatrix} \le 0. \quad (24)$$

Thus, LMI optimal solution techniques are applicable. However, their complexity will be prohibitive for large-scale problems. Therefore further research for alternative methods to solve (7b) adequately is required.

By $\mathbb{L}$ and $\mathbb{L}_r$, we always denote the original and the r-th order approximated system. The stochastic H^∞-type norm $\|\mathbb{L} - \mathbb{L}_r\|$ is computed by a binary search of the infimum of all γ such that the Riccati inequality (10) is solvable. The latter is solved via a Newton iteration as in [18]. Finally, the Lyapunov equations (2) are solved by preconditioned Krylov subspace methods described in [27].

Unfortunately, for small γ, i.e., for small approximation errors, this method of computing the error runs into numerical problems, because (10) contains the term γ^{-2}. This apparently leads to cancellation phenomena in the Newton iteration, if, e.g., $\gamma < 10^{-7}$. Therefore we mainly concentrate on cases where the error is larger, i.e., we make r sufficiently small.

A. *Type II Can be Better Than Type I*

In many examples we observe that type II reduction gives a valid error bound, but the approximation error still is better with type I. This, however, is not always true, as the example

$$(A, N, B, C^T) = \left(\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right)$$

shows. It can be easily be verified that the type I Lyapunov equations (6) are solved by

$$Q = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}.$$

The type II inequalities (7) are, e.g., solved by

$$Q = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 8 & 0 \\ 0 & 12 \end{bmatrix}.$$

TABLE I
ERROR BOUNDS AND APPROXIMATION ERRORS FOR BOTH TYPES

	σ_2	$\|\mathbb{L}-\mathbb{L}_1\|$
I	2.4853	3.9647
II	6.9282	3.5614

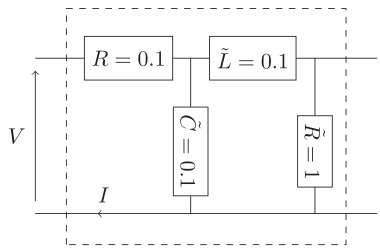

Fig. 1. Section of ladder network from [31].

If we reduce to order $r = 1$, the type I approximation error is larger than both the truncated singular value and the type II approximation error; see Table I.

B. Electrical Ladder Network With Perturbed Inductance

As our first example with a physical background, we take up the electrical ladder network described in [31], consisting of $n/2$ sections with a capacitor $\tilde{C}$, inductor $\tilde{L}$ and two resistors R and $\tilde{R}$ as depicted in Fig. 1.

But following, e.g., [32], we assume that the inductance $\tilde{L}$ is subject to stochastic perturbations. For simplicity, we replace the inverse $\tilde{L}^{-1}$ formally by $L^{-1} + \dot{w}$ in all sections. Here $L = 0.1$ and $\dot{w}$ is white noise of a certain intensity σ, where we set $\sigma = 1$, e.g., for $n = 6$, we have the system matrices

$$A = \begin{bmatrix} \frac{-1}{\tilde{C}R} & \frac{-1}{\tilde{C}} & 0 & 0 & 0 & 0 \\ \frac{1}{L} & \frac{-R\tilde{R}}{L(R+\mathrm{R})} & \frac{-\tilde{R}}{L(R+\mathrm{R})} & 0 & 0 & 0 \\ 0 & \frac{\tilde{R}}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}} & 0 & 0 \\ 0 & 0 & \frac{1}{L} & \frac{-R\tilde{R}}{L(R+\tilde{R})} & \frac{-\tilde{R}}{L(R+\tilde{R})} & 0 \\ 0 & 0 & 0 & \frac{\tilde{R}}{\tilde{C}(R+\mathrm{R})} & \frac{-1}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}} \\ 0 & 0 & 0 & 0 & \frac{1}{L} & \frac{-\tilde{R}}{L} \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{-R\tilde{R}}{R+\tilde{R}} & \frac{-\tilde{R}}{R+\tilde{R}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{-R\tilde{R}}{R+\tilde{R}} & \frac{-\tilde{R}}{R+\tilde{R}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\tilde{R} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{\tilde{C}R} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} -\frac{1}{R} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For larger n, the band structure of A and N is extended periodically. To see the behavior of our two methods, we reduce from order $n = 20$ to the orders $r = 1, 3, 5, \ldots, 19$, and compute both the theoretical bounds and the actual approximation errors in the H^∞-norm; see Fig. 2.

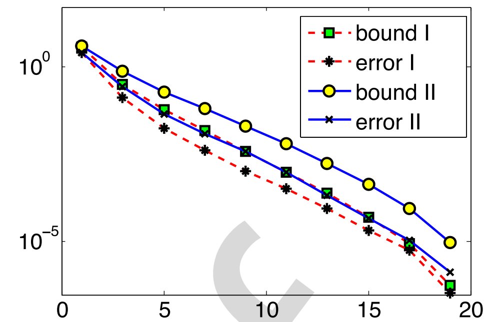

Fig. 2. In this example, for both types the bounds hold, and for all reduced orders, type I gives a smaller H^∞-error than type II.

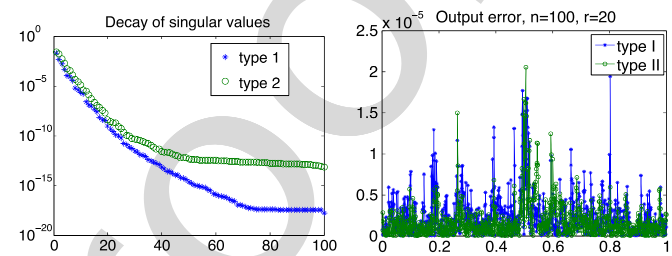

Fig. 3. Comparison of singular values and relative output error.

C. Heat Transfer Problem

As another example we consider a stochastic modification of the heat transfer problem described in [14]. On the unit square $\Omega = [0, 1]^2$, the heat equation $x_t = \Delta x$ is given with Dirichlet condition $x = u_j$, $j = 1, 2, 3$, on three of the boundary edges and a stochastic Robin condition $n \cdot \nabla x = (1/2 + \dot{w})x$ on the fourth edge (where $\dot{w}$ stands for white noise). A standard five-point finite-difference discretization on a 10×10 grid leads to a modified Poisson matrix $A \in \mathbb{R}^{100\times 100}$ and corresponding matrices $N \in \mathbb{R}^{100\times 100}$ and $B \in \mathbb{R}^{100\times 3}$. We use the input

$$u \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and choose the average temperature as the output, i.e., $C = (1/100)[1, \ldots, 1]$. We apply balanced truncation of type I and type II. For type II, an LMI-solver (MATLAB function `mincx`) is used to compute P as a solution of the LMI (24) which minimizes trace$P$ or tracePQ.

In the following figure (Fig. 3), we compare the reduced systems of order $r = 20$ for both types. The left diagram shows the decay of the singular values. Since the LMI-solver was called with tolerance level 10^{-9}, only the first about 25 singular values for type II have the correct order of magnitude. In this region, the decay for both types is roughly linear. Some analysis of this behavior for type I has been carried out in [28]. For type II, so far no theoretical results are available.

The diagram on the right displays the approximation error $\|y(t) - y_r(t)\|$ over a given time interval. For both types it has the same order of magnitude. In fact, for many examples we have observed both methods to yield very similar results.

The estimated error norm $\sum_{j=r+1}^{n} \sigma_j$ and the actual approximation error $\|\mathbb{L} - \mathbb{L}_{10}\|$ are given in Table II.

TABLE II
ERROR BOUNDS AND APPROXIMATION ERRORS FOR BOTH TYPES

	$\sum_{j=11}^{100}\sigma_j$	$\|\mathbb{L}-\mathbb{L}_{10}\|$	$\sum_{j=21}^{100}\sigma_j$	$\|\mathbb{L}-\mathbb{L}_{20}\|$
I	$4.66e-06$	$9.30e-06$	$2.00e-09$	$9.65e-09$
II	$1.75e-05$	$4.83e-06$	$1.72e-08$	$9.70e-09$

TABLE III
COMPARISON OF BOTH REDUCTION METHODS

Type	I	II
Def. of P, Q	(6)	(7)
Stability?	Yes, [24]	Yes, Thm. II.2
H^2-bound?	Yes, [20]	Yes, [33]
H^∞-bound?	No, Ex. I.3	Yes, Thm. II.4 or [19]
comput. cost	medium	high (via LMI)

As we can see, the upper error bound fails for type I, but is correct for type II. Nevertheless, judging from the H^∞ error, neither of the types seems to be preferable over the other.

D. Summary

Clearly, higher dimensional examples are required to get more insight. To this end, a more sophisticated method for the solution of (24) is needed. With general-purpose LMI-software on a standard Laptop, we hardly got higher than $n = 100$.

V. COMPARISON

Table III summarizes properties of our two methods.

As long as efficient algorithms for the solution of (7b) are not available, practical evidence favors to use the type I method in applications. Although there is no strict H^∞-type error bound for this case, in most examples the decay of singular values still roughly indicates the decay of the approximation error.

VI. CONCLUSIONS AND FUTURE WORK

We have discussed two ways of generalizing balanced truncation for stochastic linear systems. The main theoretical contributions of this paper are the preservation of asymptotic stability for type II balanced truncation proved in Theorem II.2 and the new proof of the H^∞ error bound in Theorem II.4. The efficient solution of the matrix inequality (7b) is an open issue and requires further research. The same is true for the computation of the stochastic H^∞-norm. Moreover, we are still looking for adequate interpretations of our approaches, e.g., in terms of energy minimization or Hankel operators. We hope to trigger some research in this direction.

APPENDIX A
ASYMPTOTIC MEAN SQUARE STABILITY

Consider the stochastic linear system of Itô-type

$$dx = Ax\,dt + Nx\,dw \tag{25}$$

where $w = (w(t))_{t\in\mathbb{R}_+}$ is a zero-mean real Wiener process on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ (e.g., [25], [26]).

Let $L^2_w(\mathbb{R}_+, \mathbb{R}^q)$ denote the corresponding space of nonanticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|^2_{L^2_w} := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2\,dt\right) < \infty$$

where $\mathcal{E}$ denotes expectation. For initial data $x(0) = x_0$, the solution can be written as $x(t) = \Phi(t)x_0$ with the fundamental matrix solution $\Phi(t)$, satisfying $\Phi(0) = I$

By definition, system (25) is asymptotically mean-square-stable, if $\mathcal{E}(\|x(t)\|^2) \overset{t\to\infty}{\longrightarrow} 0$, for all initial conditions x_0. In this case, for simplicity, we also call the pair (A, N) asymptotically mean-square stable.

We have the following version of Lyapunov's matrix theorem; see [23]. Here $\otimes$ denotes the Kronecker product.

Theorem A.1: The following are equivalent.

(i) System (25) is asymptotically mean-square stable.
(ii) $\max\{\Re\lambda \,\|\, \lambda \in \sigma(A\otimes I + I\otimes A + N\otimes N)\} < 0$
(iii) $\exists Y > 0 : \exists X > 0$: $A^TX + XA + N^TXN = -Y$
(iv) $\forall Y > 0 : \exists X > 0$: $A^TX + XA + N^TXN = -Y$
(v) $\forall Y \geq 0 : \exists X \geq 0$: $A^TX + XA + N^TXN = -Y$

Remark A.2: The theorem (like all other results in this paper) carries over to systems

$$dx = Ax\,dt + \sum_{j=1}^k N_jx\,dw_j$$

with more than one noise term, and many more equivalent criteria can be provided; see, e.g., [34] or [18, Th. 3.6.1].

The following theorem does not require any stability assumptions (see [18, Th. 3.2.3]). It is central in the analysis of mean-square stability.

Theorem A.3: Let

$$\alpha = \max\{\Re\lambda |\, \lambda \in \sigma(A\otimes I + I\otimes A + N\otimes N)\}.$$

Then there exists a nonnegative definite matrix $V \neq 0$, so that

$$(\mathcal{L}_A^* + \Pi_N^*)(V) = AV + VA^T + NVN^T = \alpha V.$$

We also note a simple consequence of this theorem [24, Cor. 3.2]. Here $\langle Y, V\rangle = \mathrm{trace}(YV)$ is the Frobenius inner product for symmetric matrices.

Corollary A.4: Let α, V as in the theorem. For given $Y \geq 0$ assume that

$$\exists X > 0 :\ \mathcal{L}_A(X) + \Pi_N(X) \leq -Y. \tag{26}$$

Then $\alpha \leq 0$. Moreover, if $\alpha = 0$ then $YV = VY = 0$.

APPENDIX B
STOCHASTIC BOUNDED REAL LEMMA

Now let us consider system (5) with input u and output y. If (A, N) is asymptotically mean-square stable, then (5) defines an input output operator $\mathbb{L} : u \mapsto y$ from $L^2_w(\mathbb{R}, \mathbb{R}^m)$ to $L^2_w(\mathbb{R}, \mathbb{R}^p)$, see [17]. By $\|\mathbb{L}\|$ we denote the induced operator norm, which is an analogue of the deterministic H^∞-norm. It can be characterized by the stochastic bounded real lemma.

Theorem A.5: [17] For $\gamma > 0$, the following are equivalent.

(i) System (25) is asymptotically mean-square stable and $\|\mathbb{L}\| < \gamma$.

(ii) There exists a negative definite solution $X < 0$ to the Riccati inequality

$$A^T X + XA + N^T XN - C^T C - \gamma^{-2} XBB^T X > 0.$$

(iii) There exists a positive definite solution $X > 0$ to the Riccati inequality

$$A^T X + XA + N^T XN + C^T C + \gamma^{-2} XBB^T X < 0.$$

We have stated the obviously equivalent formulations (ii) and (iii) to avoid confusion arising from different formulations in the literature. Under additional assumptions also nonstrict versions can be formulated. The following sufficient criterion is given in [18, Cor. 2.2.3] (where also the signs are changed). Unlike in the previous theorem, here asymptotic mean-square stability is assumed at the outset.

Theorem A.6: Assume that (25) is asymptotically stable in mean-square. If there exists a nonnegative definite matrix $X \geq 0$, satisfying

$$A^T X + XA + N^T XN + C^T C + \gamma^{-2} XBB^T X \leq 0$$

then $\|\mathbb{L}\| \leq \gamma$.

Appendix C
Unobservable and Unreachable Subspaces

Definition A.7: Consider system (5). A vector $v \in \mathbb{R}^n$ is called *unobservable*, if the initial condition $x(0) = v$ with $u \equiv 0$ produces the output $y \equiv 0$. The vector v is called *unreachable*, if $x(t) \neq v$ for all $t > 0$ and any solution with initial value $x(0) = 0$ and arbitrary input u.

If (A, N) is asymptotically mean-square stable, then (see [14, Th. 3.1]) the unobservable and the unreachable subspace can be characterized as the kernels of Q and P defined by

$$A^T Q + QA + N^T QN = -C^T C$$
$$AP + PA^T + NPN^T = -BB^T.$$

Theorem A.8: A state v is

(a) unobservable, if and only if $Qv = 0$.

(b) unreachable, if and only if $Pv = 0$.

In particular, the system is observable and reachable, if and only if $Q > 0$ and $P > 0$.

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Dual Pairs of Generalized Lyapunov Inequalities and Balanced Truncation of Stochastic Linear Systems

Peter Benner, Tobias Damm, and Yolanda Rocio Rodriguez Cruz

***Abstract*—We consider two approaches to balanced truncation of stochastic linear systems, which follow from different generalizations of the reachability Gramian of deterministic systems. Both preserve mean-square asymptotic stability, but only the second leads to a stochastic H^∞-type bound for the approximation error of the truncated system.**

***Index Terms*—Asymptotic mean square stability, balanced truncation, generalized Lyapunov equation, model order reduction, stochastic linear system.**

I. Introduction

OPTIMIZATION and (feedback) control of dynamical systems is often computationally infeasible for high dimensional plant models. Therefore, one tries to reduce the order of the system, so that the input-output mapping is still computable with sufficient accuracy, but at considerably smaller cost than for the original system [1]–[5]. To guarantee the desired accuracy, computable error bounds are required. Moreover, system properties which are relevant in the context of control system design like asymptotic stability need to be preserved. It has long been known that for linear time-invariant (LTI) systems the method of balanced truncation preserves asymptotic stability and provides an error bound for the L^2-induced input-output norm, i.e., the H^∞-norm of the associated transfer function; see [6], [7]. When considering model order reduction of more general system classes, it is natural to try to extend this approach. This has been worked out for descriptor systems in [8], for time-varying systems in [9]–[11], for bilinear systems in [12]–[14] and general nonlinear systems, e.g., in [15]. Yet another generalization of LTI systems is obtained considering dynamics driven by noise processes. This leads to the class of stochastic systems, which have been considered in a system theoretic context, e.g., in [16]–[18]. Quite recently, balanced truncation has also been described for linear stochastic systems of Itô type in [14], [19], and [20]. Already the formulation of the method leads to two different variants that are equivalent in the deterministic case, but not so for stochastic systems. It is natural to ask which of the above-mentioned properties of balanced truncation also hold for these variants. The aim of this paper is to answer this question.

Let us recapitulate balanced truncation for linear control systems of the form

$$\dot{x} = Ax + Bu \qquad y = Cx \qquad \sigma(A) \subset \mathbb{C}_-. \tag{1}$$

Here $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$ are the state, output, and input of the system, respectively. Moreover $\sigma(A)$ denotes the spectrum of A and $\mathbb{C}_-$ the open left half complex plane. Let

$$\mathcal{L}_A : X \mapsto A^T X + XA$$

denote the Lyapunov operator and

$$\mathcal{L}_A^* : X \mapsto AX + XA^T$$

its adjoint with respect to the Frobenius inner product $\langle Z, Y\rangle = \mathrm{trace}(Y^T Z)$. Then $\sigma(A) \subset \mathbb{C}_-$ if and only if there exists a positive definite solution X of the Lyapunov inequality $\mathcal{L}_A(X) < 0$, by Lyapunov's classical stability theorem, see, e.g., [21].

Balanced truncation means truncating a balanced realization. This realization is obtained by a state space transformation computed from the Gramians P and Q, which solve the dual pair of *Lyapunov equations*

$$\mathcal{L}_A(Q) = A^T Q + QA = -C^T C \tag{2a}$$

$$\mathcal{L}_A^*(P) = AP + PA^T = -BB^T \tag{2b}$$

or more generally the *inequalities*

$$\mathcal{L}_A(Q) \le -C^T C \qquad \mathcal{L}_A^*(P) \le -BB^T. \tag{3}$$

These (in)equalities are essential in the characterization of stability, controllability and observability of system (1). If $\det P \neq 0$, the inequalities (3) can be written as

$$\mathcal{L}_A(Q) \le -C^T C \tag{4a}$$

$$\mathcal{L}_A(P^{-1}) = P^{-1}A + A^T P^{-1} \le -P^{-1}BB^T P^{-1}. \tag{4b}$$

In the present paper we discuss extensions of (3) and (4) for stochastic linear systems.

As indicated above, the equivalent formulations (3) and (4) lead to different generalizations, if we consider Itô-type stochastic systems of the form

$$dx = Ax\,dt + Nx\,dw + Bu\,dt, \quad y = Cx \tag{5}$$

where A, B, C are as in (1) and $N \in \mathbb{R}^{n\times n}$. System (5) is asymptotically mean-square stable (e.g., [18], [22], [23]), if and

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only if there exists a positive definite solution X of the generalized Lyapunov inequality

$$(\mathcal{L}_A + \Pi_N)(X) = A^T X + XA + N^T XN < 0.$$

Here $\Pi_N : X \mapsto N^T XN$ and $\Pi_N^* : X \mapsto NXN^T$. This stability criterion indicates that in the stochastic context, the generalized Lyapunov operator $\mathcal{L}_A + \Pi_N$ takes over the role of $\mathcal{L}_A$. Substituting $\mathcal{L}_A$ by $\mathcal{L}_A + \Pi_N$ in (3) and (4), we obtain two different dual pairs of generalized Lyapunov inequalities. We call them *type I*

$$(\mathcal{L}_A + \Pi_N)(Q) = A^T Q + QA + N^T QN \le -C^T C \quad (6a)$$
$$(\mathcal{L}_A + \Pi_N)^*(P) = AP + PA^T + NPN^T \le -BB^T \quad (6b)$$

and *type II*

$$\begin{aligned}(\mathcal{L}_A + \Pi_N)(Q) &= A^T Q + QA + N^T QN \\ &\le -C^T C \quad (7a)\\ (\mathcal{L}_A + \Pi_N)(P^{-1}) &= A^T P^{-1} + P^{-1}A + N^T P^{-1} N \\ &\le -P^{-1}BB^T P^{-1}. \quad (7b)\end{aligned}$$

Note that (6) corresponds to (3) in the sense that $\mathcal{L}_A^*(P)$ has been replaced by $(\mathcal{L}_A + \Pi_N)^*(P)$, while (7) corresponds to (4), where $\mathcal{L}_A(P^{-1})$ has been replaced by $(\mathcal{L}_A + \Pi_N)(P^{-1})$. In general (if N and P do not commute), the inequalities (6b) and (7b) are not equivalent. At first glance it is not clear which generalization is more appropriate.

If the system is asymptotically mean-square stable, then for both types there are solutions $Q, P > 0$. By a suitable state space-transformation, it is possible to balance the system such that $Q = P = \Sigma > 0$ is diagonal. Consequently, the usual procedure of balanced truncation can be applied to reduce the order of (5). For simplicity, let us refer to this as *type I* or *type II balanced truncation*.

Under natural assumptions, this reduction preserves mean-square asymptotic stability. For type I, this nontrivial fact has been proven in [24]. Moreover, in [20], an H^2-error bound has been provided. However, different from the deterministic case, there is no H^∞-type error bound in terms of the truncated entries in Σ. This will be shown in Example I.3.

In contrast, for type II, an H^∞-type error bound has been obtained in [19]. In the present paper, as one of our main contributions, we show in Theorem II.2 that type II balanced truncation also preserves mean-square asymptotic stability. The proof differs significantly from the one given for type I. Using this result, we are able to give a more compact proof of the error bound, Theorem II.4, which exploits the stochastic bounded real lemma [17].

We illustrate our results by analytical and numerical examples in Section IV.

II. Type I Balanced Truncation

Consider a stochastic linear control system of Itô-type

$$dx = Ax\,dt + \sum_{j=1}^{k} N_j x\,dw_j + Bu\,dt, \quad y = Cx \quad (8)$$

where $w_j = (w_j(t))_{t\in\mathbb{R}_+}$ are uncorrelated zero-mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ (e.g., [25], [26]).

To simplify the notation, we only consider the case $k = 1$ and set $w = w_1$, $N = N_1$. But all results can immediately be generalized for $k > 1$.

Let $L_w^2(\mathbb{R}_+, \mathbb{R}^q)$ denote the corresponding space of nonanticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|_{L_w^2}^2 := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2\,dt\right) < \infty$$

where $\mathcal{E}$ denotes expectation.

Let the homogeneous equation $dx = Axdt + Nxdw$ be asymptotically mean-square-stable, i.e., $\mathcal{E}(\|x(t)\|^2) \overset{t\to\infty}{\longrightarrow} 0$, for all solutions x.

Then, by Theorem A.1, the equations

$$A^T Q + QA + N^T QN = -C^T C$$
$$AP + PA^T + NPN^T = -BB^T$$

have unique solutions $Q \ge 0$ and $P \ge 0$. If the system is observable and reachable (see Theorem A.8), then Q and P are nonsingular, and thus positive definite.

A similarity transformation

$$(A, N, B, C) \mapsto (S^{-1}AS, S^{-1}NS, S^{-1}B, CS)$$

of the system implies the contragradient transformation as

$$(Q, P) \mapsto (S^T QS, S^{-1}PS^{-T}).$$

Choosing, e.g., $S = LV\Sigma^{-1/2}$, with Cholesky factorizations $LL^T = P$, $R^T R = Q$ and a singular value decomposition $RL = U\Sigma V^T$, we obtain $S^{-1} = \Sigma^{-1/2}U^T R$ and

$$S^T QS = S^{-1}PS^{-T} = \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n).$$

After suitable partitioning

$$\Sigma = \begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix} \qquad S = [S_1 \quad S_2] \qquad S^{-1} = \begin{bmatrix}T_1\\ T_2\end{bmatrix}$$

a truncated system is given in the form

$$(A_{11}, N_{11}, B_1, C_1) = (T_1AS_1, T_1NS_1, T_1B, CS_1).$$

The following result has been proven in [24].

Theorem I.1: Let $A, N \in \mathbb{R}^{n\times n}$ satisfy

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-.$$

For a block-diagonal matrix $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2) > 0$ with $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, assume that

$$A^T\Sigma + \Sigma A + N^T\Sigma N \le 0 \text{ and } A\Sigma + \Sigma A^T + N\Sigma N^T \le 0.$$

Then, with the usual partitioning of A and N, we have

$$\sigma(I \otimes A_{11} + A_{11} \otimes I + N_{11} \otimes N_{11}) \subset \mathbb{C}_-.$$

Its implication for mean-square stability of the truncated system is immediate.

Corollary I.2: Consider an asymptotically mean square stable stochastic linear system

$$dx = Ax\,dt + Nx\,dw.$$

Assume that a matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$ is given as in Theorem I.1 and A and N are partitioned accordingly. Then the truncated system

$$dx_r = A_{11}x_r\,dt + N_{11}x_r\,dw$$

is also asymptotically mean square stable.

If the diagonal entries of Σ_2 are small, it is expected that the truncation error is small. In fact this is supported by an H^2-error bound obtained in [20]. Additionally, however, from the deterministic situation (see [2], [6]), one would also hope for an H^∞-type error bound of the form

$$\|y - y_r\|_{L^2_w(\mathbb{R}_+,\mathbb{R}^p)} \overset{?}{\leq} \alpha(\text{trace}\Sigma_2)\|u\|_{L^2_w(\mathbb{R}_+,\mathbb{R}^m)} \tag{9}$$

with some real number $\alpha > 0$. The following example shows that no such general α exists.

Example I.3: Let $A = -\begin{bmatrix}1 & 0\\ 0 & a^2\end{bmatrix}$ with $a > 1$, $N = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$, $B = \begin{bmatrix}1\\ 0\end{bmatrix}$, $C = [0 \quad 1]$.

Solving (6) with equality, we get $P = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{4a^2}\end{bmatrix}$, $Q = \begin{bmatrix}\frac{1}{4a^2} & 0\\ 0 & \frac{1}{2a^2}\end{bmatrix}$ with $\sigma(PQ) = \{1/8a^2, 1/8a^4\}$ so that $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2)$, where $\sigma_1 = 1/\sqrt{8}a$ and $\sigma_2 = 1/\sqrt{8}a^2$. The system is balanced by the transformation $S = \begin{bmatrix}2a^2 & 0\\ 0 & 1/2\end{bmatrix}^{1/4}$.

Then $CS = (1/2^{1/4})[0 \quad 1]$ so that $C_r = 0$ for the truncated system of order 1. Thus, the output of the reduced system is $y_r \equiv 0$, and the truncation error $\|\mathbb{L} - \mathbb{L}_r\|$ is equal to the stochastic H^∞-norm (see [17]) of the original system

$$\|\mathbb{L}\| = \sup_{x(0)=0, \|u\|_{L^2_w}=1} \|y\|_{L^2_w}.$$

We show now that this norm is equal to $1/\sqrt{2}a = 2a\sigma_2$. Thus, depending on a, the ratio of the truncation error and $\text{trace}\Sigma_2 = \sigma_2$ can be arbitrarily large.

According to the stochastic bounded real lemma, Theorem A.5, $\|\mathbb{L}\|$ is the infimum over all γ so that the Riccati inequality

$$\begin{aligned} 0 &< A^TX + XA + N^TXN - C^TC - \frac{1}{\gamma^2}XBB^TX\\ &= \begin{bmatrix} -2x_1 + x_3 - \frac{1}{\gamma^2}x_1^2 & -(a^2+1)x_2 - \frac{1}{\gamma^2}x_1x_2\\ -(a^2+1)x_2 - \frac{1}{\gamma^2}x_1x_2 & -2a^2x_3 - \frac{1}{\gamma^2}x_2^2 - 1 \end{bmatrix} \end{aligned} \tag{10}$$

possesses a solution $X = \begin{bmatrix}x_1 & x_2\\ x_2 & x_3\end{bmatrix} < 0$.

If a given matrix X satisfies this condition, then so does the same matrix with x_2 replaced by 0. Hence we can assume that $x_2 = 0$, and end up with the two conditions $x_3 < -(1/2a^2)$ and (after multiplying the upper left entry with $-\gamma^2$)

$$\begin{aligned} 0 &> x_1^2 + 2\gamma^2x_1 - \gamma^2x_3 = (x_1 + \gamma^2)^2 - \gamma^2(\gamma^2 + x_3)\\ &> (x_1 + \gamma^2)^2 - \gamma^2\left(\gamma^2 - \frac{1}{2a^2}\right). \end{aligned}$$

Thus necessarily $\gamma^2 > 1/2a^2$, i.e., $\gamma > 1/\sqrt{2}a$. This already proves that $\|\mathbb{L}\| \geq 1/\sqrt{2}a = 2a\sigma_2$, which suffices to disprove the existence of a general bound α in (9). Taking infima, it is easy to show that indeed $\|\mathbb{L}\| = 1/\sqrt{2}a$.

III. Type II Balanced Truncation

We now consider the inequalities (7).

Lemma II.1: Assume that $dx = Axdt + Nxdw$ is asymptotically mean-square-stable. Then inequality (7b) is solvable with $P > 0$.

Proof: By Theorem A.1, for a given $Y < 0$, there exists a $\tilde{P} > 0$, so that $A^T\tilde{P}^{-1} + \tilde{P}^{-1}A + N^T\tilde{P}^{-1}N = Y$. Then $P = \varepsilon^{-1}\tilde{P}$, for sufficiently small $\varepsilon > 0$, satisfies

$$A^TP^{-1} + P^{-1}A + N^TP^{-1}N = \varepsilon Y < -\varepsilon^2\tilde{P}^{-1}BB^T\tilde{P}^{-1}$$

so that (7b) holds even in the strict form. □

It is easy to see that like in the previous section a state space transformation

$$(A, N, B, C) \mapsto (S^{-1}AS, S^{-1}NS, S^{-1}B, CS)$$

leads to a contragradient transformation $Q \mapsto S^TQS$, $P \mapsto S^{-1}PS^{-T}$ of the solutions. That is, Q and P satisfy (7a) and (7b), if and only if S^TQS and $S^{-1}PS^{-T}$ do so for the transformed data. As before, we can assume the system to be balanced with

$$Q = P = \Sigma = \operatorname{diag}(\sigma_1 I, \ldots, \sigma_\nu I) = \begin{bmatrix}\Sigma_1 & \\ & \Sigma_2\end{bmatrix} \tag{11}$$

where $\sigma_1 > \cdots > \sigma_\nu > 0$ and $\sigma(\Sigma_1) = \{\sigma_1, \ldots, \sigma_r\}$, $\sigma(\Sigma_2) = \{\sigma_{r+1}, \ldots, \sigma_\nu\}$. Hence, we will now assume (after balancing) that a diagonal matrix Σ as in (11) is given which satisfies

$$A^T\Sigma + \Sigma A + N^T\Sigma N \leq -C^TC \tag{12a}$$

$$A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N \leq -\Sigma^{-1}BB^T\Sigma^{-1}. \tag{12b}$$

Partitioning A, N, B, C like Σ, we write the system as

$$\begin{aligned} dx_1 &= (A_{11}x_1 + A_{12}x_2 + B_1u)\,dt + (N_{11}x_1 + N_{12}x_2)\,dw\\ dx_2 &= (A_{21}x_1 + A_{22}x_2 + B_2u)\,dt + (N_{21}x_1 + N_{22}x_2)\,dw\\ y &= C_1x_1 + C_2x_2. \end{aligned}$$

The reduced system obtained by truncation is

$$dx_r = (A_{11}x_r + B_1u)\,dt + N_{11}x_r\,dw \qquad y_r = C_1x_r.$$

The index r is the number of different singular values σ_j that have been kept in the reduced system. In the following subsections, we consider matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

$\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$ as in (11), and equations of the form

$$A^T\Sigma + \Sigma A + N^T \Sigma N = -\tilde{C}^T\tilde{C} \tag{13a}$$

$$A^T\Sigma^{-1} + \Sigma^{-1} A + N^T \Sigma^{-1} N = -\tilde{B}\tilde{B}^T \tag{13b}$$

with arbitrary right-hand sides $-\tilde{C}^T\tilde{C} \leq 0$ and $-\tilde{B}\tilde{B}^T \leq 0$.

A. Preservation of Asymptotic Stability

The following theorem is the main new result of this paper.

Theorem II.2: Let A and N be given such that

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-. \tag{14}$$

Assume further that for a block-diagonal matrix $\Sigma = \text{diag}(\Sigma_1, \Sigma_2) > 0$ with $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, we have

$$A^T\Sigma + \Sigma A + N^T \Sigma N \leq 0 \tag{15a}$$

$$A^T\Sigma^{-1} + \Sigma^{-1} A + N^T \Sigma^{-1} N \leq 0. \tag{15b}$$

Then, with the usual partitioning of A and N, we have

$$\sigma(I \otimes A_{11} + A_{11} \otimes I + N_{11} \otimes N_{11}) \subset \mathbb{C}_-. \tag{16}$$

Again we have an immediate interpretation in terms of mean-square stability of the truncated system.

Corollary II.3: Consider an asymptotically mean square stable stochastic linear system

$$dx = Ax\, dt + Nx\, dw.$$

Assume that a matrix $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$ is given as in Theorem II.2 and A and N are partitioned accordingly. Then the truncated system

$$dx_r = A_{11}x_r\, dt + N_{11}x_r\, dw$$

is also asymptotically mean square stable.

Proof of Theorem II.2: Note that the inequalities (15) are equivalent to the equations (13) with appropriate right-hand sides $-\tilde{C}^T\tilde{C}$ and $-\tilde{B}\tilde{B}^T$. In accordance with the partitioning of A, N, and Σ, each matrix equation (13a) and (13b) consists of three blocks.

By way of contradiction, we assume that (16) does not hold. Then by Theorem A.3, there exist $V \geq 0$, $V \neq 0$, $\alpha \geq 0$ such that

$$A_{11}V + VA_{11}^T + N_{11}VN_{11}^T = \alpha V. \tag{17}$$

Taking the scalar product of the left upper block of (13a) with V, we obtain $0 \geq \alpha \text{trace}(\Sigma_1 V)$ whence $\alpha = 0$ and $\tilde{C}_1 V = 0$, $N_{21}V = 0$ by Corollary A.4. Hence

$$\left(A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + N_{11}^T\Sigma_1 N_{11}\right) V = 0. \tag{18}$$

Analogously, we have $\tilde{B}_1^T V = 0$.

In particular, from $N_{21}V = 0$, we get

$$(\mathcal{L}_A^* + \Pi_N^*)\left(\begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & VA_{21}^T \\ A_{21}V & 0 \end{bmatrix}.$$

We will show that $A_{21}V = 0$, which implies

$$0 \in \sigma(I \otimes A + A \otimes I + N \otimes N) \tag{19}$$

in contradiction to (14), and thus finishes the proof.

We first show that ImV is invariant under A_{11} and N_{11}. To this end, let $Vz = 0$. Then by (17)

$$0 = z^T\left(A_{11}V + VA_{11}^T + N_{11}VN_{11}^T\right)z = z^T N_{11}VN_{11}^T z$$

whence also $VN_{11}^T z = 0$, i.e., $N_{11}^T z \in \text{Ker}V$. From this, we have

$$0 = \left(A_{11}V + VA_{11}^T + N_{11}VN_{11}^T\right)z = VA_{11}^T z$$

implying $A_{11}^T z \in \text{Ker}V$. Thus, $A_{11}^T \text{Ker}V \subset \text{Ker}V$ and $N_{11}^T \text{Ker}V \subset \text{Ker}V$.

Since $\text{Ker}V = (\text{Im}V)^\top$, it follows further that ImV is invariant under A_{11} and N_{11}.

Let $V = V_1V_1^T$, where V_1 has full column rank, i.e., $\det V_1^T V_1 \neq 0$. Then by the invariance, there exist square matrices X and Y, such that

$$A_{11}V_1 = V_1 X \qquad N_{11}V_1 = V_1 Y.$$

It follows that

$$\begin{aligned} 0 &= A_{11}V_1V_1^T + V_1V_1^TA_{11}^T + N_{11}V_1V_1^TN_{11}^T \\ &= V_1(X + X^T + YY^T)V_1^T \end{aligned}$$

whence $X + X^T + YY^T = 0$. Moreover, from (18), we get

$$\begin{aligned} A_{11}^T\Sigma_1V_1 &= -\Sigma_1A_{11}V_1 - N_{11}^T\Sigma_1N_{11}V_1 \\ &= -\Sigma_1V_1X - N_{11}^T\Sigma_1V_1Y. \end{aligned} \tag{20}$$

Using this substitution in the following computation, we obtain

$$\begin{aligned} 0 \geq\, & V_1^T\Sigma_1^2\left(A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11}\right)\Sigma_1^2V_1 \\ =\, & -V_1^T\Sigma_1^3V_1X - X^TV_1^T\Sigma_1^3V_1 \\ & - V_1^T\Sigma_1^2N_{11}^T\Sigma_1V_1Y - Y^TV_1^T\Sigma_1N_{11}\Sigma_1^2V_1 \\ & + V_1^T\Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2V_1. \end{aligned} \tag{21}$$

Taking the trace in (21), we have

$$0 = \text{trace}\begin{bmatrix} V_1Y \\ V_1 \end{bmatrix}^T M \begin{bmatrix} V_1Y \\ V_1 \end{bmatrix}$$

where

$$M = \begin{bmatrix} \Sigma_1^3 & -\Sigma_1N_{11}\Sigma_1^2 \\ -\Sigma_1^2N_{11}^T\Sigma_1 & \Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2 \end{bmatrix}$$

is positive semidefinite

$$\begin{bmatrix} \Sigma_1^3 & -\Sigma_1N_{11}\Sigma_1^2 \\ -\Sigma_1^2N_{11}^T\Sigma_1 & \Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2 \end{bmatrix}\begin{bmatrix} V_1Y \\ V_1 \end{bmatrix} = 0.$$

The first block row then implies $N_{11}\Sigma_1^2 V_1 = \Sigma_1^2 V_1 Y$. From (21), using also (20) again, we thus have

$$\begin{aligned} 0 &= \left(A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11}\right)\Sigma_1^2 V_1 \\ &= -\Sigma_1 V_1 X - N_{11}^T\Sigma_1 V_1 Y + \Sigma_1^{-1}A_{11}\Sigma_1^2 V_1 + N_{11}^T\Sigma_1 V_1 Y \\ &= -\Sigma_1 V_1 X + \Sigma_1^{-1}A_{11}\Sigma_1^2 V_1 \end{aligned}$$

i.e., $A_{11}\Sigma_1^2 V_1 = \Sigma_1^2 V_1 X$. It follows that for arbitrary $k \in \mathbb{N}$, the eigenvector V in (17) can be replaced by

$$\Sigma_1^{2k} V \Sigma_1^{2k} = \Sigma_1^{2k} V_1 V_1^T \Sigma_1^{2k}$$

because

$$\begin{aligned} 0 &= \Sigma_1^2 V_1 (X + X^T + YY^T) V_1^T \Sigma_1^2 \\ &= A_{11}\left(\Sigma_1^2 V_1 V_1^T \Sigma_1^2\right) + \left(\Sigma_1^2 V_1 V_1^T \Sigma_1^2\right) A_{11}^T \\ &\quad + N_{11}\left(\Sigma_1^2 V_1 V_1^T \Sigma_1^2\right) N_{11}^T. \end{aligned}$$

Induction leads to

$$\begin{aligned} 0 = A_{11}\left(\Sigma_1^{2k} V_1 V_1^T \Sigma_1^{2k}\right) + \left(\Sigma_1^{2k} V_1 V_1^T \Sigma_1^{2k}\right) A_{11}^T \\ + N_{11}\left(\Sigma_1^{2k} V_1 V_1^T \Sigma_1^{2k}\right) N_{11}^T. \end{aligned}$$

As above, we conclude that $N_{21}\Sigma_1^{2k}V_1 = 0$, $\tilde{C}_1\Sigma_1^{2k}V_1 = 0$, and $\tilde{B}_1^T\Sigma_1^{2k}V_1 = 0$. Multiplying the lower left blocks of (13a) and (13b) with $\Sigma_1^{2(k-1)}V_1$ and $\Sigma_1^{2k}V_1$, respectively, we get

$$\begin{aligned} A_{12}^T\Sigma_1^{2k-1}V_1 + \Sigma_2 A_{21}\Sigma_1^{2(k-1)}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1 Y &= 0 \\ A_{12}^T\Sigma_1^{2k-1}V_1 + \Sigma_2^{-1} A_{21}\Sigma_1^{2k}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1 Y &= 0. \end{aligned}$$

Hence (after multiplication with Σ_2), for all $k \geq 1$, we have

$$\begin{aligned} \Sigma_2^2 A_{21}\Sigma_1^{2(k-1)}V_1 &= -\Sigma_2\left(A_{12}^T\Sigma_1^{2k-1}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1 Y\right) \\ &= A_{21}\Sigma_1^{2k}V_1. \end{aligned}$$

Applying this identity repeatedly, we get

$$A_{21}\Sigma_1^{2k}V_1 = \Sigma_2^{2k}A_{21}V_1 \quad \text{for all} \quad k \in \mathbb{N}.$$

If μ is the minimal polynomial of Σ_1^2, then $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$ implies $\det \mu(\Sigma_2^2) \neq 0$ and

$$0 = A_{21}\mu\left(\Sigma_1^2\right)V_1 = \mu\left(\Sigma_2^2\right)A_{21}V_1$$

whence $A_{21}V_1 = 0$ and also $A_{21}V = 0$. Hence we obtain the contradiction (19). □

B. Error Estimate

The following theorem has been proven in [19] using LMI-techniques. Exploiting the stability result in the previous subsection, we can give a slightly more compact proof based on the stochastic bounded real lemma, Theorem A.6.

Theorem II.4: Let A and N satisfy

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-.$$

Assume furthermore that for $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}I, \ldots, \sigma_\nu I)$ and $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, the following Lyapunov inequalities hold:

$$A^T\Sigma + \Sigma A + N^T\Sigma N \leq -C^T C$$

$$A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N \leq -\Sigma^{-1}BB^T\Sigma^{-1}.$$

If $x(0) = 0$ and $x_r(0) = 0$, then for all $T > 0$, it holds that

$$\|y - y_r\|_{L^2_w([0,T])} \leq 2(\sigma_{r+1} + \cdots + \sigma_\nu)\|u\|_{L^2_w([0,T])}.$$

Proof: We adapt a proof for deterministic systems, e.g., [2, Th. 7.9]. In the central argument we treat the case where $\Sigma_2 = \sigma_\nu I$ and show that

$$\|y - y_{\nu-1}\|_{L^2_w[0,T]} \leq 2\sigma_\nu\|u\|_{L^2_w[0,T]}. \tag{22}$$

From the left upper blocks of (13a) and (13b), we can see that also

$$A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + N_{11}^T\Sigma_1 N_{11} \leq -C_1^T C_1$$

$$A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11} \leq -\Sigma_1^{-1}B_1 B_1^T\Sigma_1^{-1}.$$

Hence we can repeat the above argument to remove $\sigma_{\nu-1}$, $\ldots, \sigma_{r+1}$ successively. By the triangle inequality we find that

$$\begin{aligned} \|y - y_r\|_{L^2_w[0,T]} &\leq \sum_{j=r}^{\nu-1}\|y_{j+1} - y_j\|_{L^2_w[0,T]} \\ &\leq 2(\sigma_\nu + \cdots + \sigma_{r+1})\|u\|_{L^2_w[0,T]}. \end{aligned}$$

which then concludes the proof.

To prove (22), we make use of the stochastic bounded real lemma. In the following let $r = \nu - 1$ and consider the error system defined by:

$$\begin{aligned} dx_e &= A_e x_e\, dt + N_e x_e\, dw + B_e u\, dt \\ y_e &= C_e x_e = y - y_r \end{aligned}$$

where

$$x_e = \begin{bmatrix} x_1 \\ x_2 \\ x_r \end{bmatrix} \qquad A_e = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{11} \end{bmatrix}$$

$$N_e = \begin{bmatrix} N_{11} & N_{12} & 0 \\ N_{21} & N_{22} & 0 \\ 0 & 0 & N_{11} \end{bmatrix} \qquad B_e = \begin{bmatrix} B_1 \\ B_2 \\ B_1 \end{bmatrix}$$

$$C_e = [C_1 \quad C_2 \quad -C_1].$$

Applying the state space transformation

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_r \end{bmatrix} = \begin{bmatrix} x_1 - x_r \\ x_2 \\ x_1 + x_r \end{bmatrix} = \underbrace{\begin{bmatrix} I_r & 0 & -I_r \\ 0 & I_{n-r} & 0 \\ I_r & 0 & I_r \end{bmatrix}}_{=S^{-1}} \begin{bmatrix} x_1 \\ x_2 \\ x_r \end{bmatrix}$$

we obtain the transformed system

$$\tilde{A}_e = S^{-1}A_eS = \begin{bmatrix} A_{11} & A_{12} & 0 \\ \frac{1}{2}A_{21} & A_{22} & \frac{1}{2}A_{21} \\ 0 & A_{12} & A_{11} \end{bmatrix}$$

$$\tilde{N}_e = S^{-1}N_eS = \begin{bmatrix} N_{11} & N_{12} & 0 \\ \frac{1}{2}N_{21} & N_{22} & \frac{1}{2}N_{21} \\ 0 & N_{12} & N_{11} \end{bmatrix}$$

$$\tilde{B}_e = S^{-1}B \begin{bmatrix} 0 \\ B_2 \\ 2B_1 \end{bmatrix}$$

$$\tilde{C}_e = C_eS = [C_1 \quad C_2 \quad 0].$$

By Theorem A.6, we have $\|\mathbb{L}_e\| \le 2\sigma_\nu$, if the Riccati inequality

$$\mathcal{R}_{\sigma_\nu}(X) = \tilde{A}_e^T X + X\tilde{A}_e + \tilde{N}_e^T X \tilde{N}_e + \tilde{C}_e^T \tilde{C}_e + \frac{1}{4\sigma_\nu^2} X\tilde{B}_e\tilde{B}_e^T X \le 0 \quad (23)$$

possesses a solution $X \ge 0$. In fact, such a solution is given by the block-diagonal matrix

$$X = \operatorname{diag}\left(\Sigma_1, 2\Sigma_2, \sigma_\nu^2\Sigma_1^{-1}\right) = \operatorname{diag}\left(\Sigma_1, 2\sigma_\nu I, \sigma_\nu^2\Sigma_1^{-1}\right) > 0.$$

To verify this, we set $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and

$$M = J(A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N + \Sigma^{-1}BB^T\Sigma^{-1})J$$

where $M \le 0$ by (13b). Considering all blocks of (13a) and (13b), a straight-forward computation yields

$$\mathcal{R}_{\sigma_\nu}(X) = \begin{bmatrix} A^T\Sigma + \Sigma A + N^T\Sigma N + C^TC & 0 \\ 0 & 0 \end{bmatrix} - \frac{\sigma_\nu}{2}\begin{bmatrix} N_{21}^T \\ 0 \\ -N_{21}^T \end{bmatrix}\begin{bmatrix} N_{21}^T \\ 0 \\ -N_{21}^T \end{bmatrix}^T + \sigma_\nu^2\begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \le 0$$

which is inequality (23). □

Example II.5: Let the system (A, N, B, C) and Q be as in Example I.3. The matrix

$$P = \begin{bmatrix} 1+\sqrt{1-p} & 0 \\ 0 & p \end{bmatrix}^{-1} > 0, \text{ where } 0 < p \le 1$$

satisfies inequality (7b). As in Example I.3, we have $\mathbb{L}_r = 0$ for the corresponding reduced system of order 1, so that the truncation error again is $1/\sqrt{2}a$, independently of $p \in]0, 1]$.

On the other hand we have

$$\sigma_2^2 = \min\sigma(PQ) = \frac{1}{4a^2(1+\sqrt{1-p})} \ge \frac{1}{8a^2}$$

with equality for $p \to 0$. Theorem II.4 thus gives the sharp error bound $2\sigma_2 = 1/\sqrt{2}a$. Note, that there is no $P > 0$ satisfying (7b).

The previous example illustrates the problem of optimizing over all solutions of inequality (7b).

IV. Numerical Examples

To compare the reduction methods, we need to compute Q, P from (6) or (7). Instead of the inequalities (6a), (6b), (7a) we can consider the corresponding equations, for which quite efficient algorithms have been developed recently, e.g., [27]–[30]. These also allow for a low-rank approximation of the solutions. In contrast we cannot replace (7b) by the corresponding equation, because this may not be solvable (see Example II.5). Even worse, we neither have any solvability or uniqueness criteria nor reliable algorithms.

Therefore, in general, we have to work with the inequality (7b), which is solvable according to Lemma II.1, but of course not uniquely solvable.

In view of our application, we aim at a solution P of (7b), so that (some of) the eigenvalues of PQ are particularly small, since they provide the error bound. Choosing a matrix $Y < 0$ and a very small ε along the lines of the proof of Lemma II.1 can be contrary to this aim. Hence some optimization over all solutions of (7b) is required.

Note also that a matrix $P > 0$ satisfies (7b), if and only if it satisfies the linear matrix inequality (LMI)

$$\begin{bmatrix} PA^T + AP + BB^T & PN^T \\ NP & -P \end{bmatrix} \le 0. \quad (24)$$

Thus, LMI optimal solution techniques are applicable. However, their complexity will be prohibitive for large-scale problems. Therefore further research for alternative methods to solve (7b) adequately is required.

By $\mathbb{L}$ and $\mathbb{L}_r$, we always denote the original and the r-th order approximated system. The stochastic H^∞-type norm $\|\mathbb{L} - \mathbb{L}_r\|$ is computed by a binary search of the infimum of all γ such that the Riccati inequality (10) is solvable. The latter is solved via a Newton iteration as in [18]. Finally, the Lyapunov equations (2) are solved by preconditioned Krylov subspace methods described in [27].

Unfortunately, for small γ, i.e., for small approximation errors, this method of computing the error runs into numerical problems, because (10) contains the term γ^{-2}. This apparently leads to cancellation phenomena in the Newton iteration, if, e.g., $\gamma < 10^{-7}$. Therefore we mainly concentrate on cases where the error is larger, i.e., we make r sufficiently small.

A. Type II Can be Better Than Type I

In many examples we observe that type II reduction gives a valid error bound, but the approximation error still is better with type I. This, however, is not always true, as the example

$$(A, N, B, C^T) = \left(\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right)$$

shows. It can easily be verified that the type I Lyapunov equations (6) are solved by

$$Q = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}.$$

The type II inequalities (7) are, e.g., solved by

$$Q = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 8 & 0 \\ 0 & 12 \end{bmatrix}.$$

TABLE I
ERROR BOUNDS AND APPROXIMATION ERRORS FOR BOTH TYPES

	σ_2	$\|\mathbb{L}-\mathbb{L}_1\|$
I	2.4853	3.9647
II	6.9282	3.5614

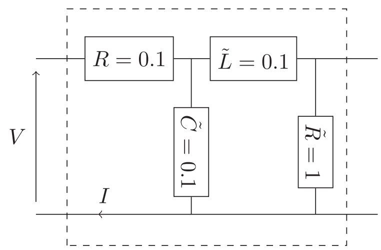

Fig. 1. Section of ladder network from [31].

If we reduce to order $r = 1$, the type I approximation error is larger than both the truncated singular value and the type II approximation error; see Table I.

B. Electrical Ladder Network With Perturbed Inductance

As our first example with a physical background, we take up the electrical ladder network described in [31], consisting of $n/2$ sections with a capacitor $\tilde{C}$, inductor $\tilde{L}$ and two resistors R and $\tilde{R}$ as depicted in Fig. 1.

But following, e.g., [32], we assume that the inductance $\tilde{L}$ is subject to stochastic perturbations. For simplicity, we replace the inverse $\tilde{L}^{-1}$ formally by $L^{-1} + \dot{w}$ in all sections. Here $L = 0.1$ and $\dot{w}$ is white noise of a certain intensity σ, where we set $\sigma = 1$, e.g., for $n = 6$, we have the system matrices

$$A = \begin{bmatrix} \frac{-1}{\tilde{C}R} & \frac{-1}{\tilde{C}} & 0 & 0 & 0 & 0 \\ \frac{1}{L} & \frac{-R\tilde{R}}{L(R+\tilde{R})} & \frac{-\tilde{R}}{L(R+\tilde{R})} & 0 & 0 & 0 \\ 0 & \frac{\tilde{R}}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}} & 0 & 0 \\ 0 & 0 & \frac{1}{L} & \frac{-R\tilde{R}}{L(R+\tilde{R})} & \frac{-\tilde{R}}{L(R+\tilde{R})} & 0 \\ 0 & 0 & 0 & \frac{\tilde{R}}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}} \\ 0 & 0 & 0 & 0 & \frac{1}{L} & \frac{-\tilde{R}}{L} \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{-R\tilde{R}}{R+\tilde{R}} & \frac{-\tilde{R}}{R+\tilde{R}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{-R\tilde{R}}{R+\tilde{R}} & \frac{-\tilde{R}}{R+\tilde{R}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\tilde{R} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{\tilde{C}R} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} -\frac{1}{R} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For larger n, the band structure of A and N is extended periodically. To see the behavior of our two methods, we reduce from order $n = 20$ to the orders $r = 1, 3, 5, \ldots, 19$, and compute both the theoretical bounds and the actual approximation errors in the H^∞-norm; see Fig. 2.

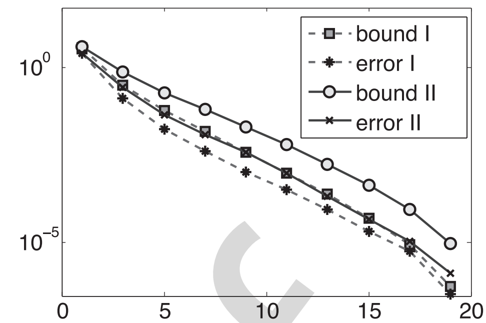

Fig. 2. In this example, for both types the bounds hold, and for all reduced orders, type I gives a smaller H^∞-error than type II.

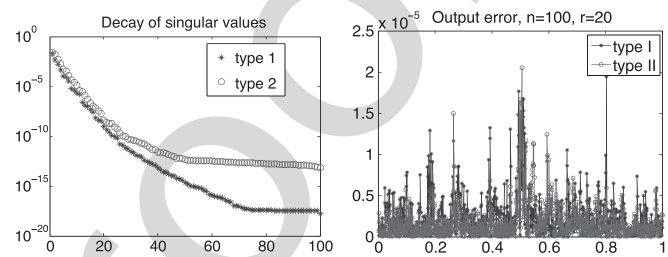

Fig. 3. Comparison of singular values and relative output error.

C. Heat Transfer Problem

As another example we consider a stochastic modification of the heat transfer problem described in [14]. On the unit square $\Omega = [0, 1]^2$, the heat equation $x_t = \Delta x$ is given with Dirichlet condition $x = u_j$, $j = 1, 2, 3$, on three of the boundary edges and a stochastic Robin condition $n \cdot \nabla x = (1/2 + \dot{w})x$ on the fourth edge (where $\dot{w}$ stands for white noise). A standard five-point finite-difference discretization on a 10×10 grid leads to a modified Poisson matrix $A \in \mathbb{R}^{100\times 100}$ and corresponding matrices $N \in \mathbb{R}^{100\times 100}$ and $B \in \mathbb{R}^{100\times 3}$. We use the input

$$u \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and choose the average temperature as the output, i.e., $C = (1/100)[1, \ldots, 1]$. We apply balanced truncation of type I and type II. For type II, an LMI-solver (MATLAB function `mincx`) is used to compute P as a solution of the LMI (24) which minimizes trace$P$ or tracePQ.

In the following figure (Fig. 3), we compare the reduced systems of order $r = 20$ for both types. The left diagram shows the decay of the singular values. Since the LMI-solver was called with tolerance level 10^{-9}, only the first about 25 singular values for type II have the correct order of magnitude. In this region, the decay for both types is roughly linear. Some analysis of this behavior for type I has been carried out in [28]. For type II, so far no theoretical results are available.

The diagram on the right displays the approximation error $\|y(t) - y_r(t)\|$ over a given time interval. For both types it has the same order of magnitude. In fact, for many examples we have observed both methods to yield very similar results.

The estimated error norm $\sum_{j=r+1}^{n} \sigma_j$ and the actual approximation error $\|\mathbb{L} - \mathbb{L}_{10}\|$ are given in Table II.

TABLE II
ERROR BOUNDS AND APPROXIMATION ERRORS FOR BOTH TYPES

	$\sum_{j=11}^{100}\sigma_j$	$\|\mathbb{L}-\mathbb{L}_{10}\|$	$\sum_{j=21}^{100}\sigma_j$	$\|\mathbb{L}-\mathbb{L}_{20}\|$
I	$4.66e-06$	$9.30e-06$	$2.00e-09$	$9.65e-09$
II	$1.75e-05$	$4.83e-06$	$1.72e-08$	$9.70e-09$

TABLE III
COMPARISON OF BOTH REDUCTION METHODS

Type	I	II
Def. of P, Q	(6)	(7)
Stability?	Yes, [24]	Yes, Thm. II.2
H^2-bound?	Yes, [20]	Yes, [33]
H^∞-bound?	No, Ex. I.3	Yes, Thm. II.4 or [19]
comput. cost	medium	high (via LMI)

As we can see, the upper error bound fails for type I, but is correct for type II. Nevertheless, judging from the H^∞ error, neither of the types seems to be preferable over the other.

D. Summary

Clearly, higher dimensional examples are required to get more insight. To this end, a more sophisticated method for the solution of (24) is needed. With general-purpose LMI-software on a standard Laptop, we hardly got higher than $n = 100$.

V. COMPARISON

Table III summarizes properties of our two methods.

As long as efficient algorithms for the solution of (7b) are not available, practical evidence favors to use the type I method in applications. Although there is no strict H^∞-type error bound for this case, in most examples the decay of singular values still roughly indicates the decay of the approximation error.

VI. CONCLUSIONS AND FUTURE WORK

We have discussed two ways of generalizing balanced truncation for stochastic linear systems. The main theoretical contributions of this paper are the preservation of asymptotic stability for type II balanced truncation proved in Theorem II.2 and the new proof of the H^∞ error bound in Theorem II.4. The efficient solution of the matrix inequality (7b) is an open issue and requires further research. The same is true for the computation of the stochastic H^∞-norm. Moreover, we are still looking for adequate interpretations of our approaches, e.g., in terms of energy minimization or Hankel operators. We hope to trigger some research in this direction.

APPENDIX A
ASYMPTOTIC MEAN SQUARE STABILITY

Consider the stochastic linear system of Itô-type

$$dx = Ax\,dt + Nx\,dw \tag{25}$$

where $w = (w(t))_{t\in\mathbb{R}_+}$ is a zero-mean real Wiener process on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ (e.g., [25], [26]).

Let $L^2_w(\mathbb{R}_+, \mathbb{R}^q)$ denote the corresponding space of nonanticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|^2_{L^2_w} := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2\,dt\right) < \infty$$

where $\mathcal{E}$ denotes expectation. For initial data $x(0) = x_0$, the solution can be written as $x(t) = \Phi(t)x_0$ with the fundamental matrix solution $\Phi(t)$, satisfying $\Phi(0) = I$

By definition, system (25) is asymptotically mean-square-stable, if $\mathcal{E}(\|x(t)\|^2) \overset{t\to\infty}{\longrightarrow} 0$, for all initial conditions x_0. In this case, for simplicity, we also call the pair (A, N) asymptotically mean-square stable.

We have the following version of Lyapunov's matrix theorem; see [23]. Here $\otimes$ denotes the Kronecker product.

Theorem A.1: The following are equivalent.

(i) System (25) is asymptotically mean-square stable.
(ii) $\max\{\Re\lambda \parallel \lambda \in \sigma(A\otimes I + I\otimes A + N\otimes N)\} < 0$
(iii) $\exists Y > 0 : \exists X > 0$: $A^TX + XA + N^TXN = -Y$
(iv) $\forall Y > 0 : \exists X > 0$: $A^TX + XA + N^TXN = -Y$
(v) $\forall Y \geq 0 : \exists X \geq 0$: $A^TX + XA + N^TXN = -Y$

Remark A.2: The theorem (like all other results in this paper) carries over to systems

$$dx = Ax\,dt + \sum_{j=1}^k N_j x\,dw_j$$

with more than one noise term, and many more equivalent criteria can be provided; see, e.g., [34] or [18, Th. 3.6.1].

The following theorem does not require any stability assumptions (see [18, Th. 3.2.3]). It is central in the analysis of mean-square stability.

Theorem A.3: Let

$$\alpha = \max\left\{\Re\lambda |\ \lambda \in \sigma(A\otimes I + I\otimes A + N\otimes N)\right\}.$$

Then there exists a nonnegative definite matrix $V \neq 0$, so that

$$(\mathcal{L}^*_A + \Pi^*_N)(V) = AV + VA^T + NVN^T = \alpha V.$$

We also note a simple consequence of this theorem [24, Cor. 3.2]. Here $\langle Y, V\rangle = \text{trace}(YV)$ is the Frobenius inner product for symmetric matrices.

Corollary A.4: Let α, V as in the theorem. For given $Y \geq 0$ assume that

$$\exists X > 0 :\ \mathcal{L}_A(X) + \Pi_N(X) \leq -Y. \tag{26}$$

Then $\alpha \leq 0$. Moreover, if $\alpha = 0$ then $YV = VY = 0$.

APPENDIX B
STOCHASTIC BOUNDED REAL LEMMA

Now let us consider system (5) with input u and output y. If (A, N) is asymptotically mean-square stable, then (5) defines an input output operator $\mathbb{L} : u \mapsto y$ from $L^2_w(\mathbb{R}, \mathbb{R}^m)$ to $L^2_w(\mathbb{R}, \mathbb{R}^p)$, see [17]. By $\|\mathbb{L}\|$ we denote the induced operator norm, which is an analogue of the deterministic H^∞-norm. It can be characterized by the stochastic bounded real lemma.

Theorem A.5: [17] For $\gamma > 0$, the following are equivalent.

(i) System (25) is asymptotically mean-square stable and $\|\mathbb{L}\| < \gamma$.

(ii) There exists a negative definite solution $X < 0$ to the Riccati inequality

$$A^T X + XA + N^T XN - C^T C - \gamma^{-2} XBB^T X > 0.$$

(iii) There exists a positive definite solution $X > 0$ to the Riccati inequality

$$A^T X + XA + N^T XN + C^T C + \gamma^{-2} XBB^T X < 0.$$

We have stated the obviously equivalent formulations (ii) and (iii) to avoid confusion arising from different formulations in the literature. Under additional assumptions also nonstrict versions can be formulated. The following sufficient criterion is given in [18, Cor. 2.2.3] (where also the signs are changed). Unlike in the previous theorem, here asymptotic mean-square stability is assumed at the outset.

Theorem A.6: Assume that (25) is asymptotically stable in mean-square. If there exists a nonnegative definite matrix $X \geq 0$, satisfying

$$A^T X + XA + N^T XN + C^T C + \gamma^{-2} XBB^T X \leq 0$$

then $\|\mathbb{L}\| \leq \gamma$.

Appendix C
Unobservable and Unreachable Subspaces

Definition A.7: Consider system (5). A vector $v \in \mathbb{R}^n$ is called *unobservable*, if the initial condition $x(0) = v$ with $u \equiv 0$ produces the output $y \equiv 0$. The vector v is called *unreachable*, if $x(t) \neq v$ for all $t > 0$ and any solution with initial value $x(0) = 0$ and arbitrary input u.

If (A, N) is asymptotically mean-square stable, then (see [14, Th. 3.1]) the unobservable and the unreachable subspace can be characterized as the kernels of Q and P defined by

$$A^T Q + QA + N^T QN = -C^T C$$
$$AP + PA^T + NPN^T = -BB^T.$$

Theorem A.8: A state v is

(a) unobservable, if and only if $Qv = 0$.

(b) unreachable, if and only if $Pv = 0$.

In particular, the system is observable and reachable, if and only if $Q > 0$ and $P > 0$.

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