Dual pairs of generalized Lyapunov inequalities and balanced truncation of stochastic linear systems

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***Abstract*—We consider two approaches to balanced truncation of stochastic linear systems, which follow from different generalizations of the reachability Gramian of deterministic systems. Both preserve mean-square asymptotic stability, but only the second leads to a stochastic H^∞-type bound for the approximation error of the truncated system.**

***Index Terms*—generalized Lyapunov equation, model order reduction, balanced truncation, stochastic linear system, asymptotic mean square stability**

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Introduction

Optimization and (feedback) control of dynamical systems is often computationally infeasible for high dimensional plant models. Therefore, one tries to reduce the order of the system, so that the input-output mapping is still computable with sufficient accuracy, but at considerably smaller cost than for the original system, [1], [2], [3], [4], [5]. To guarantee the desired accuracy, computable error bounds are required. Moreover, system properties which are relevant in the context of control system design like asymptotic stability need to be preserved. It has long been known that for linear time-invariant (LTI) systems the method of balanced truncation preserves asymptotic stability and provides an error bound for the L^2-induced input-output norm, that is the H^∞-norm of the associated transfer function, see [6], [7]. When considering model order reduction of more general system classes, it is natural to try to extend this approach. This has been worked out for descriptor systems in [8], for time-varying systems in [9], [10], [11], for bilinear systems in [12], [13], [14] and general nonlinear systems e.g. in [15]. Yet another generaliztion of LTI systems is obtained considering dynamics driven by noise processes. This leads to the class of stochastic systems, which have been considered in a system theoretic context e.g. in [16], [17], [18]. Quite recently, balanced truncation has also been described for linear stochastic systems of Itô type in [14], [19], [20]. Already the formulation of the method leads to two different variants that are equivalent in the deterministic case, but not so for stochastic systems. It is natural to ask which of the above mentioned properties of balanced truncation also hold for these variants. The aim of this paper is to answer this question.

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Let us first recapitulate balanced truncation for linear deterministic control systems of the form

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad \sigma(A) \subset \mathbb{C}_- \,. \tag{1}$$

Here $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$ are the state, output, and input of the system, respectively. Moreover $\sigma(A)$ denotes the spectrum of A and $\mathbb{C}_-$ the open left half complex plane. Let

$$\mathcal{L}_A : X \mapsto A^T X + XA$$

denote the Lyapunov operator and

$$\mathcal{L}_A^* : X \mapsto AX + XA^T$$

its adjoint with respect to the Frobenius inner product. Then $\sigma(A) \subset \mathbb{C}_-$ if and only if there exists a positive definite solution X of the Lyapunov inequality $\mathcal{L}_A(X) < 0$, by Lyapunov's classical stability theorem, see e.g. [21].

Balanced truncation means truncating a balanced realization. This realization is obtained by a state space transformation computed from the Gramians P and Q, which solve the dual pair of *Lyapunov equations*

$$\mathcal{L}_A(Q) = A^T Q + QA = -C^T C \,, \tag{2a}$$

$$\mathcal{L}_A^*(P) = AP + PA^T = -BB^T \,, \tag{2b}$$

or more generally the *inequalities*

$$\mathcal{L}_A(Q) \le -C^T C \,, \quad \mathcal{L}_A^*(P) \le -BB^T \,. \tag{3}$$

These (in)equalities are essential in the characterization of stability, controllability and observability of system (1). If $\det P \neq 0$, the inequalities (3) can be written as

$$\mathcal{L}_A(Q) \le -C^T C \,, \tag{4a}$$

$$\mathcal{L}_A(P^{-1}) = P^{-1}A + A^T P^{-1} \le -P^{-1}BB^T P^{-1} \,. \tag{4b}$$

In the present paper we discuss extensions of (3) and (4) for stochastic linear systems.

As indicated above, the equivalent formulations (3) and (4) lead to different generalizations, if we consider Itô-type stochastic systems of the form

$$dx = Ax\,dt + Nx\,dw + Bu\,dt \,, \quad y = Cx \,, \tag{5}$$

where A, B, C are as in (1) and $N \in \mathbb{R}^{n\times n}$. System (5) is asymptotically mean-square stable (e.g. [22], [23], [18]), if and only if there exists a positive definite solution X of the generalized Lyapunov inequality

$$(\mathcal{L}_A + \Pi_N)(X) = A^T X + XA + N^T XN < 0 \,.$$

Here $\Pi_N : X \mapsto N^T XN$ and $\Pi_N^* : X \mapsto NXN^T$. This stability criterion indicates that in the stochastic context the generalized Lyapunov operator $\mathcal{L}_A + \Pi_N$ takes over the role of $\mathcal{L}_A$. Substituting $\mathcal{L}_A$ by $\mathcal{L}_A + \Pi_N$ in (3) and (4), we obtain two different dual pairs of generalized Lyapunov inequalities. We call them *type I*:

$$(\mathcal{L}_A + \Pi_N)(Q) = A^T Q + QA + N^T QN \leq -C^T C \ , \quad \text{(6a)}$$
$$(\mathcal{L}_A + \Pi_N)^*(P) = AP + PA^T + NPN^T \leq -BB^T \ , \quad \text{(6b)}$$

and *type II*:

$$\begin{aligned}(\mathcal{L}_A + \Pi_N)(Q) &= A^T Q + QA + N^T QN \\ &\leq -C^T C \ , \quad &\text{(7a)}\\ (\mathcal{L}_A + \Pi_N)(P^{-1}) &= A^T P^{-1} + P^{-1} A + N^T P^{-1} N \\ &\leq -P^{-1} BB^T P^{-1} \ . \quad &\text{(7b)}\end{aligned}$$

Note that (6) corresponds to (3) in the sense that $\mathcal{L}_A^*(P)$ has been replaced by $(\mathcal{L}_A + \Pi_N)^*(P)$, while (7) corresponds to (4), where $\mathcal{L}_A(P^{-1})$ has been replaced by $(\mathcal{L}_A + \Pi_N)(P^{-1})$. In general (if N and P do not commute), the inequalities (6b) and (7b) are not equivalent. At first glance it is not clear which generalization is more appropriate.

If the system is asymptotically mean-square stable and certain observability and reachability conditions are fulfilled, then for both types there are solutions $Q, P > 0$. By a suitable state space-transformation, it is possible to balance the system such that $Q = P = \Sigma > 0$ is diagonal. Consequently, the usual procedure of balanced truncation can be applied to reduce the order of (5). For simplicity, let us refer to this as *type I* or *type II balanced truncation*.

Under natural assumptions, this reduction preserves mean-square asymptotic stability. For type I, this nontrivial fact has been proven in [24]. Moreover, in [20], an H^2-error bound has been provided. However, different from the deterministic case, there is no H^∞-type error bound in terms of the truncated entries in Σ. This will be shown in Example I.3.

In contrast, for type II, an H^∞-type error bound has been obtained in [19]. In the present paper, as one of our main contributions, we show in Theorem II.2 that type II balanced truncation also preserves mean-square asymptotic stability. The proof differs significantly from the one given for type I. Using this result, we are able to give a more compact proof of the error bound, Theorem II.4, which exploits the stochastic bounded real lemma [17].

We illustrate our results by analytical and numerical examples in Section IV.

I. Type I balanced truncation

Consider a stochastic linear control system of Itô-type

$$dx = Ax\,dt + \sum_{j=1}^{k} N_j x\,dw_j + Bu\,dt \ , \quad y = Cx \ , \quad (8)$$

where $w_j = (w_j(t))_{t\in\mathbb{R}_+}$ are uncorrelated zero mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ (e.g. [25], [26]).

To simplify the notation, we only consider the case $k = 1$ and set $w = w_1$, $N = N_1$. But all results can immediately be generalized for $k > 1$.

Let $L^2_w(\mathbb{R}_+, \mathbb{R}^q)$ denote the corresponding space of non-anticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|^2_{L^2_w} := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2 dt\right) < \infty,$$

where $\mathcal{E}$ denotes expectation.

Let the homogeneous equation $dx = Ax\,dt + Nx\,dw$ be asymptotically mean-square-stable, i.e. $\mathcal{E}(\|x(t)\|^2) \stackrel{t\to\infty}{\longrightarrow} 0$, for all solutions x.

Then, by Theorem A.1 the equations

$$A^T Q + QA + N^T QN = -C^T C \ ,$$
$$AP + PA^T + NPN^T = -BB^T \ ,$$

have unique solutions $Q \geq 0$ and $P \geq 0$. Under suitable observability and controllability conditions, Q and P are nonsingular.

A similarity transformation

$$(A, N, B, C) \mapsto (S^{-1}AS, S^{-1}NS, S^{-1}B, CS)$$

of the system implies the contragredient transformation as

$$(Q, P) \mapsto (S^T QS, S^{-1}PS^{-T}) \ .$$

Choosing e.g. $S = LV\Sigma^{-1/2}$, with Cholesky factorizations $LL^T = P$, $R^T R = Q$ and a singular value decomposition $RL = U\Sigma V^T$, we obtain $S^{-1} = \Sigma^{-1/2} U^T R$ and

$$S^T QS = S^{-1}PS^{-T} = \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) \ .$$

After suitable partitioning

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \ S = \begin{bmatrix} S_1 & S_2 \end{bmatrix}, \ S^{-1} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

a truncated system is given in the form

$$(A_{11}, N_{11}, B_1, C_1) = (T_1 AS_1, T_1 NS_1, T_1 B, CS_1) \ .$$

The following result has been proven in [24].

Theorem I.1 *Let $A, N \in \mathbb{R}^{n\times n}$ satisfy*

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_- \ .$$

For a block-diagonal matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, assume that

$$A^T\Sigma + \Sigma A + N^T \Sigma N \leq 0 \textit{ and } A\Sigma + \Sigma A^T + N\Sigma N^T \leq 0.$$

Then, with the usual partitioning of A and N, we have

$$\sigma(I \otimes A_{11} + A_{11} \otimes I + N_{11} \otimes N_{11}) \subset \mathbb{C}_- \ .$$

Its implication for mean-square stability of the truncated system is immediate.

Corollary I.2 *Consider an asymptotically mean square stable stochastic linear system*

$$dx = Ax\,dt + Nx\,dw \ .$$

Assume that a matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$ *is given as in Theorem I.1 and* A *and* N *are partitioned accordingly.*
Then the truncated system

$$dx_r = A_{11}x_r\,dt + N_{11}x_r\,dw$$

is also asymptotically mean square stable.

If the diagonal entries of Σ_2 are small, it is expected that the truncation error is small. In fact this is supported by an H^2-error bound obtained in [20]. Additionally, however, from the deterministic situation (see [6], [2]), one would also hope for an H^∞-type error bound of the form

$$\|y - y_r\|_{L^2_w(\mathbb{R}_+,\mathbb{R}^p)} \overset{?}{\leq} \alpha(\operatorname{trace}\Sigma_2)\|u\|_{L^2_w(\mathbb{R}_+,\mathbb{R}^m)} \tag{9}$$

with some number $\alpha > 0$. The following example shows that no such general α exists.

Example I.3 Let $A = -\begin{bmatrix} 1 & 0 \\ 0 & a^2 \end{bmatrix}$ with $a > 1$, $N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.
Solving (6) with equality, we get $P = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4a^2} \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{4a^2} & 0 \\ 0 & \frac{1}{2a^2} \end{bmatrix}$ with $\sigma(PQ) = \{\frac{1}{8a^2}, \frac{1}{8a^4}\}$ so that $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2)$, where $\sigma_1 = \frac{1}{\sqrt{8}a}$ and $\sigma_2 = \frac{1}{\sqrt{8}a^2}$. The system is balanced by the transformation $S = \begin{bmatrix} 2a^2 & 0 \\ 0 & 1/2 \end{bmatrix}^{1/4}$.
Then $CS = \frac{1}{2^{1/4}}\begin{bmatrix} 0 & 1 \end{bmatrix}$ so that $C_r = 0$ for the truncated system of order 1. Thus the output of the reduced system is $y_r \equiv 0$, and the truncation error $\|\mathbb{L} - \mathbb{L}_r\|$ is equal to the stochastic H^∞-norm (see [17]) of the original system,

$$\|\mathbb{L}\| = \sup_{x(0)=0, \|u\|_{L^2_w}=1} \|y\|_{L^2_w} \ .$$

We show now that this norm is equal to $\frac{1}{\sqrt{2}a} = 2a\sigma_2$. Thus, depending on a, the ratio of the truncation error and $\operatorname{trace}\Sigma_2 = \sigma_2$ can be arbitrarily large.

According to the stochastic bounded real lemma, Theorem A.5, $\|\mathbb{L}\|$ is the infimum over all γ so that the Riccati inequality

$$\begin{aligned} 0 &< A^TX + XA + N^TXN - C^TC - \frac{1}{\gamma^2}XBB^TX \qquad (10)\\ &= \begin{bmatrix} -2x_1 + x_3 - \frac{1}{\gamma^2}x_1^2 & -(a^2+1)x_2 - \frac{1}{\gamma^2}x_1x_2 \\ -(a^2+1)x_2 - \frac{1}{\gamma^2}x_1x_2 & -2a^2x_3 - \frac{1}{\gamma^2}x_2^2 - 1 \end{bmatrix} \end{aligned}$$

possesses a solution $X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} < 0$.

If a given matrix X satisfies this condition, then so does the same matrix with x_2 replaced by 0. Hence we can assume that $x_2 = 0$, and end up with the two conditions $x_3 < -\frac{1}{2a^2}$ and (after multiplying the upper left entry with $-\gamma^2$)

$$\begin{aligned} 0 &> x_1^2 + 2\gamma^2x_1 - \gamma^2x_3 = (x_1 + \gamma^2)^2 - \gamma^2(\gamma^2 + x_3)\\ &> (x_1+\gamma^2)^2 - \gamma^2(\gamma^2 - \tfrac{1}{2a^2}) \ . \end{aligned}$$

Thus necessarily $\gamma^2 > \frac{1}{2a^2}$, i.e. $\gamma > \frac{1}{\sqrt{2}a}$. This already proves that $\|\mathbb{L}\| \geq \frac{1}{\sqrt{2}a} = 2a\sigma_2$, which suffices to disprove the existence of a general bound α in (9). Taking infima, it is easy to show that indeed $\|\mathbb{L}\| = \frac{1}{\sqrt{2}a}$.

II. Type II balanced truncation

We now consider the inequalities (7).

Lemma II.1 *Assume that* $dx = Ax\,dt + Nx\,dw$ *is asymptotically mean-square-stable. Then inequality* (7b) *is solvable with* $P > 0$*.*

Proof: By Theorem A.1, for a given $Y < 0$, there exists a $\tilde{P} > 0$, so that $A^T\tilde{P}^{-1} + \tilde{P}^{-1}A + N^T\tilde{P}^{-1}N = Y$. Then $P = \varepsilon^{-1}\tilde{P}$, for sufficiently small $\varepsilon > 0$, satisfies

$$A^TP^{-1} + P^{-1}A + N^TP^{-1}N = \varepsilon Y < -\varepsilon^2\tilde{P}^{-1}BB^T\tilde{P}^{-1}$$

so that (7b) holds even in the strict form. □

It is easy to see that like in the previous section a state space transformation

$$(A, N, B, C) \mapsto (S^{-1}AS, S^{-1}NS, S^{-1}B, CS)$$

leads to a contragredient transformation $Q \mapsto S^TQS$, $P \mapsto S^{-1}PS^{-T}$ of the solutions. That is, Q and P satisfy (7a) and (7b), if and only if S^TQS and $S^{-1}PS^{-T}$ do so for the transformed data. As before, we can assume the system to be balanced with

$$Q = P = \Sigma = \operatorname{diag}(\sigma_1 I, \ldots, \sigma_\nu I) = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}, \tag{11}$$

where $\sigma_1 > \ldots > \sigma_\nu > 0$ and $\sigma(\Sigma_1) = \{\sigma_1, \ldots, \sigma_r\}$, $\sigma(\Sigma_2) = \{\sigma_{r+1}, \ldots, \sigma_\nu\}$. Hence, we will now assume (after balancing) that a diagonal matrix Σ as in (11) is given which satisfies

$$A^T\Sigma + \Sigma A + N^T\Sigma N \leq -C^TC \ , \tag{12a}$$

$$A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N \leq -\Sigma^{-1}BB^T\Sigma^{-1} \ . \tag{12b}$$

Partitioning A, N, B, C like Σ, we write the system as

$$\begin{aligned} dx_1 &= (A_{11}x_1 + A_{12}x_2)\,dt + (N_{11}x_1 + N_{12}x_2)\,dw + B_1u\,dt\\ dx_2 &= (A_{21}x_1 + A_{22}x_2)\,dt + (N_{21}x_1 + N_{22}x_2)\,dw + B_2u\,dt\\ y &= C_1x_1 + C_2x_2 \ . \end{aligned}$$

The reduced system obtained by truncation is

$$dx_r = A_{11}x_r + N_{11}x_r\,dw + B_1u\,dt \ , \quad y_r = C_1x_r \ .$$

The index r is the number of different singular values σ_j that have been kept in the reduced system. In the following subsections, we consider matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},$$

$\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$ as in (11), and equations of the form

$$A^T\Sigma + \Sigma A + N^T\Sigma N = -\tilde{C}^T\tilde{C} \tag{13a}$$

$$A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N = -\tilde{B}\tilde{B}^T \tag{13b}$$

with arbitrary right-hand sides $-\tilde{C}^T\tilde{C} \leq 0$ and $-\tilde{B}\tilde{B}^T \leq 0$.

For convenience, we write out the blocks of these equations explicitly:

$$A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + N_{11}^T\Sigma_1 N_{11} = -N_{21}^T\Sigma_2 N_{21} - \tilde{C}_1^T\tilde{C}_1 \quad (14)$$

$$A_{12}^T\Sigma_1 + \Sigma_2 A_{21} + N_{12}^T\Sigma_1 N_{11} = -N_{22}^T\Sigma_2 N_{21} - \tilde{C}_2^T\tilde{C}_1 \quad (15)$$

$$A_{22}^T\Sigma_2 + \Sigma_2 A_{22} + N_{22}^T\Sigma_2 N_{22} = -N_{12}^T\Sigma_1 N_{12} - \tilde{C}_2^T\tilde{C}_2 \quad (16)$$

$$A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1} A_{11} + N_{11}^T\Sigma_1^{-1} N_{11} = -N_{21}^T\Sigma_2^{-1} N_{21} - \tilde{B}_1\tilde{B}_1^T \quad (17)$$

$$A_{12}^T\Sigma_1^{-1} + \Sigma_2^{-1} A_{21} + N_{12}^T\Sigma_1^{-1} N_{11} = -N_{22}^T\Sigma_2^{-1} N_{21} - \tilde{B}_2\tilde{B}_1^T \quad (18)$$

$$A_{22}^T\Sigma_2^{-1} + \Sigma_2^{-1} A_{22} + N_{22}^T\Sigma_2^{-1} N_{22} = -N_{12}^T\Sigma_1^{-1} N_{12} - \tilde{B}_2\tilde{B}_2^T \quad (19)$$

A. Preservation of asymptotic stability

The following theorem is the main new result of this paper.

Theorem II.2 *Let A and N be given such that*

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_- \,. \quad (20)$$

Assume further that for a block-diagonal matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2) > 0$ with $\sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$, we have

$$A^T\Sigma + \Sigma A + N^T\Sigma N \leq 0 \quad \text{and} \quad (21a)$$

$$A^T\Sigma^{-1} + \Sigma^{-1} A + N^T\Sigma^{-1} N \leq 0 \,. \quad (21b)$$

Then, with the usual partitioning of A and N, we have

$$\sigma(I \otimes A_{11} + A_{11} \otimes I + N_{11} \otimes N_{11}) \subset \mathbb{C}_- \,. \quad (22)$$

Again we have an immediate interpretation in terms of mean-square stability of the truncated system.

Corollary II.3 *Consider an asymptotically mean square stable stochastic linear system*

$$dx = Ax\,dt + Nx\,dw \,.$$

Assume that a matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$ is given as in Theorem II.2 and A and N are partitioned accordingly. Then the truncated system

$$dx_r = A_{11}x_r\,dt + N_{11}x_r\,dw$$

is also asymptotically mean square stable.

Proof of Theorem II.2: Note that the inequalities (21) are equivalent to the equations (14) – (19) with appropriate right-hand sides $-\tilde{C}^T\tilde{C}$ and $-\tilde{B}\tilde{B}^T$.
By way of contradiction, we assume that (22) does not hold. Then by Theorem A.3, there exist $V \geq 0$, $V \neq 0$, $\alpha \geq 0$ such that

$$A_{11}V + VA_{11}^T + N_{11}VN_{11}^T = \alpha V \,. \quad (23)$$

Taking the scalar product of the equation (14) with V, we obtain $0 \geq \alpha \operatorname{trace}(\Sigma_1 V)$ whence $\alpha = 0$ and $\tilde{C}_1 V = 0$, $N_{21}V = 0$ by Corollary A.4. Hence

$$\left(A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + N_{11}^T\Sigma_1 N_{11}\right) V = 0 \,. \quad (24)$$

Analogously, we have $\tilde{B}_1^T V = 0$ by (15).
In particular, from $N_{21}V = 0$, we get

$$\begin{aligned}
&(\mathcal{L}_A^* + \Pi_N^*)\left(\begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} A_{11}V + VA_{11}^T + N_{11}VN_{11}^T & VA_{21}^T + N_{11}VN_{21}^T \\ A_{21}V + N_{21}VN_{11}^T & N_{21}VN_{21}^T \end{bmatrix} \\
&= \begin{bmatrix} 0 & VA_{21}^T \\ A_{21}V & 0 \end{bmatrix} .
\end{aligned}$$

We will show that $A_{21}V = 0$, which implies

$$0 \in \sigma(I \otimes A + A \otimes I + N \otimes N) \quad (25)$$

in contradiction to (20), and thus finishes the proof.

We first show that $\operatorname{Im} V$ is invariant under A_{11} and N_{11}. To this end let $Vz = 0$. Then by (23),

$$0 = z^T\left(A_{11}V + VA_{11}^T + N_{11}VN_{11}^T\right)z = z^T N_{11}VN_{11}^T z \,,$$

whence also $VN_{11}^T z = 0$, i.e. $N_{11}^T z \in \operatorname{Ker} V$. From this, we have

$$0 = \left(A_{11}V + VA_{11}^T + N_{11}VN_{11}^T\right)z = VA_{11}^T z \,,$$

implying $A_{11}^T z \in \operatorname{Ker} V$. Thus $A_{11}^T \operatorname{Ker} V \subset \operatorname{Ker} V$ and $N_{11}^T \operatorname{Ker} V \subset \operatorname{Ker} V$.

Since $\operatorname{Ker} V = (\operatorname{Im} V)^\perp$, it follows further that $\operatorname{Im} V$ is invariant under A_{11} and N_{11}.

Let $V = V_1V_1^T$, where V_1 has full column rank, i.e. $\det V_1^T V_1 \neq 0$. Then by the invariance, there exist square matrices X and Y, such that

$$A_{11}V_1 = V_1X \quad \text{and} \quad N_{11}V_1 = V_1Y \,.$$

It follows that

$$\begin{aligned}
0 &= A_{11}V_1V_1^T + V_1V_1^T A_{11}^T + N_{11}V_1V_1^T N_{11}^T \\
&= V_1(X + X^T + YY^T)V_1^T \,,
\end{aligned}$$

whence $X + X^T + YY^T = 0$. Moreover, from (24), we get

$$\begin{aligned}
A_{11}^T\Sigma_1 V_1 &= -\Sigma_1 A_{11}V_1 - N_{11}^T\Sigma_1 N_{11}V_1 \\
&= -\Sigma_1 V_1 X - N_{11}^T\Sigma_1 V_1 Y \,. \quad (26)
\end{aligned}$$

Using this substitution in the following computation, we obtain

$$\begin{aligned}
0 &\geq V_1^T\Sigma_1^2\left(A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11}\right)\Sigma_1^2 V_1 \\
&= V_1^T\Sigma_1^2(A_{11}^T\Sigma_1 V_1) + (A_{11}^T\Sigma_1 V_1)^T\Sigma_1^2 V_1 \\
&\quad + V_1^T\Sigma_1^2 N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2 V_1 \\
&= -V_1^T\Sigma_1^3 V_1 X - X^T V_1^T\Sigma_1^3 V_1 \quad (27) \\
&\quad - V_1^T\Sigma_1^2 N_{11}^T\Sigma_1 V_1 Y - Y^T V_1^T\Sigma_1 N_{11}\Sigma_1^2 V_1 \\
&\quad + V_1^T\Sigma_1^2 N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2 V_1 \,.
\end{aligned}$$

We will show that the right hand side has nonnegative trace. This then implies that the whole term vanishes. Note that

$$\begin{aligned}\operatorname{trace}(Y^TV_1^T\Sigma_1^3V_1Y) &= \operatorname{trace}(V_1^T\Sigma_1^3V_1YY^T)\\ &= \operatorname{trace}(-V_1^T\Sigma_1^3V_1(X+X^T))\\ &= \operatorname{trace}(-V_1^T\Sigma_1^3V_1X - X^TV_1^T\Sigma_1^3V_1)\ .\end{aligned}$$

Taking the trace in (27), we have

$$\begin{aligned}0 &\geq \operatorname{trace}\Big(Y^TV_1^T\Sigma_1^3V_1Y - V_1^T\Sigma_1^2N_{11}^T\Sigma_1V_1Y\\ &\qquad - Y^TV_1^T\Sigma_1N_{11}\Sigma_1^2V_1 + V_1^T\Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2V_1\Big)\\ &= \operatorname{trace}\begin{bmatrix} V_1Y\\ V_1\end{bmatrix}^T M \begin{bmatrix} V_1Y\\ V_1\end{bmatrix}\ .\end{aligned}$$

where

$$M = \begin{bmatrix} \Sigma_1^3 & -\Sigma_1N_{11}\Sigma_1^2\\ -\Sigma_1^2N_{11}^T\Sigma_1 & \Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2\end{bmatrix}\ .$$

The matrix M is positive semidefinite, because the upper left block is positive definite, and the corresponding Schur complement

$$\Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2 - \Sigma_1^2N_{11}^T\Sigma_1\Sigma_1^{-3}\Sigma_1N_{11}\Sigma_1^2 = 0$$

vanishes. Hence

$$\begin{bmatrix} \Sigma_1^3 & -\Sigma_1N_{11}\Sigma_1^2\\ -\Sigma_1^2N_{11}^T\Sigma_1 & \Sigma_1^2N_{11}^T\Sigma_1^{-1}N_{11}\Sigma_1^2\end{bmatrix}\begin{bmatrix} V_1Y\\ V_1\end{bmatrix} = 0$$

implying via the first block row that $N_{11}\Sigma_1^2V_1 = \Sigma_1^2V_1Y$. From (27), using also (26) again, we thus have

$$\begin{aligned}0 &= \left(A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11}\right)\Sigma_1^2V_1\\ &= -\Sigma_1V_1X - N_{11}^T\Sigma_1V_1Y + \Sigma_1^{-1}A_{11}\Sigma_1^2V_1 + N_{11}^T\Sigma_1V_1Y\\ &= -\Sigma_1V_1X + \Sigma_1^{-1}A_{11}\Sigma_1^2V_1\ ,\end{aligned}$$

i.e. $A_{11}\Sigma_1^2V_1 = \Sigma_1^2V_1X$. It follows that for arbitrary $k\in\mathbb{N}$, the eigenvector V in (23) can be replaced by

$$\Sigma_1^{2k}V\Sigma_1^{2k} = \Sigma_1^{2k}V_1V_1^T\Sigma_1^{2k}$$

because

$$\begin{aligned}0 &= \Sigma_1^2V_1\left(X+X^T+YY^T\right)V_1^T\Sigma_1^2\\ &= A_{11}\left(\Sigma_1^2V_1V_1^T\Sigma_1^2\right) + \left(\Sigma_1^2V_1V_1^T\Sigma_1^2\right)A_{11}^T\\ &\quad + N_{11}\left(\Sigma_1^2V_1V_1^T\Sigma_1^2\right)N_{11}^T\ .\end{aligned}$$

Induction leads to

$$\begin{aligned}0 &= A_{11}\left(\Sigma_1^{2k}V_1V_1^T\Sigma_1^{2k}\right) + \left(\Sigma_1^{2k}V_1V_1^T\Sigma_1^{2k}\right)A_{11}^T\\ &\quad + N_{11}\left(\Sigma_1^{2k}V_1V_1^T\Sigma_1^{2k}\right)N_{11}^T\ .\end{aligned}$$

As above, we conclude that $N_{21}\Sigma_1^{2k}V_1 = 0$, $\tilde{C}_1\Sigma_1^{2k}V_1 = 0$, and $\tilde{B}_1^T\Sigma_1^{2k}V_1 = 0$. Multiplying (15) with $\Sigma_1^{2(k-1)}V_1$ and (18) with $\Sigma_1^{2k}V_1$, we get

$$\begin{aligned}A_{12}^T\Sigma_1^{2k-1}V_1 + \Sigma_2A_{21}\Sigma_1^{2(k-1)}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1Y &= 0\ ,\\ A_{12}^T\Sigma_1^{2k-1}V_1 + \Sigma_2^{-1}A_{21}\Sigma_1^{2k}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1Y &= 0\ .\end{aligned}$$

Hence (after multiplication with Σ_2), for all $k\geq 1$, we have

$$\begin{aligned}\Sigma_2^2A_{21}\Sigma_1^{2(k-1)}V_1 &= -\Sigma_2\left(A_{12}^T\Sigma_1^{2k-1}V_1 + N_{12}^T\Sigma_1^{2k-1}V_1Y\right)\\ &= A_{21}\Sigma_1^{2k}V_1\ .\end{aligned}$$

Applying this identity repeatedly, we get

$$A_{21}\Sigma_1^{2k}V_1 = \Sigma_2^{2k}A_{21}V_1 \quad \text{for all } k\in\mathbb{N}.$$

If μ is the minimal polynomial of Σ_1^2, then $\sigma(\Sigma_1)\cap\sigma(\Sigma_2)=\emptyset$ implies $\det\mu(\Sigma_2^2)\neq 0$ and

$$0 = A_{21}\mu(\Sigma_1^2)V_1 = \mu(\Sigma_2^2)A_{21}V_1\ ,$$

whence $A_{21}V_1 = 0$ and also $A_{21}V = 0$. Hence we obtain the contradiction (25). □

B. Error estimate

The following theorem has been proven in [19] using LMI-techniques. Exploiting the stability result in the previous subsection, we can give a slightly more compact proof based on the stochastic bounded real lemma, Theorem A.6.

Theorem II.4 *Let A and N satisfy*

$$\sigma(I\otimes A + A\otimes I + N\otimes N)\subset\mathbb{C}_-\ .$$

Assume furthermore that for $\Sigma = \operatorname{diag}(\Sigma_1,\Sigma_2) > 0$ with $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}I,\ldots,\sigma_\nu I)$ and $\sigma(\Sigma_1)\cap\sigma(\Sigma_2)=\emptyset$, the following Lyapunov inequalities hold,

$$\begin{aligned}A^T\Sigma + \Sigma A + N^T\Sigma N &\leq -C^TC\ ,\\ A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N &\leq -\Sigma^{-1}BB^T\Sigma^{-1}\ .\end{aligned}$$

If $x(0)=0$ and $x_r(0)=0$, then for all $T>0$, it holds that

$$\|y-y_r\|_{L^2_w([0,T])} \leq 2(\sigma_{r+1}+\ldots+\sigma_\nu)\|u\|_{L^2_w([0,T])}\ .$$

Proof: We adapt a proof for deterministic systems e.g. [2, Theorem 7.9]. In the central argument we treat the case where $\Sigma_2 = \sigma_\nu I$ and show that

$$\|y-y_{\nu-1}\|_{L^2_w[0,T]} \leq 2\sigma_\nu\|u\|_{L^2_w[0,T]}\ . \tag{28}$$

From (14) and (17), we can see that also

$$\begin{aligned}A_{11}^T\Sigma_1 + \Sigma_1A_{11} + N_{11}^T\Sigma_1N_{11} &\leq -C_1^TC_1\ ,\\ A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11} &\leq -\Sigma_1^{-1}B_1B_1^T\Sigma_1^{-1}\ .\end{aligned}$$

Hence we can repeat the above argument to remove $\sigma_{\nu-1},\ldots,\sigma_{r+1}$ successively. By the triangle inequality we find that

$$\begin{aligned}\|y-y_r\|_{L^2_w[0,T]} &\leq \sum_{j=r}^{\nu-1}\|y_{j+1}-y_j\|_{L^2_w[0,T]}\\ &\leq 2(\sigma_\nu+\ldots+\sigma_{r+1})\|u\|_{L^2_w[0,T]}\ .\end{aligned}$$

which then concludes the proof.

To prove (28), we make use of the stochastic bounded real lemma. In the following let $r=\nu-1$ and consider the error system defined by

$$\begin{aligned}dx_e &= A_ex_e\,dt + N_ex_e\,dw + B_eu\,dt\ ,\\ y_e &= C_ex_e = y - y_r\ ,\end{aligned}$$

where

$$x_e = \begin{bmatrix} x_1 \\ x_2 \\ x_r \end{bmatrix}, \quad A_e = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{11} \end{bmatrix},$$
$$N_e = \begin{bmatrix} N_{11} & N_{12} & 0 \\ N_{21} & N_{22} & 0 \\ 0 & 0 & N_{11} \end{bmatrix}, \quad B_e = \begin{bmatrix} B_1 \\ B_2 \\ B_1 \end{bmatrix},$$
$$C_e = \begin{bmatrix} C_1 & C_2 & -C_1 \end{bmatrix}.$$

Applying the state space transformation

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_r \end{bmatrix} = \begin{bmatrix} x_1 - x_r \\ x_2 \\ x_1 + x_r \end{bmatrix} = \underbrace{\begin{bmatrix} I_r & 0 & -I_r \\ 0 & I_{n-r} & 0 \\ I_r & 0 & I_r \end{bmatrix}}_{=S^{-1}} \begin{bmatrix} x_1 \\ x_2 \\ x_r \end{bmatrix},$$

we obtain the transformed system

$$\tilde{A}_e = S^{-1} A_e S = \begin{bmatrix} A_{11} & A_{12} & 0 \\ \frac{1}{2}A_{21} & A_{22} & \frac{1}{2}A_{21} \\ 0 & A_{12} & A_{11} \end{bmatrix},$$
$$\tilde{N}_e = S^{-1} N_e S = \begin{bmatrix} N_{11} & N_{12} & 0 \\ \frac{1}{2}N_{21} & N_{22} & \frac{1}{2}N_{21} \\ 0 & N_{12} & N_{11} \end{bmatrix},$$
$$\tilde{B}_e = S^{-1} B \begin{bmatrix} 0 \\ B_2 \\ 2B_1 \end{bmatrix}, \quad \tilde{C}_e = C_e S = \begin{bmatrix} C_1 & C_2 & 0 \end{bmatrix}.$$

By Theorem A.6, we have $\|\mathbb{L}_e\| \leq 2\sigma_\nu$, if the Riccati inequality

$$\mathcal{R}_\gamma(X) = \tilde{A}_e^T X + X\tilde{A}_e + \tilde{N}_e^T X \tilde{N}_e + \tilde{C}_e^T \tilde{C}_e + \frac{1}{4\sigma_\nu^2} X \tilde{B}_e \tilde{B}_e^T X \leq 0 \quad (29)$$

possesses a solution $X \geq 0$. We will show now that the block-diagonal matrix

$$X = \operatorname{diag}(\Sigma_1, 2\Sigma_2, \sigma_\nu^2 \Sigma_1^{-1}) = \operatorname{diag}(\Sigma_1, 2\sigma_\nu I, \sigma_\nu^2 \Sigma_1^{-1}) > 0$$

satisfies (29). Partitioning $\mathcal{R}_{\sigma_\nu}(X) = \begin{bmatrix} R_{11} & R_{21}^T & R_{31}^T \\ R_{21} & R_{22} & R_{32}^T \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$, we have

$$\begin{aligned}
R_{11} &= A_{11}^T \Sigma_1 + \Sigma_1 A_{11} + N_{11}^T \Sigma_1 N_{11} + \frac{\sigma_\nu}{2} N_{21}^T N_{21} + C_1^T C_1 \\
&= A_{11}^T \Sigma_1 + \Sigma_1 A_{11} + N_{11}^T \Sigma_1 N_{11} + N_{21}^T \Sigma_2 N_{21} + C_1^T C_1 \\
&\quad - \frac{\sigma_\nu}{2} N_{21}^T N_{21} \\
R_{21} &= A_{12}^T \Sigma_1 + \sigma_\nu A_{21} + N_{12}^T \Sigma_1 N_{11} + \sigma_\nu N_{22}^T N_{21} + C_2^T C_1 \\
R_{31} &= \frac{\sigma_\nu}{2} N_{21}^T N_{21} \\
R_{22} &= 2\sigma_\nu (A_{22}^T + A_{22} + N_{22}^T N_{22}) + N_{12}^T \Sigma_1 N_{12} \\
&\quad + \sigma_\nu^2 N_{12}^T \Sigma_1^{-1} N_{12} + C_2^T C_2 + B_2 B_2^T \\
&= A_{22}^T \Sigma_2 + \Sigma_2 A_{22} + N_{22}^T \Sigma_2 N_{22} + N_{12}^T \Sigma_1 N_{12} + C_2^T C_2 \\
&\quad + \sigma_\nu^2 (A_{22}^T \Sigma_2^{-1} + \Sigma_2^{-1} A_{22} + N_{22}^T \Sigma_2^{-1} N_{22} \\
&\quad + N_{12}^T \Sigma_1^{-1} N_{12} + \Sigma_2^{-1} B_2 B_2^T \Sigma_2^{-1}) \\
R_{32} &= \sigma_\nu^2 (\Sigma_1^{-1} A_{12} + N_{11}^T \Sigma_1^{-1} N_{12}) + \sigma_\nu (A_{21}^T + N_{21}^T N_{22}) \\
&\quad + \sigma_\nu \Sigma_1^{-1} B_1 B_2^T \\
&= \sigma_\nu^2 (\Sigma_1^{-1} A_{12} + N_{11}^T \Sigma_1^{-1} N_{12} + A_{21}^T \Sigma_2^{-1} + N_{21}^T \Sigma_2^{-1} N_{22} \\
&\quad + \Sigma_1^{-1} B_1 B_2^T \Sigma_2^{-1}) \\
R_{33} &= \sigma_\nu^2 (A_{11}^T \Sigma_1^{-1} + \Sigma_1^{-1} A_{11} + N_{11}^T \Sigma_1^{-1} N_{11}) + \frac{\sigma_\nu}{2} N_{21}^T N_{21} \\
&\quad + \sigma_\nu^2 \Sigma_1^{-1} B_1 B_1^T \Sigma_1^{-1} \\
&= \sigma_\nu^2 (A_{11}^T \Sigma_1^{-1} + \Sigma_1^{-1} A_{11} + N_{11}^T \Sigma_1^{-1} N_{11} \\
&\quad + \Sigma_1^{-1} B_1 B_1^T \Sigma_1^{-1} + N_{21}^T \Sigma_2^{-1} N_{21}) - \frac{\sigma_\nu}{2} N_{21}^T N_{21}
\end{aligned}$$

With the permutation matrix $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ we define

$$M = J(A^T \Sigma^{-1} + \Sigma^{-1} A + N^T \Sigma^{-1} N + \Sigma^{-1} B B^T \Sigma^{-1}) J,$$

where $M \leq 0$ by (13b). Using (14) – (19), we have

$$\mathcal{R}_{\sigma_\nu}(X) = \left[\begin{array}{c|c} A^T\Sigma + \Sigma A + N^T \Sigma N + C^T C & 0 \\ \hline 0 & 0 \end{array}\right] - \frac{\sigma_\nu}{2} \begin{bmatrix} N_{21}^T \\ 0 \\ -N_{21}^T \end{bmatrix} \begin{bmatrix} N_{21}^T \\ 0 \\ -N_{21}^T \end{bmatrix}^T + \sigma_\nu^2 \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & M \end{array}\right] \leq 0,$$

which is inequality (29). □

Example II.5 Let the system (A, N, B, C) and Q be as in Example I.3. The matrix

$$P = \begin{bmatrix} 1 + \sqrt{1-p} & 0 \\ 0 & p \end{bmatrix}^{-1} > 0, \text{ where } 0 < p \leq 1,$$

satisfies inequality (7b). As in Example I.3, we have $\mathbb{L}_r = 0$ for the corresponding reduced system of order 1, so that the truncation error again is $\frac{1}{\sqrt{2a}}$, independently of $p \in]0, 1]$.

On the other hand we have

$$\sigma_2^2 = \min \sigma(PQ) = \frac{1}{4a^2(1 + \sqrt{1-p})} \leq \frac{1}{8a^2},$$

with equality for $p \to 0$. Theorem II.4 thus gives the sharp error bound $2\sigma_2 = \frac{1}{\sqrt{2a}}$. Note, that there is no $P > 0$ satisfying the *equation* (7b).

The previous example illustrates the problem of optimizing over all solutions of inequality (7b).

III. Numerical Examples

To compare the reduction methods we need to compute Q, P from (6) or (7). Instead of the inequalities (6a), (6b), (7a) we can consider the corresponding equations, for which quite efficient algorithms have been developed recently, e.g. [27], [28], [29], [30]. These also allow for a low-rank approximation of the solutions. In contrast we cannot replace (7b) by the corresponding equation, because this may not be solvable (see Example II.5). Even worse, we do not have any solvability or uniqueness criteria nor reliable algorithms.

Therefore, in general, we have to work with the inequality (7b), which is solvable according to Lemma II.1, but of course not uniquely solvable.

In view of our application, we aim at a solution P of (7b), so that (some of) the eigenvalues of PQ are particularly small, since they provide the error bound. Choosing a matrix $Y < 0$ and a very small ε along the lines of the proof of Lemma II.1 can be contrary to this aim. Hence some optimization over all solutions of (7b) is required.

Note also that a matrix $P > 0$ satisfies (7b), if and only if it satisfies the linear matrix inequality (LMI)

$$\begin{bmatrix} PA^T + AP + BB^T & PN^T \\ NP & -P \end{bmatrix} \le 0 \,. \tag{30}$$

Thus, LMI optimal solution techniques are applicable. However, their complexity will be prohibitive for large-scale problems. Therefore further research for alternative methods to solve (7b) adequately is required.

By $\mathbb{L}$ and $\mathbb{L}_r$, we always denote the original and the r-th order approximated system. The stochastic H^∞-type norm $\|\mathbb{L} - \mathbb{L}_r\|$ is computed by a binary search of the infimum of all γ such that the Riccati inequality (10) is solvable. The latter is solved via a Newton iteration as in [18]. Finally, the Lyapunov equations (2) are solved by preconditioned Krylov subspace methods described in [27].
Unfortunately, for small γ, i.e. for small approximation errors, this method of computing the error runs into numerical problems, because (10) contains the term γ^{-2}. This apparently leads to cancellation phenomena in the Newton iteration, if e.g. $\gamma < 10^{-7}$. Therefore we mainly concentrate on cases where the error is larger, that is we make r sufficiently small.

A. Type II can be better than type I

In many examples we observe that type II reduction gives a valid error bound, but the approximation error still is better with type I. This, however, is not always true, as the example

$$(A, N, B, C^T) = \left(\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right)$$

shows. It can easily be verified that the type I Lyapunov equations (6) are solved by

$$Q = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix} .$$

The type II inequalities (7) are e.g. solved by

$$Q = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 8 & 0 \\ 0 & 12 \end{bmatrix} .$$

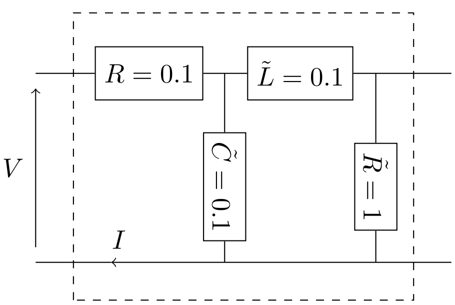
Reduction to order $r = 1$ gives the following error bounds and approximation errors for both types:

	σ_2	$\|\mathbb{L} - \mathbb{L}_1\|$
I	2.4853	3.9647
II	6.9282	3.5614

As we see, the type I approximation error is larger than both the truncated singular value and the type II approximation error.

B. An electrical ladder network with perturbed inductance

As our first example with a physical background, we take up the electrical ladder network described in [31], consisting of $n/2$ sections with a capacitor $\tilde{C}$, inductor $\tilde{L}$ and two resistors R and $\tilde{R}$ as depicted here.



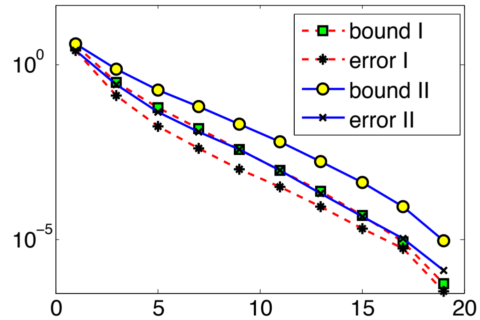
But following e.g. [32], we assume that the inductance $\tilde{L}$ is subject to stochastic perturbations. For simplicity, we replace the inverse $\tilde{L}^{-1}$ formally by $L^{-1} + \dot{w}$ in all sections. Here $L = 0.1$ and $\dot{w}$ is white noise of a certain intensity σ, where we set $\sigma = 1$. E.g. for $n = 6$, we have the system matrices

$$A = \begin{bmatrix} \frac{-1}{\tilde{C}R} & \frac{-1}{\tilde{C}} & 0 & 0 & 0 & 0 \\ \frac{1}{L} & \frac{-R\tilde{R}}{L(R+\tilde{R})} & \frac{-\tilde{R}}{L(R+\tilde{R})} & 0 & 0 & 0 \\ 0 & \frac{\tilde{R}}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}} & 0 & 0 \\ 0 & 0 & \frac{1}{L} & \frac{-R\tilde{R}}{L(R+\tilde{R})} & \frac{-\tilde{R}}{L(R+\tilde{R})} & 0 \\ 0 & 0 & 0 & \frac{\tilde{R}}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}(R+\tilde{R})} & \frac{-1}{\tilde{C}} \\ 0 & 0 & 0 & 0 & \frac{1}{L} & \frac{-\tilde{R}}{L} \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{-R\tilde{R}}{R+\tilde{R}} & \frac{-\tilde{R}}{R+\tilde{R}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{-R\tilde{R}}{R+\tilde{R}} & \frac{-\tilde{R}}{R+\tilde{R}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\tilde{R} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{\tilde{C}R} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$
$$C = \begin{bmatrix} -\frac{1}{R} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

For larger n, the band structure of A and N is extended periodically. To see the behaviour of our two methods, we reduce from order $n = 20$ to the orders $r = 1, 3, 5, \dots, 19$, and compute both the theoretical bounds and the actual approximation errors in the H^∞-norm. The results are shown in the following figure.

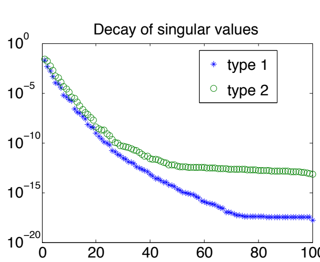


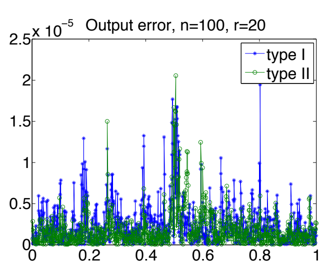
In this example, for both types the bounds hold, and for all reduced orders, type I gives a better approximation than type II.

C. A heat transfer problem

As another example we consider a stochastic modification of the heat transfer problem described in [14]. On the unit square $\Omega = [0, 1]^2$ the heat equation $x_t = \Delta x$ is given with Dirichlet condition $x = u_j$, $j = 1, 2, 3$ on three of the boundary edges and a stochastic Robin condition $n \cdot \nabla x = (1/2 + \dot{w})x$ on the fourth edge (where $\dot{w}$ stands for white noise). A standard 5-point finite difference discretization on a 10×10 grid leads to a modified Poisson matrix $A \in \mathbb{R}^{100\times 100}$ and corresponding matrices $N \in \mathbb{R}^{100\times 100}$ and $B \in \mathbb{R}^{100\times 3}$. We use the input $u \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and choose the average temperature as the output, i.e. $C = \frac{1}{100}[1, \dots, 1]$. We apply balanced truncation of type I and type II. For type II, an LMI-solver (MATLAB® function `mincx`) is used to compute P as a solution of the LMI (30) which minimizes $\operatorname{trace} P$ or $\operatorname{trace} PQ$.

In the following two figures, we compare the reduced systems of order $r = 20$ for both types. The left figure shows the decay of the singular values. Since the LMI-solver was called with tolerance level 10^{-9}, only the first about 25 singular values for type II have the correct order of magnitude. The right figure shows the approximation error $\|y(t) - y_r(t)\|$ over a given time interval. For both types it has the same order of magnitude. In fact, for many examples we have observed both methods to yield very similar results.





We have computed the estimated error norm and the actual approximation error for both types:

	$\sum_{j=11}^{100} \sigma_j$	$\|\mathbb{L} - \mathbb{L}_{10}\|$	$\sum_{j=21}^{100} \sigma_j$	$\|\mathbb{L} - \mathbb{L}_{20}\|$
I	$4.66e-06$	$9.30e-06$	$2.00e-09$	$9.65e-09$
II	$1.75e-05$	$4.83e-06$	$1.72e-08$	$9.70e-09$

As we can see, the upper error bound fails for type I, but is correct for type II. Nevertheless, judging from the H^∞ error, neither of the types seems to be preferable over the other.

D. Summary

Clearly, higher dimensional examples are required to get more insight. To this end a more sophisticated method for the solution of (30) is needed. With general purpose LMI-software on a standard Laptop, we hardly got higher than $n = 100$.

IV. Conclusions

We have compared two types of balanced truncation for stochastic linear systems, which are related to different Gramian type matrices P and Q. The following table collects properties of these reduction methods.

Type	I	II
Def. of P, Q	(6)	(7)
Stability?	Yes, [24]	Yes, Thm. II.2
H^2-bound?	Yes, [20]	no result
H^∞-bound?	No, Ex. I.3	Yes, Thm. II.4 or [19]

The main contributions of this paper are the preservation of asymptotic stability for type II balanced truncation proved in Theorem II.2 and the new proof of the H^∞ error bound in Theorem II.4. The efficient solution of (7b) is an open issue and requires further research. The same is true for the computation of the stochastic H^∞-norm.

Appendix

Asymptotic mean square stability

Consider the stochastic linear system of Itô-type

$$dx = Ax\,dt + Nx\,dw\,, \tag{31}$$

where $w = (w(t))_{t\in\mathbb{R}_+}$ is a zero mean real Wiener process on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ (e.g. [25], [26]).
Let $L^2_w(\mathbb{R}_+, \mathbb{R}^q)$ denote the corresponding space of non-anticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|^2_{L^2_w} := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2 dt\right) < \infty,$$

where $\mathcal{E}$ denotes expectation. By definition, system (31) is asymptotically mean-square-stable, if $\mathcal{E}(\|x(t)\|^2) \overset{t\to\infty}{\longrightarrow} 0$, for all initial conditions $x(0) = x_0$.

We have the following version of Lyapunov's matrix theorem, see [23]. Here $\otimes$ denotes the Kronecker product.

Theorem A.1 *The following are equivalent.*

(i) *System* (31) *is asymptotically mean-square stable.*
(ii) $\max\{\Re\lambda \mid \lambda \in \sigma(A \otimes I + I \otimes A + N \otimes N)\} < 0$
(iii) $\exists Y > 0 : \exists X > 0$: $A^T X + XA + N^T XN = -Y$
(iv) $\forall Y > 0 : \exists X > 0$: $A^T X + XA + N^T XN = -Y$
(v) $\forall Y \geq 0 : \exists X \geq 0$: $A^T X + XA + N^T XN = -Y$

Remark A.2 The theorem (like all other results in this paper) carries over to systems

$$dx = Ax\,dt + \sum_{j=1}^{k} N_j x\,dw_j$$

with more than one noise term, and many more equivalent criteria can be provided, see e.g. [33] or [18, Theorem 3.6.1].

The following theorem does not require any stability assumptions (see [18, Theorem 3.2.3]). It is central in the analysis of mean-square stability.

Theorem A.3 *Let*

$$\alpha = \max\{\Re\lambda \mid \lambda \in \sigma(A \otimes I + I \otimes A + N \otimes N)\} .$$

Then there exists a nonnegative definite matrix $V \neq 0$, such that

$$(\mathcal{L}_A^* + \Pi_N^*)(V) = AV + VA^T + NVN^T = \alpha V .$$

We also note a simple consequence of this theorem [24, Corollary 3.2]. Here $\langle Y, V\rangle = \operatorname{trace}(YV)$ is the Frobenius inner product for symmetric matrices.

Corollary A.4 *Let α, V as in the theorem. For given $Y \geq 0$ assume that*

$$\exists X > 0 :\ \mathcal{L}_A(X) + \Pi_N(X) \leq -Y . \tag{32}$$

Then $\alpha \leq 0$. Moreover, if $\alpha = 0$ then $YV = VY = 0$.

The stochastic bounded real lemma

Now let us consider system (5) with input u and output y. If system (31) is asymptotically mean-square stable, then (5) defines an input output operator $\mathbb{L} : u \mapsto y$ from $L^2_w(\mathbb{R}, \mathbb{R}^m)$ to $L^2_w(\mathbb{R}, \mathbb{R}^p)$, see [17]. By $\|\mathbb{L}\|$ we denote the induced operator norm, which is an analogue of the deterministic H^∞-norm. It can be characterized by the stochastic bounded real lemma.

Theorem A.5 *[17] For $\gamma > 0$, the following are equivalent.*

(i) *System* (31) *is asymptotically mean-square stable and $\|\mathbb{L}\| < \gamma$.*

(ii) *There exists a negative definite solution $X < 0$ to the Riccati inequality*

$$A^T X + XA + N^T XN - C^T C - \gamma^{-2} XBB^T X > 0 .$$

(iii) *There exists a positive definite solution $X > 0$ to the Riccati inequality*

$$A^T X + XA + N^T XN + C^T C + \gamma^{-2} XBB^T X < 0 .$$

We have stated the obviously equivalent formulations (ii) and (iii) to avoid confusion arising from different formulations in the literature. Under additional assumptions also non-strict versions can be formulated. The following sufficient criterion is given in [18, Corollary 2.2.3] (where also the signs are changed). Unlike in the previous theorem, here asymptotic mean-square stability is assumed at the outset.

Theorem A.6 *Assume that* (31) *is asymptotically stable in mean-square. If there exists a nonnegative definite matrix $X \geq 0$, satisfying*

$$A^T X + XA + N^T XN + C^T C + \gamma^{-2} XBB^T X \leq 0 ,$$

then $\|\mathbb{L}\| \leq \gamma$.

References

[1] G. Obinata, B. Anderson, Model Reduction for Control System Design, Springer, 2001.

[2] A. C. Antoulas, Approximation of large-scale dynamical systems, Vol. 6 of Advances in Design and Control, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.

[3] P. Benner, V. Mehrmann, D. C. Sorensen (Eds.), Dimension Reduction of Large-Scale Systems, Vol. 45 of Lecture Notes in Computational Science and Engineering, Springer-Verlag, 2005.

[4] W. H. Schilders, H. A. van der Vorst, J. Rommes (Eds.), Model Order Reduction: Theory, Research Aspects and Applications, Vol. 13 of Mathematics in Industry, Springer-Verlag, 2008.

[5] U. Baur, P. Benner, L. Feng, Model order reduction for linear and nonlinear systems: A system-theoretic perspective, Arch. Comput. Method. E. 21 (4) (2014) 331–358.

[6] B. C. Moore, Principal component analysis in linear systems: controllability, observability, and model reduction, IEEE Trans. Autom. Control AC-26 (1981) 17–32.

[7] L. Pernebo, L. M. Silverman, Model reduction via balanced state space representations, IEEE Trans. Autom. Control AC-27 (2) (1982) 382–387.

[8] T. Stykel, Analysis and numerical solution of generalized Lyapunov equations, Ph.D. thesis, Technische Universität Berlin (2002).

[9] A. Shokoohi, L. M. Silverman, P. M. Van Dooren, Linear time-variable systems: balancing and model reduction, IEEE Trans. Automat. Contr. AC-28 (8) (1983) 810–822.

[10] E. Verriest, T. Kailath, On generalized balanced realizations, IEEE Trans. Automat. Contr. AC-28 (8) (1983) 833–844.

[11] H. Sandberg, A. Rantzer, Balanced truncation of linear time-varying systems, IEEE Trans. Automat. Contr. 49 (2) (2004) 217–229.

[12] S. A. Al-Baiyat, M. Bettayeb, U. M. Al-Saggaf, New model reduction scheme for bilinear systems, Int. J. Systems Sci. 25 (1994) 1631–1642.

[13] W. S. Gray, J. Mesko, Energy functions and algebraic Gramians for bilinear systems, in: Preprints of the 4th IFAC Nonlinear Control Systems Design Symposium, Enschede, The Netherlands, 1998, pp. 103–108.

[14] P. Benner, T. Damm, Lyapunov equations, energy functionals, and model order reduction of bilinear and stochastic systems, SIAM J. Control Optim. 49 (2) (2011) 686–711.

[15] J. M. A. Scherpen, Balancing for nonlinear systems, Syst. Control Lett. 21 (2) (1993) 143–153.

[16] W. M. Wonham, Random differential equations in control theory, in: A. T. Bharucha-Reid (Ed.), Probab. Methods Appl. Math., Vol. 2, Academic Press, New York - London, 1970, pp. 131–212.

[17] D. Hinrichsen, A. J. Pritchard, Stochastic H_∞, SIAM J. Control Optim. 36 (5) (1998) 1504–1538.

[18] T. Damm, Rational Matrix Equations in Stochastic Control, no. 297 in Lecture Notes in Control and Information Sciences, Springer, 2004.

[19] T. Damm, P. Benner, Balanced truncation for stochastic linear systems with guaranteed error bound, in: Proceedings of MTNS-2014, Groningen, The Netherlands, 2014, pp. 1492–1497.

[20] M. Redmann, P. Benner, Model reduction for stochastic systems, Preprint MPIMD/14-03, Max Planck Institute Magdeburg (2014).

[21] F. R. Gantmacher, The Theory of Matrices (Vol. II), Chelsea, New York, 1959.

[22] D. L. Kleinman, On the stability of linear stochastic systems, IEEE Trans. Autom. Control AC-14 (1969) 429–430.

[23] R. Z. Khasminskij, Stochastic Stability of Differential Equations, Sijthoff & Noordhoff, Alphen aan den Rijn, NL, 1980.

[24] P. Benner, T. Damm, M. Redmann, Y. Rocio Rodriguez Cruz, Positive operators and stable truncation, Linear Algebra Appl. doi:10.1016/j.laa.2014.12.005, in press. published electronically, Dec. 30, 2014.

[25] L. Arnold, Stochastic Differential Equations: Theory and Applications. Translation., John Wiley and Sons Inc., New York etc., 1974.

[26] B. Oeksendal, Stochastic Differential Equations, 5th Edition, Springer-Verlag, 1998.

[27] T. Damm, Direct methods and ADI-preconditioned Krylov subspace methods for generalized Lyapunov equations, Numer. Lin. Alg. Appl. 15 (9) (2008) 853–871.
[28] P. Benner, T. Breiten, Low rank methods for a class of generalized Lyapunov equations and related issues., Numer. Math. 124 (3) (2013) 441–470.
[29] D. Kressner, P. Sirković, Greedy low-rank methods for solving general linear matrix equations, Technical report, ANCHP, MATHICSE, EPF Lausanne, Switzerland (2014).
[30] S. Shank, V. Simoncini, D. Szyld, Efficient low-rank solutions of generalized Lyapunov equations, Report 14-11-10, Department of Mathematics Temple University, Philadelphia, PA 19122 (2014).
[31] S. Gugercin, A. Antoulas, A survey of model reduction by balanced truncation and some new results, Int. J. Control 77 (8) (2004) 748–766.
[32] V. A. Ugrinovskii, I. R. Petersen, Absolute stabilization and minimax optimal control of uncertain systems with stochastic uncertainty, SIAM J. Control Optim. 37 (4) (1999) 1089–1122.
[33] H. Schneider, Positive operators and an inertia theorem, Numer. Math. 7 (1965) 11–17.