

# Higher spin representations of $K(E_{10})$

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We review the recently constructed non-trivial fermionic representations of the infinite-dimensional subalgebra  $K(\mathfrak{e}_{10})$  of the hyperbolic Kac–Moody algebra  $\mathfrak{e}_{10}$ . These representations are all unfaithful (and more specifically, of finite dimension). In addition we present their decompositions under the various finite-dimensional subgroups associated with some maximal supergravities in dimensions  $D \leq 11$ , and the projectors for ‘spin- $\frac{7}{2}$ ’ which have not been given before. Those representations that have not been derived from supergravity still have to find a role and a proper physical interpretation in the conjectured correspondence between  $E_{10}$  and M-theory. Nevertheless, they provide novel mathematical structures that could shed some light on fundamental questions in supergravity and on the possible role of  $K(E_{10})$  as an ‘R-symmetry’ of M-theory, and perhaps also on the algebra  $\mathfrak{e}_{10}$  itself.

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# 1 Introduction

The hyperbolic Kac–Moody algebra  $\mathfrak{e}_{10}$  has been conjectured to generate an underlying symmetry of M-theory [1, 2] and its (maximal compact) subalgebra  $K(\mathfrak{e}_{10})$  (fixed by the Chevalley involution) plays the role of the generalised R-symmetry transformations [3–7]. In the  $\mathfrak{e}_{10}$  conjecture the constrained null motion of a spinning particle on the symmetric space  $E_{10}/K(E_{10})$  is equivalent to the dynamics of supergravity or even M-theory. This conjecture thus far has only been verified for a finite set of generators of the infinite-dimensional algebra  $\mathfrak{e}_{10}$  both in the bosonic and fermionic sector [2, 4, 5, 8–10]. However, it has thus far proved impossible to construct a spinning particle action on  $E_{10}/K(E_{10})$  that has one-dimensional local supersymmetry, as was explained at length in Ref. [11].

One major obstacle when constructing a supersymmetric  $E_{10}$ -model is the disparity between the bosonic and fermionic degrees of freedom that are used: The bosons are associated with the infinitely many directions of the symmetric space  $E_{10}/K(E_{10})$  whereas the fermions used in Refs. [4, 5] were constructed out of a finite-dimensional (hence unfaithful) representation of dimension 320 of the R-symmetry group  $K(E_{10})$ . [12] It therefore appears necessary to construct larger, preferably infinite-dimensional, fermionic representations of  $K(E_{10})$  and this is the topic we will pursue in the present contribution that is partially based on our paper Ref. [13].

We develop a new formalism for constructing representations of  $K(\mathfrak{e}_{10})$  and exhibit new irreducible examples of dimensions 1728 and 7040, respectively. We refer to them as ‘higher spin representations’ although their spin is not necessarily higher from a space-time point of view but rather when viewed from the (truncated) Wheeler–DeWitt superspace of metrics. This point will be explained in more detail below. We will see that only the **7040** contains also genuine higher spin fields from the space-time perspective. Our formalism gives the action of an infinite number of  $K(\mathfrak{e}_{10})$  generators that are labelled by the positive real roots of  $\mathfrak{e}_{10}$ . Since the representations are finite-dimensional and therefore necessarily unfaithful, an infinite number of these generators will be represented by the same operator on the representation space.

Let us emphasize that a proper understanding of the fermionic sector will be essential for further progress with understanding the role of  $E_{10}$  in M-theory, something that is unlikely in our opinion to be achievable if one restricts attention to the bosonic sector only. On top of the (unknown) representation theory of  $K(E_{10})$  this might quite possibly require some novel type of bosonisation, as is suggested by the fact that  $E_{10}$  seems to ‘know everything’ about the fermions of maximal supergravity that we have learnt from supersymmetry (in particular, the structure of the bosonic and fermionic multiplets). Equally important, the actual *physics* of the quantised theory with fermions is likely to differ very much from that of the purely bosonic system, as is obvious from the example of supersymmetric quantum cosmology investigated in Ref. [14].

## 2 $\mathfrak{e}_{10}$ and $K(\mathfrak{e}_{10})$

The (split real) Lie algebra  $\mathfrak{e}_{10}$  is a hyperbolic Kac–Moody Lie algebra [15]. Its only known definition is in terms of generators and relations. There are 30 generators  $(e_i, f_i, h_i)$  for  $i = 1, \dots, 10$  and each triple generates an  $\mathfrak{sl}(2, \mathbb{R})$  subalgebra of  $\mathfrak{e}_{10}$ . The full set of defining relations

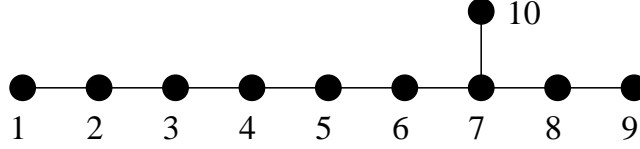


Figure 1: The Dynkin diagram of  $\mathfrak{e}_{10}$  with labelling of nodes.

is given by

$$\begin{aligned}
 [h_i, h_j] &= 0, & [h_i, e_j] &= A_{ij}e_j, & [h_i, f_j] &= -A_{ij}f_j, \\
 [e_i, f_j] &= \delta_{ij}h_i, & (\text{ad } e_i)^{1-A_{ij}}e_j &= 0, & (\text{ad } f_i)^{1-A_{ij}}f_j &= 0.
 \end{aligned} \tag{1}$$

Here,  $A_{ij}$  are the elements of the symmetric Cartan matrix associated with the  $\mathfrak{e}_{10}$  Dynkin diagram shown in figure 1. The Cartan matrix is of Lorentzian signature and there are roots  $\alpha$  of the algebra with norms  $\alpha^2 = 2 - 2k$  for  $k \in \mathbb{N}_0$ . The roots with  $\alpha^2 = 2$  are called real roots and they have multiplicity one; all others are imaginary and have higher multiplicity.

The subalgebra  $K(\mathfrak{e}_{10})$  is generated by the ‘compact’ combinations

$$x_i = e_i - f_i \tag{2}$$

which are invariant under the Cartan-Chevalley involution

$$\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h_i) = -h_i \tag{3}$$

The relations satisfied by these elements are in general not homogeneous (unlike the standard relations in the Chevalley–Serre presentation for the  $e_i$  and  $f_i$  above). Depending on whether two nodes  $i$  and  $j$  are connected by a line in the Dynkin diagram or not one has two cases

$$\begin{aligned}
 [x_i, x_j] &= 0 & \text{if } i \text{ and } j \text{ are not connected} \\
 [x_i, [x_i, x_j]] + x_j &= 0 & \text{if } i \text{ and } j \text{ are connected}
 \end{aligned} \tag{4}$$

We will refer to these as the Berman–Serre relations; these relations were studied in a more general context in Ref. [16]. The algebra  $K(\mathfrak{e}_{10})$  is then defined as the free Lie algebra over the generators  $\{x_i\}$  subject to the relations (4). The task of finding representations of  $K(\mathfrak{e}_{10})$  is tantamount to finding matrices or operators that satisfy these relations.

Since all simple generators  $x_i$  are associated with real simple roots (of multiplicity one) one can also rephrase these relations more generally for any real roots by considering a generator  $J(\alpha)$  for any (positive) real root  $\alpha$ . Using a basis of simple roots  $\alpha_i$  of the root lattice one then has  $x_i = J(\alpha_i)$  as particular case. The relations (4) are then equivalent for real roots  $\alpha$  and  $\beta$  obeying  $\alpha \cdot \beta \in \{-1, 0, 1\}$

$$\begin{aligned}
 [J(\alpha), J(\beta)] &= \epsilon_{\alpha, \beta} J(\alpha + \beta), & \text{if } \alpha \cdot \beta &= -1, \\
 [J(\alpha), J(\beta)] &= -\epsilon_{\alpha, -\beta} J(\alpha - \beta), & \text{if } \alpha \cdot \beta &= +1, \\
 [J(\alpha), J(\beta)] &= 0, & \text{if } \alpha \cdot \beta &= 0,
 \end{aligned} \tag{5}$$

and  $\epsilon_{\alpha,\beta} \in \{-1, 1\}$  is a certain cocycle on the  $\mathfrak{e}_{10}$  root lattice that satisfies

$$\epsilon_{\alpha,\beta} = -\epsilon_{\beta,\alpha} = -\epsilon_{-\alpha,-\beta}, \quad \epsilon_{\alpha+\beta,-\beta} = \epsilon_{\alpha,\beta}. \quad (6)$$

The restriction on the inner product in the commutation is to make sure that  $\alpha \mp \beta$  is a real root or no root at all, such that one does not have to worry about multiplicities from imaginary roots on the right-hand side. By contrast  $\epsilon_{\alpha,\beta}$  can be defined for any pair of elements  $(\alpha, \beta)$  of the root lattice.

To the root lattice of  $\mathfrak{e}_{10}$  one can also associate elements  $\Gamma(\alpha)$  of the  $\mathfrak{so}(10)$  Clifford algebra of real  $(32 \times 32)$  matrices such that [13]

$$\Gamma(\alpha)\Gamma(\beta) = \epsilon_{\alpha,\beta}\Gamma(\alpha + \beta) = -\epsilon_{\alpha,-\beta}\Gamma(\alpha - \beta). \quad (7)$$

With these rules it is then not hard to verify that

$$J(\alpha) = \frac{1}{2}\Gamma(\alpha) \quad (8)$$

provides a representation of  $K(\mathfrak{e}_{10})$  for all real roots  $\alpha$ . This 32-dimensional representation is known as the Dirac-spinor of  $K(\mathfrak{e}_{10})$ . By choosing a particular basis of the root lattice, called wall basis, one could exhibit [13] that the  $x_i$  for  $i = 1, \dots, 9$  are just the usual spin representation  $x_i = \frac{1}{2}\Gamma^{i i+1}$  of  $\mathfrak{so}(10)$  but we will not use this here.

### 3 Tensors and spinors on Wheeler–DeWitt mini-superspace

The space of diagonal spatial metrics in 11 space-time dimensions is a Lorentzian ten-dimensional space in the Hamiltonian treatment of general relativity. This space is actually a finite-dimensional truncation of the full Wheeler–DeWitt ‘superspace’ (*alias* the ‘moduli space of 10-geometries’) to the finite-dimensional subspace of diagonal scale factors (the negative direction that renders this metric indefinite is associated with the scaling mode of the metric). We choose a basis  $e_{\mathbf{a}}$  for this ten-dimensional space ( $\mathbf{a}, \mathbf{b}, \dots = 1, \dots, 10$ ) with inner products

$$e_{\mathbf{a}} \cdot e_{\mathbf{b}} = G_{\mathbf{ab}} \quad (9)$$

where  $G_{\mathbf{ab}}$  is the Lorentzian DeWitt superspace metric restricted to the space of metric scale factors; more explicitly, it follows from the Einstein–Hilbert action that

$$G_{\mathbf{ab}} = \delta_{\mathbf{ab}} - 1 \quad \Rightarrow \quad G^{\mathbf{ab}} = \delta_{\mathbf{ab}} - \frac{1}{9} \quad (10)$$

This Lorentzian space can be identified with the Lorentzian space spanned by the roots of  $\mathfrak{e}_{10}$ . In the remainder we do not require the explicit form of  $G_{\mathbf{ab}}$  of (10).

Our ansatz for fermionic representations of  $K(\mathfrak{e}_{10})$  then consists in considering tensor-spinors  $\phi_A^{\mathbf{a}_1 \dots \mathbf{a}_n} = \phi_A^{(\mathbf{a}_1 \dots \mathbf{a}_n)}$  that are completely symmetric in their  $n$  tensor indices and also carry a spinor index  $A = 1, \dots, 32$  of  $\mathfrak{so}(10)$ . The Dirac-spinor discussed in the preceding section then simply corresponds to  $n = 0$ . We will also consider the case when  $\phi_A^{\mathbf{a}_1 \dots \mathbf{a}_n}$  is traceless in its tensor indices.

Since the tensor indices are those of a Lorentzian  $\mathfrak{so}(1,9)$  space while the spinor index belongs to the Euclidean  $\mathfrak{so}(10)$  subalgebra of  $K(\mathfrak{e}_{10})$  our approach could be termed hybrid. Certainly one cannot take simple  $\Gamma$ -traces of  $\phi_A^{a_1 \dots a_n}$  because  $\mathbf{a}, \mathbf{b}, \dots$  are not  $\text{SO}(10)$  indices, so the only option to render the tensor-spinor irreducible is to make it traceless in its indices  $\mathbf{a}_1, \mathbf{a}_2, \dots$ .

The generators  $J(\alpha)$  of  $K(\mathfrak{e}_{10})$  are then given by combinations of an object acting on the tensor indices and gamma matrices acting on the spinor index. More precisely, we make the ansatz

$$J(\alpha)\phi_A^{a_1 \dots a_n} = -2X(\alpha)^{a_1 \dots a_n}{}_{b_1 \dots b_n} \Gamma(\alpha)_{AB} \phi_B^{b_1 \dots b_n}. \quad (11)$$

Due to the known properties (5) of the  $\Gamma(\alpha)$  under commutation, checking the consistency relations (7) then can be reduced to checking the following conditions on the tensors  $X(\alpha)$  for real roots [13]

$$\begin{aligned} \{X(\alpha), X(\beta)\} &= \frac{1}{2}X(\alpha \pm \beta), & \text{if } \alpha \cdot \beta = \mp 1, \\ [X(\alpha), X(\beta)] &= 0, & \text{if } \alpha \cdot \beta = 0. \end{aligned} \quad (12)$$

Note that there is no  $\epsilon_{\alpha, \beta}$  in these relations as it is already taken care of by the  $\Gamma(\alpha)$ . The Dirac-spinor corresponds to the solution  $X(\alpha) = \frac{1}{4}$  to these equations.

Another  $K(\mathfrak{e}_{10})$  representation that has been known from supergravity considerations is the case  $n = 1$  that corresponds to the  $D = 11$  gravitino and has dimension 320 [4–6]. In our language it corresponds to the solution

$$X(\alpha)^a{}_b = -\frac{1}{2}\alpha^a \alpha_b + \frac{1}{4}\delta_b^a, \quad (13)$$

where  $\alpha^a$  are the components of the root  $\alpha$  with respect to the basis  $e_a$ , i.e.,  $\alpha = \sum_a \alpha^a e_a$ . ‘Typewriter font’ indices are raised and lowered with the Lorentzian  $G_{ab}$ .

For the gravitino (or vector-spinor) one can find a rewriting in terms of pure  $\mathfrak{so}(10)$  representation by letting [17]

$$\psi_A^a = \sum_B \Gamma_{AB}^a \phi_B^a \quad (\text{no sum on } a). \quad (14)$$

The object on the left is then a standard vector-spinor of  $\mathfrak{so}(10)$ . A similar simple and explicit rewriting into  $\mathfrak{so}(10)$  representations is not known for the new representations we discuss below.

We also note that due to the unfaithfulness of the representations, one obtains (infinite-dimensional) ideals in  $K(\mathfrak{e}_{10})$ , leading to the result that  $K(\mathfrak{e}_{10})$  is not a simple Lie algebra. The quotient Lie algebras  $\mathfrak{q}$  of  $K(\mathfrak{e}_{10})$  by the ideals of the **32** and **320** have been analysed and are given by  $\mathfrak{q}_{32} \cong \mathfrak{so}(32)$  and  $\mathfrak{q}_{320} \cong \mathfrak{so}(288, 32)$ . It may seem surprising that the ‘compact’  $K(\mathfrak{e}_{10})$  admits a non-compact quotient in the **320** representation but this is not a contradiction due to the infinite-dimensionality of  $K(\mathfrak{e}_{10})$ . For the higher spin representations below, the quotients have not been worked out.

## 4 Higher spin representations

In Ref. [13] two further solutions to (12) were found that correspond to the values  $n = 2$  and  $n = 3$  (corresponding to spin  $s = \frac{5}{2}$  and  $s = \frac{7}{2}$ , respectively [18]). These representations go beyond supergravity as there appears to be no supergravity model from which they would be derivable. For spin  $s = \frac{5}{2}$  ( $n = 2$ ) the corresponding tensors are given by

$$X(\alpha)^{a_1 a_2}_{b_1 b_2} = \frac{1}{2} \alpha^{a_1} \alpha^{a_2} \alpha_{b_1} \alpha_{b_2} - \alpha^{(a_1} \delta_{(b_1}^{a_2)} \alpha_{b_2)} + \frac{1}{4} \delta_{b_1}^{(a_1} \delta_{b_2}^{a_2)} \quad (15)$$

and for  $n = 3$  (spin- $\frac{7}{2}$ ) by

$$\begin{aligned} X(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3} &= -\frac{1}{3} \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} + \frac{3}{2} \alpha^{(a_1} \alpha^{a_2} \delta_{(b_1}^{a_3)} \alpha_{b_2} \alpha_{b_3)} - \frac{3}{2} \alpha^{(a_1} \delta_{(b_1}^{a_2} \delta_{b_2}^{a_3)} \alpha_{b_3)} \\ &+ \frac{1}{4} \delta_{(b_1}^{(a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3)} + \frac{1}{12} (2 - \sqrt{3}) \alpha^{(a_1} G^{a_2 a_3)} G_{(b_1 b_2} \alpha_{b_3)} \\ &+ \frac{1}{12} (-1 + \sqrt{3}) \left( \alpha^{(a_1} \alpha^{a_2} \alpha^{a_3)} G_{(b_1 b_2} \alpha_{b_3)} + \alpha^{(a_1} G^{a_2 a_3)} \alpha_{(b_1} \alpha_{b_2} \alpha_{b_3)} \right). \end{aligned} \quad (16)$$

These expressions can be found and verified analytically. We have also extended the search for solutions of this type for  $n \leq 10$  with the ansatz above but have not found any additional solutions so far.

The spin- $\frac{5}{2}$  solution as given is of dimension  $\frac{10 \times 11}{2} \times 32 = 1760$ . It turns out that this representation is reducible since the subspace spanned by the trace  $G_{ab} \phi_A^{ab}$  is invariant. This trace transforms in the spin- $\frac{1}{2}$  representation of dimension 32, leaving an irreducible 1728-dimensional representation of  $K(\mathfrak{e}_{10})$ . By contrast, the spin- $\frac{7}{2}$  representation of dimension  $\frac{10 \times 11 \times 12}{6} \times 32 = 7040$  is irreducible as given.

In the next two sections, we investigate further properties of the new higher spin representations.

## 5 Projectors and Weyl group action

The  $K(\mathfrak{e}_{10})$  generators  $J(\alpha)$  are defined for all positive roots  $\alpha$  of  $\mathfrak{e}_{10}$ . As the roots  $\alpha$  are space-like elements in a Lorentzian ten-dimensional space, they have a stabiliser of type  $\mathfrak{so}(1, 8) \subset \mathfrak{so}(1, 9)$ . This stability algebra can be used to decompose the ‘polarisation tensor’  $X(\alpha)$  into irreducible pieces for a fixed  $\alpha$ . The irreducible  $\mathfrak{so}(1, 8)$  terms are given by projectors  $\Pi^{(j)}(\alpha)$ , such that tensor  $X(\alpha)$  can be expressed in terms of these projectors. [19] This rewriting greatly facilitates the exponentiation of the corresponding matrices, and will make it easy to work out the exponentiated (Weyl) group actions.

### 5.1 Projectors for spin- $\frac{5}{2}$

For  $n = 2$ , the polarisation tensor  $X(\alpha)^{a_1 a_2}_{b_1 b_2}$  lies in the reducible **55** of  $\mathfrak{so}(1, 9)$ , where we work for simplicity with the reducible representation of dimension 1760 given in (15). The decomposition of  $X(\alpha)$  under the regularly embedded  $\mathfrak{so}(1, 8)$  is

$$\mathbf{55} \rightarrow \mathbf{54} \oplus \mathbf{1} \rightarrow (\mathbf{44} \oplus \mathbf{9} \oplus \mathbf{1}') \oplus \mathbf{1}. \quad (17)$$

The splitting of the singlets here has been done in such a way that  $\mathbf{1}$  corresponds to the  $\mathfrak{so}(1,9)$  singlet corresponding to the trace with  $G_{ab}$ . One can check that the following are complete orthonormal projectors on the various pieces

$$\begin{aligned}
\Pi^{(44)}(\alpha)^{a_1 a_2}_{b_1 b_2} &= \frac{2}{9} \alpha^{a_1} \alpha^{a_2} \alpha_{b_1} \alpha_{b_2} - \alpha^{(a_1} \delta_{(b_1}^{a_2)} \alpha_{b_2)} + \delta_{b_1}^{(a_1} \delta_{b_2}^{a_2)} \\
&\quad + \frac{1}{18} (\alpha^{a_1} \alpha^{a_2} G_{b_1 b_2} + G^{a_1 a_2} \alpha_{b_1} \alpha_{b_2}) - \frac{1}{9} G^{a_1 a_2} G_{b_1 b_2}, \\
\Pi^{(9)}(\alpha)^{a_1 a_2}_{b_1 b_2} &= -\frac{1}{2} \alpha^{a_1} \alpha^{a_2} \alpha_{b_1} \alpha_{b_2} + \alpha^{(a_1} \delta_{(b_1}^{a_2)} \alpha_{b_2)}, \\
\tilde{\Pi}^{(1)}(\alpha)^{a_1 a_2}_{b_1 b_2} &= \frac{1}{10} G^{a_1 a_2} G_{b_1 b_2}, \\
\tilde{\Pi}^{(1')}(\alpha)^{a_1 a_2}_{b_1 b_2} &= \frac{5}{18} \alpha^{a_1} \alpha^{a_2} \alpha_{b_1} \alpha_{b_2} - \frac{1}{18} (\alpha^{a_1} \alpha^{a_2} G_{b_1 b_2} + G^{a_1 a_2} \alpha_{b_1} \alpha_{b_2}) \\
&\quad + \frac{1}{90} G^{a_1 a_2} G_{b_1 b_2}.
\end{aligned} \tag{18}$$

In terms of these, the tensor  $X(\alpha)$  takes the form

$$X(\alpha)^{a_1 a_2}_{b_1 b_2} = \left( \frac{1}{4} \tilde{\Pi}^{(1)}(\alpha) + \frac{1}{4} \tilde{\Pi}^{(1')}(\alpha) - \frac{3}{4} \Pi^{(9)}(\alpha) + \frac{1}{4} \Pi^{(44)}(\alpha) \right)^{a_1 a_2}_{b_1 b_2}. \tag{19}$$

What is important here is that the coefficients of all projectors are of the form  $\frac{2k+1}{4}$  with  $k \in \mathbb{Z}$ . This implies that when one constructs the ‘Weyl group’ generator

$$w_\alpha = e^{\frac{\pi}{2} J(\alpha)} \tag{20}$$

acting in the representation is idempotent in the eighth power. Weyl group has been put into inverted commas above because this is more correctly an element of a covering of the Weyl group that has been dubbed the spin-extended Weyl group [17, 20]. Acting on spinor representations, the characteristic feature is that only the eighth power  $w_\alpha^8 = \mathbb{1}$  whereas one normally has the fourth power for the covering of the Weyl on bosonic representations [15].

## 5.2 Projectors for spin- $\frac{7}{2}$

In this case, the polarisation tensor  $X(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3}$  is in the (reducible) totally symmetric  $\mathbf{220}$  of  $\mathfrak{so}(1,9)$ . This decomposes under  $\mathfrak{so}(1,8)$  as

$$\mathbf{220} \rightarrow \mathbf{210} \oplus \mathbf{10} \rightarrow (\mathbf{156} \oplus \mathbf{44} \oplus \mathbf{9} \oplus \mathbf{1}) \oplus (\mathbf{9} \oplus \mathbf{1}). \tag{21}$$

There are two singlets and two vectors of  $\mathfrak{so}(1,8)$  appearing in the decomposition and some associated freedom in constructing the orthonormal projectors. We choose a particular combination

of these representations as follows [21]

$$\begin{aligned}
\Pi^{(156)}(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3} &= -\frac{1}{11} \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} + \frac{15}{22} \alpha^{(a_1} \alpha^{a_2} \delta_{(b_1}^{a_3)} \alpha_{b_2} \alpha_{b_3)} \\
&\quad - \frac{3}{2} \alpha^{(a_1} \delta_{(b_1}^{a_2} \delta_{b_2}^{a_3)} \alpha_{b_3)} + \delta_{(b_1}^{(a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3)} \\
&\quad - \frac{3}{44} \left( \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{(b_1} G_{b_2 b_3)} + \alpha^{(a_1} G^{a_2 a_3)} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} \right) \\
&\quad + \frac{3}{22} \left( \alpha^{(a_1} \alpha^{a_2} \delta_{(b_1}^{a_3)} G_{b_2 b_3)} + G^{(a_1 a_2} \delta_{(b_1}^{a_3)} \alpha_{b_2} \alpha_{b_3)} \right) \\
&\quad - \frac{3}{11} G^{(a_1 a_2} \delta_{(b_1}^{a_3)} G_{b_2 b_3)} + \frac{3}{22} \alpha^{(a_1} G^{a_2 a_3)} G_{(b_1 b_2} \alpha_{b_3)} \\
\Pi^{(44)}(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3} &= \frac{1}{3} \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} - \frac{3}{2} \alpha^{(a_1} \alpha^{a_2} \delta_{(b_1}^{a_3)} \alpha_{b_2} \alpha_{b_3)} \\
&\quad + \frac{3}{2} \alpha^{(a_1} \delta_{(b_1}^{a_2} \delta_{b_2}^{a_3)} \alpha_{b_3)} - \frac{1}{6} \alpha^{(a_1} G^{a_2 a_3)} G_{(b_1 b_2} \alpha_{b_3)} \\
&\quad + \frac{1}{12} \left( \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{(b_1} G_{b_2 b_3)} + \alpha^{(a_1} G^{a_2 a_3)} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} \right) \tag{22}
\end{aligned}$$

for the (unique) two biggest representations,

$$\begin{aligned}
\Pi^{(9)}(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3} &= -\frac{9}{22} \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} + \frac{9}{11} \alpha^{(a_1} \alpha^{a_2} \delta_{(b_1}^{a_3)} \alpha_{b_2} \alpha_{b_3)} \\
&\quad + \frac{3}{44} \left( \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{(b_1} G_{b_2 b_3)} + \alpha^{(a_1} G^{a_2 a_3)} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} \right) \\
&\quad - \frac{3}{22} \left( \alpha^{(a_1} \alpha^{a_2} \delta_{(b_1}^{a_3)} G_{b_2 b_3)} + G^{(a_1 a_2} \delta_{(b_1}^{a_3)} \alpha_{b_2} \alpha_{b_3)} \right) \\
&\quad + \frac{1}{44} G^{(a_1 a_2} \delta_{(b_1}^{a_3)} G_{b_2 b_3)} - \frac{1}{88} \alpha^{(a_1} G^{a_2 a_3)} G_{(b_1 b_2} \alpha_{b_3)} \\
\Pi^{(9')}(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3} &= \frac{1}{4} G^{(a_1 a_2} \delta_{(b_1}^{a_3)} G_{b_2 b_3)} - \frac{1}{8} \alpha^{(a_1} G^{a_2 a_3)} G_{(b_1 b_2} \alpha_{b_3)} \tag{23}
\end{aligned}$$

for the vectors and finally for the singlets

$$\begin{aligned}
\Pi^{(1)}(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3} &= \frac{1}{12} \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} \\
&\quad + \frac{1}{24} (-1 - \sqrt{3}) \left( \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{(b_1} G_{b_2 b_3)} + \alpha^{(a_1} G^{a_2 a_3)} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} \right) \\
&\quad + \frac{1}{24} (2 + \sqrt{3}) \alpha^{(a_1} G^{a_2 a_3)} G_{(b_1 b_2} \alpha_{b_3)} \\
\Pi^{(1')}(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3} &= \frac{1}{12} \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} \\
&\quad + \frac{1}{24} (-1 + \sqrt{3}) \left( \alpha^{a_1} \alpha^{a_2} \alpha^{a_3} \alpha_{(b_1} G_{b_2 b_3)} + \alpha^{(a_1} G^{a_2 a_3)} \alpha_{b_1} \alpha_{b_2} \alpha_{b_3} \right) \\
&\quad + \frac{1}{24} (2 - \sqrt{3}) \alpha^{(a_1} G^{a_2 a_3)} G_{(b_1 b_2} \alpha_{b_3)} \tag{24}
\end{aligned}$$



The tensor  $X(\alpha)$  reads as follows in this basis

$$X(\alpha)^{a_1 a_2 a_3}_{b_1 b_2 b_3} = \left( \frac{5}{4} \Pi^{(1)}(\alpha) - \frac{3}{4} \Pi^{(1')}(\alpha) + \frac{1}{4} \Pi^{(9)}(\alpha) + \frac{1}{4} \Pi^{(9')}(\alpha) - \frac{3}{4} \Pi^{(44)}(\alpha) + \frac{1}{4} \Pi^{(156)}(\alpha) \right)^{a_1 a_2 a_3}_{b_1 b_2 b_3}. \quad (25)$$

Again, it is important that the coefficients of all the orthonormal projectors are of the form  $\frac{2k+1}{4}$  such that we are dealing with a genuine fermionic representation of  $K(E_{10})$ .

## 6 Branching under subalgebras

The infinite-dimensional Lie algebra  $K(\mathfrak{e}_{10})$  has infinitely many finite-dimensional subalgebras. [22] Of these are of particular interest to us the following, all of which can be obtained by deleting a single node from the  $\mathfrak{e}_{10}$  Dynkin diagram:

- |     |   |                  |                           |
|-----|---|------------------|---------------------------|
| (a) | $\mathfrak{so}(10)$                         | deleting node 10 | SUGRA in $D = 11$         |
| (b) | $\mathfrak{so}(2) \oplus \mathfrak{so}(16)$ | deleting node 2  | SUGRA in $D = 3$          |
| (c) | $\mathfrak{so}(9) \oplus \mathfrak{so}(2)$  | deleting node 8  | IIB SUGRA in $D = 10$     |
| (d) | $\mathfrak{so}(9) \oplus \mathfrak{so}(9)$  | deleting node 9  | Doubled SUGRA in $D = 10$ |

The last case requires some explanation. In Ref. [8] the decomposition of  $\mathfrak{e}_{10}$  under its  $\mathfrak{so}(9, 9)$  subalgebra was studied and shown to correspond to both type IIA and type IIB theory since the Ramond–Ramond potentials occurred in a spinor representation of  $\mathfrak{so}(9, 9)$  that can be read either as all even or all odd forms; similarly, the fermions arrange themselves correctly for the two theories [24]. In investigations of double field theory the same structure appears [25] and we have therefore dubbed this T-duality agnostic decomposition as ‘doubled SUGRA.’

There are some additional subtleties associated with the global assignment of fermionic and bosonic representations at the group level. More precisely, the  $\mathfrak{so}(16)$  is the Lie algebra of  $Spin(16)/\mathbb{Z}_2$ . The  $\mathbb{Z}_2$  is not diagonally embedded in the center  $\mathbb{Z}_2 \times \mathbb{Z}_2$  but as one of the factors; this entails that the representations  $\mathbf{16}_v$  and  $\mathbf{128}_c$  are spinorial (that is, they transform with a factor  $(-1)$  upon rotation by  $2\pi$ ), whereas the  $\mathbf{128}_s$  is tensorial [26]. Moreover, the  $\mathbf{16}$  spinor of  $Spin(9) = [Spin(9) \times Spin(9)]_{\text{diag}}$  is identified with the (spinorial)  $\mathbf{16}_v$  of  $Spin(16)$ . The diagonal  $Spin(9)$  also lies as a regular subgroup in  $Spin(9)$  as it corresponds to the dimensional reduction from  $D = 11$  to  $D = 10$  (over a spatial direction).

The decompositions of the spin- $\frac{1}{2}$  and spin- $\frac{3}{2}$  representations were already given in Ref. [24], while the decompositions of the new higher spin representations under the various subalgebras have not been given previously. To find the relevant decompositions for spin- $\frac{5}{2}$  and spin- $\frac{7}{2}$  is actually rather involved, and can only be done on a computer. The main problem here is that the  $K(E_{10})$  representations are not highest or lowest weight representations (it is not even clear whether  $K(E_{10})$  admits any analog of such representations), so the customary tools of representation theory cannot be applied. However, the subrepresentations obtained after descending to any finite-dimensional subgroup *are* highest or lowest weight representations, so

given any of the above subgroups, one must first identify the corresponding highest or lowest weights. For instance, for the  $\text{spin-}\frac{7}{2}$  representation this requires (amongst other things) the (simultaneous) diagonalisation of various  $7040 \times 7040$  matrices. It seems clear that for yet higher dimensional realisations such a procedure would become impractical very quickly unless better methods are developed.

### 6.1 Branching the spin $s = \frac{1}{2}$ and $s = \frac{3}{2}$ representations

These were already understood in previous work [24]. The fractions  $\frac{1}{2}$  and  $\frac{3}{2}$  in the decompositions (b) and (c) below correspond to the  $\mathfrak{so}(2) \cong \mathfrak{u}(1)$  charges. In these cases all the representations form doublets of  $\mathfrak{so}(2)$  that can also be thought of as complex one-dimensional representations of  $\mathfrak{u}(1)$ . This has to be taken into account when checking the dimension count of the decompositions.

$$\begin{aligned}
\mathbf{32} & \xrightarrow{a} \mathbf{32} \\
& \xrightarrow{b} \left( \frac{1}{2}, \mathbf{16}_v \right) \\
& \xrightarrow{c} \left( \mathbf{16}, \frac{1}{2} \right) \\
& \xrightarrow{d} (\mathbf{16}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{16})
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\mathbf{320} & \xrightarrow{a} \mathbf{288} \oplus \mathbf{32} \\
& \xrightarrow{b} \left( \frac{1}{2}, \mathbf{128}_c \right) \oplus \left( \frac{1}{2}, \mathbf{16}_v \right) \oplus \left( \frac{3}{2}, \mathbf{16}_v \right) \\
& \xrightarrow{c} \left( \mathbf{16}, \frac{3}{2} \right) \oplus \left( \mathbf{128}, \frac{1}{2} \right) \oplus \left( \mathbf{16}, \frac{1}{2} \right) \\
& \xrightarrow{d} (\mathbf{9}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{9}) \oplus (\mathbf{1}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{1})
\end{aligned} \tag{27}$$

Since these are the ‘physical’ fermions of maximal supergravity, let us briefly comment on their interpretation.

The  $\mathbf{32}$  representation of  $K(\mathfrak{e}_{10})$  corresponds to the 32 supersymmetry generators of maximal supergravity. We see that in the decomposition (a) relevant for  $D = 11$  supergravity one obtains a single generator consistent with  $\mathcal{N} = 1$  supersymmetry. In the decomposition (b) one obtains an  $\mathfrak{so}(2)$  doublet of sixteen generators (in the vector of  $\mathfrak{so}(16)$ ); the  $\mathfrak{so}(2)$  corresponds to the spatial part of the  $\mathfrak{so}(1, 2)$  Lorentz symmetry of which the doublet is the irreducible spinor and the sixteen components correspond to  $\mathfrak{so}(16)$  R-symmetry of maximal  $\mathcal{N} = 16$  supersymmetry in  $D = 3$  dimensions. The decomposition (c) gives an  $\mathfrak{so}(2)$  R-symmetry doublet of spinors of the spatial  $\mathfrak{so}(9)$  Lorentz symmetry in  $D = 10$  in agreement with the supersymmetry generators

of chiral type IIB supergravity. More specifically, the appearance of this  $U(1)$  effectively ‘complexifies’ the  $SO(9)$  representation, in line with the chirality of the type IIB fermions. The last decomposition (*d*) is consistent with a type IIA formulation of doubled supergravity [8, 25].

The decompositions of the **320** representation of  $K(\mathfrak{e}_{10})$  can be interpreted similarly [24]. For example, the decomposition (*b*) gives the 128 physical fermions in  $D = 3$  together with components associated with the non-propagating gravitino that is needed when formulating  $\mathcal{N} = 16$  supergravity in  $D = 3$ . We also note again that in the type IIB decomposition (*c*) one always obtains doublets of the R-symmetry  $\mathfrak{so}(2)$ , in accord with the chirality of the underlying fermionic multiplets.

## 6.2 Branching of the spin- $\frac{5}{2}$ representation

The decomposition under the various subalgebras is

$$\begin{aligned}
\mathbf{1728} &\xrightarrow{a} \mathbf{1120} \oplus 2 \times \mathbf{288} \oplus \mathbf{32} \\
&\xrightarrow{b} \left(\frac{1}{2}, \mathbf{560}_v\right) \oplus \left(\frac{1}{2}, \mathbf{128}_c\right) \oplus 2 \times \left(\frac{1}{2}, \mathbf{16}_v\right) \oplus \left(\frac{3}{2}, \mathbf{128}_c\right) \oplus \left(\frac{3}{2}, \mathbf{16}_v\right) \\
&\xrightarrow{c} \left(\mathbf{432}, \frac{1}{2}\right) \oplus 2 \times \left(\mathbf{128}, \frac{1}{2}\right) \oplus 2 \times \left(\mathbf{16}, \frac{1}{2}\right) \oplus \left(\mathbf{128}, \frac{3}{2}\right) \oplus \left(\mathbf{16}, \frac{3}{2}\right) \\
&\xrightarrow{d} (\mathbf{36}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{36}) \oplus (\mathbf{9}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{9}) \oplus \\
&\quad \oplus (\mathbf{128}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{128}) \oplus (\mathbf{1}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{1}) \tag{28}
\end{aligned}$$

From the  $\mathfrak{so}(10)$  decomposition we see that the space-time spin of this  $K(\mathfrak{e}_{10})$  representation is not really higher than  $3/2$  since the **1120** corresponds to an *anti*-symmetric tensor-spinor of  $\mathfrak{so}(10)$  with two tensor indices. The  $\mathbf{560}_v$  of  $\mathfrak{so}(16)$  that arises is the anti-symmetric three-form. Similar to the  $\mathbf{16}_v$  discussed above, this is actually a spinorial representation with the correct assignment when lifted to the group  $Spin(16)/\mathbb{Z}_2$ . The **432** of  $\mathfrak{so}(9)$  that arises in case (*c*) is the tensor-spinor with two antisymmetric indices.

## 6.3 Branching of the spin- $\frac{7}{2}$ representation

Under the subalgebras listed above, the  $K(\mathfrak{e}_{10})$  spin- $\frac{7}{2}$  representation of dimension 7040 decomposes as

$$\begin{aligned}
\mathbf{7040} &\xrightarrow{a} \mathbf{2400} \oplus \mathbf{1440} \oplus 2 \times \mathbf{1120} \oplus 3 \times \mathbf{288} \oplus 3 \times \mathbf{32} \\
&\xrightarrow{b} \left(\frac{1}{2}, \mathbf{1920}_s\right) \oplus \left(\frac{3}{2}, \mathbf{560}_v\right) \oplus \left(\frac{1}{2}, \mathbf{560}_v\right) \oplus \left(\frac{3}{2}, \mathbf{128}_c\right) \oplus 2 \times \left(\frac{1}{2}, \mathbf{128}_c\right) \\
&\quad \oplus \left(\frac{5}{2}, \mathbf{16}_v\right) \oplus 2 \times \left(\frac{3}{2}, \mathbf{16}_v\right) \oplus 3 \times \left(\frac{1}{2}, \mathbf{16}_v\right) \\
&\xrightarrow{c} \left(\mathbf{768}, \frac{1}{2}\right) \oplus \left(\mathbf{576}, \frac{1}{2}\right) \oplus \left(\mathbf{432}, \frac{3}{2}\right) \oplus 2 \times \left(\mathbf{432}, \frac{1}{2}\right)
\end{aligned}$$

$$\begin{aligned}
& \oplus 2 \times \left( \mathbf{128}, \frac{3}{2} \right) \oplus 4 \times \left( \mathbf{128}, \frac{1}{2} \right) \oplus \left( \mathbf{16}, \frac{5}{2} \right) \oplus 2 \times \left( \mathbf{16}, \frac{3}{2} \right) \oplus 4 \times \left( \mathbf{16}, \frac{1}{2} \right) \\
& \xrightarrow{d} (\mathbf{128}, \mathbf{9}) \oplus (\mathbf{128}, \mathbf{1}) \oplus (\mathbf{16}, \mathbf{84}) \oplus (\mathbf{16}, \mathbf{36}) \oplus 2 \times (\mathbf{16}, \mathbf{9}) \oplus 2 \times (\mathbf{16}, \mathbf{1}) \\
& \oplus (\mathbf{9}, \mathbf{128}) \oplus (\mathbf{1}, \mathbf{128}) \oplus (\mathbf{84}, \mathbf{16}) \oplus (\mathbf{36}, \mathbf{16}) \oplus 2 \times (\mathbf{9}, \mathbf{16}) \oplus 2 \times (\mathbf{1}, \mathbf{16}) \quad (29)
\end{aligned}$$

As already mentioned, it is a non-trivial task to work out these decompositions in practice. As a further test we have also checked that the further decompositions of the  $\mathfrak{so}(2) \oplus \mathfrak{so}(16)$  and  $\mathfrak{so}(10)$  representations under their common  $\mathfrak{so}(8)$  subalgebra coincide (for  $\mathfrak{so}(16)$  this subalgebra is obtained after descending first to the diagonal subalgebra  $[\mathfrak{so}(8) \oplus \mathfrak{so}(8)]_{\text{diag}}$ ). Similarly, there is another  $\mathfrak{so}(8)$  that is common to the  $\mathfrak{so}(10)$  decomposition (a), to the type IIB decomposition (c) and to the  $\mathfrak{so}(9) \oplus \mathfrak{so}(9)$  decomposition in (d) and that corresponds to the spatial rotations of maximal  $D = 9$  supergravity. The further branching of (a), (c) and (d) to this common subgroup has been checked to be consistent. Moreover, we have verified that the common  $\mathfrak{so}(9)$  of the type IIB decomposition (c) and the T-duality agnostic decomposition (d) gives the same representations.

Let us finally highlight some new features arising here, that have no analog for spin  $s \leq \frac{5}{2}$ .

- In the  $\mathfrak{so}(10)$  decomposition (a) one sees the **2400** that corresponds to a tensor-spinor that is antisymmetric in *three* tensor indices. The **1440** is a tensor-spinor with two *symmetric* tensor indices; since the  $\mathfrak{so}(10)$  is the spatial rotation group of  $D = 11$  supergravity, this means that the spin- $\frac{7}{2}$  of  $K(\mathfrak{e}_{10})$  contains genuinely higher spin representations also from a space-time perspective!
- Under the  $\mathfrak{so}(2) \oplus \mathfrak{so}(16)$  decomposition (b) one finds the vector-spinor of  $\mathfrak{so}(16)$  with 1920 components. Note that consistent with the spinorial nature of the  $K(E_{10})$  representation it is the  $\mathbf{1920}_s$ , where the spinorial double-valued aspect of  $Spin(16)/\mathbb{Z}_2$  is carried by the vector index and not by the  $s$ -type spinor index.
- The **768** appearing in the  $\mathfrak{so}(9) \oplus \mathfrak{so}(2)$  decomposition (c) is the anti-symmetric three-form tensor-spinor of  $\mathfrak{so}(9)$ . By contrast the **576** is a tensor-spinor with two *symmetric* tensor indices and therefore this  $K(\mathfrak{e}_{10})$  representation also contains fermionic higher spin fields from the type IIB perspective.
- The **84** in the  $\mathfrak{so}(9) \oplus \mathfrak{so}(9)$  decomposition (d) is the anti-symmetric three-form of  $\mathfrak{so}(9)$ ; the **36** is the anti-symmetric two-form already encountered above.

We also note that the  $\mathfrak{so}(2)$  eigenvalues can become larger and larger the bigger the  $K(\mathfrak{e}_{10})$  representation becomes.

## 7 Outlook

There are two pressing questions arising out of our work. The first concerns the possible physical role of the new  $K(E_{10})$  representations. In particular, one may wonder whether they are of

relevance to overcoming the difficulties in constructing a supersymmetric  $E_{10}$  model that were encountered in Ref. [11]. It is conceivable that in order to make progress both the supersymmetry constraint and the propagating fermions will have to be assigned to representations of  $K(E_{10})$  different from the ones used so far (and in particular incorporate spatial gradients in one form or another). Let us also note that one can easily couple the new fermion representations to the bosonic  $E_{10}/K(E_{10})$  sigma model, namely by adding a Dirac-like term  $\propto \Psi D_t \Psi$  to the bosonic action, where  $D_t \equiv \partial_t + \sum_{\alpha,r} Q^r(\alpha) J^r(\alpha)$  is the  $K(E_{10})$  covariant derivative, and  $Q^r(\alpha)$  the  $K(\mathfrak{e}_{10})$ -connection as computed from the bosonic sigma model in the standard way. Of course, there remains the question whether one can define a new supersymmetry that makes the combined action supersymmetric at least at low levels.

Secondly, the very existence of the two new higher spin representations for  $s = \frac{5}{2}$  and  $s = \frac{7}{2}$  which cannot be explained from maximal supergravity, strongly suggests that these constitute only the tip of the iceberg of the unexplored representation theory of  $K(\mathfrak{e}_{10})$ . Although our (limited) search for new examples has not been successful so far, we expect there to exist an infinite tower of such realisations of higher and higher spin, which are less and less unfaithful with increasing spin, but which can occur only at ‘sporadic’ values of the spin, because the simultaneous decomposability under all the subgroups analysed in the foregoing section puts very tight constraints on such new representations. [27] We reiterate that working out these decompositions is currently a tedious task due to the lack of general methods for studying the representation theory of  $K(\mathfrak{e}_{10})$ . An explicit construction of further examples and, more ambitiously, a systematic understanding of their structure would afford an entirely new method to explore the root spaces associated with timelike imaginary roots, and thus one of the main obstacles towards a better understanding of  $\mathfrak{e}_{10}$ . One step forward might be the understanding of the decomposition of tensor products of  $K(\mathfrak{e}_{10})$  representations. We thus hope that our investigations help to clarify the structure of this enigmatic object and maybe also the elusive Kac–Moody algebra  $\mathfrak{e}_{10}$  itself.

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- [18] In the remainder we will not always put quotation marks when we talk of ‘spin’. We trust that readers will understand that this terminology is to be taken with a grain of salt, cf. Ref. [13].
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