Higher Spin Black Holes in Three Dimensions: Comments on Asymptotics and Regularity

Máximo Bañados[a], Rodrigo Canto[a] and Stefan Theisen[b]

[a]Departamento de Física, P. Universidad Católica de Chile, Santiago 22, Chile

[b] Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), 14476 Golm, Germany

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Abstract

In the context of (2+1)–dimensional $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ Chern-Simons theory we explore issues related to regularity and asymptotics on the solid torus, for stationary and circularly symmetric solutions. We display and solve all necessary conditions to ensure a regular metric and metric-like higher spin fields. We prove that holonomy conditions are necessary but not sufficient conditions to ensure regularity, and that Hawking conditions do not necessarily follow from them. Finally we give a general proof that once the chemical potentials are turn on – as demanded by regularity – the asymptotics cannot be that of Brown-Henneaux.

1 Introduction

Higher spin black holes in three dimensions have been analyzed in great detail recently — see [1] for a review of the relevant features — mostly in their Chern-Simons formulation, i.e. in terms of the properties of flat connections on a manifold with the topology of a solid torus. While a metric and higher spin fields can be introduced in terms of the connections, discussing properties of the black hole in terms of the metric encounters difficulties. This is because the symmetry group is now much larger than the diffeomorphisms group and this makes the definition of a horizon ambiguous.

In particular, it has been known for a long time that for Chern-Simons black holes the radial coordinate can be gauged away and most of their properties can be studied from connections at a fixed radius [2]. Nevertheless, at the end of the day one should be able to write spacetime fields, globally well defined on the manifold under consideration. The aim of this note is to discuss the issue of horizon regularity and asymptotics, introducing a radial coordinate in an appropriate way.

As already alluded to, the use of the word 'horizon' is tricky because in higher spin theories there is no global definition for it. It is a gauge dependent concept. Without attempting to solve this problem, what we do is to consider $sl(N,\mathbb{R})$ gauge fields on a solid torus. For the case $N = 2$ one knows that these fields can be interpreted as black holes. The question is whether extending to $N > 2$, one can think of these solutions as

black holes carrying some extra charges, and whether for small values of these charges, they can be seen as perturbations of the $N = 2$ case. Finally, we are also interested in the question whether these solutions can be regarded as perturbations of AdS space for large r, consistently with regularity on the whole solid torus.

The correct asymptotic conditions in terms of a radial coordinate were analyzed in detail in [3]. Precise fall-off behaviour for all fields is described, ensuring the W_3 symmetry acting as extended higher spin diffeomorphisms. We shall prove here that there is an incompatibility between regularity and AdS asymptotics, as described in [3]. In other words, the conditions spelled out there cannot be achieved simultaneously with regularity at the center of the solid torus, within the class of stationary and circularly symmetric solutions.

Before plunging into details, we briefly summarize the content of this note. In the next section we carefully consider the issue of regularity at the center of the solid torus and how this restricts the near-horizon structure of the metric and higher spin fields. In Sec. 3 we analyze the general structure of solutions near the horizon and characterize them in terms of the eigenvalues of the holonomies. We explicitly solve the holonomy conditions by expressing the chemical potentials in terms of the spin-2 and spin-3 charges in Section 4. The simplest regular solution with four charges is constructed in Section 5. The asymptotics of this solution changes if one switches off the spin-3 charge. In the last section we show that this a general feature: we give an argument which shows that regularity at the horizon clashes with AdS asymptotics of the solution. Most of our discussion is specific to $N = 3$, but can be generalized to arbitrary N.

2 Regularity in the vicinity of the 'horizon'

We are concerned with $sl(N, \mathbb{R})$ gauge fields on the solid torus. We parametrize this manifold by three coordinates with ranges,

$$0 \leq \quad t \quad < 2\pi \tag{1}$$

$$0 \leq \quad r \quad < \infty \tag{2}$$

$$0 \leq \quad \phi \quad < 2\pi \tag{3}$$

The Euclidean time coordinate t is periodic and describes the contractible cycle. The surfaces ϕ=constant are planes parametrized by r, t, which are polar-like coordinates around the center of the solid torus. We use the convention where $r = 0$ describes the center of the solid torus. We loosely call this surface 'the horizon' since for $N = 2$ one recovers the 2+1 BTZ black hole [4].

Regularity of gauge fields is often expressed as a condition on holonomies over the contractible cycle in the (Euclidean) t-direction. This is a necessary but not a sufficient condition. Further restrictions come from local properties at the center of the solid torus. This problem exists irrespective of the presence of higher spin fields and is, for example, already encountered in the three-dimensional pure gravity theory.

Riemann curvature technology, available for spin two, allows to analyze singularity/regularity questions related to the metric $g_{\mu\nu}$. But there is no analogous structure

for higher spin fields. Therefore, in order to establish regularity/irregularity of these fields, we need to consider them directly.

The basic idea is to express all components of a given field in a regular set of coordinates (e.g. Cartesian), and demand the components to be regular and well-defined (i.e. they should have unique values) [5]. These conditions are not new to black hole physics. The new ingredient here are the higher spin fields. We shall analyze the conditions on the components of the spin-3 field $g_{\mu\nu\rho}$ that lead to a sensible and regular solution. The generalization to higher spins $N > 3$ is straightforward.

Regularity of $r^2 dt^2 + dr^2$ (with $0 \le t < 2\pi$) is well-known. It is simply the metric on the Euclidean plane expressed in polar coordinates (which do not cover the whole plane because they are singular at the origin).

Even though this combination is a regular rank-2 tensor, it is not often stressed that both $r\,dt$ and dr are individually *singular* one-forms. We explain this point in detail. The extension to spin-3 is then straightforward.

The transformation between Cartesian (regular) and polar (irregular) coordinates $r = \sqrt{x^2 + y^2}$ and $t = \arctan(y/x)$ implies the relationship between the one-forms

$$dr = \frac{xdx + ydy}{\sqrt{x^2 + y^2}} \qquad dt = \frac{xdy - ydx}{x^2 + y^2} \qquad (4)$$

While r is well defined at the origin $x = y = 0$, t is not as x/y is undefined there. Regularity of any tensor component requires that it is finite and that it has a unique value. Uniqueness forbids undetermined expressions of the form $0/0$. Clearly dt violates the finiteness condition while $r dt$ violates the uniqueness condition. To see this very explicitly, approach the origin along a path with slope λ, i.e. set $y = \lambda\, x$. This reveals that $r\, dt$ is finite but it depends on λ. The same is true for dr and, for that matter, for any $0/0$ expression.[1].

In the same way it is straightforward to see that $ds^2 = \alpha^2 r^2 dt^2 + dr^2$, with α a constant, is singular except for $\alpha = 1$. It is flat away from the origin where it has a conical singularity if $\alpha \ne 1$. If $\alpha = n$ is an integer one finds an excess of $2\pi(n-1)$. A vector parallel transported around the origin comes back to itself, and the geometry appears to be regular. This is however not the case. This is one example (we shall see others) where topological conditions are not sufficient for regularity. They are merely necessary conditions.

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[1]As a byproduct of this analysis consider the Reissner-Nordstrom black hole in $d = 4$ with metric and gauge field (near the horizon at $r = r_0$)

$$\begin{aligned} ds^2 &= \left(1 - \frac{r}{r_0}\right) dt^2 + \frac{dr^2}{1 - \frac{r}{r_0}} + r^2 d\Omega \\ A &= \left(\frac{q}{r} - \frac{q}{r_0}\right) dt \end{aligned} \qquad (5)$$

The constant piece in the gauge field is added in order to have $A(r = r_0) = 0$. One may naively conclude that the coefficient of dt in A must vanish linearly to have regularity. But one must be careful with the nature of coordinates. The Schwarzschild coordinate r is not a proper coordinate. The proper radial coordinate ρ for this metric is related to r by $\rho = 2\sqrt{r - r_0}$. Thus, when writing the solution in terms of the proper coordinate one finds the gauge field $A \sim \rho^2 dt$, which is indeed regular (while $\rho\, dt$ is not).

Turning now to the spin-3 field, the most general Ansatz which guarantees finiteness at the origin $r = 0$ is

$$
\begin{aligned}
ds_3^3 = \; & f_1 \, r^3 dt^3 + f_2 \, dr^3 + f_3 \, d\phi^3 + f_4 \, r^2 dt^2 dr + f_5 \, r^2 dt^2 d\phi + f_6 \, dr^2 r dt \\
& + f_7 \, dr^2 d\phi + f_8 \, d\phi^2 r dt + f_9 \, d\phi^2 dr + f_{10} \, r dt dr d\phi
\end{aligned} \tag{6}
$$

The coefficients f_i are constants. Higher orders in r have been suppressed; they are not restricted by regularity conditions. Recall that ϕ is the coordinate on the non-contractible cycle of the solid torus and therefore $d\phi$ is regular. This means, in particular, that f_3 can be non-zero at the horizon. Expressing the dr and dt via (4) in terms of dx and dt and requiring the absence of $0/0$ expressions, reveals that for the spin-3 field to be regular at $r = 0$ it must have the form

$$
ds_3^3 = f_5 (r^2 dt^2 + dr^2) d\phi + f_3 d\phi^3 \tag{7}
$$

All solutions found in the literature so far have precisely this structure, namely

$$
ds_3^3 = d\phi \times ds^2 \, (\text{black hole}) \tag{8}
$$

even away from the origin.

Note that the shift function, characterizing angular momentum, does not appear at leading order in the near horizon geometry. This issue affects the spin-2 and spin-3 fields. For the metric one expects a term $N^\phi dt d\phi$. For reasons explained above, dt is singular and a necessary condition for regularity is that N^ϕ must vanish at the horizon. We thus see that imposing not only finiteness but also single-valuedness imply that N^ϕ must vanish at least quadratically at the horizon. This is of course the case for the BTZ metric, as can be verified directly and also for the Kerr solution.

3   General near horizon analysis

We now analyze the general structure of solutions near the horizon. We turn on all charges. In particular, we do not set the charges in a_ϕ and b_ϕ equal, which would correspond to 'non-rotating' solutions.[2] This case was mostly considered so far in the literature. An important lesson learned from the analysis of this section is that holonomy conditions imposed on gauge fields do not imply regularity on the metric-like fields. To be specific we consider $N = 3$, but it will be clear that all our results are valid for any N.

We consider solutions of $F_{\mu\nu} = 0$

$$
\begin{aligned}
A_t &= g_1^{-1} a_t \, g_1 & B_t &= g_2^{-1} b_t \, g_2 \\
A_\phi &= g_1^{-1} a_\phi \, g_1 & B_\phi &= g_2^{-1} b_\phi \, g_2 \\
A_r &= g_1^{-1} \partial_r g_1 & B_r &= g_2^{-2} \partial_r g_2
\end{aligned} \tag{9}
$$

where

$$
[a_t, a_\phi] = 0 \qquad [b_t, b_\phi] = 0 \tag{10}
$$

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[2]For background and notation see e.g. [6].

Here $g_1, g_2 \in SL(3)$ are functions of r and $a_t, \ldots, b_\phi \in sl(3)$ are constant. From these two connections we form the dreibein $e_\mu = \frac{1}{2}(A_\mu - B_\mu)$ and the spin-2 and spin-3 fields [7]

$$g_{\mu\nu} = \text{Tr}(e_\mu e_\nu) \qquad g_{\mu\nu\rho} = \text{Tr}(e_{(\mu} e_\nu e_{\rho)}) \tag{11}$$

We seek group elements g_1 and g_2 and matrices a_t, b_t, a_ϕ, b_ϕ such that $g_{\mu\nu}$ and $g_{\mu\nu\rho}$ are regular. The regularity conditions can be split into three classes, on which we elaborate below.

1. **Topological:** a_t and b_t must have trivial holonomies on the contractible cycle

$$e^{\oint a_t dt} = e^{\oint b_t dt} = \mathbb{1} \tag{12}$$

2. **Finiteness:** Since the 1-form dt is singular, the time component of the vielbein $e_\mu dx^\mu$ must satisfy

$$e_t = 0 \qquad \text{at} \qquad r = 0 \tag{13}$$

3. **Single-valuedness:** As discussed above, in the vicinity of the horizon at $r = 0$, the metric and spin-3 field must read[3],

$$
\begin{aligned}
ds_2^2 = & \quad f_1(r^2 dt^2 + dr^2) + f_2 d\phi^2 + \text{ (higher powers of } r) \\
ds_3^3 = & \left(f_3(r^2 dt^2 + dr^2) + f_4 d\phi^2 \right) d\phi + \text{ (higher powers of } r)
\end{aligned}
\tag{14}
$$

where f_i are constants. All other components must vanish at $r = 0$ at a sufficiently high order. Special attention will be given to whether holonomy conditions suffice to guarantee the Hawking conditions

$$\frac{g_{tt}}{g_{rr}} = \frac{g_{tt\phi}}{g_{rr\phi}} = r^2 \tag{15}$$

as required by regularity.

Let us study conditions 1., 2. and 3. one by one.

3.1 Holonomy conditions and diagonal fields

Eqn. (12) means that a_t and b_t must have trivial holonomy and live in the Jordan class of diagonal matrices, with $i\times$(integer eigenvalues). This means there exists constant matrices M_1, M_2 such that

$$
\begin{aligned}
a_t = & \quad M_1^{-1} a_t^{(d)} M_1 \qquad a_t^{(d)} = i \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & -p_1 - p_2 \end{pmatrix} \\
b_t = & \quad M_2^{-1} b_t^{(d)} M_2 \qquad b_t^{(d)} = i \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & -q_1 - q_2 \end{pmatrix}
\end{aligned}
\tag{16}
$$

---

[3]Recall that angular momentum $(dt d\phi)$ only appears at $\mathcal{O}(r^2)$ at the horizon.

where, to satisfy condition (12), p_1, p_2, q_1, q_2 must be integers. We shall see below that further regularity conditions demand $q_1 = p_1$ and $q_2 = p_2$, up to trivial permutations of the eigenvalues.

Note also (10), i.e. that a_t commutes with a_ϕ, and b_t commutes with b_ϕ. This implies that M_1, M_2 also make a_ϕ, b_ϕ diagonal, that is

$$
\begin{aligned}
a_\phi &= M_1^{-1} a_\phi^{(d)} M_1 & a_\phi^{(d)} &= \begin{pmatrix} -2\lambda_1 & 0 & 0 \\ 0 & \lambda_1 + \lambda_2 & 0 \\ 0 & 0 & \lambda_1 - \lambda_2 \end{pmatrix} \\
b_\phi &= M_2^{-1} b_\phi^{(d)} M_2 & b_\phi^{(d)} &= \begin{pmatrix} -2\rho_1 & 0 & 0 \\ 0 & \rho_1 + \rho_2 & 0 \\ 0 & 0 & \rho_1 - \rho_2 \end{pmatrix}
\end{aligned}
\tag{17}
$$

The four charges T_1, W_1, T_2, W_2 are now parameterized in terms of the four eigenvalues $\lambda_1, \lambda_2, \rho_1, \rho_2$. The exact relation is not necessary for this analysis. We shall come back to this point in Sec. 4 where the above parametrization, which was introduced in [8], proves very convenient.

Note that for generic values of the two charges T and W the three eigenvalues of a_ϕ are nondegenerate and as long as the discriminant of the characteristic equation is positive, i.e. $4T^3 - 27W^2 > 0$, they are all real. In this case the matrix M_1 which diagonalizes a_ϕ (and also a_t) is also real. The condition on the discriminant is also the condition for the black hole to be above extremality [9].

3.2 A finite vielbein on the torus: $e_t = 0$ at $r = 0$

The 1-form dt is singular and must always appear multiplied by a function that vanishes at the horizon. This condition may seem natural and easy to implement. Note however, that as N increases, the number of conditions grows rapidly. In the $sl(N) \times sl(N)$ theory, there are $N-1$ metric-like fields $g_{\mu\nu}, g_{\mu\nu\rho}, g_{\mu\nu\rho\sigma}$, etc. Each of these fields has components involving the 1-form dt at various orders. To give some examples,

$$
\begin{aligned}
ds^2 &= \cdots + g_{tt}\, dt^2 + g_{t\phi}\, dt d\phi + \cdots \\
ds^3 &= \cdots + g_{ttt}\, dt^3 + g_{tt\phi}\, dt^2 d\phi + \cdots \\
ds^4 &= \cdots + g_{tttt}\, dt^4 + g_{ttt\phi}\, dt^3 d\phi + g_{tt\phi\phi}\, dt^2 d\phi^2 + \cdots
\end{aligned}
\tag{18}
$$

where

$$
g_{\mu_1 \dots \mu_n} = \mathrm{Tr}(e_{(\mu_1} \cdots e_{\mu_n)})
\tag{19}
$$

Imposing the necessary conditions to achieve regularity term by term may become quite tedious. Fortunately, the gauge formulation allows a solution that applies to all components at once.

The idea is to implement regularity on the connections, A_μ, B_μ rather than on the metric-like fields. But there is a subtlety which we now discuss. In principle, the connections $A_\mu(x)dx^\mu$ and $B_\mu(x)dx^\nu$ can be regular functions if they satisfy

$$
A_t(r) \to 0 \qquad B_t(r) \to 0 \qquad \text{as} \qquad r \to 0
\tag{20}
$$

(plus other conditions from single-valuedness). The trouble is that stationary and circularly symmetric solutions cannot fulfill this property. Recall

$$A_t(r) = g^{-1}(r)\, a_t\, g(r) \tag{21}$$

where a_t is constant. Therefore, if $A_t(r_0) = 0$ at any point r_0, then invertibility of g implies $a_t = 0$ and hence $A_t(r) = 0$ at all points. The same holds for B_t.

This obstruction has nothing to do with higher spin fields and shows up for $sl(N)$, including the very well-known case $N = 2$. The static, circularly symmetric connections associated to BTZ black holes satisfy the following properties. The dreibein, the difference of both connections,

$$e_\mu = \frac{1}{2}(A_\mu - B_\mu) \tag{22}$$

is regular at the horizon, $e_t = 0$ at $r = 0$, but the spin connection

$$w_\mu = \frac{1}{2}(A_\mu + B_\mu)$$

is singular. This means that the metric field is regular but not the connection. Of course the curvature is again regular. Therefore, as long as we are interested in regularity of the metric-like fields $g_{\mu_1\mu_2\ldots}$, it suffices to impose

$$e_t(r) \to 0 \qquad \text{as} \qquad r \to 0. \tag{23}$$

One may wonder whether (23) implies conditions that could over-determine the problem. This is not case. From (22) we see that (23) implies

$$g_1^{-1}a_t g_1 - g_2^{-1}b_t g_2 = 0 \qquad \Rightarrow \qquad b_t = S^{-1}a_t S \tag{24}$$

where $S := g_1 g_2^{-1}$. In words, our regularity condition imply that a_t and b_t must be in the same $sl(N, \mathbb{R})$ class. We know already that a_t and b_t are both in the diagonal class because they must satisfy (12). Condition (24) now implies that they must satisfy

$$\text{Tr}(a_t^2) = \text{Tr}(b_t^2) \qquad \text{Tr}(a_t^3) = \text{Tr}(b_t^3) \tag{25}$$

These conditions can be solved by setting

$$q_1 = p_1 \qquad \text{and} \qquad q_2 = p_2 \tag{26}$$

in (16), that is

$$b_t^{(d)} = a_t^{(d)} \tag{27}$$

3.3   Single valued fields on the torus

We now address condition (14) and build fields with all desired properties at the center of the solid torus. Collecting the information from the two previous paragraphs, we first note that the vielbein components can be written as

$$e_t = V^{-1}\left(U^{-1}a_t^{(d)}U - a_t^{(d)}\right)V$$
$$e_\phi = V^{-1}\left(U^{-1}a_\phi^{(d)}U - b_\phi^{(d)}\right)V \tag{28}$$
$$e_r = V^{-1}\left(U^{-1}\partial_r U\right)V$$

where $V = M_2\, g_2$ and $U = M_1\, g_1\, g_2^{-1} M_2^{-1}$ (and we have used (27)). Since the components $g_{\mu\nu}, g_{\mu\nu\rho}$ are extracted from the traces $\text{Tr}(e^2)$ and $\text{Tr}(e^3)$, the matrix V is irrelevant and can be set to one with no loss of generality. From now on we assume

$$V = \mathbb{1}$$

The horizon (center of the solid torus) is defined as the point where $e_t = 0$. Inspecting (28) we see that this requires that $U(r = 0)$ must commute with $a_t^{(d)}$. Then, in the vicinity of the horizon we expand, without loss of generality,

$$U = \mathbb{1} + rX_1 + r^2 X_2 + r^3 X_3 + \cdots \tag{29}$$

where X_1, X_2, X_3 are, at this point, general matrices, only constrained by the condition $\det U = 1$.

To implement conditions (14) requires a bit more work, but is straightforward. We plug (29) into (28), ignoring the factor V, and compute the traces (11) to first order in r. This gives expressions for $g_{\mu\nu}$ and $g_{\mu\nu\rho}$. To impose all regularity conditions (14) at $r = 0$ we shall not need to go beyond the linear term in (29), i.e. X_2, X_3 and the higher orders are not constrained by the regularity conditions at the horizon. To comply with $\det(U) = 1$ we set the matrix X_1 to be

$$X_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & -x_1 - x_5 \end{pmatrix} \tag{30}$$

The general expressions obtained are long and not worth displaying here, but we show some components that should be zero, which implies conditions on X_1. For example,

$$g_{r\phi} = 2\, x_1(\lambda_2 - \rho_2 - 3\,\lambda_1 + 3\,\rho_1) + 4\, x_5(\lambda_2 - \rho_2) + \mathcal{O}(r)\,. \tag{31}$$

From (14) we know that this component must vanish at $r = 0$. This is our first condition on the allowed group elements. One can set $g_{r\phi} = 0$ to order $\mathcal{O}(1)$ by solving (31) with respect to x_5

$$x_5 = \frac{3\,(\lambda_1 - \rho_1) - (\lambda_2 - \rho_2)}{\lambda_2 - \rho_2}\, \frac{x_1}{2} \tag{32}$$

No other conditions follow from (14).

We now move to the spin-3 field. First, one finds a non-zero value for

$$g_{\phi\phi r} = 3\, x_1 \big(3(\lambda_1 - \rho_1) - (\lambda_2 - \rho_2)\big)\big(3(\lambda_1 - \rho_1) + (\lambda_2 - \rho_2)\big) + \mathcal{O}(r)\,. \tag{33}$$

According to (14) this component must vanish at $r = 0$. The only solution, keeping all charges unconstrained is,

$$x_1 = 0 \tag{34}$$

Next, we note the components

$$g_{ttt} = -3(p_1 - p_2)(p_1 + 2p_2)(p_2 + 2p_1)(x_2 x_6 x_7 - x_3 x_4 x_8)\, r^3 + \mathcal{O}(r^4)$$
$$g_{ttr} = -9(p_1^2 + p_2^2 + p_1 p_2)(x_2 x_6 x_7 + x_3 x_4 x_8)\, r^2 + \mathcal{O}(r^3) \tag{35}$$

which, as discussed before, should vanish at the order displayed. For generic integers p_1, p_2 this requires

$$x_2 x_6 x_7 = 0 \qquad \text{and} \qquad x_3 x_4 x_8 = 0 \tag{36}$$

and there are no further conditions from the regularity of g_3.

The equations (36) can be solved in various ways. For instance for $\{x_3 = x_6 = 0\}$ and $\{x_7 = x_8 = 0\}$ one finds

$$\frac{g_{tt}}{g_{rr}} = \frac{g_{tt\phi}}{g_{rr\phi}} = (p_2 - p_1)^2 \, r^2 + \mathcal{O}(r^3) \tag{37}$$

For $\{x_2 = x_8 = 0\}$ and $\{x_4 = x_6 = 0\}$ one obtains

$$\frac{g_{tt}}{g_{rr}} = \frac{g_{tt\phi}}{g_{rr\phi}} = (2 \, p_1 + p_2)^2 \, r^2 + \mathcal{O}(r^3) \tag{38}$$

and for $\{x_2 = x_3 = 0\}$ and $\{x_4 = x_7 = 0\}$

$$\frac{g_{tt}}{g_{rr}} = \frac{g_{tt\phi}}{g_{rr\phi}} = (2 \, p_2 + p_1)^2 \, r^2 + \mathcal{O}(r^3) \tag{39}$$

Clearly, they follow from each other by permutation of the eigenvalues of a_t and b_t. The three remaining minimal solutions (with only one factor from each triplet in (36) set to zero) give ratios g_{tt}/g_{rr} and $g_{tt\phi}/g_{rr\phi}$ which (i) depend on some of the remaining non-zero $x's$ and (ii) are not equal. We will discuss one of them further in Section 5.

For the solutions displayed above regularity can be achieved in various ways: (a) $p_2 = p_1 \pm 1$, (b) $p_2 = -2 \, p_1 \pm 1$ and (c) $p_1 = -2 \, p_2 \pm 1$. But clearly there are other possibilities which lead to a conical excess. Even for $p_1 = 1, p_2 = 0$, which is the case mostly discussed in the literature, there are two options, one leading to a regular solution and the other to a conical surplus by 2π.

4 Solution of the holonomy conditions

So far we have parametrized the solutions of the holonomy conditions by two integer eigenvalues of a_t, p_1 and p_2, and we have also parametrized a_ϕ by its two eigenvalues, λ_1 and λ_2. We have used that a_t and a_ϕ can be simultaneously diagonalized, but we have not established relations between the eigenvalues and physical quantities, i.e. charges and chemical potentials which enter the thermodynamic description developed in [6]. When that description is used, the holonomy conditions are solved by expressing the charges in terms of the chemical potentials, which amounts to solving cubic equations (for $N = 3$). To identify the physical charges in our solutions, we use, following [6], the Casimirs of a_ϕ. This allows for a generic solution of the holonomy conditions which expresses the chemical potentials through the charges which are parametrized by λ_1 and λ_2 and which is compatible with the analysis in [6].

We will give a brief constructive description of how to arrive at this simple generic solution of the holonomy conditions. Consider the usual principally embedded $sl(2) \rightarrow$

$sl(3)$ gauge connections in highest weight gauge[4]

$$a_\phi = \begin{pmatrix} 0 & T/2 & W \\ 1 & 0 & T/2 \\ 0 & 1 & 0 \end{pmatrix} \tag{40}$$

and[5]

$$a_t = i\,\mu\,a_\phi + i\,\nu\left(a_\phi^2 - \frac{1}{3}\mathrm{Tr}(a_\phi^2)\mathbb{1}\right) \tag{41}$$

T and W are the charges and μ and ν the chemical potentials. Using (17) we can parametrize the charges in terms of the a_ϕ eigenvalues as

$$\begin{aligned}
T &= \frac{1}{2}\mathrm{Tr}(a_\phi^2) = 3\,\lambda_1{}^2 + \lambda_2{}^2 \\
W &= \frac{1}{3}\mathrm{Tr}(a_\phi^3) = -2\,\lambda_1\,(\lambda_1 - \lambda_2)\,(\lambda_1 + \lambda_2)
\end{aligned} \tag{42}$$

If we now combine a_t in (40) with (16) and (17), we can solve for μ, ν in terms of λ_1, λ_2:

$$\begin{aligned}
\mu_{(p_1,p_2)} &= \frac{1}{2}\frac{\left(3\,\lambda_1{}^2 - 6\,\lambda_1\,\lambda_2 - \lambda_2{}^2\right)p_1}{\lambda_2\,(3\,\lambda_1 - \lambda_2)\,(3\,\lambda_1 + \lambda_2)} + \frac{\left(3\,\lambda_1{}^2 - \lambda_2{}^2\right)p_2}{\lambda_2\,(3\,\lambda_1 - \lambda_2)\,(3\,\lambda_1 + \lambda_2)} \\
\nu_{(p_1,p_2)} &= \frac{3}{2}\frac{(\lambda_1 + \lambda_2)\,p_1}{\lambda_2\,(3\,\lambda_1 - \lambda_2)\,(3\,\lambda_1 + \lambda_2)} + \frac{3\lambda_1\,p_2}{\lambda_2\,(3\,\lambda_1 - \lambda_2)\,(3\,\lambda_1 + \lambda_2)}
\end{aligned} \tag{43}$$

It is worth stressing that these expressions for the chemical potentials are the general solutions of the holonomy conditions. They are labeled by the two integers p_1, p_2 which parameterize all possible holonomy branches, while λ_1, λ_2 parametrizes the charges.

The above result is purely algebraic. We will now demonstrate its consistency with the thermodynamic analysis of [6]. From (42) it is clear that the eigenvalues are implicitly defined in terms of the charges, i.e. $\lambda_1(T, W), \lambda_2(T, W)$. Consider

$$G[T, W]_{(p_1,p_2)} = -3\,p_1\,\lambda_1 + (p_1 + 2\,p_2)\,\lambda_2 \tag{44}$$

Using the implicit definitions in (42), one finds

$$\begin{aligned}
\mu &\equiv \frac{\partial G[T, W]}{\partial T} \\
&= \frac{\partial G}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial T} = \frac{1}{2}\frac{\left(3\,\lambda_1{}^2 - 6\,\lambda_1\,\lambda_2 - \lambda_2{}^2\right)p_1}{\lambda_2\,(3\,\lambda_1 - \lambda_2)\,(3\,\lambda_1 + \lambda_2)} + \frac{\left(3\,\lambda_1{}^2 - \lambda_2{}^2\right)p_2}{\lambda_2\,(3\,\lambda_1 - \lambda_2)\,(3\,\lambda_1 + \lambda_2)} \\
\nu &\equiv \frac{\partial G[T, W]}{\partial W} \\
&= \frac{\partial G}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial W} = \frac{3}{2}\frac{(\lambda_1 + \lambda_2)\,p_1}{\lambda_2\,(3\,\lambda_1 - \lambda_2)\,(3\,\lambda_1 + \lambda_2)} + \frac{3\lambda_1\,p_2}{\lambda_2\,(3\,\lambda_1 - \lambda_2)\,(3\,\lambda_1 + \lambda_2)}
\end{aligned} \tag{45}$$

---

[4]A similar analysis applies to the second $SL(3)$ factor, where we start with $b_\phi = \begin{pmatrix} 0 & 1 & 0 \\ \bar{T}/2 & 0 & 1 \\ \bar{W} & \bar{T}/2 & 0 \end{pmatrix}$.

[5]As discussed in [10], it is convenient to introduce chemical potentials in the time component of the fields. In this way the time evolution generates an 'allowed gauge transformation'.

in agreement with (43). In fact, if (45) are satisfied, then the eigenvalues of a_t are

$$\text{Eig}(a_t) = 2\pi i\, (p_1, p_2, -p_1 - p_2) \tag{46}$$

and hence the holonomy condition (12) holds.

Analogously, let $F[\mu, \nu]_{(p_1, p_2)}$ be the following implicit function of the chemical potentials,

$$F[\mu, \nu]_{(p_1, p_2)} = \frac{1}{2} \frac{\left(3\lambda_1^4 + 30\lambda_1^3\lambda_2 - 6\lambda_1\lambda_2^3 + \lambda_2^4 - 12\lambda_1^2\lambda_2^2\right) p_1}{\lambda_2\, (3\lambda_1 - \lambda_2)\,(3\lambda_1 + \lambda_2)} \tag{47}$$

$$+ \frac{\left(3\lambda_1^4 + \lambda_2^4 - 12\lambda_1^2\lambda_2^2\right) p_2}{\lambda_2\, (3\lambda_1 - \lambda_2)\,(3\lambda_1 + \lambda_2)} \tag{48}$$

where it is now understood to use equations (43) to implicitly define the eigenvalues in terms of the chemical potentials, i.e., $\lambda_1(\mu, \nu), \lambda_2(\mu, \nu)$. It then follows from (42)

$$
\begin{aligned}
T &= \frac{\partial F[\mu, \nu]}{\partial \mu} = \frac{\partial F}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial \mu} = 3\lambda_1^2 + \lambda_2^2 \\
W &= \frac{\partial F[\mu, \nu]}{\partial \nu} = \frac{\partial F}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial \nu} = -2\lambda_1\,(\lambda_1 - \lambda_2)\,(\lambda_1 + \lambda_2)
\end{aligned} \tag{49}
$$

in agreement with (42). It goes without saying that $F[\mu, \nu]$ and $G[T, W]$ are, respectively, the canonical and micro-canonical actions that can be computed with the methods discussed in [6]. Here our goal is to study properties of the solutions and we refer the interested reader to this reference for more information on how to derive $F[\mu, \nu]$ and $G[T, W]$ from the Chern-Simons action.

If we impose in addition to the holonomy condition that the solution has a BTZ limit in which $W \to 0$ as $\nu \to 0$, then we see from (45) and (49) that $\nu = W = 0$ leads to

$$0 = \lambda_1\,(\lambda_1 - \lambda_2)\,(\lambda_1 + \lambda_2) \quad , \quad 0 = (\lambda_1 + \lambda_2)\,p_1 + 2\lambda_1\,p_2 \tag{50}$$

This only possesses solutions with one unconstrained charge or, equivalently, one unconstrained λ_i eigenvalue, if one of the integer eigenvalues of a_t, p_1, p_2 or $p_1 + p_2$, is zero.

5   A simple exact regular solution

We now turn to the construction of a simple exact regular solution which has a BTZ limit. As stated in section 4, such solutions must belongs to the class of a_t matrices with at least one eigenvalue equal to zero. In particular, we will use the class with $p_1 = 1, p_2 = 0$, i.e., with eigenvalues $(2\pi, 0, -2\pi)$. This is the class which has mostly been considered in the literature, starting from [11]. Our exact solution will be constructed using what we will call the 'minimal prescription', which is given by

$$U = e^{rX_1} \tag{51}$$

where X_1 is the matrix (30), partially determined by the single-valuedness conditions on the solid torus. It is 'minimal' in the sense that the group element is determined by the

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same algebra element X_1 at all orders in r. As explained in Section 3.3, we need to set $x_1 = x_5 = 0$. Then, for the particular values $p_1 = 1, p_2 = 0$, besides the branches shown in (37) and (39) which are valid for generic p_i's, [6], we can choose $x_3 = x_7 = 0$. This solves the equations (36) and gives

$$\frac{g_{tt}}{g_{rr}} = \frac{g_{tt\phi}}{g_{rr\phi}} = r^2 + \mathcal{O}(r^3) \tag{52}$$

There are no further conditions from regularity at the horizon and therefore the parameters x_2, x_4, x_6, x_8 are left free. However, if we then construct the solution using (51), we find that these parameters enter the metric fields only through the two products $x_2 x_4$ and $x_6 x_8$ and that the metric fields are of the general form

$$\begin{aligned}
ds^2 &= g_{rr}\, dr^2 + g_{tt}\, dt^2 + g_{t\phi}\, dt\, d\phi + g_{\phi\phi}\, d\phi^2 \\
ds^3 &= d\phi \times (g_{rr\phi}\, dr^2 + g_{tt\phi}\, dt^2 + g_{t\phi\phi}\, dt\, d\phi + g_{\phi\phi\phi}\, d\phi^2)
\end{aligned} \tag{53}$$

All other components vanish. Furthermore, the component g_{rr} is always a constant[7] and we can achieve $g_{rr} = 1$ either by rescaling r or by the choice

$$x_6\, x_8 = 4 - x_2\, x_4 \tag{54}$$

If we then demand that the metric fields are described by four independent charges and, at the same time, the existence of a BTZ limit in ds^2, we need to set $x_2\, x_4 = 2$.

This being done, the non-vanishing components of the metric are[8]

$$\begin{aligned}
g_{rr} &= 1 \\
g_{tt} &= \sinh^2(r) \\
g_{t\phi} &= \frac{i}{2}\, (-\rho_2 + 3\,\rho_1 - \lambda_2 + 3\,\lambda_1)\sinh^2(r) \\
g_{\phi\phi} &= \frac{1}{4}\, (3\,\rho_1 - \rho_2)\,(\lambda_2 - 3\,\lambda_1)\sinh^2(r) - \frac{9}{16}\, (\rho_1 + \rho_2)\,(\lambda_1 + \lambda_2)\sinh^2(2\,r) \\
&\quad + \frac{1}{4}\, (\lambda_2 - \rho_2)^2 + \frac{3}{4}\, (\lambda_1 - \rho_1)^2
\end{aligned} \tag{55}$$

---

[6]On the branches (37) and (39) the minimal description does not lead to field configurations with four independent parameters. A non-minimal description might, but a systematic analysis is involved and beyond the scope of this note.

[7]It is worth mentioning that with the minimal prescription the exact form of the metric fields is of the form (53), independent of the branch.

[8]The factors of i in the components with an odd power of dt are due to our use of Euclidean signature.

while the spin-3 field is

$$
\begin{aligned}
g_{rr\phi} =& \frac{1}{8}\,(\lambda_1 + \lambda_2 - \rho_1 - \rho_2)\\
g_{tt\phi} =& \frac{1}{32}\,(\lambda_1 + \lambda_2 - \rho_1 - \rho_2)\left(3\,\sinh^2(2\,r) - 8\,\sinh^2(r)\right)\\
g_{t\phi\phi} =& \frac{i}{8}\,(\lambda_1 + \lambda_2 - \rho_1 - \rho_2)\,(\rho_2 - 3\,\rho_1 + \lambda_2 - 3\,\lambda_1)\sinh^2(r)\\
& - \frac{3\,i}{8}\,(\lambda_1\,\rho_2 - \lambda_2\,\rho_1)\sinh^2(2\,r)\\
g_{\phi\phi\phi} =& \frac{3}{64}\left(\lambda_2\,\rho_2^2 + 3\,\lambda_1^2\rho_2 - 3\,\lambda_1\,\rho_1^2 - \lambda_2^2\rho_2 - \lambda_2^2\rho_1 + 3\,\lambda_1^2\rho_1 + \lambda_1\rho_2^2\right.\\
& \left. + 6\,\lambda_2\,\rho_1\,\rho_2 - 6\,\lambda_1\,\lambda_2\,\rho_2 - 6\,\lambda_1\,\lambda_2\,\rho_1 + 6\,\lambda_1\,\rho_1\,\rho_2 - 3\,\lambda_2\,\rho_1{}^2\right)\sinh^2(2\,r)\\
& - \frac{1}{16}\,(3\,\rho_1 - \rho_2)\,(\lambda_2 - 3\,\lambda_1)\,(\lambda_1 + \lambda_2 - \rho_1 - \rho_2)\sinh^2(r)\\
& - \frac{1}{8}\,(\lambda_1 - \rho_1)\,(-\lambda_2 + \lambda_1 - \rho_1 + \rho_2)\,(\lambda_1 + \lambda_2 - \rho_1 - \rho_2)
\end{aligned}
\tag{56}
$$

In the BTZ limit, where $W_1 \to 0$ as $\nu_1 \to 0$ and $W_2 \to 0$ as $\nu_2 \to 0$, the spin-3 field is zero. As discussed at the end of section 4, for this particular case, this happens when $\lambda_1 \to -\lambda_2$ and $\rho_1 \to -\rho_2$ at the same time.

The above solution is written in a proper radial coordinate such that $g_{rr} = 1$. Often the use of Schwarzschild-like coordinates $r \to \ln(r)$, where the horizon is at $r = 1$, is more convenient. In these coordinates the asymptotic expansion (as $r \to \infty$) is

$$
\begin{aligned}
g_{rr} =& \frac{1}{r^2}\\
g_{tt} =& \frac{1}{4}r^2 + \mathcal{O}(1)\\
g_{t\phi} =& \frac{1}{8}\,i\,(-\rho_2 + 3\,\rho_1 - \lambda_2 + 3\,\lambda_1)\,r^2 + \mathcal{O}(1)\\
g_{\phi\phi} =& -\frac{9}{64}\,(\rho_1 + \rho_2)\,(\lambda_1 + \lambda_2)\,r^4 + \mathcal{O}(r^2)
\end{aligned}
\tag{57}
$$

At large r the leading term $(\sim r^4)$ of $g_{\phi\phi}$ vanishes in the BTZ limit, i.e. these solutions are not small deformations of the BTZ black hole or, in other words, the limit is not smooth. This is a general feature and will be discussed in the next section.

6   A "no go theorem": asymptotics vs regularity

So far we have concentrated on the near horizon properties, and we have imposed conditions on the group element U to achieve regularity for a wide range of possibilities. A natural question which now arises is whether the group element U can be extended across the manifold, maintaining stationary and circularly symmetric fields such that the solution is asymptotically AdS? The answer is no, as we shall show.

By asymptotically AdS we mean solutions where the asymptotic form of the metric (in Schwarzschild-like coordinates) is

$$ds^2 = ds^2_{\text{AdS}} + \frac{1}{r}(\text{corrections}) \tag{58}$$

Most importantly, the AdS radius of ds^2_{AdS} must not depend on the charges. This last requirement cannot be fulfilled when the spin-3 charge is active.

It is important to mention that this problem only arises when we impose the asymptotic condition at infinity and, at the same time, require regularity in the interior. Asymptotically AdS solutions which carry all charges can of course be constructed (see [3, 12, 7]). The trouble arises when, in addition, we impose regularity at the horizon. If one does so, the spin-3 source appears at the leading term in the metric and the value of the cosmological constants jumps when this source is turned on/off. This phenomenon was first observed by Gutperle and Kraus in [11], for a particular choice of the r-dependent group element. In [11] this was interpreted holographically as the effect of turning on an irrelevant operator in the UV theory. We prove here that there exists no choice of group element such that the cosmological constant is stable[9]. Again we will restrict to $N = 3$.

A distinguished family of solutions to (10) are the (anti-) chiral fields

$$a_t = a_\phi \qquad \text{and} \qquad b_t = -b_\phi \tag{59}$$

where a_ϕ and b_ϕ as in Section 4 and with the group element

$$g_1 = g_2^{-1} = \begin{pmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{r} \end{pmatrix} \tag{60}$$

These solutions belong to the principal embedding $sl(2) \hookrightarrow sl(3)$ and carry all desired W_3 features.

The spacetime connections, built as in (9), have the form

$$
A_\phi = \begin{pmatrix} 0 & T/(2r) & W/r^2 \\ r & 0 & T/(2r) \\ 0 & r & 0 \end{pmatrix} = A_t \tag{61}
$$

$$
= \text{background} + \text{fluctuations (subleading as } r \to \infty) \tag{62}
$$

They have the desired structure: a charge-independent background + subleading terms carrying the charges. This solution cannot be extended, however, all the way to the horizon.

Regularity at the center of the solid torus requires the holonomy conditions to be satisfied and thus the sources, chemical potentials, have to be turned on. Given a_ϕ, the general solution to (10) is a_t as given in (41).

When $\nu \neq 0$, the matrix a_t in (41) is no longer proportional to a_ϕ. In particular, it has non-zero components in all of its

---

[9]This problem does not arise if one works in the so called diagonal embedding where no spin-3 charge is present.

entries[10]

$$a_t \simeq \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \mu & a_{22} & a_{23} \\ \nu & \mu & a_{33} \end{pmatrix} \tag{63}$$

Here, we have explicitly displayed the values for the lower left corner because they are the relevant ones for our analysis, and also the simplest ones. All other coefficients a_{ij} are functions of T, W, μ and ν.

Given a_t in (63), the spacetime field A_t, using the group element displayed in (60), is

$$A_t = \begin{pmatrix} a_{11} & a_{12}/r & a_{13}/r^2 \\ \mu\,r & a_{22} & a_{23}/r \\ \nu\,r^2 & \mu\,r & a_{33} \end{pmatrix} \tag{64}$$

For large r, the components in the upper triangle become 'fluctuations'. The components in the lower triangle grow with r. We clearly see that the spin-2 source μ appears at order r while the spin-3 source ν appears at order r^2. This is the effect observed in [11].

If one turns ν on/off, the asymptotics of the metric changes and the cosmological constant jumps by a factor of four, i.e. the radius of the asymptotic AdS_3 decreases by a factor of two.

A natural question is whether the choice (60) is mandatory. Perhaps a more general group element can avoid the term $\nu\,r^2$ in the lower corner? The question we ask is whether one can build solutions that are simultaneously regular at the horizon and which have the desired asymptotics. This is not the case, as we now prove.

Consider a general group element $g(r) \in SL(3)$

$$g = \begin{pmatrix} \alpha_1(r) & \alpha_2(r) & \alpha_3(r) \\ \alpha_4(r) & \alpha_5(r) & \alpha_6(r) \\ \alpha_7(r) & \alpha_8(r) & \alpha_9(r) \end{pmatrix} \tag{65}$$

(One of these functions is fixed by the requirement $\det(g_1) = 1$, but we shall not need to impose this explicitly.) We write again a general solution a_t as in (41).

Given $g(r)$, A_t, A_ϕ are uniquely defined by (9). It is useful to give names to some of their components,

$$A_\phi = \begin{pmatrix} \star & \star & \star \\ f_2(r) & X(r) & \star \\ o_2(r) & f_1(r) & \star \end{pmatrix} \qquad A_t = \begin{pmatrix} \star & \star & \star \\ f_4(r) & \star & \star \\ o_1(r) & f_3(r) & \star \end{pmatrix} \tag{66}$$

The symbol $\star$ represents functions which do not enter the argument. We would like to have solutions approaching (61) at infinity, thus we need to find functions $\alpha_i(r)$ such that, asymptotically,

$$o_1(r), o_2(r), X(r) \to 0 \tag{67}$$

together with

$$f_1(r), f_2(r) \to r \tag{68}$$

---

[10]here we use Lorentzian signature, hence there are no factors of i.

It turns out that conditions (67) and (68) are inconsistent.

To show this we invert the problem and write the functions $\alpha_i(r)$ in the group element in terms of $f_1, f_2, f_3, f_4, o_1, o_2$ and X. Although long, it is easy to show that $\alpha_2(r), \alpha_5(r), \alpha_6(r), \alpha_8(r), \alpha_9(r)$ can all be expressed as functions of $f_1(r), f_2(r), f_3(r), o_1(r), o_2(r)$. The explicit formulae are emposing and not useful, so we don't display them here. The important point is that the following relation follows,

$$\nu\Big(f_1(r)f_2(r) - o_2(r)X(r)\Big) + \mu\, o_2(r) - o_1(r) = 0 \tag{69}$$

In other words, no matter what the group element $g(r)$ is, the components of A_ϕ, A_t are related by (69).

The obstruction is now clear. Conditions (67) demand o_1, o_2, X to vanish while (68) demand f_1, f_2 to diverge. But this is inconsistent with (69). If the spin-3 source is set to zero, then $\nu = 0$ and o_1 and o_2 can be set to zero.

In conclusion, for generic values of the charges T and W, regularity of the horizon requires $\nu \neq 0$, and this is inconsistent with a field asymptotically AdS of the form (61).

7   Acknowledgements

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